Note on Deriving the State Transition Model for the Meteorites Project

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1 A First Attempt

From the PDF and a review of the meteorites code, we are given a motion model with terms a, b, c such that:

$$x_t = c + b\Delta t + a\Delta t^2 \tag{1}$$

where:

c is the initial position,
b is the initial velocity and
a is a scaled constant acceleration or a pseudo acceleration.
Δt is the time since the beginning of the meteorite's life.

One note here pertaining to the simulation is that since the Δt is considered from the beginning of the meteorite life to the present, the position estimate is not computed by integrating over each time step, but rather the current position is found directly from the a, b, c and the Δt. For this reason, the values of a, b and c are truly constant.

That said, it makes sense to choose states such that $\bar{x} = [x, \dot{x}, \ddot{x}]^T$ as this is a simple and standard state space used for a constant acceleration model in one dimension. To create the Kalman Filter dynamics model, F, used to propagate the state $\bar{x}$ forward in time, we must find a consistent set of differential equations from which to construct it. Based on equation (1), we know that the acceleration will be the same throughout the entire simulation, so our state dynamics should end up with a constant acceleration model.

A reasonable first approach is to assume the position $x = c$, velocity $\dot{x} = b$, and acceleration $\ddot{x} = a$. So that we have the equation representing the motion over any time interval:

$$x_t = x_{t-1} + \dot{x}\Delta t + \ddot{x}_{t-1}\Delta t^2 \tag{2}$$

To get the remaining equations that are consistent with the first, take the derivatives of equation (2) with respect to time. The first derivative is:

$$\dot{x}_t = \dot{x}_{t-1} + \ddot{x}_{t-1}2\Delta t \tag{3}$$

and the second derivative with respect to time is:

$$\ddot{x}_t = \ddot{x}_{t-1}2 \tag{4}$$

By inspection, we can construct each row of F from equations (2-4) and then write out the state propagation from one time step to the next as:

$$\bar{x}_t = F\bar{x}_{t-1} \tag{5}$$

which is:

$$\begin{bmatrix} x_t \\ \dot{x}_t \\ \ddot{x}_t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & \Delta t^2 \\ 0 & 1 & 2\Delta t \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \dot{x}_{t-1} \\ \ddot{x}_{t-1} \end{bmatrix} \tag{6}$$

But now a problem becomes apparent with this result, namely, we no longer have a constant acceleration. At each time step the acceleration is doubled due to the coefficient value of 2. So how do we get a proper and consistent dynamics model?

2 A Second Approach

Recall that a constant acceleration model (think of the example of a falling object from physics) is:

$$x_t = x_{t-1} + \dot{x}_{t-1}\Delta t + \ddot{x}_{t-1}\frac{\Delta t^2}{2} \tag{7}$$

Taking the first derivative with respect to time of (7) results in:

$$\dot{x}_t = \dot{x}_{t-1} + \ddot{x}_{t-1}\Delta t \tag{8}$$

and the second derivative with respect to time (acceleration) is:

$$\ddot{x}_t = \ddot{x}_{t-1} \tag{9}$$

The entire state transition model from equations (7-9) is:

$$\begin{bmatrix} x_t \\ \dot{x}_t \\ \ddot{x}_t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \dot{x}_{t-1} \\ \ddot{x}_{t-1} \end{bmatrix} \tag{10}$$

So this approach results in a true constant acceleration model with a consistent set of differential equations.

3 But will it work?

Will it work? That is, is the model correct, or a close enough approximation, given the model presented in the PDF? Yes it will work and here is why.

First of all, often the dynamics model is an approximation of the true model and the noise covariance Q is used to deal with these mismatches. Also, the terms a, b, c specified in the model are implemented as constants for each meteorite and the Δt is the entire life of the meteorite from the beginning of its life until the current time. Therefore, since the KF is estimating and correcting over 1 time step intervals, one could consider the filter to be approximating the solution over such short intervals that the error is insignificant.

But that is not the answer in this case. It actually turns out that the constant acceleration model is a correct model of the process if we recognize a as scaled acceleration such that:

$$\ddot{x} = 2a. \tag{11}$$

So, given that the Kalman Filter correctly estimates the acceleration $\ddot{x}$, it will be able to correctly transition the state from one time step to the next. To repeat, in another way. The filter will estimate $2a$ rather than a directly, and from this it will be able to correctly predict the next position in time.