Effect of Signal Sparsity, Signal Density and Noise Level on Rotational Equivariant Features for Image Classification

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Abstract

Current convolution methods for feature learning are already translation equivariant i.e. translation in input image produces a proportionate translation in feature maps. However, this is not true for rotational equivariance. A lot of recent research has been focussed on ensuring rotational equivariance for the same. Our research focusses on characterizing effect of different input image parameters, like sparsity, density and noise, on effectiveness of the learnt rotational equivariant features proposed in the recent work of Harmonic Convolutions. Feature maps learnt using Harmonic Convolutions exhibit equivariance to patch-wise translation and 360 degree rotation. These variant of normal convolutions use parameter-efficient and low computational complexity representation, thereby encoding complicated rotational equivariance within the network. In this paper, we show the effectiveness of rotational equivariance features for image classification as the sparsity, density and noise levels of the input image vary.

1. Introduction

2. Related Work

3. Problem Analysis

Harmonic Convolutions hard bake 360 degree rotational equivariance into their feature representations by restricting the convolution filters to be from the circular harmonics family. In the following sections, we will discuss the properties of circular harmonics which help the network learn rotation equivariant features. To reiterate, rotational equivariance implies that a particular rotation in input image produces a proportionate rotation in feature maps.

3.1. Circular Harmonics Equivariance

We describe an image using polar co-ordinates r and ϕ as F(r , ϕ). This can be understood better by looking at Figure 1. Please note that this is just to depict the polar co-ordinates of a filter. In actual setting, the filter will be applied patch-wise to the image and therefore, a single image patch will be described using this method.
We will now show that there exists a filter W_m such that the cross-correlation of F with W_m yields a rotationally equivariant feature map. This condition is satisfied when W_m is a circular harmonic of the form W_m = R(r) $e^{i(m\phi+\beta)}$ for some m belonging to integers. Consider the rotation of original image by θ which leads to a new image F(r, $\phi-\theta$). The cross-correlation for the rotated image is

$$\begin{aligned}[W * F(r, \phi-\theta)] &= \int W(r, \phi)\, F(r, \phi-\theta)\, dr\, d\phi \\ &= \int W(r, \phi' + \theta)\, F(r, \phi')\, dr\, d\phi'\end{aligned}$$

where $\phi' = \phi-\theta$. If we replace W_m to be of the form R(r) $e^{i(m\phi+\beta)}$, then the integral transforms as:

$$\begin{aligned}[W * F(r, \phi-\theta)] &= \int R(r)\, e^{i(m(\phi'+\theta)+\beta)}\, F(r, \phi')\, dr\, d\phi' \\ &= e^{im\theta} \int R(r)\, e^{i(m\phi'+\beta)}\, F(r, \phi')\, dr\, d\phi'\end{aligned}$$

When rotation θ = 0, then $\phi = \phi'$. Therefore, the above equation can be written as

$$\begin{aligned}&= e^{im\theta} \int R(r)\, e^{i(m\phi+\beta)}\, F(r, \phi)\, dr\, d\phi \\ &= e^{im\theta}\, [W * F(r, \phi)]\end{aligned}$$

Hence, we observe that the cross-correlation of the rotated signal F(r, $\phi-\theta$) with harmonic filter W_m = R(r) $e^{i(m\phi+\beta)}$ is equal to the response at 0 rotation [W * F(r, ϕ)] , multiplied by a complex phase shift $e^{im\theta}$. Thus, we have shown that cross-correlation with W_m yields a rotationally equivariant feature mapping when W_m is a circular harmonic.

3.2. Chain rule for cross-correlation of circular harmonics

In this section, we will show that the rotation order of a feature map, that we obtain after subsequent cross-correlations in each layer, is equal to the sum of the rotation orders of

Figure 1. An image describing polar co-ordinates with image center as origin

the filters in the chain. Let us consider cross-correlation of a θ rotated image F(r, $\phi - \theta$) with a filter W_m and then followed by another cross-correlation with W_n. As shown previously, we write the response of first cross-correlation as $e^{im\theta}$[W * F(r, ϕ)] . Then we can write the chain rule of harmonic cross-correlation as:

$$\begin{aligned} W_n * [W_m * \text{F(r, } \phi - \theta)] &= W_n * e^{im\theta}[W_m * \text{F(r, } \phi)] \\ &= e^{im\theta}(W_n * [W_m * \text{F(r, } \phi)]) \\ &= e^{im\theta} e^{in\theta} \; [\; W_n * [\; W_m * \text{F}]] \\ &= e^{i(m+n)\theta} \; [\; W_n * [\; W_m * \text{F}]] \end{aligned}$$

Therefore, we see that chained cross-correlation results in a summation of the rotation orders of the individual filters W_m and W_n.

3.3. Circular harmonic filter architecture

As we described in the previous sections, a Harmonic filter is defined as W_m = R(r) $e^{i(m\phi+\beta)}$, so that it can capture the rotation equivariance while image classification. We also showed that whenever a test image is rotated, the cross-correlation of the rotated image with the learned filter, results in a product of $e^{im\theta}$ with [W * F(r, ϕ)], where [W * F(r, ϕ)] is the original non-rotated test image. Hence, the classifier is able to learn that any $e^{im\theta}$ multiple of the original cross-correlation is the same image rotated by θ, and thereby classify it correctly. Therefore, this clearly defined mapping helps achieve rotation equivariance of filters, which in turn leads to global rotation invariance of the network in predicting rotated images. This is the reason, the network is able to classify rotated images with equally good accuracy.

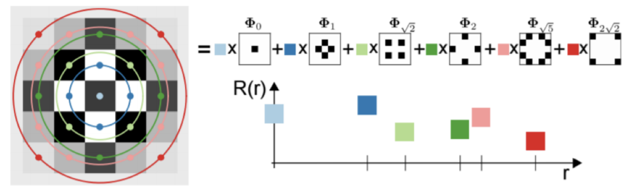

Figure 2. Filter Architecture showing mapping of concentric harmonic filter profile to the cells of a square filter

We will now delve deeper into the architecture of a Harmonic filter. The learnable parameters of a harmonic filter W_m = R(r) $e^{i(m\phi+\beta)}$, are the radial profile R(r), which is a function of the radius r from the origin, and the per-filter phase offset β. For a n*n filter, the number of radial profile elements is equal to the number of rings of equal distance from the center of the filter. An example of this filter can be seen in Figure 2. This filter is a 5 * 5 filter with 6 rings of equal distance from the center of the filter. The smallest ring is just a single point. It is called a 5*5 filter because we are mapping the radial profile to the elements of a square filter with has length 5 and hence dimensions 5*5. And hence, this filter has 6 radial profile terms and 1 phase offset to learn. This is in contrast to a regular convolution filter which has 25 parameters to learn (because of 5*5 dimension of the filter).

Now, that we have explained the architecture of the harmonic filters, there is one very important characteristic to notice. Because we are learning a complete radial profile which depends only on the radius, the filter values for a particular radial profile are same, with only difference in sign. This can be perceived as an advantage as well as a disadvantage. Advantage because this kind of constraint enables the network to learn a rotation invariant classifier. And a disadvantage because the constraint doesn't let the network learn the filter freely during training , so that it can perform the best. In the next sections, we will see how this constraining of filters serves as an advantage as well as a disadvantage when looking at different applications.

To reiterate, till now the paper gives the reader a background of how Harmonic Networks work and how they are able to hard-bake rotation equivariance. More reference on this can be found in the paper [GIVE REFERENCE HERE]. From now on, we will discuss our main focus which describes how change in signal sparsity, signal density and noise level affects rotational equivariant features.

4. Experiments

5. Conclusion

6. Future Work

7. References

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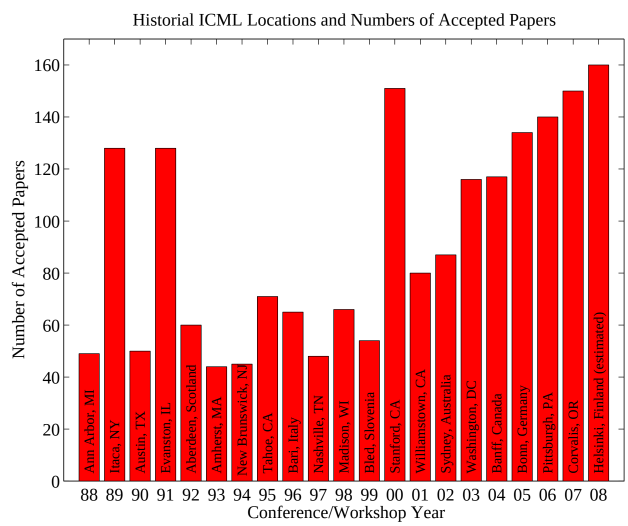

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Algorithm 1 Bubble Sort

```
Input: data x_i, size m
repeat
  Initialize noChange = true.
  for i = 1 to m − 1 do
    if x_i > x_{i+1} then
      Swap x_i and x_{i+1}
      noChange = false
    end if
  end for
until noChange is true
```

Table 1. Classification accuracies for naive Bayes and flexible Bayes on various data sets.

DATA SET	NAIVE	FLEXIBLE	BETTER?
BREAST	95.9± 0.2	96.7± 0.2	√
CLEVELAND	83.3± 0.6	80.0± 0.6	×
GLASS2	61.9± 1.4	83.8± 0.7	√
CREDIT	74.8± 0.5	78.3± 0.6	
HORSE	73.3± 0.9	69.7± 1.0	×
META	67.1± 0.6	76.5± 0.5	√
PIMA	75.1± 0.6	73.9± 0.5	
VEHICLE	44.9± 0.6	61.5± 0.4	√

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