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A TIMOSHENKO BEAM ELEMENT†

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A Timoshenko beam finite element which is based upon the exact differential equations of an infinitesimal element in static equilibrium is presented. Stiffness and consistent mass matrices are derived. Convergence tests are performed for a simply-supported beam and a cantilever. The effect of the shear coefficient on frequencies is discussed and a study is made of the accuracy obtained when analysing frameworks with beams.

1. INTRODUCTION

In the Bernoulli–Euler theory of flexural vibrations of beams only the transverse inertia and elastic forces due to bending deflections are considered. As the ratio of the depth of the beam to the wavelength of vibration increases the Bernoulli–Euler equation tends to overestimate the frequency. The applicability of this equation can be extended by including the effects of the shear deformation and rotary inertia of the beam. The equation which includes these secondary effects was derived by Timoshenko [1, 2] and solutions of it for various boundary conditions have been published [3–5]. The dynamic stiffness matrix method has been used [6, 7] for studying the dynamic behaviour of two- and three-dimensional frameworks comprised of Bernoulli–Euler or Timoshenko beams.

The stiffness and mass matrices for the simple beam finite element were published in 1963 by Leckie and Lindberg [8]. Their analysis was based on the exact differential equations of an infinitesimal element in static equilibrium. In the present paper Leckie and Lindberg's work is extended to include shear deformation and rotary inertia in the analysis. When these secondary effects are taken into account, it is important to be clear which rotation is to be used at the ends of the finite element model. The authors have chosen this rotation to be that of the cross section of the beam, i.e. the rotation of the neutral axis plus the shearing angle. With this it is possible to impose the correct boundary conditions at say a clamped or free end of a beam.

Bending and shearing deformations were considered separately by Kapur [9] for deriving stiffness and consistent mass matrices for a Timoshenko beam. A cubic displacement function was assumed for the bending deformation (the displacements at each node being one translation and one rotation). A similar assumption was made for shear deformations, also allowing a translational and rotational displacement at each node. The resulting element matrices were of order 8×8 and no stiffness coupling was permitted between bending and shear deformations. The results presented by Kapur agreed very well with exact frequencies for simply-supported beams and cantilevers. However, in general structures, where all beams are not collinear, complications arise with Kapur's element in coupling up forces and displacements, and each node must be treated specially. Kapur's frequencies converged

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much more rapidly than some other answers obtained with a finite element Timoshenko beam published by Archer [10]. Kapur wrote that Archer's element could not represent the exact boundary conditions at a clamped or free end of a beam. Mid-side nodes were used in the work reported in reference [11] to improve the rate of convergence of a beam with shear deformation and rotary inertia. The respective authors of both reference [11] and reference [12] (who have also published stiffness and mass matrices for a Timoshenko beam) do not state which rotations were used in formulating their matrices.

In the present paper the matrices for a Timoshenko beam are derived. A section is included describing how the boundary conditions are satisfied and convergence tests are carried out on a simply-supported beam and a cantilever. The significance of the K factor, which must be used when shear deformation is allowed, is then discussed. Finally, the accuracy which can be obtained when analysing a two-dimensional structure with beams is studied.

2. DERIVATION OF ELEMENT MATRICES

2.1. DIFFERENTIAL EQUATIONS

Consider an infinitesimal element of beam of length δx and flexural rigidity EI. The element is in static equilibrium under the forces shown in Figure 1.†

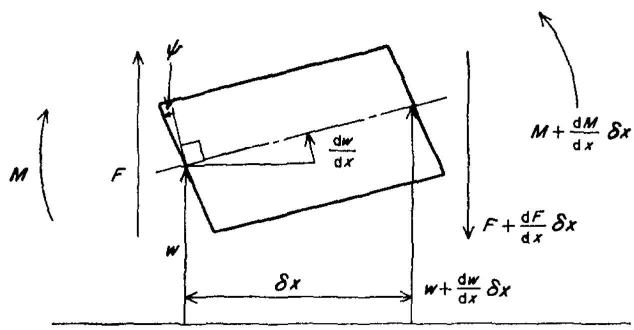

Figure 1. Forces and displacements on infinitesimal element of beam.

The shear angle, ψ, is measured as positive in an anticlockwise direction from the normal to the midsurface to the outer face of the beam.

Now

$$F/A = GK\psi. \tag{1}$$

The static equilibrium relations are

$$\frac{dM}{dx} = F, \tag{2}$$

$$\frac{dF}{dx} = 0. \tag{3}$$

The rotation of the cross section in an anticlockwise direction (see Figure 1) is

$$\theta = \frac{dw}{dx} + \psi. \tag{4}$$

The stress–strain relation in bending is

$$\frac{d^2 w}{dx^2} + \frac{d\psi}{dx} = \frac{M}{EI}. \tag{5}$$

† A list of notation used is given in the Appendix.

Solution of equations (1), (2), (3) and (5) yields

$$F = \alpha_1, \tag{6}$$

$$\psi = \frac{\alpha_1}{GKA}, \tag{7}$$

$$M = \alpha_1 x + \alpha_2, \tag{8}$$

$$w = \frac{1}{EI}\left(\alpha_1 \frac{x^3}{6} + \alpha_2 \frac{x^2}{2} + \alpha_3 x + \alpha_4\right). \tag{9}$$

2.2. ELEMENT STIFFNESS MATRIX

The above four equations can now be applied to the finite element by substituting the end conditions (see Figure 2) in them.

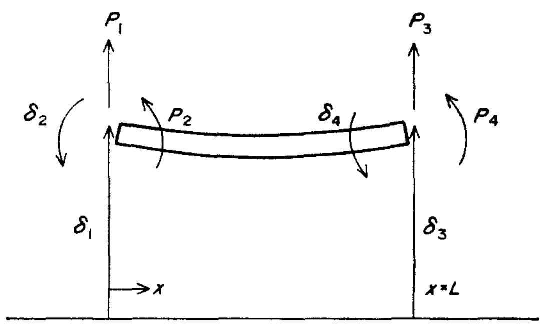

Figure 2. Generalized forces and displacements on finite element.

The rotations at the ends of the beam, δ_2 and δ_4, can be expressed as rotations of the cross section by using equation (4). The displacements δ_1 to δ_4 can be related to the constants α_1 to α_4 by

$$\{\boldsymbol{\delta}_i\} = \frac{1}{EI}[\mathbf{X}]\{\boldsymbol{\alpha}_i\}, \tag{10}$$

where

$$[\mathbf{X}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \beta & 0 & 1 & 0 \\ \dfrac{L^3}{6} & \dfrac{L^2}{2} & L & 1 \\ \dfrac{L^2}{2}+\beta & L & 1 & 0 \end{bmatrix},$$

$\{\boldsymbol{\delta}_i\} = \{w_1 \theta_1 w_2 \theta_2\}_i^T$ and $\beta = EI/GKA$. Similarly the forces can be related to the constants by

$$\{\mathbf{P}_i\} = [\mathbf{Y}]\{\boldsymbol{\alpha}_i\}, \tag{11}$$

where

$$[\mathbf{Y}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ L & 1 & 0 & 0 \end{bmatrix}$$

and the elements of $\{\mathbf{P}_i\}$ are defined in Figure 2. Substituting for $\{\boldsymbol{\alpha}_i\}$ from equation (10) in equation (11) gives

$$\{\mathbf{P}_i\} = EI[\mathbf{Y}][\mathbf{X}]^{-1}\{\boldsymbol{\delta}_i\}$$

or

$$\{\mathbf{P}_i\} = [\mathbf{S}]\{\boldsymbol{\delta}_i\},$$

where [S] is the element stiffness matrix

$$[\mathbf{S}] = \frac{EI}{L(L^2+12\beta)}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2+12\beta & -6L & 2L^2-12\beta \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2-12\beta & -6L & 4L^2+12\beta \end{bmatrix}.$$

If $\beta = 0$ the above matrix reduces to the element stiffness matrix derived by Leckie and Lindberg [8] for a thin beam.

2.3. ELEMENT MASS MATRIX

The kinetic energy of a finite element, T, vibrating at circular frequency, ω, including the effects of shear deformation and rotary inertia, can be written down as

$$T = \frac{\omega^2}{2}\int_0^L \left[\rho A w^2 + \rho I\left(\psi + \frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]\mathrm{d}x.$$

Substituting for w from equation (9), ψ from equation (7) and $\{\boldsymbol{\alpha}_i\}$ from equation (10) and integrating with respect to x then gives

$$T = \tfrac{1}{2}\rho A\omega^2\{\boldsymbol{\delta}_i\}^T[\mathbf{X}]^{-T}[\mathbf{H}][\mathbf{X}]^{-1}\{\boldsymbol{\delta}_i\}$$

where [H] may be found readily as

$$[\mathbf{H}] = \begin{bmatrix} \frac{L^7}{252}+\gamma\left(\frac{L^5}{20}+\frac{L^3}{3}\beta+L\beta^2\right) & & & \text{SYMMETRICAL} \\ \frac{L^6}{72}+\gamma\left(\frac{L^4}{8}+\frac{L^2}{2}\beta\right) & \frac{L^5}{20}+\frac{\gamma L^3}{3} & & \\ \frac{L^5}{30}+\gamma\left(\frac{L^3}{6}+L\right) & \frac{L^4}{8}+\frac{\gamma L^2}{2} & \frac{L^3}{3}+\gamma L & \\ \frac{L^4}{24} & \frac{L^3}{6} & \frac{L^2}{2} & L \end{bmatrix}.$$

Here $\gamma = I/A$. The element mass matrix is then

$$\rho A[\mathbf{X}]^{-T}[\mathbf{H}][\mathbf{X}]^{-1}$$

which can be set up easily on a computer.

3. BOUNDARY CONDITIONS

The two types of boundary condition which give rise to some difficulty are clamped or free ends.

In the case of the clamped end of a beam $\delta_1 = \delta_2 = 0$ (see Figure 2) and some shear strain must be possible. In our case this shear strain (ψ_1) is provided by the reaction at the wall (P_1) to give $\psi_1 = P_1/GKA$ from equation (1).

For a free end of a beam with no point load at the end the shear strain must be zero. Since at the free end of a beam the vertical reaction is zero, then that part of the rotation of the cross section, by using equations (1) and (4), which gives the shear strain, will also be zero.

4. CONVERGENCE TEST

In this section the finite element derived in section 2 is tested for convergence on a simply-supported beam and a cantilever. The frequencies are compared with exact values computed by using references [3] and [6] for the simply-supported beam and cantilever respectively. In reference [9] a similar test was carried out to compare the performance of a Timoshenko beam element with that given in reference [10]. Insufficient information was given in reference [9] to enable the present authors to make a direct comparison. However, the value of k/l used by Kapur and Archer was used to compute our frequencies. The exact frequencies are given in non-dimensional form as C in Table 1, with

$$C = \omega_n(\rho l^4/Ek^2)^{1/2},$$

where ω_n is the nth natural frequency.

TABLE 1

Percentage error in frequencies between finite element and exact Timoshenko theory for a simply-supported beam and cantilever

	Degrees of freedom on full beam									
	Simply-supported beam				Cantilever					
Mode	2	4	8	16	2	4	8	16	a†	b†
1	20·11	1·79	0·39	0·09	1·01	0·25	0·06	0·01	11·5	5·77
2	69·13	37·36	3·99	0·98	67·10	3·63	1·23	0·31	38	35·3
3	—	56·69	12·08	3·19	—	50·86	5·68	1·54	72	68
4	—	42·07	32·23	6·81	—	64·19	10·78	3·69	108	107

† Columns a and b give the percentage difference between the exact Timoshenko and Bernoulli–Euler theories for the simply-supported beam and cantilever respectively. $\nu = 0{\cdot}3$, $K = 0{\cdot}85$, $k/l = 0{\cdot}08$. Exact values of C computed by the authors for the first four modes are 8·8397, 28·461, 51·498 and 75·364 for the simply-supported beam and 3·3241, 16·289, 36·708 and 58·279 for the cantilever, where $C = \omega_n(\rho l^4/Ek^2)^{1/2}$.

Table 1 shows the results for four idealizations and also gives an indication of the effect of shear deformation and rotary inertia on the first few modes by giving the percentage difference in frequencies between the Timoshenko and Bernoulli–Euler theories. The element presented shows itself to converge from above onto exact answers given by a solution of Timoshenko's equation. The rate of convergence of the element depends on the depth to wavelength ratio of the beam under consideration, i.e. on the relative importance of bending to shear deformation. In the dynamic case the deflection curve for an element is still the static curve. Consequently the shear force is constant along the length so that the strain energy stored by shear deformation is also constant. During vibration the shear force on an element is not constant and so the element gives a poorer representation of shear deformation than bending deformation.

5. THE SHEAR COEFFICIENT

The finite element presented converges from above onto frequencies which are obtained from a solution of Timoshenko's equation. The question arises as to the accuracy of these frequencies compared with exact answers obtained from the elasticity equations. Cowper [13] has obtained an exact solution for a simply-supported beam. In his paper values of fundamental frequency computed by using Timoshenko's equation for two different values of the shear coefficient are compared with exact values for a variation of beam depth to length ratio up to one. The discrepancy between frequencies was shown to depend on the value of K used and to increase with increasing depth to length ratio of the beam. The discrepancy was greatest when the value $K = \frac{2}{3}$ was used; here K was regarded as the ratio of the average shear stress on a cross-section to the shear stress at the centroid, a parabolic distribution of shear stress being assumed. Far better accuracy was achieved by using

$$K = \frac{10(1+\nu)}{12+11\nu}.$$

For $\nu = 0{\cdot}3$, $K = 0{\cdot}85$. This formula was derived by Cowper in an earlier paper [14], and it is a by-product of an analysis whereby Timoshenko's beam equations were derived by integration of the three-dimensional elasticity equations.

TABLE 2

Errors in the Timoshenko theory for the first natural frequency of a simply-supported beam

	Percentage error between exact [13] and finite element									
	h/l									
K	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·66	−0·350	−1·247	−2·385	−3·540	−4·606	−5·552	−6·377	−7·092	−7·711	−8·245
0·85	−0·020	−0·079	−0·162	−0·258	−0·358	−0·456	−0·549	−0·636	−0·714	−0·783
0·86	−0·006	−0·028	−0·063	−0·111	−0·165	−0·223	−0·279	−0·334	−0·384	−0·429

Finite element frequencies were obtained from a ten element idealization of a half simply-supported beam

$$\text{percentage error} = \left(\frac{\omega_{\text{finite element}}}{\omega_{\text{exact}}} - 1\right) \times 100$$

Material properties for the analysis were $E = 30 \times 10^6$ lb f/in², $\nu = 0{\cdot}3$, $\rho g = 0{\cdot}283$ lb f/in³.
($E = 206{\cdot}8$ GN/m², $\rho = 7{\cdot}83 \times 10^3$ kg/m³)

Table 2 gives this comparison between the exact and finite element frequencies for the two values of K used by Cowper and also for a value obtained using a formula derived by Goodman [15] in which K must satisfy

$$\frac{16(1-\mu)}{K^3} - \frac{8(3-2\mu)}{K^2} + \frac{8}{K} - 1 = 0,$$

where $\mu = (1-2\nu)/2(1-\nu)$. For $\nu = 0{\cdot}3$, $K = 0{\cdot}86$. The above expression ensures that the frequency given by the Timoshenko equation is correct in the limit of zero wavelength.

It appears from these results that exact frequencies can be approached provided the appropriate value of K is used.

6. ACCURACY OBTAINED WHEN ANALYSING A TWO-DIMENSIONAL STRUCTURE WITH BEAMS

The accuracy of frequencies when analysing two-dimensional structures with beams does not appear to have been reported in the literature. The object of this section is to carry out such a study on the natural frequencies of in-plane vibration of a single bay portal frame.

There have been no exact frequencies obtained by using two-dimensional theory published for a moderately thick portal frame; therefore the finite element method was used to provide some very accurate frequencies.

6.1. SIMPLY-SUPPORTED BEAM USING PLANE STRESS ELEMENTS

In order to assess the validity of using finite elements to idealize a portal frame a preliminary study was carried out to compare the fundamental frequency of a simply-supported beam idealized with a 17-node plane stress finite element [16] with exact frequencies obtained by using Cowper's equation [14]. One half of a beam of total length 100 in was idealized by using a mesh of two elements through the depth of the beam and ten along half the length

TABLE 3

Percentage error in fundamental frequency between a plane stress finite element analysis of a simply-supported beam and exact frequencies

h/l	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
% error	0·003	0·020	0·065	0·145	0·268	0·435	0·643	0·892	1·177	1·495

Exact frequencies (Hz) for simply-supported beam obtained by using equation (5) of reference [13] (which also apply to Table 2) were 90·308, 172·772, 243·083, 300·707, 347·135, 384·418, 414·467, 438·854, 458·806, and 475·266 with $E = 30 \times 10^6$ lbf/in², $\nu = 0{\cdot}3$ and $\rho g = 0{\cdot}283$ lbf/in³.

of the beam.† The depth of the beam was varied from 10 to 100 in with the length held constant. With boundary conditions similar to those imposed in Cowper's theory the number of degrees of freedom on the half beam was 399. Continuous reduction [17] was used to reduce the eigenvalue problem solved to 62. Table 3 shows the percentage error between the finite element frequencies and those from Cowper's equation. The small percentage errors obtained between the frequencies suggested that the method could be used to analyse at least one-dimensional structures and that reducing the eigenvalue problem did not effect the fundamental frequency very much.

6.2. PORTAL FRAME ANALYSIS

6.2.1. *Plane stress elements*

The 17-node element used for the simply-supported beam analysis was also used for the portal frame. Half the portal frame was idealized, and by suitably changing the boundary conditions, anti-symmetrical and symmetrical modes of vibration for a full portal frame were simulated. Two different meshes of elements were used (see Figure 3) and use was again made of the front solution to reduce the size of the eigenvalue problem solved from 316 to 39 and 696 to 69 for the coarse and fine meshes, respectively.

The leg length of the three sides of the portal frame was kept constant at 10 in and the breadth of the legs was varied from 1 to 9 in. The length of the legs was taken as the length to the centre of the corner joints as shown in Figure 4.

6.2.2. *Beams*

The portal frame was analysed by using both Bernoulli–Euler and Timoshenko beam theories. In each case the length of the legs of the portal frame was as defined in Figure 4. The stretching energy of the legs was taken into account in both Bernoulli–Euler and Timoshenko cases. The portal frame frequencies were computed by using the Timoshenko beam elements derived in section 2 of this paper and also by using the dynamic stiffness matrix method for the Bernoulli–Euler and Timoshenko beam theories. For the beam finite element analysis a half portal frame was idealized with 15 equal length elements giving 44 and 43 degrees of freedom for the antisymmetrical and symmetrical modes of vibration,

† 1 in = 2·54 cm.

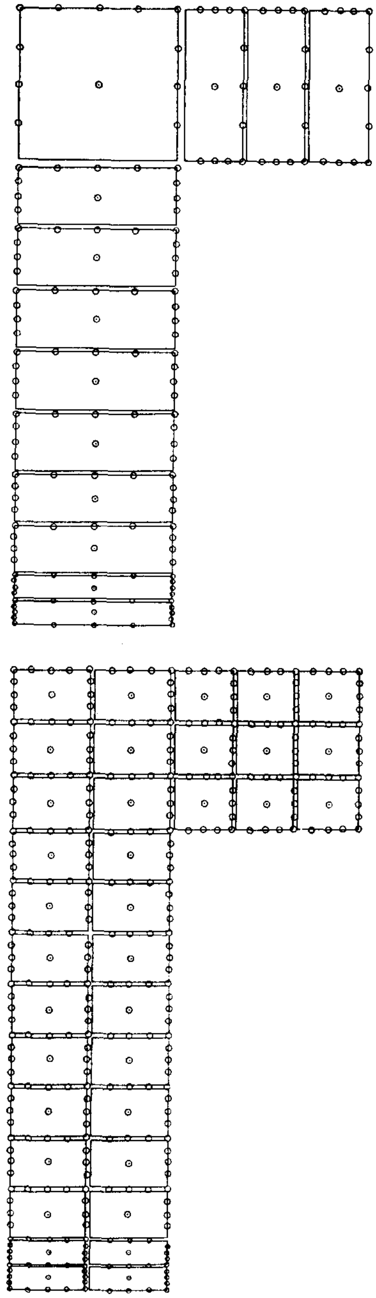

Figure 3. Plane stress idealizations of portal frames.

respectively. In Table 4 percentage errors between finite element and dynamic stiffness matrix frequencies are given for the first six modes of vibration for portal frame leg thickness from 1 to 9 in.

6.2.3. *Comparison between beam and plane stress frequencies*

In Figure 5 the results described in sections 6.2.1 and 6.2.2 have been plotted for the first anti-symmetrical and symmetrical modes of vibration for the single bay portal frame.

The graph shows that refining the mesh has changed the frequency of the anti-symmetrical mode to a greater extent than that of the symmetrical mode especially as the leg thickness is increased. This is because the deformation in the region of the joint is much more complicated in the first antisymmetrical than in the first symmetrical mode. In the case of the coarser mesh idealization this region of the structure did not contain a sufficient number of elements to represent the deformation properly.

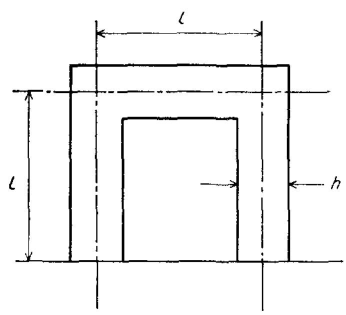

Figure 4. Portal frame dimensions.

TABLE 4

Percentage errors in frequency between Timoshenko finite element and dynamic stiffness matrix methods for in-plane vibrations of a portal frame with a leg length of 10 *in*

Mode	Percentage error in frequency between Timoshenko finite element and dynamic stiffness matrix methods, h/l 0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
1	<0·0001†	<0·0001†	<0·0001†	<0·0001†	<0·0001†	0·0001†	0·0005†	0·0007†	0·001†
2	0·015	0·044	0·067	0·079	0·084	0·087	0·089	0·090	0·091
3	0·036†	0·101	0·138	0·167	0·190	0·125†	0·123†	0·121†	0·118†
4	0·044	0·106†	0·140†	0·127†	0·126†	0·207	0·219	0·229	0·235†
5	0·132†	0·129†	0·158†	0·219†	0·253†	0·274†	0·287†	0·292†	0·293
6	0·219	0·163	0·200	0·227	0·245	0·256	0·262	0·266	0·267

† Antisymmetrical modes; others symmetrical. Half portal frame idealized by using 15 equal length elements giving 44 and 43 degrees of freedom for antisymmetrical and symmetrical modes of vibration, respectively. $E = 30 \times 10^6$ lbf/in^2, $\nu = 0{\cdot}3$, $K = 0{\cdot}85$ and $\rho g = 0{\cdot}283$ lbf/in^3.

In fact the plane stress finite element frequencies are not exact for the following reasons. (i) The element used was a conforming one which allows a cubic variation in strain along each of its sides. It is certain that if more elements had been used, and the same size eigenvalue problem solved, lower frequencies would have been obtained. (ii) The eigenvalue reduction technique used, which reduces freedoms out on a static basis by using certain master freedoms, will have raised the frequencies slightly from their true value for the model taken. The reasons why the beam frequencies can be expected to differ from the plane stress results are as follows. (iii) In both beam theories a linear axial and bending stress distribution across the thickness of the beam is implied. (iv) In the Bernoulli–Euler theory shear deformation and rotary inertia are not allowed. (v) For the Timoshenko beam it was not certain what value of the shear coefficient (K) was the correct one to use ($K = 0{\cdot}85$ was used). (vi) The corner of the portal frame was assumed to be rigid. (vii) When two beams are "joined" at right angles there is a geometric misrepresentation of the structure at this point.

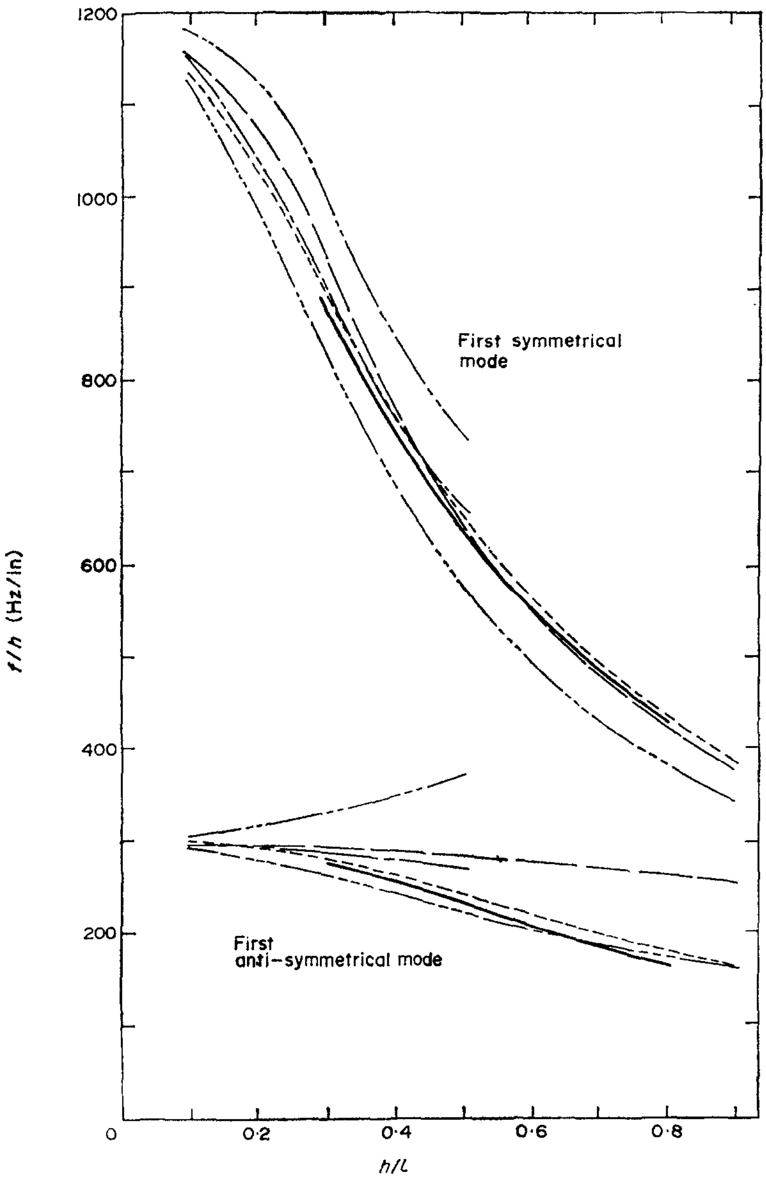

Figure 5. Natural frequencies of portal frames obtained by using plane stress finite element and dynamic stiffness matrix idealizations. ——, Fine mesh plain stress; – – –, coarse mesh plain stress; — —, Bernoulli–Euler; — - - —, Bernoulli–Euler (Barkan); — - - - —, Timoshenko; — — —, Timoshenko (Barkan).

Figure 5 shows that the Timoshenko beam theory gives a better estimation of the first antisymmetrical mode of the portal frame than the Bernoulli–Euler theory and *vice versa* for the symmetrical mode. It is not certain why this is so. It can only be said that it seems that beam theory is inadequate for the general analysis of frameworks in which the depths of the beams are of the same order as the distance between joints. It is suggested that reasons (vi) and (vii) are the most important contributors to the error. Barkan [18] has suggested that an allowance can be made for joint flexibility by using a modified length for the beams. He has published a graph which can be used to give the modified values. This graph was used by the present authors to obtain new, shorter, leg lengths for portal frame leg thicknesses from 1 to 5 in. The new lengths were then used for computing frequencies using the dynamic stiffness matrix method for both Bernoulli–Euler and Timoshenko beam theories. Barkan's book does not indicate if his theory is meant to be applied to static analyses only, or whether it may be used in dynamic problems, i.e. the mass of the missing portion of the frame must be allowed for. The authors assumed that the theory could only be used for static problems and applied a simple correction multiplier to the frequencies to allow for this. The results obtained are plotted on Figure 5. The effect of Barkan's theory on the answers is that neither

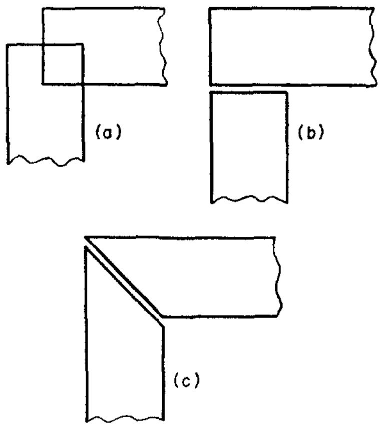

Figure 6. Idealization of joints.

the Bernoulli–Euler nor Timoshenko theories give a good estimation of the first anti-symmetrical mode and that the Timoshenko beam theory is now superior to the Bernoulli–Euler theory for the first symmetrical mode. It seems that Barkan's theory over-corrects the frequencies and that modified leg lengths should depend on the mode of vibration being examined.

The geometric misrepresentation which is incurred when two beams are joined at right angles (see Figure 6(a)) is perhaps the largest contributor to the error in frequencies. This problem could be overcome by formulating beam finite elements with ends similar to those shown in Figure 6(b) and (c). This would be satisfactory for the type of joints analysed in this paper, but for the general analysis of frameworks a whole series of beams with special ends would be needed.

7. CONCLUSIONS

A beam element which includes the effects of shear deformation and rotary inertia has been presented which converges from above onto a solution of Timoshenko's equation. It has been shown that the element will converge onto an exact solution of the elasticity equations for a simply-supported beam provided that the correct value of the shear coefficient is used. By using the rotation of the cross section as the angular displacement of the beam it has been possible to satisfy all types of boundary conditions. Accurate frequencies were computed for a single bay portal frame using a plane stress finite element. These results showed that the Bernoulli–Euler and Timoshenko beam theories were generally unsatisfactory when the depth of the portal frame legs were of the same order as the distance between the joints.

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APPENDIX

NOTATION

A cross-sectional area of beam
E elastic modulus
F shear force
G shear modulus
f frequency of vibration
g acceleration due to gravity (386·4 in/s^2 in all numerical work)
h total depth of beam
I second moment of area of cross section about a principal axis normal to the plane of bending
K shear coefficient
k radius of gyration of cross section about a principal axis normal to the plane of bending
L length of beam element
l total length of beam or length of portal frame leg

M bending moment
P_i ith generalized force
w transverse deflection of beam
x distance along length of beam
α_i constants of integration
δ_i ith generalized displacement
ν Poisson's ratio
θ rotation of cross-section
ρ mass density
ψ shear strain
ω circular frequency