Analytical Hierarchy Process (AHP)

Pair_Wise Comparison Using Triangular Fuzzy Sets:

$$A = \begin{array}{c} \\ \text{apple} \\ \text{banana} \\ \text{cherry} \end{array} \begin{array}{c} \begin{array}{ccc} \text{apple} & \text{banana} & \text{cherry} \end{array} \\ \begin{bmatrix} 1 & 1/3 & 5 \\ 3 & 1 & 7 \\ 1/5 & 1/7 & 1 \end{bmatrix} \end{array}$$

Sum: $\frac{21}{5}$ $\quad \frac{31}{21}$ $\quad 13$

1. Converting matrix A to triangular fuzzy matrix A:

$$A = \begin{bmatrix} 1 & 1/3 & 5 \\ 3 & 1 & 7 \\ 1/5 & 1/7 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} (1,1,1) & 1/3 & (4,5,6) \\ (2,3,4) & (1,1,1) & (6,7,8) \\ 1/5 & 1/7 & (1,1,1) \end{bmatrix}$$

what about fractional numbers?!

$$5 \to (4,5,6) \Rightarrow 1/5 \to \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right)$$
$$3 \to (2,3,4) \Rightarrow 1/3 \to \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$$
$$7 \to (6,7,8) \Rightarrow 1/7 \to \left(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}\right)$$

$\Rightarrow$ Fuzzified Matrix A is now like below:

$$A = \begin{bmatrix} (1,1,1) & (1/4, 1/3, 1/2) & (4,5,6) \\ (2,3,4) & (1,1,1) & (6,7,8) \\ (1/6, 1/5, 1/4) & (1/8, 1/7, 1/6) & (1,1,1) \end{bmatrix}$$

Sum: $\left(\frac{19}{6}, \frac{21}{5}, \frac{21}{4}\right)$ $\left(\frac{11}{8}, \frac{31}{21}, \frac{5}{3}\right)$ $(11, 13, 15)$.

now we should normalize Matrix A:

normalized Matrix A: $$\begin{bmatrix} \left(\frac{4}{21}, \frac{5}{21}, \frac{6}{19}\right) & \left(\frac{3}{20}, \frac{7}{31}, \frac{4}{33}\right) & \left(\frac{4}{15}, \frac{5}{13}, \frac{6}{11}\right) \\ \left(\frac{8}{21}, \frac{5}{7}, \frac{24}{19}\right) & \left(\frac{3}{5}, \frac{21}{3}, \frac{8}{11}\right) & \left(\frac{2}{5}, \frac{7}{13}, \frac{8}{11}\right) \\ \left(\frac{2}{63}, \frac{1}{21}, \frac{3}{38}\right) & \left(\frac{3}{40}, \frac{3}{31}, \frac{4}{33}\right) & \left(\frac{1}{15}, \frac{1}{13}, \frac{1}{11}\right) \end{bmatrix}$$

normalization process (Calculation part):

$$(1,1,1) \div \left(\frac{19}{6}, \frac{21}{5}, \frac{21}{4}\right) = \left(\frac{4}{21}, \frac{5}{21}, \frac{6}{19}\right)$$

$$(2,3,4) \div \left(\frac{19}{6}, \frac{21}{5}, \frac{21}{4}\right) = \left(\frac{8}{21}, \frac{5}{7}, \frac{24}{19}\right)$$

$$\left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) \div \left(\frac{19}{6}, \frac{21}{5}, \frac{21}{4}\right) = \left(\frac{2}{63}, \frac{1}{21}, \frac{3}{38}\right)$$

$$\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right) \div \left(\frac{11}{8}, \frac{31}{21}, \frac{5}{3}\right) = \left(\frac{3}{20}, \frac{7}{31}, \frac{4}{33}\right)$$

$$(1,1,1) \div \left(\frac{11}{8}, \frac{31}{21}, \frac{5}{3}\right) = \left(\frac{3}{5}, \frac{21}{31}, \frac{8}{11}\right)$$

$$\left(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}\right) \div \left(\frac{11}{8}, \frac{31}{21}, \frac{5}{3}\right) = \left(\frac{3}{40}, \frac{3}{31}, \frac{4}{33}\right)$$

$$(4,5,6) \div (11,13,15) = \left(\frac{4}{15}, \frac{5}{13}, \frac{6}{11}\right)$$

$$(6,7,8) \div (11,13,15) = \left(\frac{2}{5}, \frac{7}{13}, \frac{8}{11}\right)$$

$$(1,1,1) \div (11,13,15) = \left(\frac{1}{15}, \frac{1}{13}, \frac{1}{11}\right)$$

now we should come to priority vector w:

$$w = \frac{1}{3} \times \begin{bmatrix} \left(\frac{4}{21}, \frac{5}{21}, \frac{6}{19}\right) + \left(\frac{3}{20}, \frac{7}{31}, \frac{4}{33}\right) + \left(\frac{4}{15}, \frac{5}{13}, \frac{6}{11}\right) \\ \left(\frac{8}{21}, \frac{5}{7}, \frac{24}{19}\right) + \left(\frac{3}{5}, \frac{21}{31}, \frac{8}{11}\right) + \left(\frac{2}{5}, \frac{7}{15}, \frac{8}{11}\right) \\ \left(\frac{2}{63}, \frac{1}{21}, \frac{3}{38}\right) + \left(\frac{3}{40}, \frac{3}{31}, \frac{4}{33}\right) + \left(\frac{1}{15}, \frac{1}{13}, \frac{1}{11}\right) \end{bmatrix} = \begin{bmatrix} 0.270902 \\ 0.669869 \\ 0.076200 \end{bmatrix}$$

the calculation above is after defuzzification process which is like below:

$$\to \frac{1}{3} \begin{bmatrix} \left(\frac{361}{1680} + \frac{7181}{25389} + \frac{56}{171}\right) \\ \left(\frac{643}{1260} + \frac{1815}{2821} + \frac{568}{627}\right) \\ \left(\frac{1937}{30240} + \frac{1873}{25389} + \frac{365}{3762}\right) \end{bmatrix} = \begin{bmatrix} 0.270902 \\ 0.669869 \\ 0.07620 \end{bmatrix} = \begin{bmatrix} 0.270902 \\ 0.669869 \\ 0.076200 \end{bmatrix}$$

$$\Rightarrow \lambda_{max} = \frac{21}{5}(0.270902) + \frac{31}{21}(0.669869) + 13(0.076200) = 3.11724$$

$$\Rightarrow CI = \frac{\lambda_{max} - n}{n-1} = \frac{3.11724 - 3}{3-1} = \frac{0.11724}{2} = 0.05862$$

$$\Rightarrow CR = \frac{CI}{RI} = \frac{0.05862}{0.58} = 10.1\% > 10$$