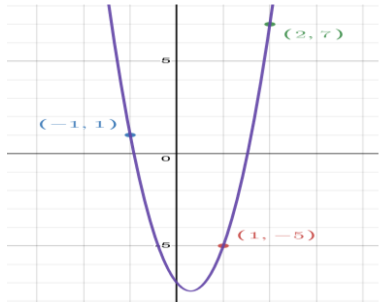
Solution Mid-Term Exam

Question 1:

Find the quadratic interpolant for the three distinct points $(\mathbf{1}, -\mathbf{5}), (-\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{7})$ **(10)**



Solution1.

The graph of a quadratic polynomial $p(x) = ax^2 + bx + c$ is a parabola, and we use the given data points to determine the coefficients a, b, and c as follows. Requiring

$$[A \mid \mathbf{y}] = \begin{bmatrix} x_1^2 & x_1 & 1 & | & y_1 \\ x_2^2 & x_2 & 1 & | & y_2 \\ x_3^2 & x_3 & 1 & | & y_3 \end{bmatrix}.$$

Setting up linear system (3), we find that its augmented matrix is (verify)

$$[A \mid \mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -5 \\ 1 & -1 & 1 & 1 \\ 4 & 2 & 1 & 7 \end{array}\right].$$

Solving this linear system, we obtain (verify)

$$a = 5, \quad b = -3, \quad c = -7.$$

Thus the quadratic interpolant is $p(x) = 5x^2 - 3x - 7$, and its graph is given in Figure 2.1. The asterisks represent the three data points. ■

Question 2: Decode the encrypted message **TBC CUG**, where encryption is **(10)**

applied by following matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$$

Solution 2: First compute inverse of $\boldsymbol{A}$ using row operations or Ajoint method

(I am using linear algebra toolkit)

$$A^{-1} = \begin{bmatrix} 2 & -4 & -1 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

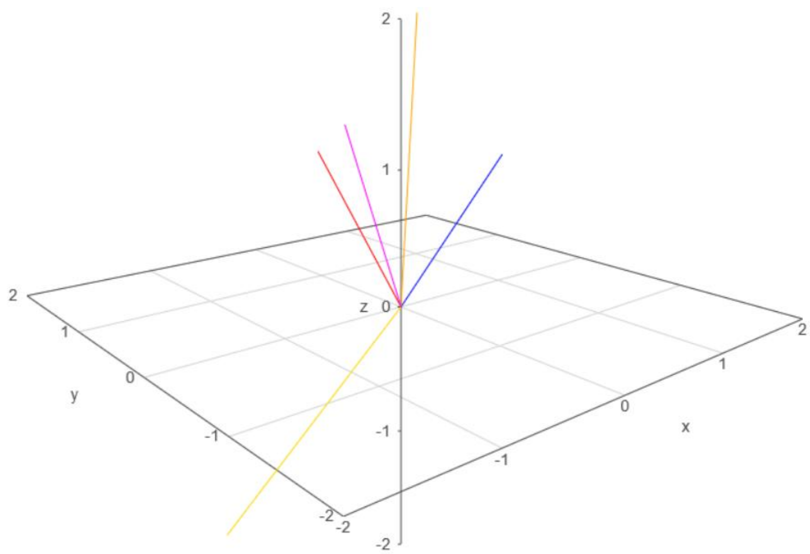
$\begin{bmatrix} T \\ B \\ C \end{bmatrix} = \begin{bmatrix} 20 \\ 2 \\ 3 \end{bmatrix}$ and $= \begin{bmatrix} C \\ U \\ G \end{bmatrix} = \begin{bmatrix} 3 \\ 21 \\ 7 \end{bmatrix}$

$$A^{-1}\begin{bmatrix} T \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 & -4 & -1 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 20 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 29 \\ 15 \\ -13 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 13 \end{bmatrix} = \begin{bmatrix} C \\ O \\ M \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} C \\ U \\ G \end{bmatrix} = \begin{bmatrix} 2 & -4 & -1 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 21 \\ 7 \end{bmatrix} = \begin{bmatrix} -85 \\ -25 \\ 46 \end{bmatrix} = \begin{bmatrix} 19 \\ 1 \\ 20 \end{bmatrix} = \begin{bmatrix} S \\ A \\ T \end{bmatrix}$$

Question 3: Find the basis for the vector space R_3 spanned by the vectors **(10)**

$v_1 = [1 \quad 0 \quad 1], v_2 = [0 \quad 1 \quad 1], v_3 = [1 \quad 1 \quad 2], v_4 = [1 \quad 2 \quad 1], v_5 = [-1 \quad 1 \quad -2].$



Solution3:

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 = 0 - - - (1)$$

$$(1) \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 & -2 & 0 \end{array}\right] R_3 - R_1 \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 - R_2 \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{array}\right] \frac{R_3}{-2} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Thereforeset $T = \{v_1, v_2, v_4\}$is L.I. subset of S.

(1) Now, number of elements in set $T = \{v_1, v_2, v_4\}$ are 3.The dimension of vector space $R_3 = 3$, guarantees Span T $=R_3$.Hence set $T = \{v_1, v_2, v_4\}$form basis for R_3.

Question 4: Using properties of the determinants, show that **(10)**

$$\begin{vmatrix} a-3 & a & a \\ a & a-3 & a \\ a & a & a-3 \end{vmatrix} = 27(a-1)$$

Solution4: Consider

$$L.H.S = \begin{vmatrix} a-3 & a & a \\ a & a-3 & a \\ a & a & a-3 \end{vmatrix} = \begin{vmatrix} 3a-3 & a & a \\ 3a-3 & a-3 & a \\ 3a-3 & a & a-3 \end{vmatrix} C_1 + (C_2 + C_3)$$

$$= (3a-3)\begin{vmatrix} 1 & a & a \\ 1 & a-3 & a \\ 1 & a & a-3 \end{vmatrix} \text{taking } (3a-3) \text{common from } C_1$$

$$= (3a-3)\begin{vmatrix} 1 & a & a \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{vmatrix} R_2 - R_1;\ \ R_3 - R_1$$

Expand with Row 1

$$= (3a-3)\begin{vmatrix} -3 & 0 \\ 0 & -3 \end{vmatrix} = 27(a-1) = R.H.S$$

Question 5: **(10)**

Let $V = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : abcd = 0\right\}$ with ordinary addition and scalar multiplication. Is V a vector space or not?

Solution 5: Given $V = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : abcd = 0\right\}$

(a) Let A and $B \in V$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -5 & 9 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 4 \\ -3 & 6 \end{bmatrix}$. Since the product (1)(4)(-3)(6) is not zero.

Hence $A + B$ *does not belong to* V.

Set V is not closed under addition and does not form a vector space.