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GAUSS AND GAUSS-JORDAN ELIMINATION METHODS FOR SOLVING SYSTEM OF LINEAR EQUATIONS: COMPARISONS AND APPLICATIONS

Adenegan, Kehinde Emmanuel[1]* and Aluko, Tope Moses [1]

[1]Department of Mathematics, Adeyemi College of Education, Ondo.

Abstract: *This paper examines the comparisons between the Gauss and Gauss-Jordan methods for solving system of linear equations. Various terminologies were discussed, some problems were considered in conjunction with the afore-mentioned methods used in solving system of linear equations. It was noted very remarkably from the solved problems that both Gaussian and Gauss-Jordan methods gave the same answers. Equally, when the same system is pivoted or partially pivoted, same answers are readily obtained. This necessarily implies that, since the same system of linear equations is rearranged leading to its matrix form to be transformed as the rows' element obviously changed, the resultant solutions are still the same. The paper also explicitly reveals that the Gauss/Gaussian and Gauss-Jordan elimination methods could be applied to different systems of linear equations arising in fields of study like Physics, Business, Economics, Chemistry, etc.*

Keywords: Matrix coefficient, augmented matrix, Echelon, Upper triangular matrix, undetermined system, Degenerated equation.

INTRODUCTION

In linear algebra, Gaussian elimination is an algorithm for solving system of linear equations, finding the rank of a matrix and calculating the inverse of an invertible square matrix. Gaussian elimination is considered as the workhorse of computational science for the solution of a system of linear equations. Carl Friedrich Gauss, a great 19th century mathematician suggested this elimination method as a part of his proof of a particular theorem. Computational scientists use this "proof" as a direct computational method.

Gaussian elimination is a systematic application of elementary row operations to a system of linear equations in order to convert the system to upper triangular form. Once the coefficient matrix is in upper triangular form, we use back substitution to find a solution. Gaussian elimination places zeros below each pivot in the matrix starting with the top row and working downwards. Matrices containing zeros below each pivot are said to be in row echelon form.

The process of Gaussian elimination has two parts. The first part (forward elimination) reduces a given system to either triangular or echelon form or results in a *degenerated equation* with no solution, indicating the system has no solution. This is accomplished through the use of elementary row operations. The second step uses back substitution to find the solution of system of linear equation. Another point of view which turns out to be very useful to analyze the algorithm is that Gaussian elimination computes matrix decomposition. The three elementary row operation used in the Gaussian elimination (multiplying rows, switching rows and adding multiples of rows to other rows) amount to multiplying the original matrix with invertible matrix from the left. The first part of algorithm computes LU *decomposition* (decomposition into a lower and upper triangular matrix), while the

*Corresponding Author. E-mail:akehinde2012@yahoo.com

second part writes the original matrix as the product of a uniquely determined invertible matrix and a uniquely determined reduced row echelon matrix.

The Gauss – Jordan method is a modification of the Gaussian elimination. It is named after Carl Friedrich Gauss and Wilhelm Jordan because it is a variation of Gaussian elimination as Jordan described in 1887 while Gaussian elimination places zeros below each pivot in the matrix starting with the top row and working downwards, Gauss – Jordan elimination method goes a step further by placing zeroes above and below each pivot. Every matrix has a reduced row echelon form and Gauss – Jordan elimination is guaranteed to find it.

The paper aims at investigating the methods of solving system of linear equations using Gauss and Gauss – Jordan elimination methods, compare and contrast the two methods and at finding out application of the methods to other fields of study.

LITERATURE REVIEW

It is not surprising that the beginning of matrices and determinants should arise through the study of linear systems. The Babylonians studied problems that led to simultaneous linear equations and some of these are preserved in clay tablet that survive. The Chinese between 200BC and 100BC came much closer to matrices than the Babylonians. Indeed, it is fair to say that the text nine chapters on the Mathematics Art written during the Han Dynasty give the first known example of matrix methods.

Cardan, in Art Magna (1545), gives a rule for solving a system of two linear equations which he called regular de modo and which is called mother of rules. This rule gives what essentially is Crammer's rule for solving a 2 x 2 system. The idea of a determinant appeared in Japan and Europe at almost exactly same time although Seki in Japan certainly published first. In 1683, Seki wrote *method of solving the dissimulated problem* that contains matrix methods written as tables in exactly the Chinese methods describe above were constructed. Without having any word that corresponds to 'determinant', Seki still introduced determinants and gave general methods for calculating them based on examples. Using his 'determinant', Seki was able to find determinants of 2 x 2, 3 x 3, 4 x 4 and 5 x 5 matrices and applied them to solve equations but not system of linear equation.

In the 1730's, Maclaurin wrote Treatise of Algebra although it was not published until 1748, two years after his death. It contains the first published results on determinant proving Crammer's rule for 2 x 2 and 3 x 3 systems and indicating how the 4 x 4 case would work. Crammer gave the general rule for *nxn* systems in a paper introduction to the analysis of algebraic curves (1750). It arose out of a desire to find the equation of a plane curve passing through a number of given points.

In 1764, Bezout gave methods of calculating determinants, as did Vandermonde in 1771. In 1772, Laplace claimed that the method introduced by Crammer and Bezout were impractical and in a paper where he studied the orbits of the inner planets, he discussed the solution of system of linear equation without actually calculating it by using determinants. Rather surprising Laplace used the word 'resultant' for what we now call the determinant. Surprisingly since it is the same word as used by Leibniz yet Laplace must have been unaware of Leibniz's work. Laplace gave the expansion of a determinant that is now named after him.

Jacques strum gave a generalization of eigen value problem in the context of solving system of ordinary differential equations. In fact the concept of an eigen value appeared 80 years earlier again in work on systems if linear differential equations by O' Alembert studying the motion of a string with mass attached to it at various points.

The first to use term 'matrix' was Sylvester in 1850. Sylvester defined a matrix to be an oblong arrangement of terms and saw it as something that led to various determinants from square arrays contained within it. After living America and returning to England in 1851, Sylvester became a Lawyer and met Cayley, a fellow lawyer who shared his interest in mathematics. Cayley quickly saw the significance of matrix concept and by 1853 Cayley had published a note giving for the first time, the inverse of matrix.

Frobenius, in 1878, wrote an important work on matrices on linear substitutions and linear forms although he seemed unaware of Cayley's work. Frobenius in his paper dealt with coefficient of forms and does not use the term matrix. However, he proved important results on canonical matrices as representatives of equivalence classes of matrices. He cites Kronecker and Weierstrass as having considered special cases of his results in 1874 and 1868 respectively. Frobenius also proved the general result that a matrix satisfies its characteristic equation. This 1878 paper by Frobenius also contained the definition of rank of a matrix that he used in his work on canonical forms and the definition of orthogonal matrices.

The method of Gaussian elimination appears in chapter eight, rectangular Array of the Chinese mathematics text, Jiuzhang Suanshu of the nine chapters on the Mathematical Art. Its use is illustrated in eighteen problems with two to five equations. The first reference to the book by this title is dated to 179BC but parts of it were written as approximately 150BC. It was commenced on by Liu Hui in the 3rd century. The method of Europe stems from the notes of Isaac Newton. In 1670, he wrote that all algebra books known to him lacked a lesson for solving simultaneously equation which Newton then supplied.

Cambridge University eventually published the notes as Arithmetical University in 1707 long after Newton left academic life. The notes were widely initiated, which made (what is now called) Gaussian elimination a standard lesson in algebra textbooks by the end of the 18th century. Carl Frederick Gauss in 1810 devised a notation for symmetric elimination that was adopted in the 19th century by professional hand computers to solve the normal equations of least squares problems.

Gauss developed Gaussian elimination around 1880 and used it to solve least squares problems in celestial computations and later in computation to measure the earth and its surface (the branch of applied mathematics) concerned with measuring or determining the shape of the earth or with locating exactly points on the earth's surface is called geodesy. Even though, Gauss's name is associated with this technique for successively eliminating variables from systems of linear equations. For years, Gaussian elimination was considered part of the development of the geodesy, not mathematics. The first appearance of Gauss-Jordan elimination in print was in handbook or geodesy written by Wilhelm-Jordan.

In matrix analysis and linear algebra, Carl D.Meyor (2000), writes "Although there has been some confusion as to which Jordan should receive credit for this algorithm, it is seems clear that the method was in fact introduced by a geodesist named Wilhelm Jordan (1842-1899) and not by the more well known mathematician Marie Ennemond Camille Jordan (1838-1992), whose name is often mistakenly associated with the technique, but who is otherwise correctly credited with other important topic in matrix analysis, the Jordan Canonical form being the most notable". Having identified the right Jordan is not the end of the problem, for A.S. Household writes, in the theory of matrices in numerical analysis (1964, P. 141) "The Gauss-Jordan method, so called seems to have been described first by Clasen (1888) since it can be regarded as a modification of Gaussian elimination, the name Gauss is properly applied, but that of Jordan seems to be due to an error, since the method was described only in the third edition of his Hanbuch der Vermes Sungskunde, prepared after his death". These claims were examined by S.C. Althoen and R. Mcluaghlin (1987). American Mathematical monthly, 94, 130-142. They concluded that Household was correct about Clasen and his 1888 publication but mistaken about Jordan who was very much alive when the third edition of his book appeared in 1888. They added that the "germ of the idea" was already present in the second edition of 1877 (This entry was contributed by John Aldrich).

The process of Gaussian elimination has two parts. The first part (forward elimination) reduces a given system to either triangular or echelon form, or results in a degenerated equation with no solution, indicating the system has no solution. This is accomplished through the use of elementary row operations. The second step uses back substitution to find the solution of system of linear equation.

Stated equivalently for matrices, the first part reduces a matrix to row echelon form using elementary row operations while the second reduces it to reduced row echelon form. Another point of view which turns out to be very useful to analyze the algorithm is that Gaussian elimination computes matrix decomposition. The 3 elementary row operations used in the Gaussian elimination (multiplying rows, switching rows and adding multiples of rows to other rows) amount to multiplying matrix with invertible matrix from the left.

THE GAUSS AND GAUSS-JORDAN ELIMINATION METHODS OF SOLVING SYSTEM OF LINEAR EQUATIONS

A system of linear equations (or linear system) is a collection of linear equation involving the same set of variables. A solution of a linear system is an assignment of values to the variable $x_1, x_2, x_3, ..., x_n$ equivalently, $\{x_i\}^n \; i=1$ such that each of the equation is satisfied. The set of all possible solutions is called the solution set. A linear system solution may behave in any one of three possible ways:

1. The system has a infinitely many solutions
2. The system has a single unique solution
3. The system has no solution

For three variables, each linear equation determines a plane in three dimensional space and the solution set is the intersection of these planes. Thus, the solution set may be a line, a single point or the empty set.

For variables, each linear equation determines a hyperplane in n-dimensional space. The solution set is the intersection of these hyper-planes, which may be a flat of any dimension. The solution set for two equations in three variables is usually a line.

In general, the bahaviour of a linear system is determined by the relationship between the number of equations and the number of unknowns. Usually, a system with fewer equations than unknowns has infinitely many solutions such system is also known as an **undetermined system.** A system with the same number of equation and unknowns has a single unique solution. A system with more equations than unknowns has no solution.There are many methods of solving linear system which include: Substitution methods, elimination methods, matrix inversion methods, graphical methods, crammer's methods, Gaussian elimination methods, Gauss-Jordan elimination methods etc.In this paper, the Gauss and Gauss-Jordan elimination methods shall be considered.

GAUSSIAN/GAUSS ELIMINATION METHOD

Gaussian elimination is a systematic application of elementary row operations to a system of linear equations in order to convert the system to upper triangular form. Once the coefficient matrix is in upper triangular form, we use back substitution to find a solution. The general procedure for Gaussian elimination can be summarized in the following steps:

- Write the augmented matrix for the system of linear equation.
- Use elementary operation on {A/b} to transform A into upper triangular form. If a zero is located on the diagonal, switch the rows until a non zero is in that place. If you are unable to do so, stop; the system has either infinite or no solution.
- Use back substitution to find the solution of the problem

Consider the system of equation in matrix form below.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Step 1: Write the above as augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

Step 2: Eliminate x_1 from the 2nd and 3rd equations by subtracting multiples $m_{21} = \frac{a_{21}}{a_{11}} \; and \; m_{31} = \frac{a_{31}}{a_{11}}$ of row 1 from rows 2 and 3 producing equivalent system:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{array}\right)$$

Step 3: Eliminate x_2 from 3rd equation by subtracting multiples $m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}}$ of row 2 from row 3 producing matrix system:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{array}\right)$$

For an illustration, using the steps above, solve the system

$$x_1 + 2x_2 - 3x_3 = 3$$
$$2x_1 - x_2 - x_3 = 11$$
$$3x_1 + 2x_2 + x_3 = -5$$

This can be written as,

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

The augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$$

Now subtract $\frac{2}{1}$ times the first row from the second row and times $\frac{3}{1}$ the first row from the third row which gives;

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{array}\right)$$

Now subtract $\frac{-4}{-5}$, i.e. $\frac{4}{5}$ times the second row from the third row. Then we have

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & 0 & 6 & -18 \end{array}\right)$$

Note that as a result of these steps, the matrix of coefficient of x has been reduced to a triangular matrix.

Finally, we detach the right-hand column back to its original position:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -18 \end{pmatrix}$$

Then by 'back substitution', starting from the bottom row we get:

$$x_1 = 2; x_2 = -4; x_3 = -3$$

The same example and others can also be considered using the Gaussian elimination method without pivoting (as directly solved above) and with partial pivoting by considering the magnitude of the elements in the first column i.e. $|3| > |2| > |1|$, changing their row's position according to magnitude, we will still obtain the same answer.

GAUSS-JORDAN ELIMINATION

Gauss-Jordan elimination is a modification of Gaussian elimination. Again we are transforming the coefficient matrix into another matrix that is much easier to solve and the system represented by the new augmented matrix has the same solution set as the original system of linear equations. In Gauss-Jordan elimination, the goal is transform the coefficient matrix into a diagonal matrix and the zeros are introduced into the matrix one column at a time. We work to eliminate the elements both above and below the diagonal elements of a given column in one passes through the matrix. The general procedure for Gauss-Jordan elimination can be summarized in the following steps:

i. Write the augmented matrix for a system of linear equation
ii. Use elementary row operation on the augmented matrix [A/b] to the transform A into diagonal form. If a zero is located on the diagonal. Switch the rows until a non-zero is in that place. If you are unable to do so, stop; the system has either infinite or no solution.
iii. By dividing the diagonal elements and a right-hand side's element in each row by the diagonal elements in the row, make each diagonal elements equal to one.

Given a system of equation 4 x 4 matrix of the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Step 1: Write the above as augmented matrix, we have

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right)$$

Step 2: Eliminate x_1 from the 2nd and 3rd and 4th equation by subtracting multiples $m_{21} = \frac{a_{21}}{a_{11}}$, $m_{31} = \frac{a_{31}}{a_{11}}$ and $m_{41} = \frac{a_{41}}{a_{11}}$ of row 1, from 2,3 and 4 respectively producing

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ 0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)} \end{array}\right)$$

Step 3: eliminate x_2 from the 1st, 3rd and 4th equation by subtracting multiples

$$m_{12} = \frac{a_{12}}{a_{22}^{(2)}}, m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}}, m_{42} = \frac{a_{42}^{(2)}}{a_{22}^{(2)}}$$ of row 2 from row 1, 3 and 4 to produce

$$\left(\begin{array}{cccc|c} a_{11}^{(3)} & 0 & a_{13}^{(3)} & a_{14}^{(3)} & b_1^{(3)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & 0 & a_{43}^{(3)} & a_{44}^{(3)} & b_4^{(3)} \end{array}\right)$$

Step 4: Eliminate x_3 from the 1st, 2nd and 4th equations by subtracting multiples $m_{13} = \frac{a_{13}^{(3)}}{a_{33}^{(3)}}, m_{23} = \frac{a_{23}^{(3)}}{a_{33}^{(3)}}, m_{43} = \frac{a_{43}^{(3)}}{a_{33}^{(3)}}$ of row 3 from row 1, 2 and 4 producing

$$\left(\begin{array}{cccc|c} a_{11}^{(4)} & 0 & 0 & a_{14}^{(4)} & b_1^{(4)} \\ 0 & a_{22}^{(4)} & 0 & a_{24}^{(4)} & b_2^{(4)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & 0 & 0 & a_{44}^{(4)} & b_4^{(4)} \end{array}\right)$$

From which we finally solve for x_1, x_2, x_3 and x_4 from the resulting simultaneous equations from above.

For the purpose of comparison, let us solve the problem given in (3.1) using Gauss-Jordan elimination method. i.e.

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

The augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$$

We now eliminate x_1 from the 2nd and 3rd equation by subtracting multiple $m_{21} = \frac{a_{21}}{a_{11}} = 2, m_{31} = \frac{a_{31}}{a_{11}} = 3$ of row 1 from row 2 and 3 respectively producing.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{array}\right)$$

Eliminate x_2 from 1st and 3rd equation by subtracting multiple $m_{12} = \frac{a_{12}}{a_{22}} = \frac{-2}{5}$, *and* $m_{32} = \frac{a_{32}}{a_{22}} = \frac{4}{5}$ of row 2 from row 1 and row 3

We have

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & -5 & 5 & 5 \\ 0 & 0 & 6 & -18 \end{array}\right)$$

Eliminate x_3 from 1st and 2nd equation by subtracting multiple $m_{13} = \frac{a_{13}}{a_{33}} = \frac{-1}{6}$, *and* $m_{23} = \frac{a_{23}}{a_{33}} = \frac{5}{6}$ of row 3 from row 1 and row 3.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -5 & 0 & 20 \\ 0 & 0 & 6 & -18 \end{array}\right)$$

This finally gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$$

For more illustration, consider the following system of equations using Gauss-Jordan Elimination (i) without pivoting (ii) with partial pivoting

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 2 & 4 & 3 & 5 \\ 3 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \\ -1 \end{pmatrix}$$

We represent the above in an augmented matrix form (i) without pivoting

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -2 \\ 2 & 4 & 3 & 5 & -2 \\ 3 & 1 & 1 & 1 & -1 \end{array}\right)$$

We now eliminate x_1 from the 2nd, 3rd, and 4th equation by subtracting multiples

$m_{21} = \dfrac{a_{21}}{a_{11}} = -1, m_{31} = \dfrac{a_{31}}{a_{11}} = 2$ and $, m_{41} = \dfrac{a_{41}}{a_{11}} = 3$ of row 1 from row 2, 3 and 4 respectively producing

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 & -1 \\ 0 & 6 & 1 & 7 & -4 \\ 0 & 4 & -2 & 4 & -4 \end{array}\right)$$

Eliminate x_2 from equations (1), (3) and (4) by subtracting multiples

$m_{12} = \dfrac{a_{12}}{a_{22}^{(2)}} = \dfrac{1}{2}, m_{32} = \dfrac{a_{32}}{a_{22}^{(2)}} \dfrac{6}{-2} = -3$ and $, m_{42} = \dfrac{a_{42}^{(2)}}{a_{22}^{(2)}} = \dfrac{4}{-2} = -2$ of row 2 from row 1, 3 and 4 respectively producing

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3/2 \\ 0 & -2 & 2 & 0 & -1 \\ 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 2 & 4 & -6 \end{array}\right)$$

Eliminate x_3 from equation (1), (2) and (4) by subtracting multiples

$m_{13} = \dfrac{a_{13}}{a_{33}} = \dfrac{0}{7} = 0, m_{23} = \dfrac{a_{23}}{a_{33}} = \dfrac{2}{7}$ and $, m_{43} = \dfrac{a_{43}}{a_{33}} = \dfrac{2}{7}$ of row 3 from row 1, 2 and 4 to produce

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3/2 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & 2 & -4 \end{array}\right)$$

Eliminate x_4 from equation (1), (2) and (3) and by subtracting multiples

$m_{14} = \dfrac{a_{14}}{a_{44}} = -\dfrac{1}{2}, m_{24} = \dfrac{a_{24}}{a_{44}} = -1$ and $, m_{34} = \dfrac{a_{34}}{a_{44}} = \dfrac{7}{2}$ of row 4 from row 1, 2 and 3 producing

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & -2 & 0 & 0 & -3 \\ 0 & 0 & 7 & 0 & 7 \\ 0 & 0 & 0 & 2 & -4 \end{array}\right)$$

$$\begin{array}{l} R_2 = -1/2\, R_2 \\ R_3 = 1/7\, R_3 \\ R_4 = 1/2\, R_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & -3/-2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\therefore x_1 = -1/2, x_2 = 3/2, x_3 = 1 \text{ and } x_4 = -2$$

Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \\ -2 \end{pmatrix}$$

(ii) With partial pivoting

Since $|3| > |2| > |1| > |-1|$, then we have the same answer following the same process.

APPLICATIONS OF GAUSS AND GAUSS-JORDAN METHODS OF SOLVING SYSTEM OF LINEAR EQUATIONS

Gauss and Gauss-Jordan methods of solving system of linear equations can be used in various subject areas and are used to solve for unknowns. With *n* equations, the highest possible number of unknowns that can be solved for is *n*. complications occur when the number of unknowns is unequal to the number of equations in the system.

Linear system of equations with more or less equations than unknowns occurs quite often. When there are fewer equations than unknowns, it is impossible solve for all the unknowns algebraically. Because of this, there usually infinite many solutions for each variable (it is hard to narrow it down). When there are more equations than unknowns (an over determined system), there are usually no solutions. This occurs because there may be solution that satisfy two of the three equations, but not necessarily all three. The only way there will be solution to an over determined system (for example, one with three equations and two unknowns) would be if all the equations are equivalent, if two of the three equations are equivalent and the other equation intersects each other at a single point.

Some of the areas where Gauss and Gauss-Jordan method can be applied are Business and Economics, Physics, Biology, Chemistry, Engineering etc.

Application in Business

Gauss and Gauss-Jordan methods of solving linear system can be used in Business Education. As a matter of fact Business and Economics are interrelated.

Bolade Nig-Limited a garment industry manufactures three shirts styles. Each shirt requires the services of three departmental stores shown below.

STYLES

	A	B	C
Cutting	0.2	0.4	0.3
Sewing	0.3	0.5	0.4
Packaging	0.1	0.2	0.1

The cutting, sewing and packaging have available maximum of 1160, 1560 and 480 hours repressively. You are required to:

i. Formulate the information into linear equation forms.
ii. Using Gaussian and Gauss-Jordan elimination methods to find the number of shirt that must be produced each week for the plant to operate at full capacity.

Let x represents style A
Let y represents style B
Let z represents style C

$$0.2x + 0.4y + 0.3z = 1160 \qquad (i)$$
$$0.3x + 0.5y + 0.4z = 1560 \qquad (ii)$$
$$0.1x + 0.2y + 0.1z = 480 \qquad (iii)$$

Solving yields x=1200,y=800 and z=2000 (all in units)

Application in Economics

Gauss and Gauss-Jordan methods can be used to find the equilibrium price and quantity to be supplied in a given market. Let's say there are two products: orange juice and water which are interrelated. Let P_1 and q_1 represents the price and quantity demanded respectively for product 1 (orange juice) and P_1 and q_2 represents the same for product 2 (water).

Demand supply

$$\text{Product 1}: P_1 = 2000 - 3q_1 - 2q_2,\ q_1 = 100 + 2q_1 + q_2$$
$$\text{Product 2}: P_2 = 2800 - q_1 - 4q_2,\ q_2 = 200 + 3q_1 + 2q_2$$

For equilibrium to be achieved, both price expressions must be equal, so the following equations are obtained.

$$\text{Product 1}: 2000 - 3q_1 - 2q_2 = 100 + 2q_1 + q_2$$
$$\text{Product 2}: 2800 - q_1 - 4q_2 = 200 + 3q_1 + 2q_2$$

The above expressions when reframed and solved give q_1=200, q_2 = 300.
Hence P_1 = ₦800, P_2 = ₦1,400.
Finding the equilibrium price and quantity in a market is very important to an Economist since beyond these values, nothing will be sold at a profit.

Applications in Physics

Similar problems as above can occur or be formulated and both Gauss and Gauss-Jordan methods can be readily applied.

CONCLUSION

The study has concluded that when performing calculation by hand, Gauss-Jordan method is more preferable to Gaussian elimination version because it avoids the need for back substitution. Besides, it was noted very remarkably that both Gauss elimination method and Gaussian-Jordan elimination method gave the same answers for each worked example and both with pivoting and partial pivoting equally gave the same answers. This necessarily implies that since the system of linear equations still remains the same despite that the equations are re-arranged leading to its matrix form to be transformed as the rows' elements obviously changed, the resultant solutions are still the same. The importance of Gauss and Gauss-Jordan elimination method can not be over emphasized due to its relevance to different field of studies (pure Science i.e. Physics, Biology, Chemistry, Mathematics etc. social science i.e. Economics, Geography, Business Education etc.).

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