Methods of Determination of Orbital Trajectory of a Body from an Initial-Value Problem

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Abstract

We describe and analyse two possible methods to determine the trajectories (orbit) and future positions and velocities of an orbiting body (for example, a comet) from the limited data of its initial position vector and velocity (an initial-value problem). These methods are analysed and compared using data from the hyperbolic comet ISON to provide an example of their working.

The calculation of trajectory of comets has been worked on by many scientists from Kepler to Newton, and has been solved for a two-body system. The present research attempts to explore the trajectory of the comet through an unconventional methods using two position vectors in order to predict the orbital trajectory of the comet. The first method involves processing this data through certain recurrence relations derived from Newton's laws of motion and gravitation. The second method involves the use of astrodynamics equations, such as the vis-viva equation, in order to calculate the Keplerian orbital elements describing trajectories, such as the semi-major and minor axis, eccentricity, etc.

Though the methods suffer from various limitations, such as e.g. not accounting for perturbations in the orbit, they tend to produce similar, and approximately correct, results. Using two position vectors for the comet ISON on October 1st 2013, the time of perihelion was estimated to within 8425 seconds of the observed perihelion time (to within 0.5% error).

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1 Introduction

1.1 Celestial Mechanics

Celestial mechanics deals with the motion of objects in outer space in general, applying the principles of physics to astronomical objects such as comets, stars, planets and so on, producing "ephemeris data". Classical mechanics includes primarily orbital mechanics, which may range from astrodynamics, dealing with artificial satellite trajectories, to motions of major, minor and natural satellites (planetary dynamics). Ephemeris data, calculated by JPL HORIZONS Ephemeris Generator[1], is an example of an application of planetary dynamics for prediction of the Solar System objects motion.

1.2 Orbit Determination

Orbit determination is the process involving the use of a collection of measurements obtained by various observations to describe the orbit or trajectory of a body by calculating a set of orbital elements that can give its position and velocity at a particular instant of time.

[1]https://ssd.jpl.nasa.gov/?ephemerides

1.3 Inaccuracies in Orbit Determination

Orbiting bodies often do not experience only one force. For example, comets experience numerous forces that make the determination of an accurate orbit exponentially harder. The forces that a comet experiences include:

- **Gravitational Force of the Sun** - The strongest force in our solar system, due to which all comets orbit around the Sun. This is the primary force affecting the comet during orbit.
- **Gravitational Force of Other Celestial Bodies** - Every celestial body, ranging from planets to moons, cause deviations in the comets orbit. These forces are assumed to be negligible in this case, as they are often minute compared to that of the Sun.
- **Non-Gravitational Forces** - Comets emit gas and dust as they approach the Sun. This emission do cause deviations in the comets trajectory, though these are very small and hard to detect (Yeomans, 1994).

1.4 Comet C/2012 ISON

Comet C/2012 ISON was a comet discovered by Vitali Nevski and Artyon Novichonok using the 0.4-meter reflector of the International Scientific Optical Network near Kislovodsk, Russia (Nevski et al., 2012). This Sun-grazing comet however disintegrated just before perihelion, due to the Sun's heat and tidal forces, besides having passed within the Sun's Roche limit (Knight & Walsh, 2013). This study uses data from the comet ISON observations in order to compare the proposed methods.

1.5 The Orbital Elements

To determine the orbit of a body, we can use Keplerian orbital elements (Vallado & McClain, 2013). The first two of the orbital elements define the size and shape of the orbit, the next three define its orientation in space, and the last one defines the position of the body in the orbit at a specific time.

- **Eccentricity** (e) - All Kepler orbits are a type of a conic section circle, ellipse, parabola or hyperbola. Which of these orbits the comet follows is determined by the eccentricity of the orbit. A circular orbit has an eccentricity of 0, an elliptical orbit has an eccentricity greater than 0, a parabolic trajectory has an eccentricity of 1, and a hyperbolic trajectory has an eccentricity greater than 1.
- **Semi-major Axis** (a) - The semi-major axis of the orbit or trajectory is similar to that of an ellipse in general. For a hyperbolic trajectory, it is considered to be the distance from the point from the perihelion (closest approach to Sun) to the point where the asymptotes of the hyperbola intersect.
- **Inclination** (i) - This refers to the inclination of the orbit with respect to the plane of the ecliptic, the plane in the sky that the Sun follows over the course of the year. It may also be considered the plane in which the planets revolve in, as

all the planets revolve within 6 degrees of it.

- **Longitude of the Ascending Node (Ω)** - The longitude of the ascending node is the angle from the First Point of Aries (0,0 for ecliptic coordinates) to the point where the comet meets the ecliptic plane from below (ascending node).

- **Argument of Periapsis (ω)** - The argument of periapsis is the angle between the ascending node and the perihelion.

- **True Anomaly (ν)** - The true anomaly defines the position of the comet in the orbit. It is the angle between the comet and the perihelion in the plane of the orbit.

1.6 Source Data

The source data used here is the initial position vector $\mathbf{r}$ of the body in orbit and its initial velocity $\mathbf{v}$; the orbit is propagated from this initial data by using two proposed methods, one involving recursive generation of the orbit using the equations of motion, and another using Keplerian orbital elements. The aim is to predict the ideal orbit of the comet based on the data from the initial-value problem.

The source data used for the initial value has been taken from NASA JPL's HORIZONS Ephemeris Generator.

1.7 Nomenclature

$\mu = GM$ - standard gravitational parameter

$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ - body's position vector

$\mathbf{v}$ - body's velocity vector
$\mathbf{a}$ - body's acceleration vector
t - time
$\mathbf{r_0}$ - initial position vector
$\mathbf{v_0}$ - initial velocity vector
$\mathbf{v_n}$ - velocity vector of nth recursion
$\mathbf{R_n}$ - displacement vector from gravitational body n
$\mathbf{h}$ - specific angular momentum vector
$\mathbf{e}$ - eccentricity vector
$\mathbf{F_{net}}$ - net force vector
$\hat{\mathbf{p}}$ - unit vector of 'p', where p is any vector
M - mean anomaly
E, F - eccentric anomaly, hyperbolic anomaly
$\|p\|$ - magnitude of 'p', where p is any vector
b - latitude (heliocentric ecliptic coordinates)
l - longitude (heliocentric ecliptic coordinates)
β - latitude (geocentric ecliptic coordinates)
γ - longitude (geocentric ecliptic coordinates)

2 Method 1 - Recursive Generation of Trajectory

2.1 Assumptions

This method to generate the orbit relies on few key assumptions in the calculation of future positions and velocities.

- **Constant Acceleration over time span δt** - The model assumes that in a very short interval of time δt the change in the acceleration on the body is negligible.

- **All Significant Forces are Considered** - The model also assumes that all significant gravitational forces have been taken into account in the calculation of the net force on the orbiting body.
- **Gravity is the Sole Force** - The model does not take into account any forces besides gravity that may perturb the trajectory of the body, for example, outgassing of a comet.

2.2 Equations of Motion

The equations of motion describe the motion of a body. For a constant acceleration in any direction, the equations of motion for displacement, velocity and acceleration vectors are similar to those for a constant linear, translational acceleration. The equations of motion that will be used in order to create recurrence relations are as follows:

$\mathbf{v} = \mathbf{v_0} + \mathbf{a}t$,

$\mathbf{r} = \mathbf{r_0} + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$.

Since we are assuming that $\mathbf{a}$ changes negligibly in the relatively short time span δt, we may use these equations (requiring constant acceleration) to predict the motion in the short time span δt.

2.3 Net Force on the Orbiting Body

A body in orbit is subject to a variety of gravitational forces, besides the most significant gravitational force from the object it is orbiting. If $\mathbf{R}$ is the displacement between the bodies and $\mathbf{r}$ the position vector of the object, the force of gravity between them is given by:

$$\mathbf{F} = -\frac{GMm}{\|R\|^2}\hat{\mathbf{R}} \tag{1}$$

Since we are considering the force vector on the orbiting body, the direction of the gravitational force vector is opposite to that of the displacement vector $\mathbf{R}$, resulting in the negative sign. The overall gravitational force on the body at a particular instant is given by:

$$\begin{aligned}\mathbf{F_{net}} &= \left(-\frac{GM_1m}{\|R_1\|^2}\hat{\mathbf{R_1}}\right) + \left(-\frac{GM_2m}{\|R_2\|^2}\hat{\mathbf{R_2}}\right) \\ &\quad + \left(-\frac{GM_3m}{\|R_3\|^2}\hat{\mathbf{R_3}}\right) + \dots \\ &= -\sum_{i=1}^{n}\frac{GM_im}{\|R_i\|^2}\hat{\mathbf{R_i}}\,.\end{aligned} \tag{2}$$

2.4 Recurrence Relations

The initial velocity vector $\mathbf{v_0}$ of the body and initial displacement vector $\mathbf{r_0}$ are known to us, as well as all displacement vectors $\mathbf{R}$ between the body and other gravitational objects. When the net force $\mathbf{F_{net}}$ of all significant gravitational bodies calculated for an instant δt, the net acceleration vector can be found, along with the change in velocity and displacement during δt,

$$\mathbf{a_n} = \frac{(\mathbf{F_n})_{\mathbf{net}}}{m} = -\sum_{i=1}^{n}\frac{GM_i}{\|(R_n)_i\|^2}(\hat{\mathbf{R_n}})_{\mathbf{i}}\,, \tag{3}$$

$$\mathbf{v_{n+1}} = \mathbf{v_n} + \mathbf{a_n}(\delta t)\,, \tag{4}$$

$$\mathbf{r_{n+1}} = \mathbf{v_n}(\delta t) - \frac{1}{2}\mathbf{a_n}(\delta t)^2\,. \tag{5}$$

With the new position vector $\mathbf{r_{n+1}}$ known, the displacement vector $\mathbf{R_{n+1}}$ between the body and other gravitational objects can once again be calculated for the next time interval, allowing us to calculate the new acceleration as in Equation 3. These recurrence relations may be performed using a computer, or a spreadsheet software, to determine the future positions, velocities and acceleration of the orbiting body.

If the body is under the primary influence of the sun, the ecliptic coordinates of the body in spherical form can be obtained from the rectangular ecliptic coordinates. From rectangular ecliptic coordinates at a a certain point in time, the calculation of ecliptic spherical coordinates, where b is the latitude and l is the longitude, is done as follows,

$$b = \sin^{-1}\left(\frac{r_z}{\|r\|}\right), \tag{6}$$

$$l = \cos^{-1}\left(\frac{r_x}{\|r\|\cos(b)}\right). \tag{7}$$

If $r_y < 0$, since $l \in [0, 2\pi]$,

$$l = 2\pi - \cos^{-1}\left(\frac{r_x}{\|r\|\cos(b)}\right). \tag{8}$$

2.5 Limitations

- The future positions of all other significant objects causing gravitational forces must also be known and calculated in order to find $\mathbf{R}$ for the model. Hence, this method works better when there are fewer massive objects that can perturb the orbit.
- The method requires thousands of calculations to be performed and is dependent on computational capability.
- The method requires assumptions (such as acceleration being constant over δt) that may lead to inaccuracy as the error accumulates over time.

3 Method 2 - Keplerian Orbital Elements

3.1 Assumptions

This method also requires a few key assumptions, but no longer requires the assumption of acceleration being constant for a short time interval δt.

- **Only Two Bodies in System** - This method assumes that there are only two bodies in the system that have gravitational effects on each other, and one is much more massive than the other (such as a star and a comet).
- **Gravity is the Sole Force** - This method does not take into account forces besides gravity.

3.2 Determination of Semi-Major Axis, Momentum and Energy

The semi-major axis is one of the orbital elements. It can be calculated from an initial-value problem by taking the well-known vis-viva equation, which relates the magnitude of position and velocity of an orbiting body, and is derived from the conservation of energy and angular momentum. Many of the following equations are well known orbital dynamics equations. The semi-major axis a may be positive or negative, depending on the eccentricity of the orbit: elliptical, hyperbolic, or parabolic (Vallado & McClain, 2013),

$$\|v\|^2 = \mu\left(\frac{2}{\|r\|} - \frac{1}{a}\right), \tag{9}$$

$$a = \frac{\mu\|r\|}{2\mu - \|v\|^2\|r\|}. \tag{10}$$

The specific relative angular momentum $\mathbf{h}$ is defined as the angular momentum $\mathbf{L} = m\mathbf{v}\times\mathbf{r}$ per unit mass m,

$$\mathbf{h} = \frac{\mathbf{L}}{m} = \mathbf{r} \times \mathbf{v}. \tag{11}$$

The specific orbital energy can be calculated from the semi-major axis and gravitational parameter (Vallado & McClain, 2013),

$$\varepsilon = -\frac{\mu}{2a}. \tag{12}$$

The specific orbital energy is positive for hyperbolic orbits, zero for parabolic orbits, and negative for elliptic orbits.

3.3 Shape and Orientation of Orbit in Space

The orbital elements, except those relating position and time from Section 1.5, can be calculated using the information from that has been derived in section 3.2. The eccentricity vector $\mathbf{e}$ can be calculated using the position, velocity and angular momentum vector (Vallado & McClain, 2013),

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|r\|}. \tag{13}$$

The inclination of the orbit i, longitude of the ascending node Ω and the argument of periapsis ω can also be calculated using angular momentum, after defining an ascending node vector $\mathbf{n}$ (Vallado & McClain, 2013),

$$\mathbf{n} = \hat{\mathbf{k}} \times \mathbf{h}, \tag{14}$$

$$i = \cos^{-1}\left(\frac{h_z}{\|h\|}\right), \tag{15}$$

$$\Omega = \cos^{-1}\left(\frac{n_x}{\|n\|}\right). \tag{16}$$

If $n_y < 0$, since $\Omega \in [0, 2\pi]$,

$$\Omega = 2\pi - \cos^{-1}\left(\frac{n_x}{\|n\|}\right), \tag{17}$$

$$\omega = \frac{\mathbf{n}\cdot\mathbf{e}}{\|n\|\|e\|}. \tag{18}$$

If $e_z < 0$, since $\omega \in [0, 2\pi]$,

$$\omega = 2\pi - \frac{\mathbf{n}\cdot\mathbf{e}}{\|n\|\|e\|}. \tag{19}$$

3.4 Position and Time of Body in Orbit

The calculation of true anomaly ν at the initial position can be done using the following orbit equation (Vallado & McClain, 2013),

$$\|r\| = \frac{a(1-e^2)}{1+e\cos(\nu)}, \tag{20}$$

The calculation of eccentric/hyperbolic anomaly varies depending on the eccentricity of the orbit. If elliptical, we calculate the eccentric anomaly E, and if hyperbolic, the hyperbolic anomaly F. Furthermore, this can be used to calculate a useful angular parameter called the mean anomaly M,

$$E = \cos^{-1}\left(\frac{e + \cos(\nu)}{1 + e\cos(\nu)}\right), \tag{21}$$

$$F = \cosh^{-1}\left(\frac{e + \cos(\nu)}{1 + e\cos(\nu)}\right), \tag{22}$$

$$M = E - e\sin(E)\,, \tag{23}$$

$$M = e\sinh(F) - F\,. \tag{24}$$

Finally, the time t until perihelion (the closest approach of an orbiting body to the central body), can be found as follows (Vallado & McClain, 2013):

$$t = M\sqrt{\frac{|a|^3}{\mu}}\,. \tag{25}$$

When all orbital elements, as well as the time until perihelion, are found, the trajectory of the orbiting body is determined.

3.5 Estimation of Future Positions and Velocities

Mean anomaly after a certain time t can be calculated from mean anomaly at the initial position, as shown below,

$$M = M_0 + t\sqrt{\frac{\mu}{|a|^3}}\,. \tag{26}$$

Calculation of new true anomaly from the new mean anomaly requires solution of the Kepler's equation which is transcendental. Numerical analysis techniques, such as the Newton-Raphson method or approximate series expansions, can be used to calculate these. Otherwise, the answer can be solved numerically by a graphing software.

For the elliptical orbit, an example of a series expansion to estimate true anomaly is

$$\nu \approx M + 2e\sin M + 1.25e^2\sin 2M\,. \tag{27}$$

From true anomaly, the magnitude of the position vector of the orbiting body can be calculated using the orbit equation (Equation 20), and the velocity using the vis-viva equation (Equation 9),

$$\|r\| = \frac{a(1-e^2)}{1 + e\cos(\nu)}\,,$$

$$\|v\|^2 = \mu\left(\frac{2}{\|r\|} - \frac{1}{a}\right).$$

3.6 Limitations

- **Producing Future Positions and Velocities** - While not requiring computation, this method suffers from being difficult to solve future positions and velocities of the orbiting body easily due to transcendental equations. Furthermore, even if future $\|r\|$ and $\|v\|$ may be found, the vector directions are difficult to obtain.

- **Solves Only A Two-Body Problem** - The method only can take into account a system of two bodies and, therefore, suffers from inability to solve perturbations from other gravitational bodies, besides any additional non-gravitational perturbations.

4 Comparison of the methods in determining the orbit of comet C/2012 ISON

4.1 Source Data and Coordinate Transformations

The models require the availability of data of an initial-value problem. To test these two models, we used two position vectors of Comet C/2012 ISON for October 1, 2013 at Universal Time 22:43:18 and 23:03:55, acquired from JPL's HORIZONS Ephemeris Generator. The time span between these observations is 1237 seconds. Below are the two position vectors taken from JPL, at 22:43:18 and 23:03:55 in the form of ecliptic spherical coordinates, that provide the data necessary for the initial-value condition to be provided.

$$\mathbf{R_1} \equiv \begin{bmatrix} R \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 3.1757723 \times 10^{11} \\ 3.0119287^\circ \\ 140.9443874^\circ \end{bmatrix} ,$$

$$\mathbf{R_2} = \begin{bmatrix} 3.1751296 \times 10^{11} \\ 3.0114493^\circ \\ 140.9523988^\circ \end{bmatrix} .$$

We may convert the spherical ecliptic coordinates in the form (R, β, γ) to rectangular coordinates using the following transformation equations (similar to Equations 6 and 7 rearranged).

$$R_x = \|r\| \cos(\beta) \cos(\gamma), \tag{28}$$

$$R_y = \|r\| \cos(\beta) \sin\gamma, \tag{29}$$

$$R_z = \|r\| \sin(\beta). \tag{30}$$

These are the geocentric rectangular coordinates of the comet. Using heliocentric rectangular coordinates of the Earth position relative to the Solar System barycenter from JPL Horizons, and appropriate geocentric ecliptic to heliocentric ecliptic transformations, we shift the origin of the coordinate system to the barycenter of the Solar System to obtain the position vector (Vallado & McClain, 2013).

$$\mathbf{r} = \mathbf{R} + \mathbf{r_{earth}} \, .$$

This provides position vectors in rectangular heliocentric coordinates, as shown below. The velocity vector can be calculated from this data. The assumption made is that velocity is constant over small intervals such as $\delta t = 1237\,\mathrm{s}$, and hence velocity $\mathbf{v}$ can also be calculated,

$$\mathbf{r_1} = \begin{bmatrix} -9.8249745 \times 10^{10} \\ 2.2251597 \times 10^{11} \\ 1.6685951 \times 10^{10} \end{bmatrix} ,$$

$$\mathbf{r_2} = \begin{bmatrix} -9.8234310 \times 10^{10} \\ 2.2247802 \times 10^{11} \\ 1.6679921 \times 10^{10} \end{bmatrix} ,$$

$$\mathbf{v} = \frac{\mathbf{r_2} - \mathbf{r_1}}{t} = \begin{bmatrix} 1.2477128 \times 10^4 \\ -3.0675133 \times 10^4 \\ -4.8746831 \times 10^3 \end{bmatrix} .$$

Therefore, we obtain the source data to use in the two methods.

4.2 Results and Comparison of Both Methods

In Method 1, only the Sun (the most significant gravitational force on the comet) is taken into account, and the time over which the acceleration is constant is taken to be $(\delta t) = 750\,\mathrm{s}$ (step size). The results of the trajectory prediction by both methods are listed in the table in Figure 1. Method 1 does not produce certain orbital elements in its process of predicting future trajectory, positions and velocities of the comet, so elements such as inclination, longitude of ascending node, and argument of periapsis (uniquely produced by Method 2) have not been added.

With Method 1, the future position and velocity vectors of the comet can also be predicted. However, in Method 2, the calculation of displacements and velocities requires numerically solving Kepler' Equation which is transcendental. For this reason, only 20 data points (at 250000 second intervals) have been calculated numerically and used in Method 2, whereas Method 1 calculates the position vectors and velocities for every interval of $\delta t = 750\,\mathrm{s}$. The results are plotted with respect to time (in seconds) since the time of the position vector data equal to UT=22:43:18, October 1st, 2013 (which shall be considered the epoch of observation).

The displacement–time, velocity–time and acceleration–time tables and graphs obtained from both methods are also shown in Figures 2, 3 and 4.

5 Conclusion

5.1 Evaluation of Results

Both methods produced very similar orbital elements for comet ISON, and, to a certain extent, similar velocity change with time. However, the obtained distance to the Sun (delta) with time is slightly different. Comparing 'time until perihelion' with actually observed time of the comet's perihelion, we find that the time until perihelion of Method 2 is different by about 8425 seconds (0.17%) from the observed value (which is 4953720 seconds since UT=22:43:18 October 1, 2013). The time until perihelion obtained by Method 1 differs by about 17220 seconds (0.34%).

5.2 Conclusion

The two proposed methods to calculate the comet trajectory might be useful in order to determine an initial trajectory of a body such as a comet or asteroid from their initial position vector and velocity data, and are possible methods to receive rough estimates of the comet trajectory with a limited number of observations.

Predicting orbital trajectories and, hence the positions in the sky, of various Solar System objects, as well as satellites, would be invaluable for astronomers and astrophysicists investigating space, and can additionally be used to predict possible collisions, and gather knowledge about the origins of some bodies such as e.g. comets.

5.3 Scope for Further Work

- Method 1 had a degree of inaccuracy in the calculation of comet ISON's trajectory due to the fact that only the Sun was considered as a gravitational body. This can be improved by adding the effects of various other gravitational objects provided we know their variation with time.
- Using a system with high computational capability, the step size δt could be made

Comparison of Method 1 and Method 2

Orbital Elements/ Results	Method 1	Method 2
Eccentricity (e)	1.00056485315212	1.00050137810302
Perihelion Distance (q) (in AU)	0.0131712231685369	0.011532299012302
Time Until Perihelion (t) (in s)	4936500	4945294.56058056
Semi-Major Axis (a) (in AU)	-25.2011757545193	-27.6695691424324
Specific Angular Momentum (h) (in m^2/s^2)	7.517970244994855E+14	6.766458063406568E+14
Maximum Velocity (at Perihelion) (in m/s)	381542.482084268	392210.948501245
Inclination (i) (in degrees)	-	69.4550146488665
Longitude of The Ascending Node (Ω) (in degrees)	-	295.296580856069
Argument of Periapsis (ω) (in degrees)	-	346.018108012809

Figure 1: Comparison of orbital elements calculated by Methods 1 and 2.

smaller, allowing Method 1 to produce even more accurate results.

- Perturbations to the trajectories of orbiting bodies should be accounted for using other methods and a greater number of observations.

References

D. A. Vallado and W. D. McClain. *Fundamentals of Astrodynamics and Applications*, Second edition (Microcosm Press, Hawthorne, CA, 2013), pp. 20-33, 151-198.

D. K. Yeomans. *A Review of Comets and Nongravitational Forces*. Symposium - International Astronomical Union, 160, 241 (1994).

M. M. Knight and K. J. Walsh. *Will comet ISON (C/2012 S1) survive perihelion?* The Astrophysical Journal Letters, 776, L5 (2013).

V. Nevski, A. Novichonok, O. Burhonov, et al. *Comet C/2012 S1 (Ison)*. Central Bureau Electronic Telegrams, 3238, 1 (2012).

Comparison of Method 1 and Method 2's Displacement and Velocity with Time

Time Since Epoch	Method 1 (Displacement)	Method 2 (Displacement)	Method 1 (Velocity)	Method 2 (Velocity)
0	2.43792E+11	2.43792E+11	33472.45219	33472.45219
250000	2.35387E+11	2.35389E+11	34049.37193	34047.91209
500000	2.26838E+11	2.26838E+11	34667.51536	34666.27689
750000	2.18131E+11	2.18129E+11	35334.44788	35333.44446
1000000	2.09226E+11	2.09249E+11	36059.55926	36056.54160
1250000	2.00188E+11	2.00183E+11	36844.77065	36844.27885
1500000	1.90893E+11	1.90914E+11	37710.37471	37707.44033
1750000	1.81372E+11	1.81422E+11	38665.52485	38659.57439
2000000	1.71633E+11	1.71684E+11	39724.35310	39717.99073
2250000	1.61618E+11	1.61671E+11	40912.17700	40905.24038
2500000	1.51261E+11	1.51348E+11	42263.45347	42251.38362
2750000	1.40581E+11	1.40672E+11	43811.51414	43797.59951
3000000	1.29490E+11	1.29588E+11	45618.71926	45602.19679
3250000	1.17882E+11	1.18024E+11	47778.64830	47751.19458
3500000	1.05727E+11	1.05882E+11	50413.35790	50378.28617
3750000	9.28507E+10	9.30244E+10	53753.44389	53706.04860
4000000	7.89970E+10	7.92406E+10	58227.10077	58141.99797
4250000	6.39403E+10	6.41821E+10	64660.22924	64545.39659
4500000	4.68037E+10	4.71743E+10	75493.33145	75210.13215
4750000	2.58045E+10	2.64471E+10	101519.05006	100322.61464
5000000	1.43870E+10	1.04567E+10	164824.81623	159393.70362

Figure 2: Comparison of displacement and velocity calculated by Methods 1 and 2.

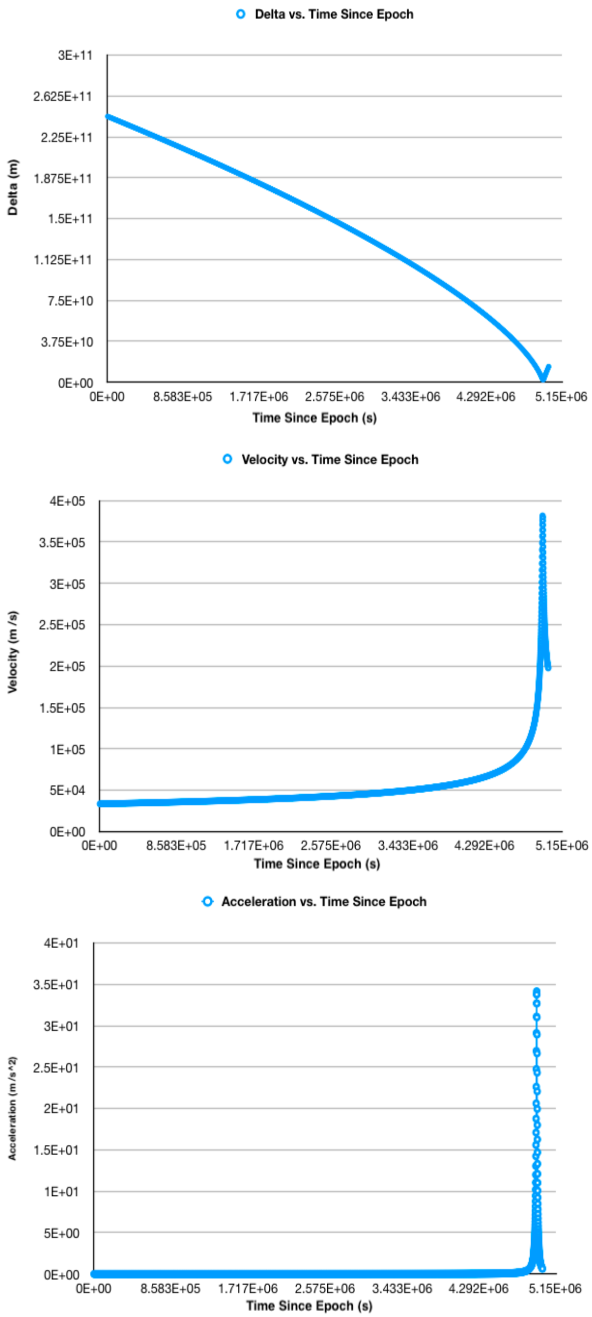

Figure 3: Results of Method 1. *Top*: Distance (Delta) (in m) of comet ISON from the central body (Sun) vs time since epoch in sec (initial time). *Middle*: Velocity of comet ISON vs time since epoch. *Bottom*: Acceleration of comet ISON vs time since epoch.

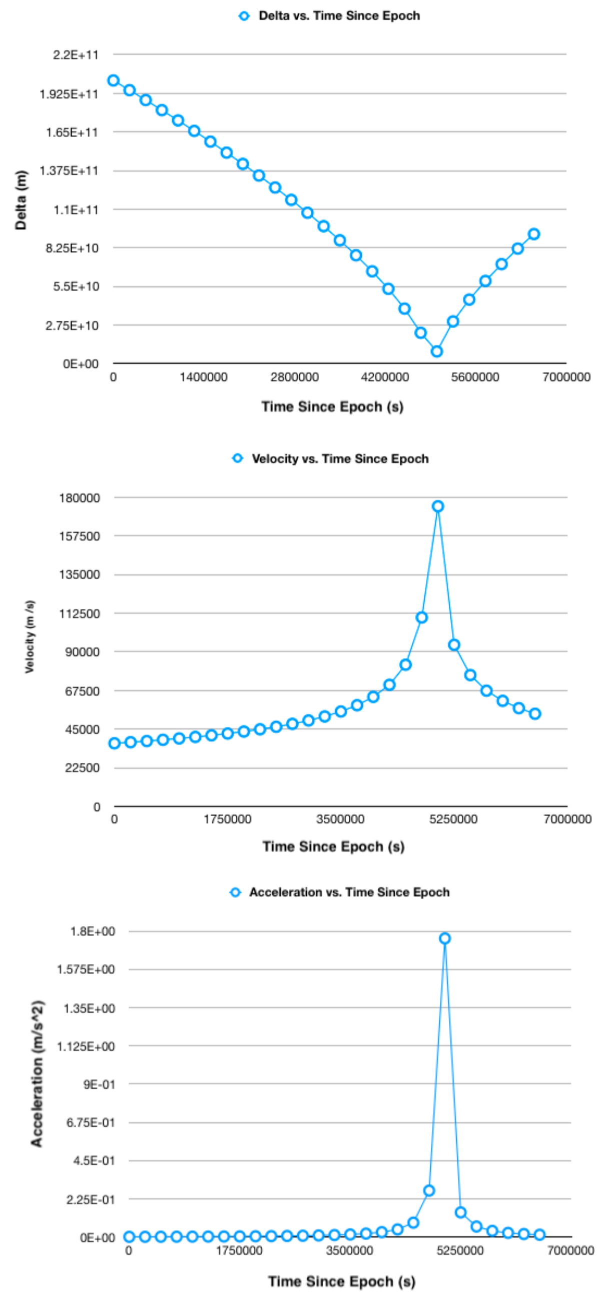

Figure 4: Results of Method 2. *Top*: Distance (Delta) (in m) of comet ISON from the central body (Sun) vs time since epoch in sec (initial time). *Middle*: Velocity of comet ISON vs time since epoch. *Bottom*: Acceleration of comet ISON vs time since epoch.