Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

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Lecture Outline

- Counting
  - Product Rule
  - Sum Rule
  - Product and Sum rule mix questions
  - Inclusion-Exclusion Principle
  - The Pigeonhole Principle

Counting Applications

- Counting has many applications in computer science and mathematics.

- For example,
  - Counting the number of operations used by an algorithm to study its time complexity
  - Counting the successful outcomes of experiments
  - Counting all the possible outcomes of experiments
  - …

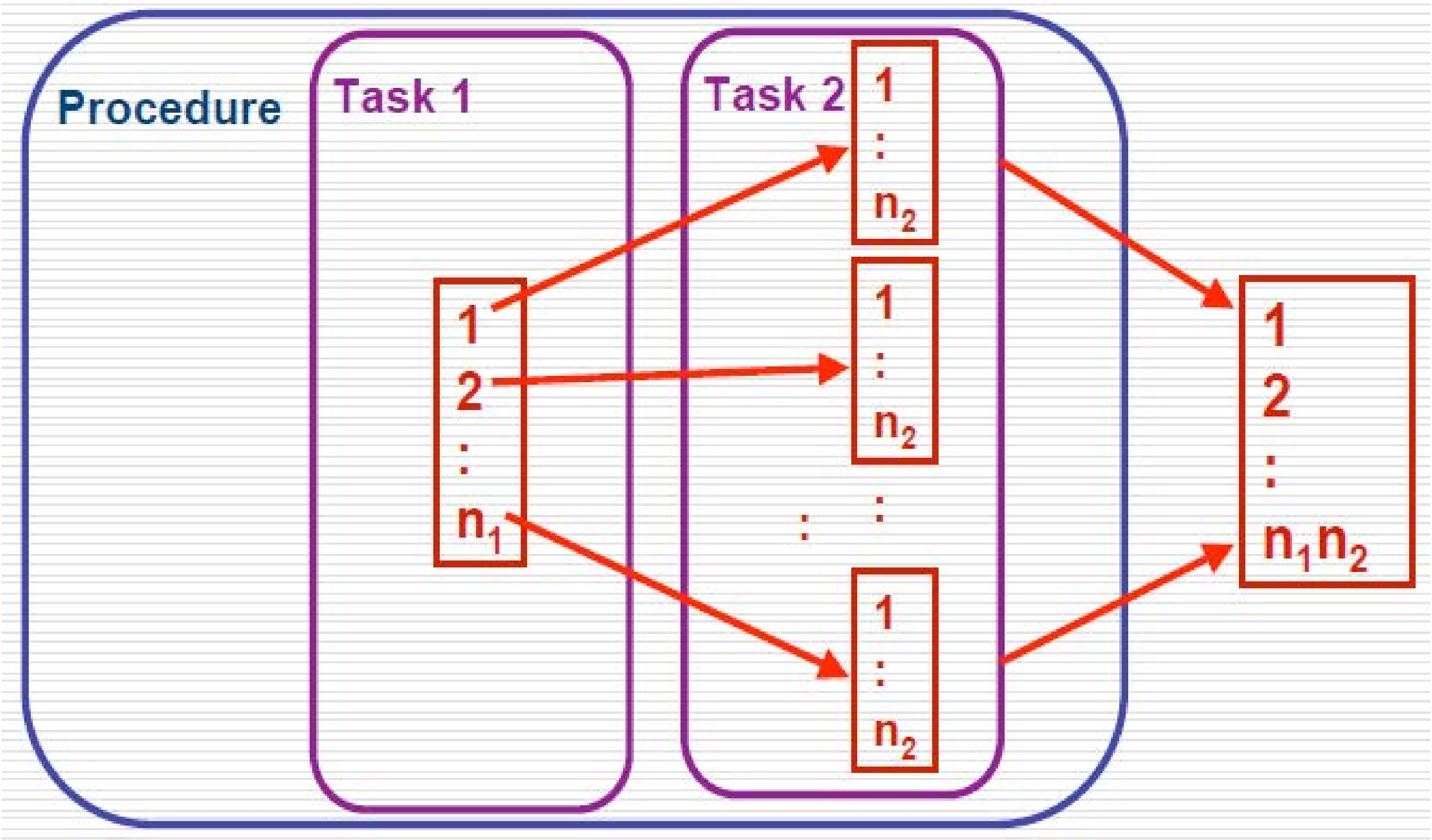
Basic Counting Principles

- Two basic counting principles
  - The product rule
  - The sum rule

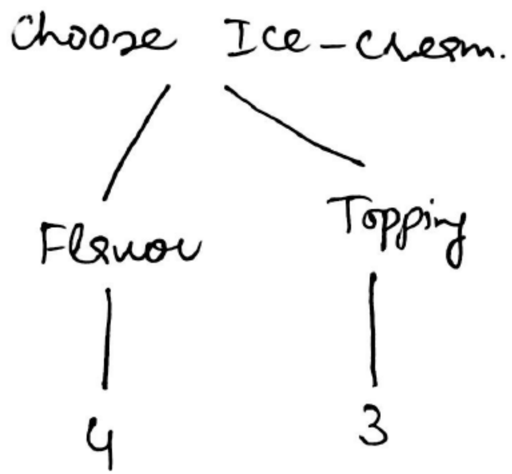
The Product Rule

- Also called the multiplicative rule.
- Suppose that a procedure can be broken into a sequence of two tasks.
- Assume there are n_1 ways to do the first task.
- Assume for each of these ways of doing the first task, there are n_2 ways to do the second task.
- So, there are $n_1 n_2$ ways to do the procedure.
  - This applies when doing the “procedure” is made up of separate tasks
  - We must make one choice AND a second choice

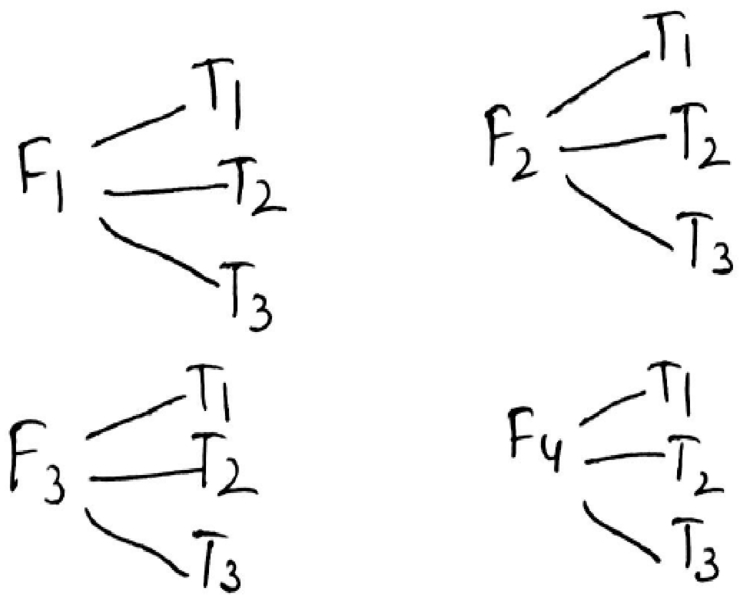
The Product Rule



Example

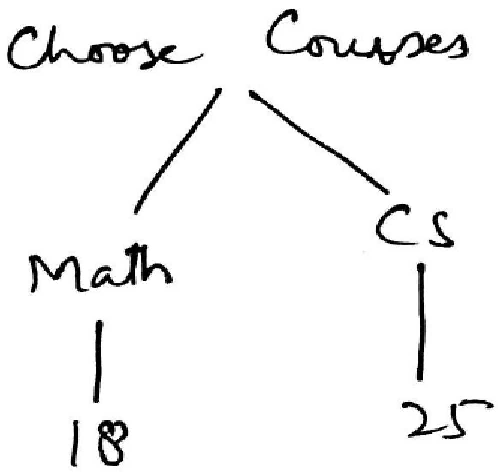


$4 \times 3 = 12$ ways



Example

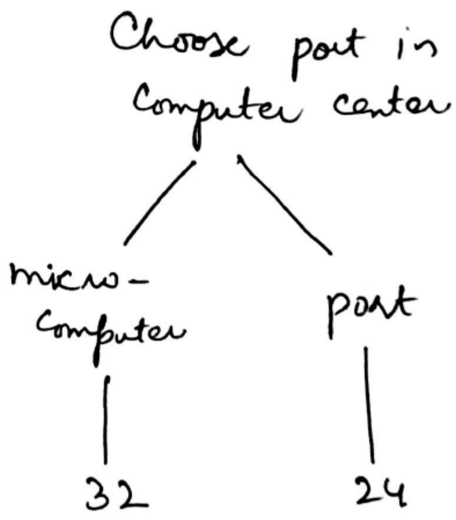
- There are 18 math majors and 25 CS majors
- How many ways are there to pick one math major and one CS major?



- Solution:
- Break the procedure into tasks
  - Task 1: Math major
  - Task 2: CS major
  - By product rule, There are 18 * 25 = 450 ways to pick courses.

Example

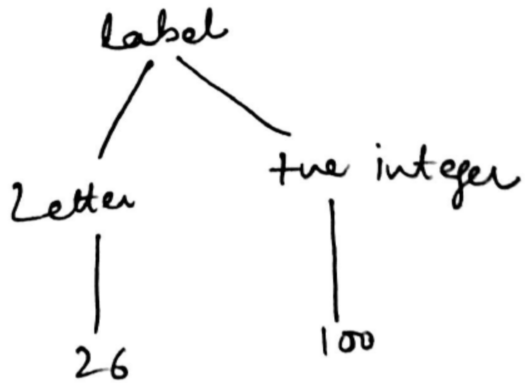
- There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there.



- Solution:
- Break the procedure into tasks
  - Task 1: Choosing a microcomputer
  - Task 2: Choosing a port
  - By product rule, There are 32*24 = 768 ways to choose a port to a microcomputer.

Example

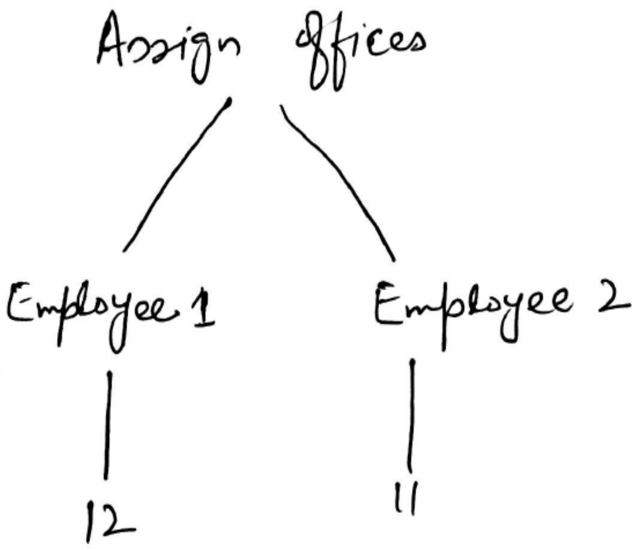
- The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. How many chairs can be labeled differently?



- Solution:
- Break the procedure into tasks
  - Task 1: assigning one of the 26 letters
  - Task 2: assigning one of the 100 possible integers
  - By product rule, There are 26*100 = 2600 ways to assign labels to the chairs.

Example

- A new company with just two employees, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?



- Solution:
- Break the procedure into tasks
  - Task 1: assigning an office to employee 1
  - Task 2: assigning an office to employee 2
  - By product rule, There are 12 *11 = 132 ways to assign offices to two employees.

Extended Version of The Product Rule

- A procedure can be broken down into a sequence of tasks $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_\mathrm{m}$
- Assume each task T_i $(\mathrm{i} = 1,2, \ldots, \mathrm{m})$, can be done in n_i different ways, regardless of how the previous tasks were done.
- The procedure can be done in $\mathrm{n}_1 . \mathrm{n}_2 \ldots \mathrm{n}_\mathrm{m}$ different ways.

Example

- How many different bit strings of length seven are there?
- Solution:
- Break the procedure into tasks
  - Task 1: assigning bit 1 to 0 or 1
  - Task 2: assigning bit 2 to 0 or 1
  - …
  - Task 7: assigning bit 7 to 0 or 1

b_7	b_6	b_5	b_4	b_3	b_2	b_1
2	2	2	2	2	2	2

- Count different ways of doing each task and then use the product rule
  - Each task can be done in 2 different ways.
  - By product rule, There are $2^7 = 128$ different bit strings of length seven.

Example

- How many uppercase English letter strings of length three are there?

$$\begin{matrix} L_1 & L_2 & L_3 \\ 26 & 26 & 26 \end{matrix} = 26^3$$

- Other versions of above question:

$$\begin{matrix} L_1 & L_2 & L_3 \\ 26 & 25 & 24 \end{matrix} \text{(Same letter not repeated)}$$

$$\begin{matrix} A & L_2 & L_3 \\ 1 & 26 & 26 \end{matrix} \text{(Start with Letter A)}$$

$$\begin{matrix} A & L_2 & L_3 \\ 1 & 25 & 24 \end{matrix} \text{(Start with Letter A and same letter not repeated)}$$

Example

- The format of telephone numbers in North America is specified by a numbering plan.
  - Let X denote a digit between 0 and 9.
  - Let N denote a digit between 2 and 9.
  - Let Y denote a digit between 0 and 1.
  - In the old plan, The format of telephone numbers is NYX-NNX-XXXX.
  - In the new plan, The format of telephone numbers is NXX-NXX-XXXX.

X	N	Y
10	8	2

- How many north American telephone numbers are possible under the old plan and under the new plan.

Example

In the old plan, the formats of the area code, office code, and station code are *NYX*, *NNX*, and *XXXX*, respectively, so that telephone numbers had the form *NYX*-*NNX*-*XXXX*.

X	N	Y
10	8	2

Solution:

- $8 * 2 * 10 = 160$ area codes with format *NYX.*
- $8 * 8 * 10 = 640$ office codes with format *NNX*.
- $10 * 10 * 10 * 10 = 10{,}000$ station codes with format *XXXX*.
- Consequently, applying the product rule again, it follows that under the old plan, $160 * 640 * 10{,}000 = 1{,}024{,}000{,}000$
- Under the new plan, $800 * 800 * 10{,}000 = 6{,}400{,}000{,}000$

Example

- How many strings of 4 decimal digits, do not contain the same digit twice?

- Solution:

$$\begin{array}{cccc} d_4 & d_3 & d_2 & d_1 \\ 10 & 9 & 8 & 7 \end{array}$$

$$10 * 9 * 8 * 7 = 5040\ ways$$

Example

- How many different license plates are available if each plate contains a sequence of three letters followed by three digits?

$$\begin{array}{cccccc} L_1 & L_2 & L_3 & D_1 & D_2 & D_3 \\ 26 & 26 & 26 & 10 & 10 & 10 \end{array}$$

- $26 * 26 * 26 * 10 * 10 * 10 =$ $17{,}576{,}000$ possible license plates

Example

- If repetitions of letters is not allowed:

L_1	L_2	L_3	D_1	D_2	D_3
26	25	24	10	10	10

$$26*25*24*10*10*10$$

- If repetitions of letters and digits are not allowed:

L_1	L_2	L_3	D_1	D_2	D_3
26	25	24	10	9	8

$$26*25*24*10*9*8$$

- If sequence of letters is LHE and repetitions of digits are not allowed:

L	H	E	D_1	D_2	D_3
1	1	1	10	9	8

$$1*1*1*10*9*8$$

Example

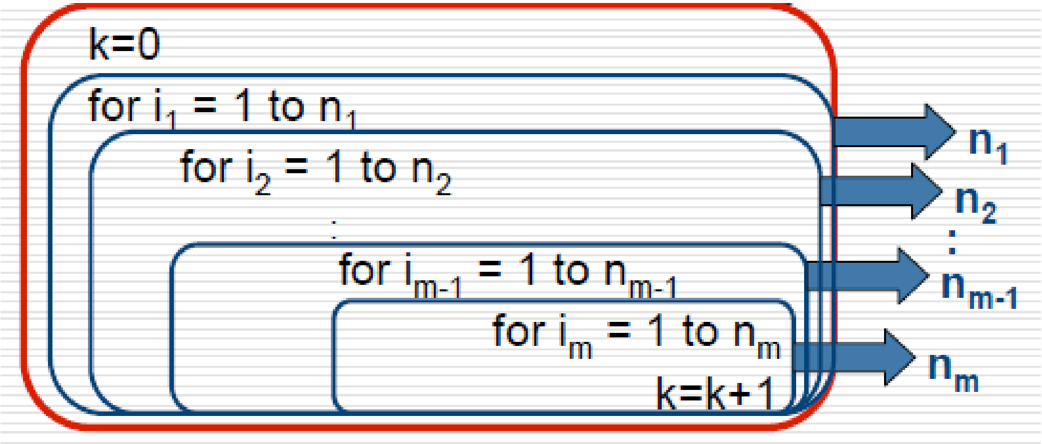
- What is the value of k after the following code has been executed?

$$\text{int } k = 0;$$
$$for(\text{int } i = 1;\ i \le 7;\ i{+}{+})$$
$$for(\text{int } j = 1;\ j \le 8;\ j{+}{+})$$
$$for(\text{int } n = 1;\ n \le 10;\ n{+}{+})$$
$$k = k + 1;$$

- Solution:
- $7 * 8 * 10 = 560$

Example

- What is the value of **k** after the following code has been executed?

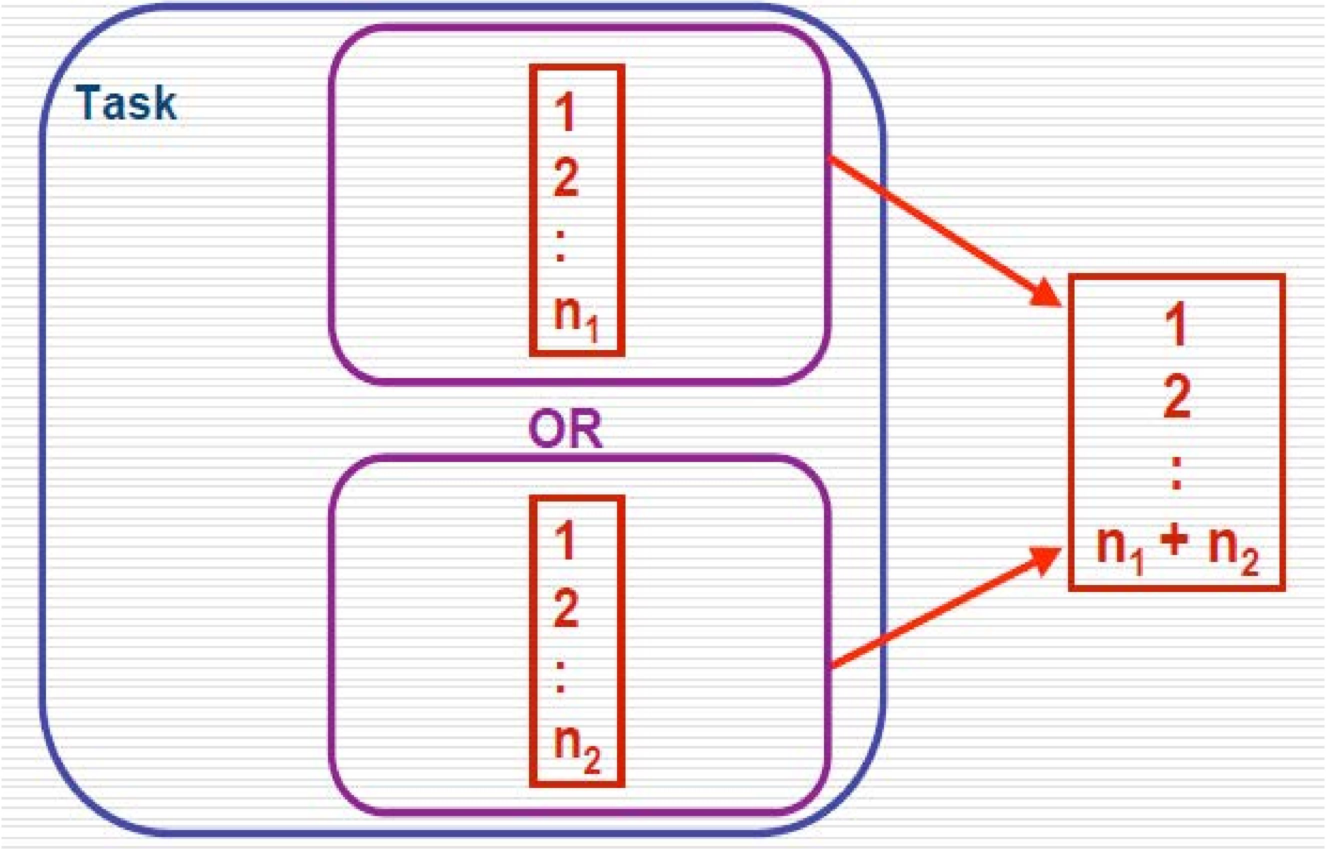


- Solution:
- Task i: traversing the i-th loop. (1≤i≤m)
- By the product rule, the nested loops traversed $n_1 n_2 \ldots n_m$ times.
- So the final value of k is $n_1 n_2 \ldots n_m$.

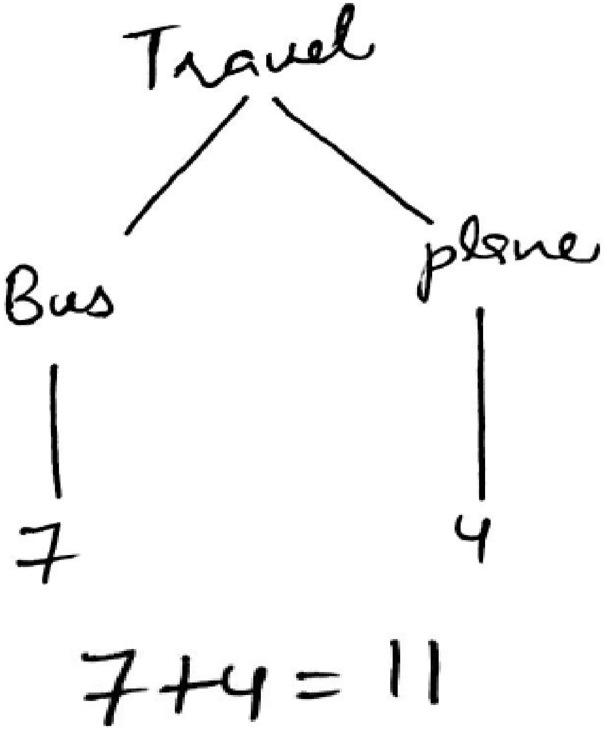
The Sum Rule

- Also called the addition rule.
- Assume a task can be done either in one of n_1 ways or in one of n_2 ways.
- Assume none of the set of n_1 ways is the same as any of the set n_2 ways.
- If these tasks can be done at the same time, then there are $n_1 + n_2$ ways to do the task.
    - We must make one choice OR a second choice.

The Sum Rule

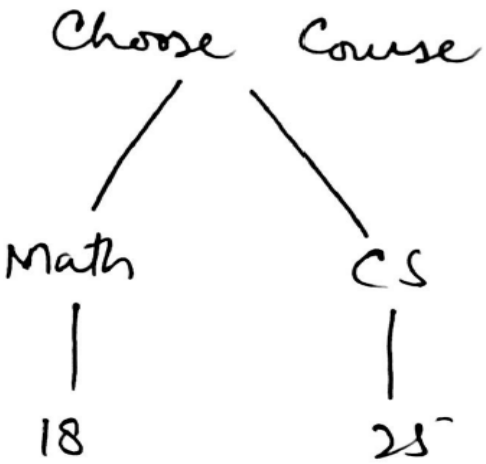


Example



Example

- There are 18 math majors and 25 CS majors
- How many ways are there to pick one math major or one CS major?
- Solution:

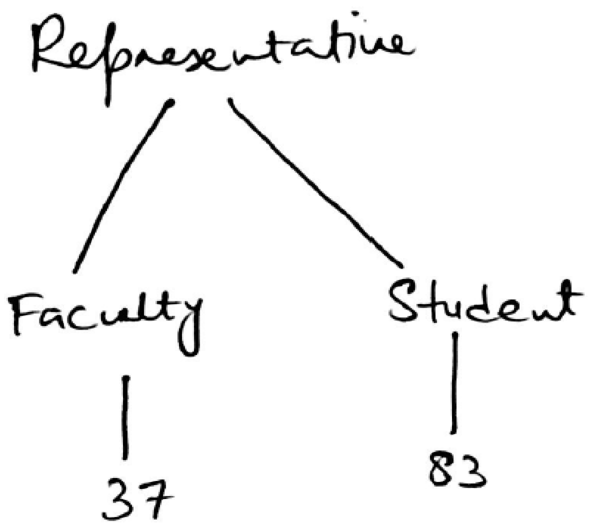


  - There are 18 ways to pick Math major
  - There are 25 ways to pick CS major

- By the sum rule, there are 18 + 25= 43 different ways to pick courses.

Example

- Assume there are 37 members of the mathematics faculty and 83 mathematics majors.
- Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee.
- How many different ways to choose this representative?
- Solution:
- 37+83=120



Extended Version of The Sum Rule

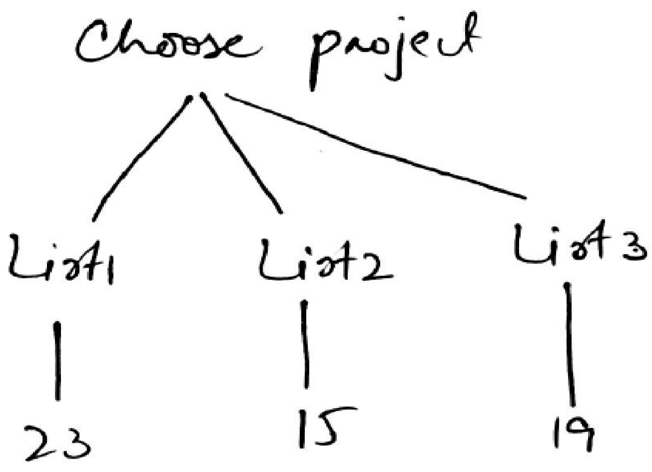
- Suppose a task can be done in one of n_1 ways, in one of n_2 ways, …, or in one of n_m ways.
- Assume none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$.
- The task can be done in $\mathrm{n}_1 + \mathrm{n}_2 + \cdots + \mathrm{n}_\mathrm{m}$ different ways.

Example

- A student can choose a computer project from one of three lists. The three lists contains 23, 15 and 19 possible projects. No project is on more than one list.
- How many possible projects are there to choose from?

- Solution:

$$23 + 15 + 19 = 57$$



Example

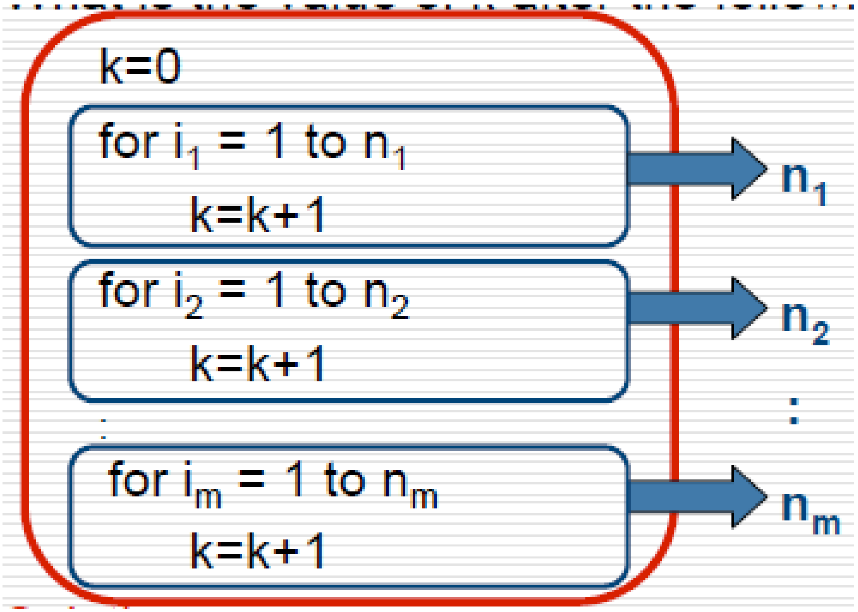
- What is the value of k after the following code has been executed?

$$
\begin{aligned}
&\text{int } k = 0;\\
&for(\text{int } i = 1;\ i \le 7;\ i{+}{+})\\
&\qquad k = k + 1;\\
&for(\text{int } j = 1;\ j \le 8;\ j{+}{+})\\
&\qquad k = k + 1;\\
&for(\text{int } n = 1;\ n \le 10;\ n{+}{+})\\
&\qquad k = k + 1;
\end{aligned}
$$

- Solution:
- $7 + 8 + 10 = 25$

Example

- What is the value of k after the following code has been executed?



- Solution:
- Task i: traversing the i-th loop. ($1 \leq i \leq m$)
- By the sum rule, the final value of k is $\mathrm{n}_1 + \mathrm{n}_2 + \cdots + \mathrm{n}_\mathrm{m}$.

The Product Rule and The Sum Rule

- Some complicated counting problems can be solved using both the product rule and the sum rule.

Example

- In a version of the computer language BASIC, the name of a variable is a string of one character or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter. How many different variable names are there in this version of BASIC?

Example

$$\begin{array}{ccc} L & D & A = L + D \\ 26 & 10 & 26 + 10 = 36 \end{array}$$

- In a version of the computer language BASIC, the name of a variable is a string of one character or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter. How many different variable names are there in this version of BASIC?
- V1 (L): variable name with one character. (int a)
- V2 (LA): variable name with two alphanumeric characters. (int aa; int a1;)
- $V1 = 26,\ V2 = 26 * 36 = 936$
- $V1 + V2 = 26 + 936 = 962$

Example

- How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

$$D_1 \quad D_2 \quad D_3 \quad L_1 \quad L_2 \quad L_3$$

$$+$$

$$L_1 \quad L_2 \quad L_3 \quad D_1 \quad D_2 \quad D_3$$

$$10^3 * 26^3 + 26^3 * 10^3$$

Example

- How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

$$L_1 \quad L_2 \quad D_1 \quad D_2 \quad D_3 \quad D_4$$

$$+$$

$$D_1 \quad D_2 \quad L_1 \quad L_2 \quad L_3 \quad L_4$$

$$26^2 * 10^4 + 10^2 * 26^4$$

The Inclusion-Exclusion Principle

- Suppose a task can be done in n_1 or in n_2 ways.
- However, some of the set of n_1 ways are the same as some of the n_2 other ways.
- To count the number of ways to the task, we add n_1 and n_2 and subtract the number of ways that is common in n_1 ways and n_2 ways.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- How many bit strings of length eight either start with a 1 bit or end with two bits 00?

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- How many bit strings of length eight either start with a 1 bit or end with two bits 00?

Solution:

	b_8	b_7	b_6	b_5	b_4	b_3	b_2	b_1	
$2^7 =$	2	2	2	2	2	2	2	1	start with 1 bit
$2^6 =$	1	1	2	2	2	2	2	2	End with 00's bits
$2^5 =$	1	1	2	2	2	2	2	1	Common ways

- By inclusion-exclusion principles, the number of such strings is $2^7 + 2^6 - 2^5 = 160$.

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- How many bit strings of length eight either start with a 1 bit or end with two bits 00?

Solution:

- Task 1: bit strings of length eight starts with 1.
- Task 2: bit strings of length eight ends with 00.
  - By the product rule, the number of task 1 is 2^7.
  - By the product rule, the number of task 2 is 2^6.
  - Common in task 1 and task 2 and the number of them are 2^5.
  - By inclusion-exclusion principles, the number of such strings is $2^7 + 2^6 - 2^5 = 160$.

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- How many uppercase English letter strings of length three either start with letter A or end with letter C?

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- How many uppercase English letter strings of length three either start with letter A or end with letter C?

Solut

	L_1	L_2	L_3	
start with letter A	1	26	26	$= 26^2$
End with letter C	26	26	1	$= 26^2$
Common way	1	26	1	$= 26$

- By inclusion-exclusion principles, the number of such strings are:

$$26^2 + 26^2 - 26 = 1326.$$

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- How many uppercase English letter strings of length three either start with letter A or end with letter C?

Solution:

- Task 1: Strings start with letter A.
- Task 2: Strings end with letter C.
  - By the product rule, the number of task 1 is 26^2.
  - By the product rule, the number of task 2 is 26^2.
  - Common in task 1 and task 2 and the number of them are 26.
  - By inclusion-exclusion principles, the number of such strings are:

$$26^2 + 26^2 - 26 = 1326.$$

Example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- A computer company receives 350 applications for a job. Suppose that 220 of them majored in computer science, 147 of them majored in business and 51 of them majored both in computer science and business. How many of these applicants majored in neither computer science nor business?

$$Total = 350$$

$$CS = 220$$

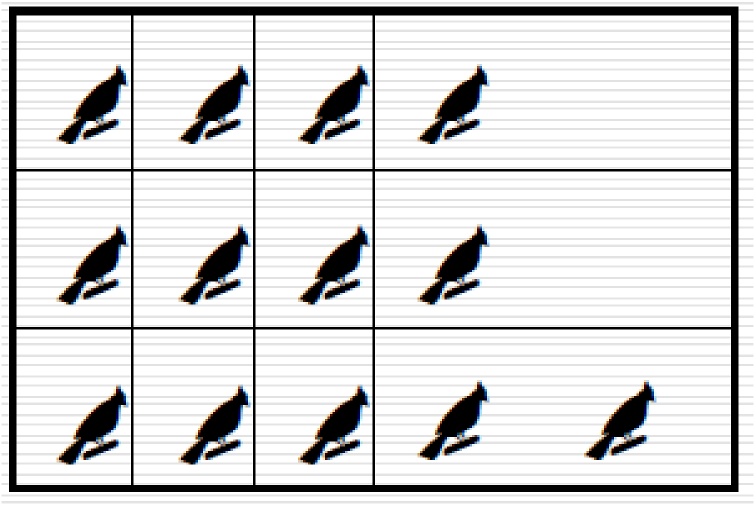
$$B = 147$$

$$CS \cap B = 51$$

- Solution:
- By inclusion-exclusion principle, 220+147-51=316.
- So, the number of applicants majored in neither computer science nor business is 350-316=34.

The Pigeonhole Principle

- Assume 13 pigeons fly into 12 pigeonholes to rest.
- A least one of 12 pigeonholes must have at least two pigeons in it.



- If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.

The Pigeonhole Principle

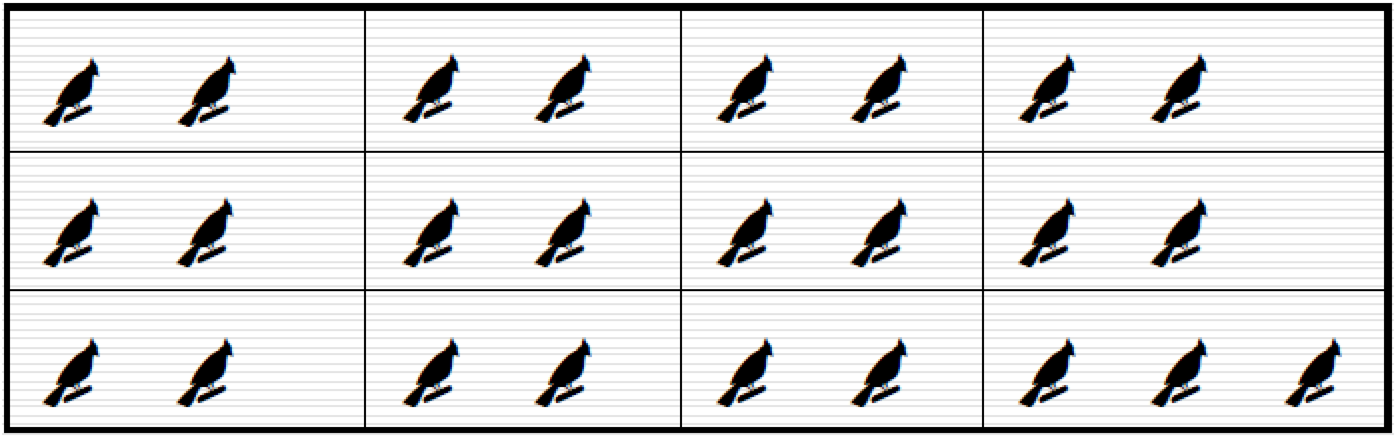
- K $\subseteq \mathbf{Z}^{+}$
- Assume k+1 or more objects are placed into k boxes.
- So, there is at least one box containing two or more of the objects.

Example

- Among a group of 367 people (randomly chosen), there must be at least two with the same birthday, because there are only 365 possible birthdays.
- In any group of 27 English words (randomly chosen), there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
- Among a set of 15 or more students, at least 3 are born on the same day of the week.

The Generalized Pigeonhole Principle

- Assume 25 pigeons fly into 12 pigeonholes to rest.
- At least one of 12 pigeonholes must have at least three pigeons in it.



- When the number of objects exceeds a multiple of the number of boxes.

The Generalized Pigeonhole Principle

- Assume N objects are placed into k boxes.
- So, there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example

- Show among 100 people there are at least 9 who were born in the same month.
- Solution:
- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each month, we have a box that contains persons who was born in that month.
  - The number of boxes is 12 and the number of objects is100.
  - By the generalized pigeonhole principle, at least one of these boxes contains at least $\lceil 100/12 \rceil$ = 9 persons.
  - So, there must be at least 9 persons who were born in the same month.

Example

A	B	C	D	F
5	5	5	5	6

- What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F.
- Solution:
- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each grade, we have a box that contains students who got that grade.
  - The number of boxes is 5, by the generalized pigeonhole principle, to have at least 6 (= $\lceil N/5 \rceil$) students at the same box, the total number of the students must be at least N = 5 . 5 + 1 = 26.

Exercise Questions

Chapter # 6

Topic # 6.1

Q 1,2,3,4,5,6,7,8,9,10,11,28,29,30,31,32,33-a,b

Chapter Reading

Chapter # 6

Topic # 6.1(The Basics of Counting)

Topic # 6.2 (The Pigeonhole Principle)