Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

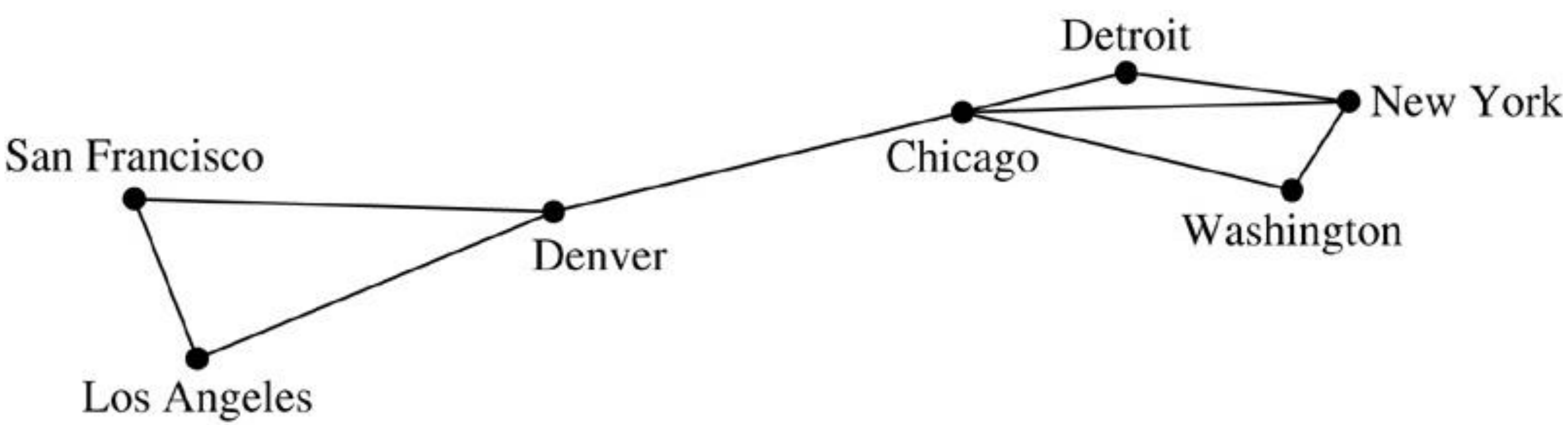
Mahwish Waqas

Lecture Outline

- Graphs
  - Undirected Graph
  - Directed Graph
  - Graph Models
  - Graph Terminologies
  - Degree of a vertex

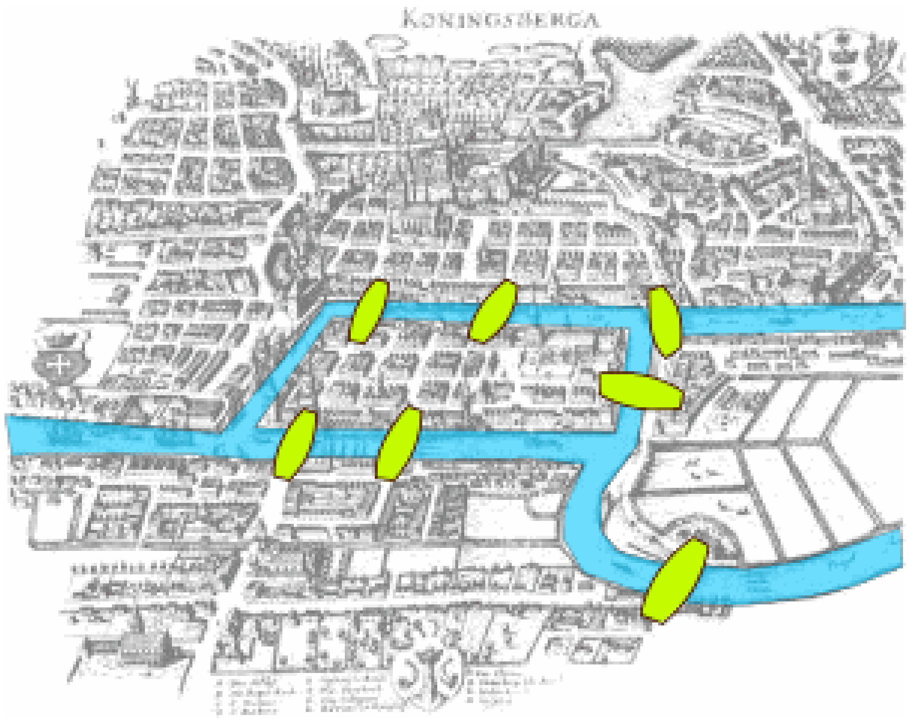
Graphs

- What are graphs?
  - A class of discrete structures useful for representing relations among objects.
  - Vertices (nodes) connected by edges.
  - Theory about graphs can be used to solve a lot of important problems



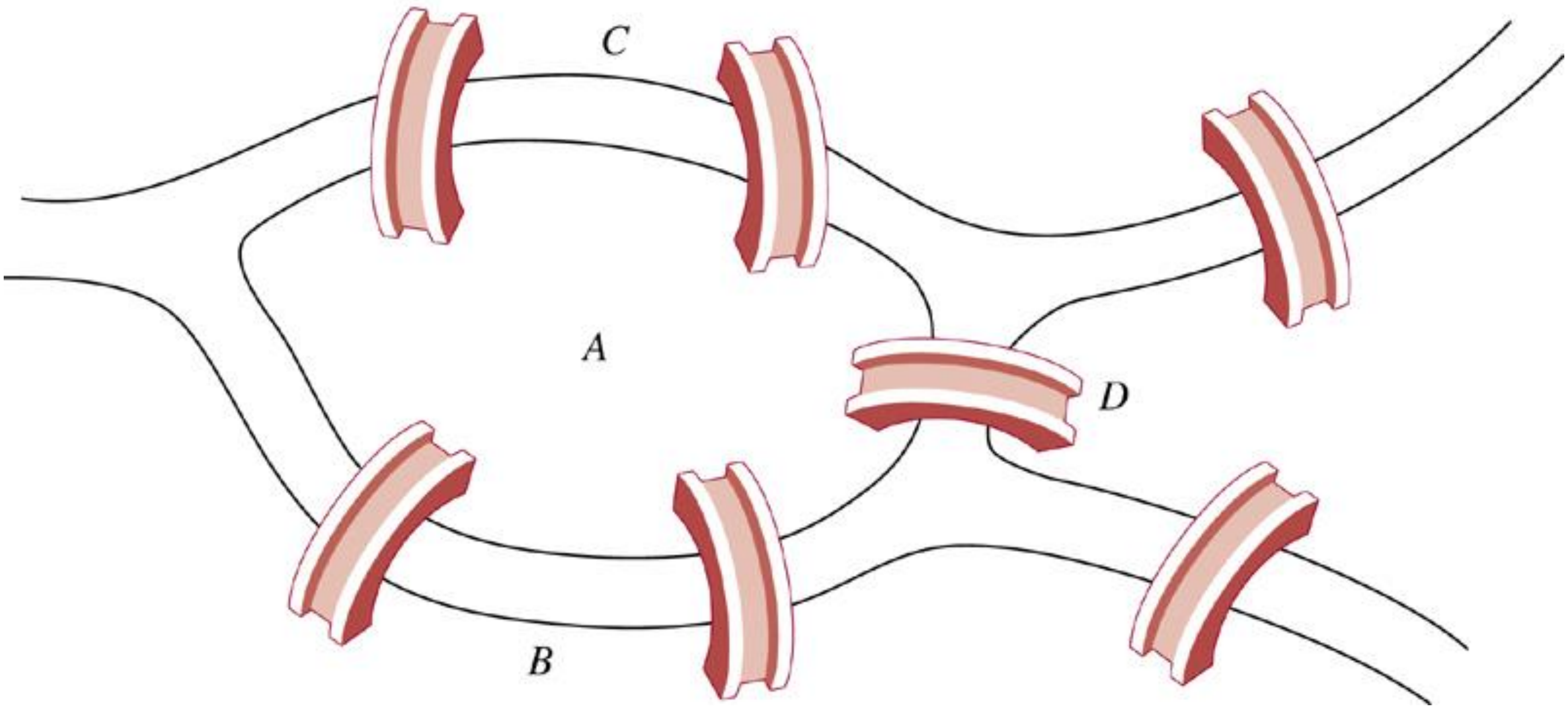
The First Graph Theory

- The first graph theory paper by Leonhard
- Euler in 1736: Seven bridges of Königsberg,
- A town with 7 bridges and 4 pieces of land…



The Origin of Graph Theory

- Can we travel each bridge exactly once and return to the starting point?



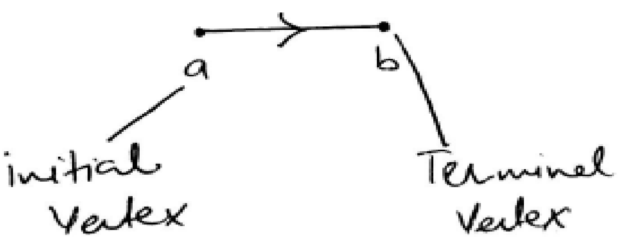
Graphs

- Graph
  - Directed
  - Undirected

Graphs

- Graph
- Directed Edge

Un-directed Edge



Ordered pair

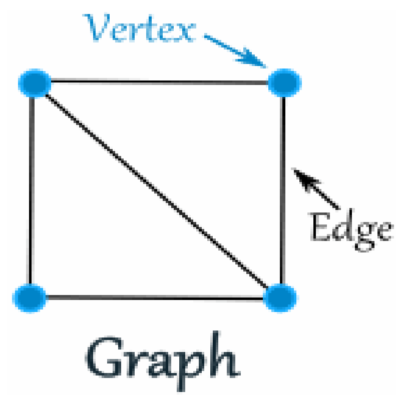
(a,b)

Un-ordered pair

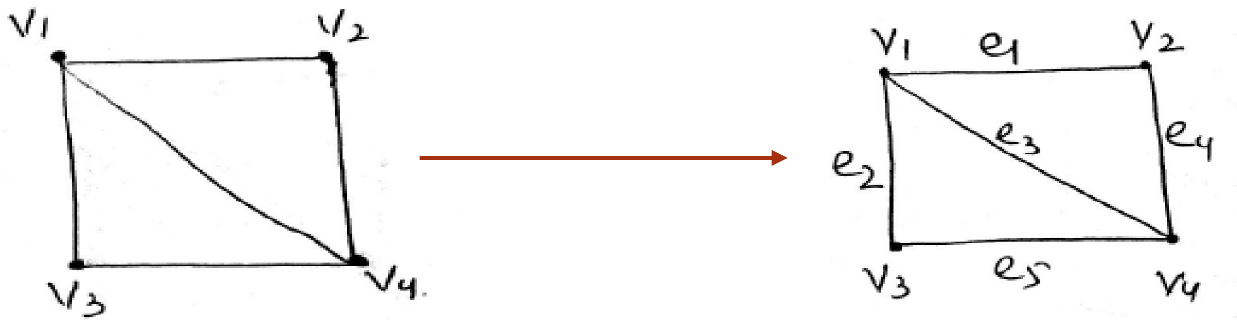
$\{a,b\}$

Definition - Graphs

- A graph G = (V, E) is defined by a set of vertices V , and a set of edges E consisting of ordered or unordered pairs of vertices from V.
- Thus a graph G = (V, E)
  - V = set of vertices
  - E = set of edges = subset of V $\times$ V
- Each edge has either one or two vertices associated with it, called its endpoints.
- An edge is said to connect its endpoints.



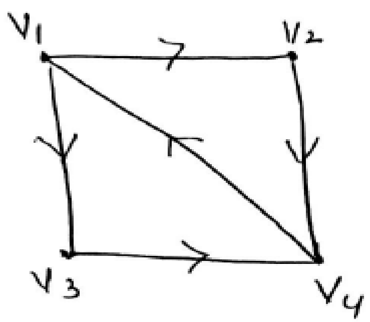
Graphs

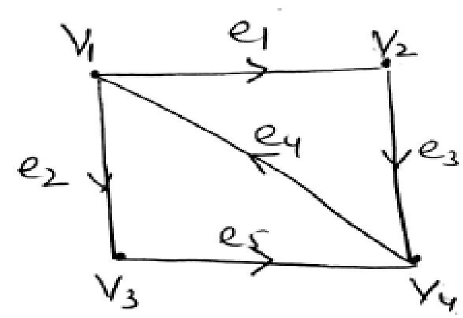


$$V = \{v_1, v_2, v_3, v_4\}$$

$$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}\}.$$

Graphs



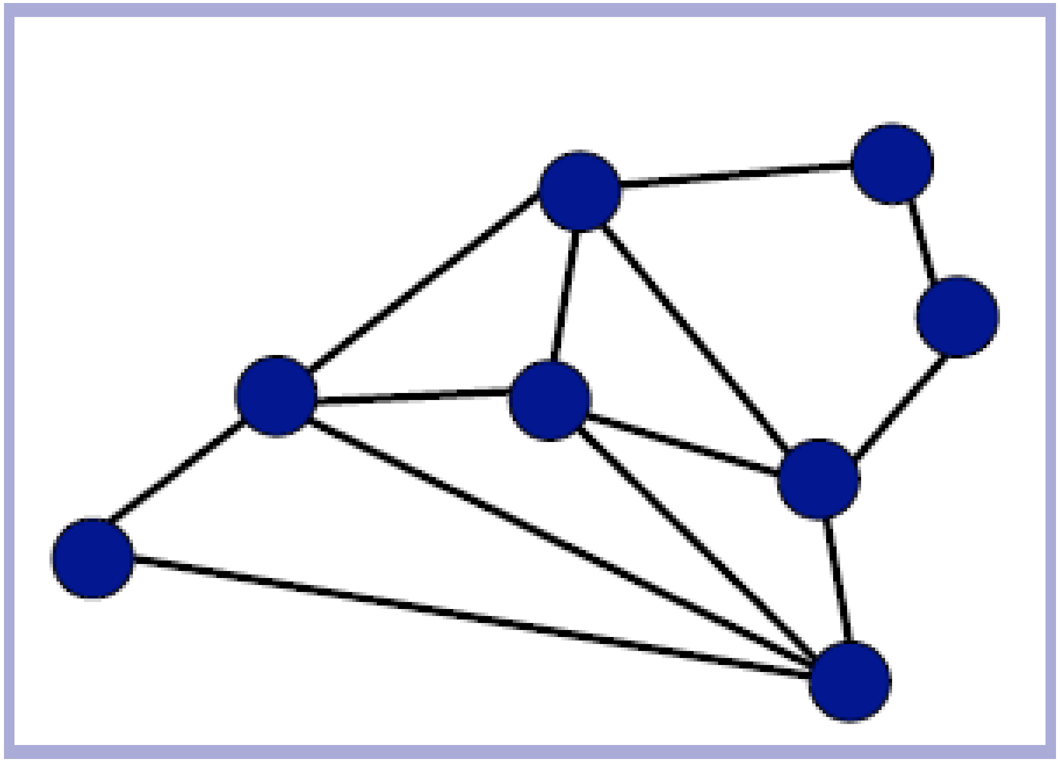


$$V = \{V_1, V_2, V_3, V_4\}$$

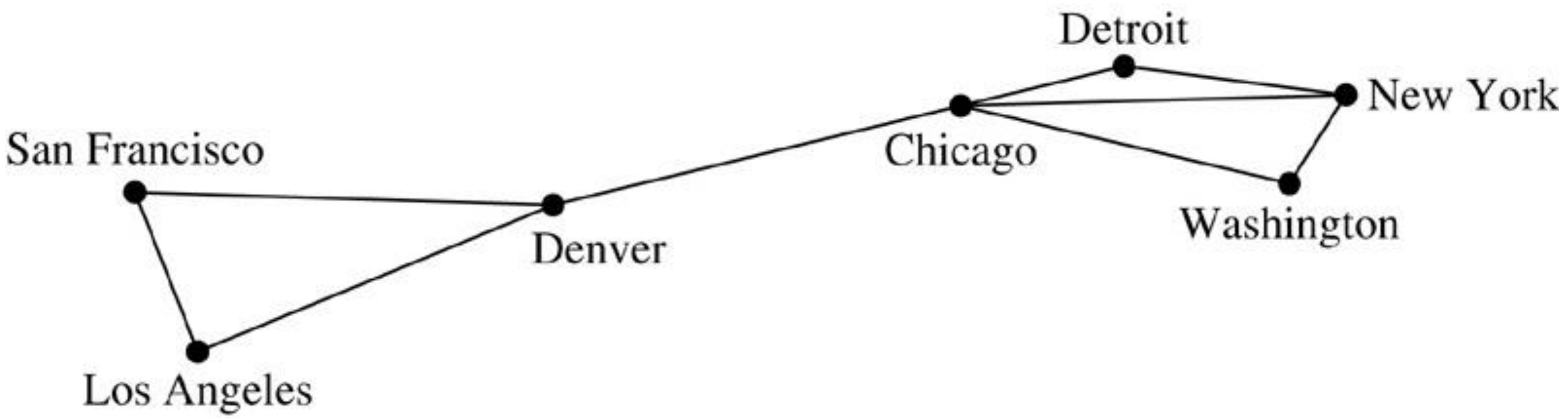
$$E = \{(V_1, V_2), (V_1, V_3), (V_2, V_4), (V_4, V_1), (V_3, V_4)\}$$

Simple Graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.



Example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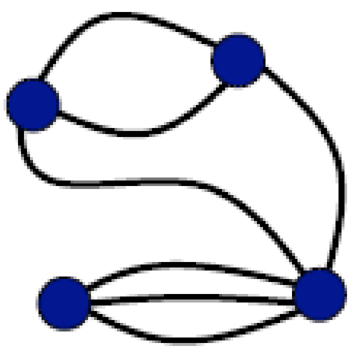
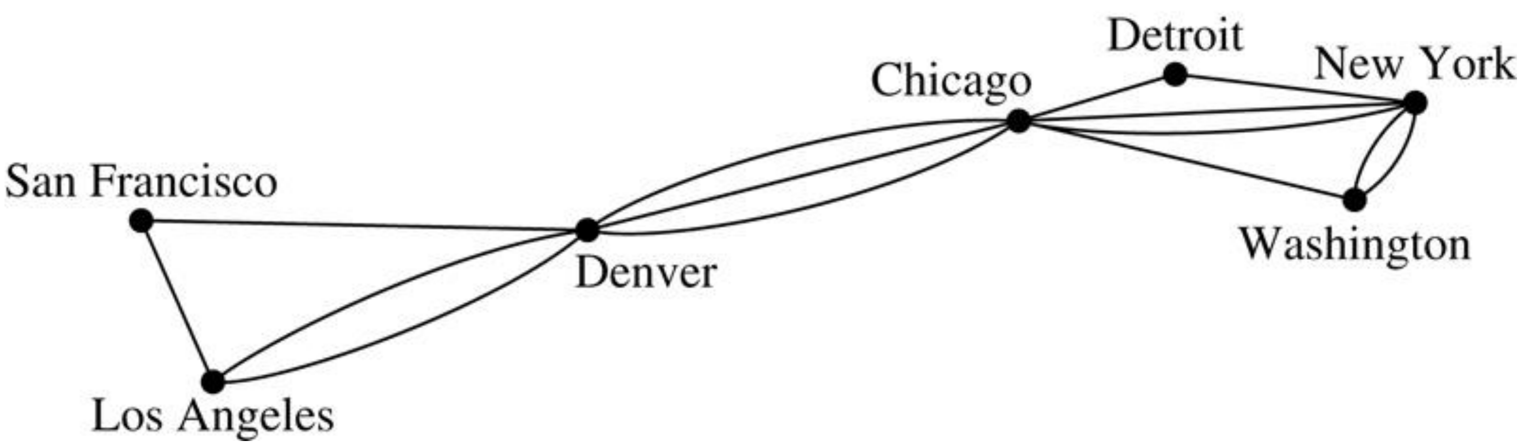
Multigraph

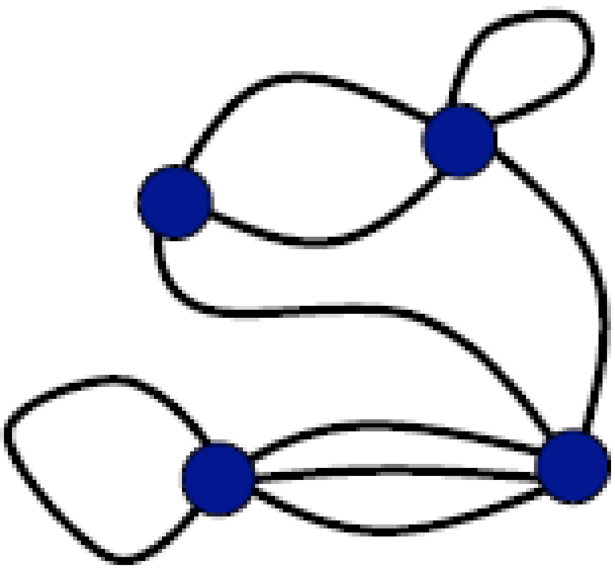
- A multigraph: multiple edges connecting the same nodes
- E.g., nodes are cities, edges are segments of major highways.



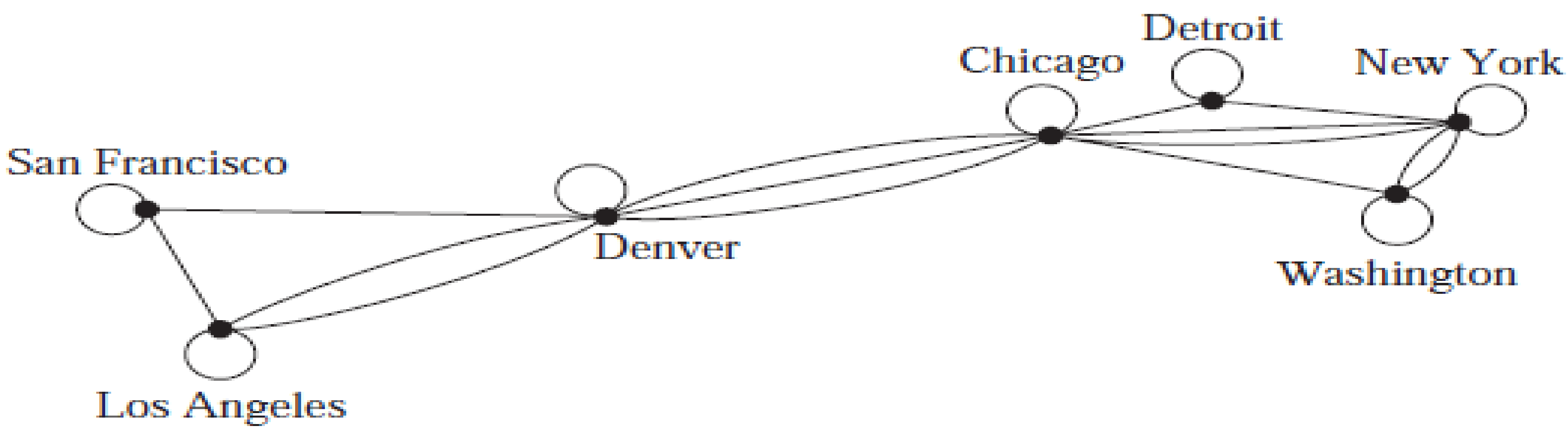


Pseudographs

- Pseudograph: Like a multigraph, but edges connecting a node to itself are allowed.



Example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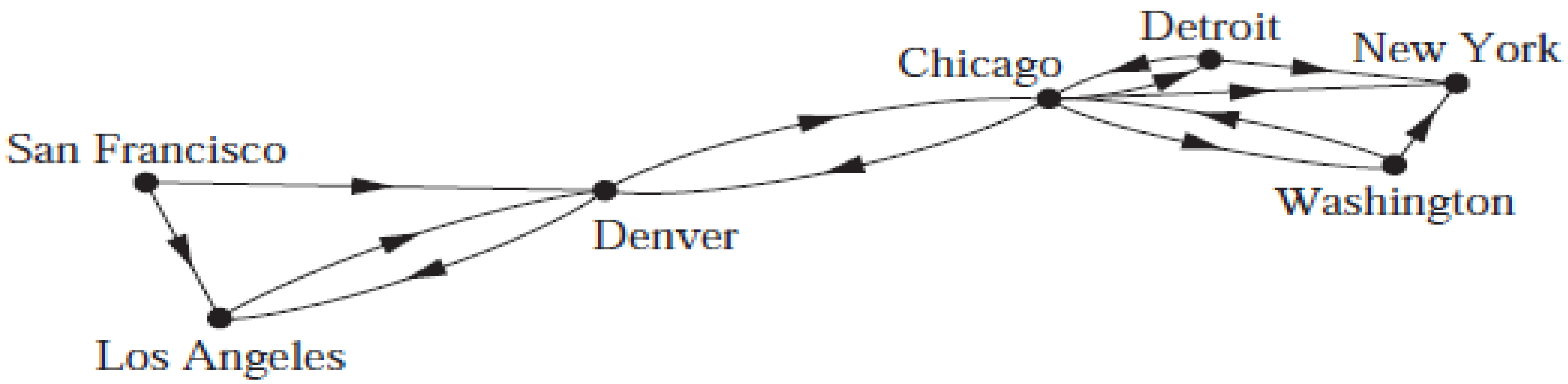
Directed Graphs

- A directed graph (or digraph) (V,E) consists of a set of vertices V and a set of directed edges E on V. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (***u,v***) is said to start at ***u*** and end at ***v***.

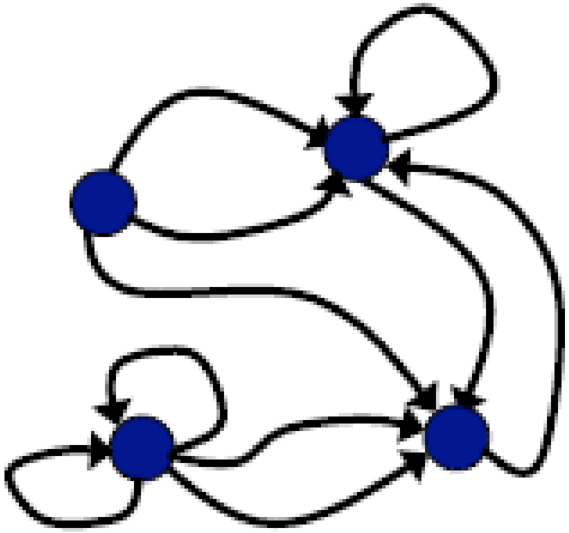
Simple Directed Graph

A directed graph that has no loops and has no multiple directed edges is called a **simple directed graph**.

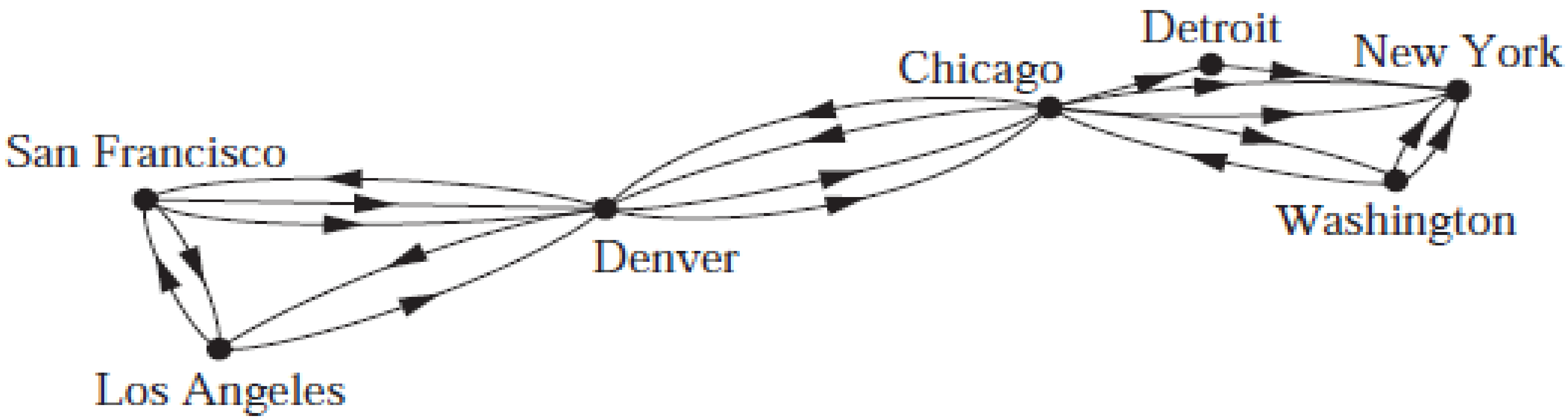


Directed Multigraphs

- A directed multigraph has directed parallel edges.



Example



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Types of Graphs: Summary

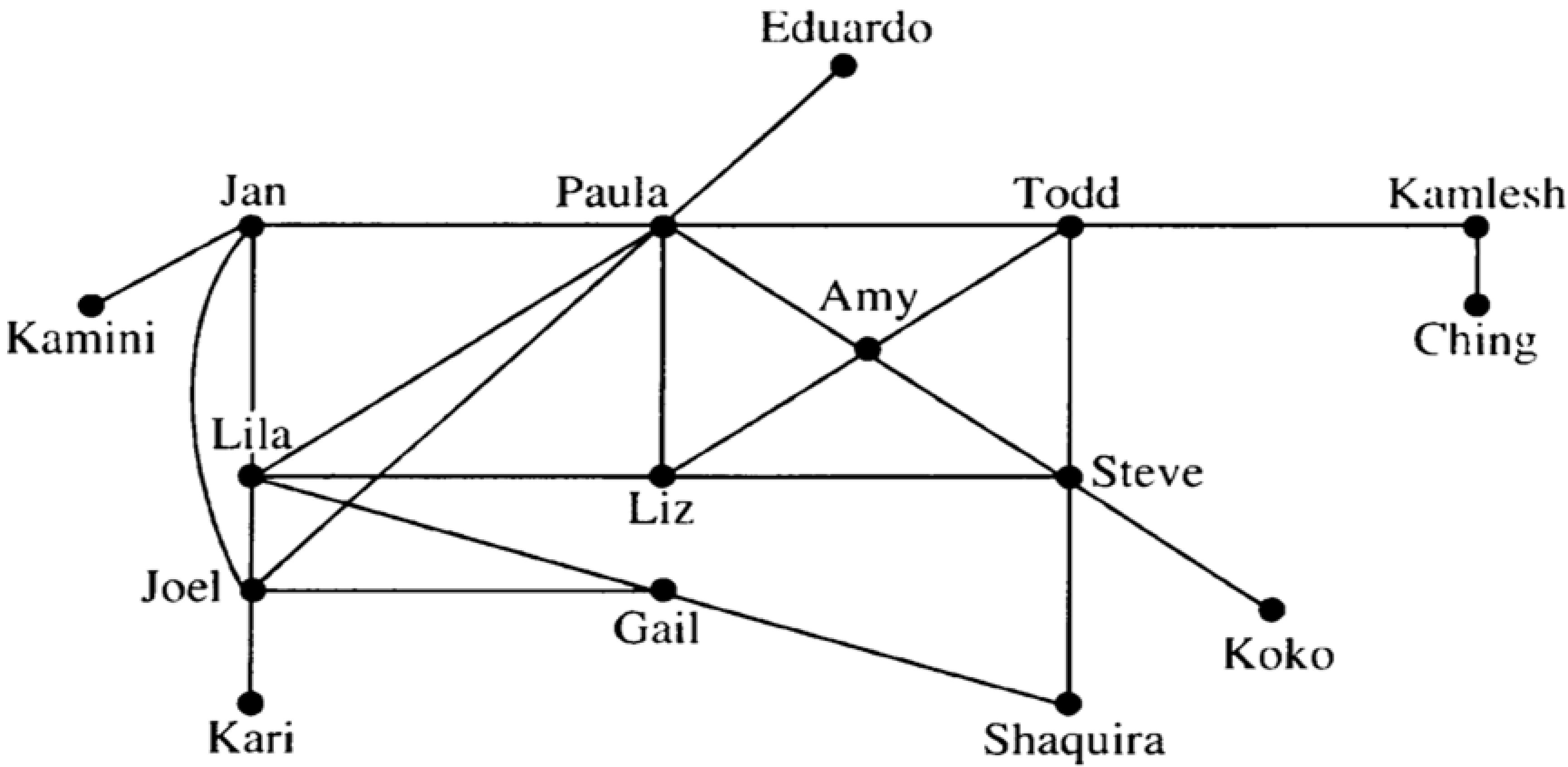
- Summary of the book's definitions.
- Keep in mind this terminology is not fully standardized across different authors...

Type	Edges	Multiple Edges Allowed?	Loops Allowed?
Simple Graphs	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes

Graph Models: Acquaintanceship Graph

- Represent whether two people know each other, that is, whether they are acquainted.
- Each person in a particular group of people is represented by a vertex.
- Undirected edge is used to connect two people when these people know each other.
- No multiple edges are used.
- Usually no loops are used. (If we want to include the notion of self-knowledge, we would include loops.)

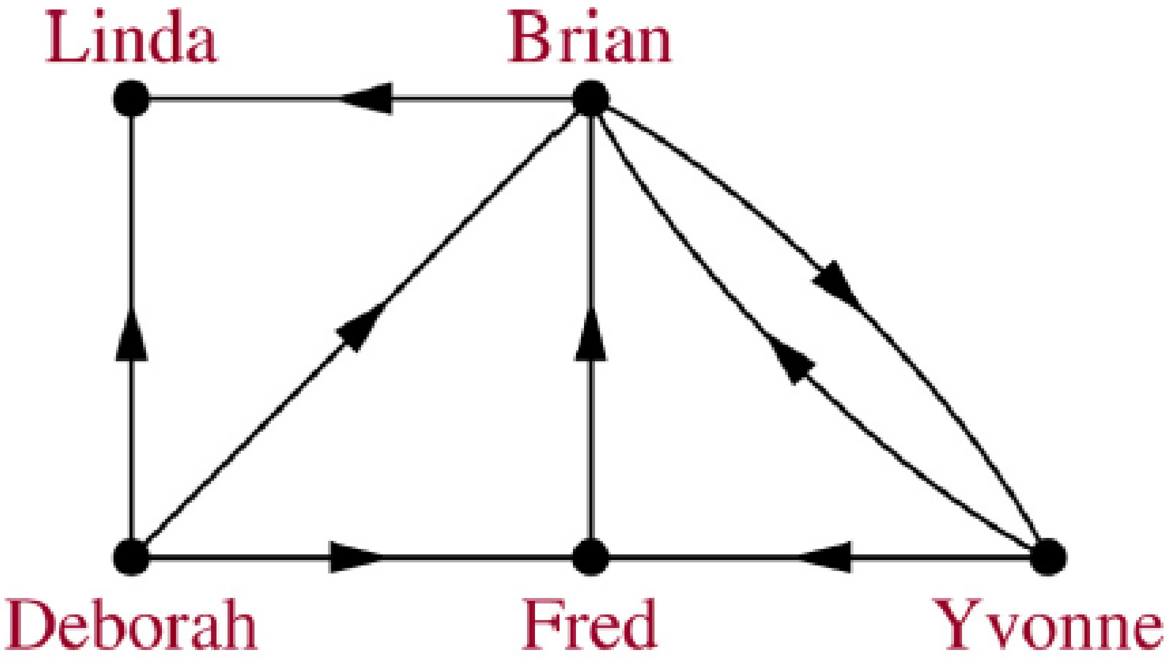
Example



An Acquaintanceship Graph.

An Influence Graph

- A directed edge (a, b) means a can influence b.
- E.g. (Fred, Brian) means Fred can influence Brian.

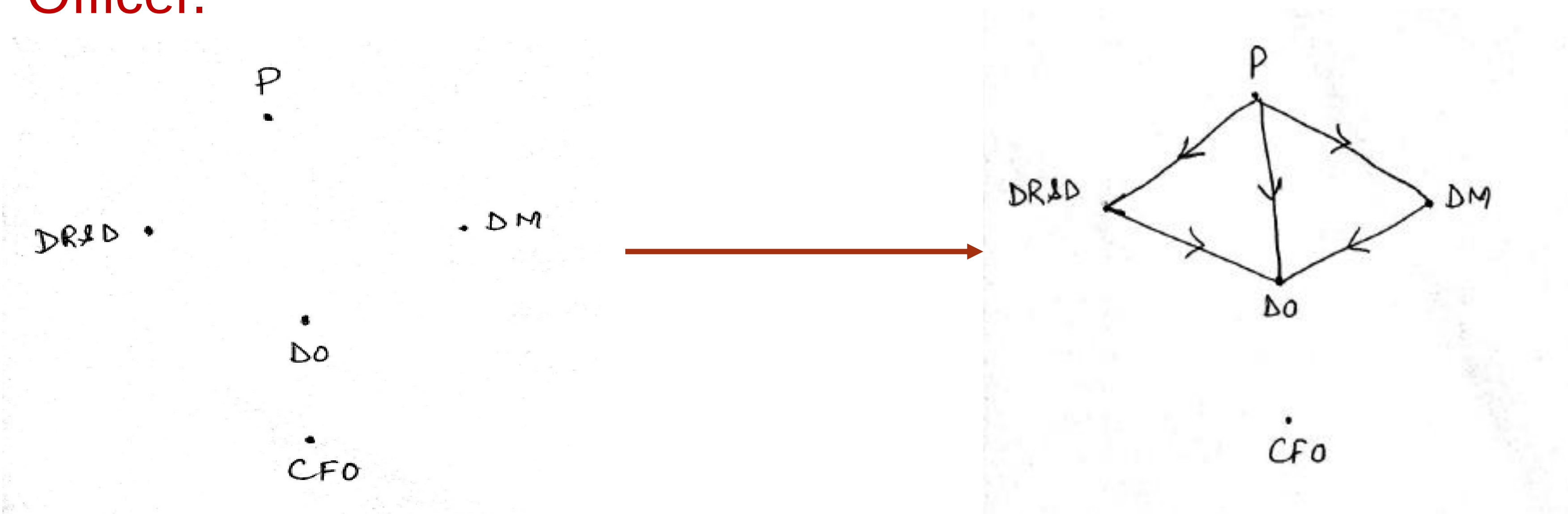


Example

- Construct an influence graph for the board members of a company if the President can influence the Director of Research and Development, the Director of Marketing and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

Example

The President can influence the Director of Research and Development, the Director of Marketing and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

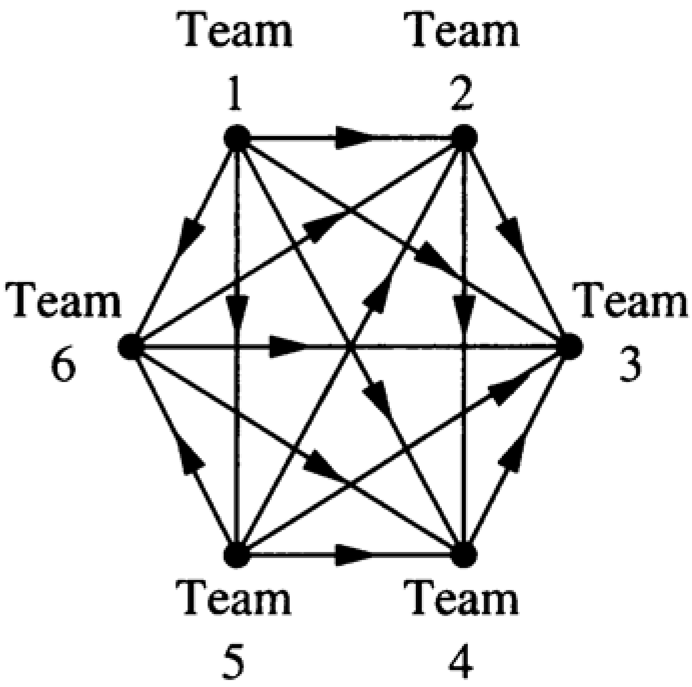


Round Robin Tournaments

- A tournament where each team plays each other team exactly once.
- Such tournaments can be modeled using directed graphs where each team is represented by a vertex. Note that (a, b) is an edge if team ‘a’ beats team ‘b’. This graph is a simple directed graph, containing no loops or multiple directed edges.

Example

Team 1 is undefeated in this tournament, and Team 3 is winless.



Intersection Graph

- The intersection graph of a collections of sets $A_1, A_2, A_3, \dots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a non empty intersection.
- $A_1 = \{0, 2, 4, 6, 8\}$
- $A_2 = \{0, 1, 2, 3, 4\}$
- $A_3 = \{1, 3, 5, 7, 11\}$
- $A_4 = \{1, 4, 9, 16, 25\}$

Example

- $A_1 = \{0, 2, 4, 6, 8\}$
- $A_2 = \{0, 1, 2, 3, 4\}$
- $A_3 = \{1, 3, 5, 7, 11\}$
- $A_4 = \{1, 4, 9, 16, 25\}$

$A_1 \cap A_2 = \{0, 2, 4\}$

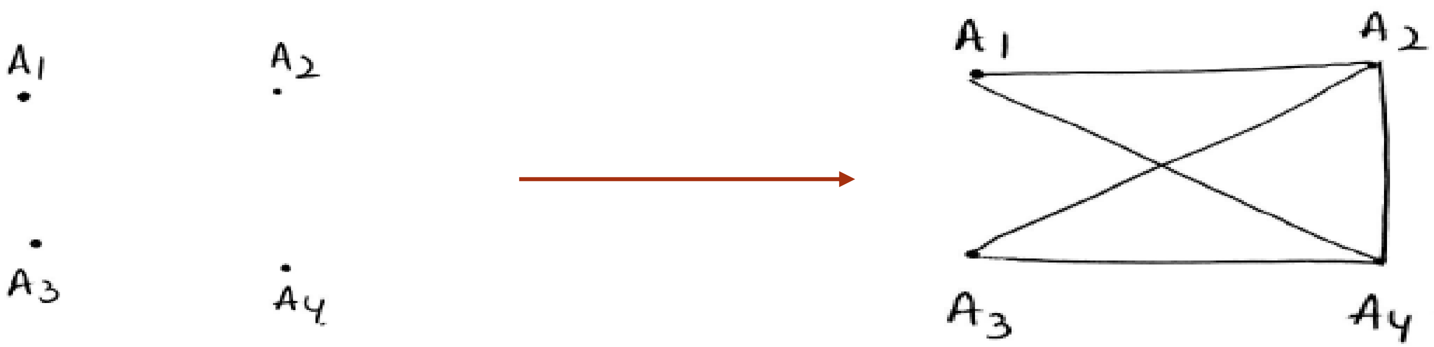
$A_1 \cap A_3 = \emptyset$

$A_1 \cap A_4 = \{4\}$

$A_2 \cap A_3 = \{1, 3\}$

$A_2 \cap A_4 = \{1, 4\}$

$A_3 \cap A_4 = \{1\}$

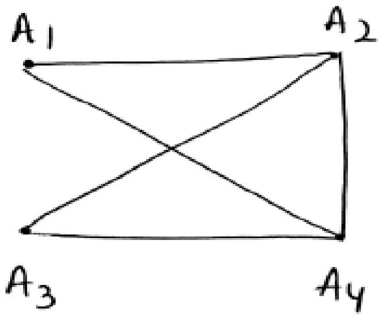


Graph Terminologies

- Adjacent:

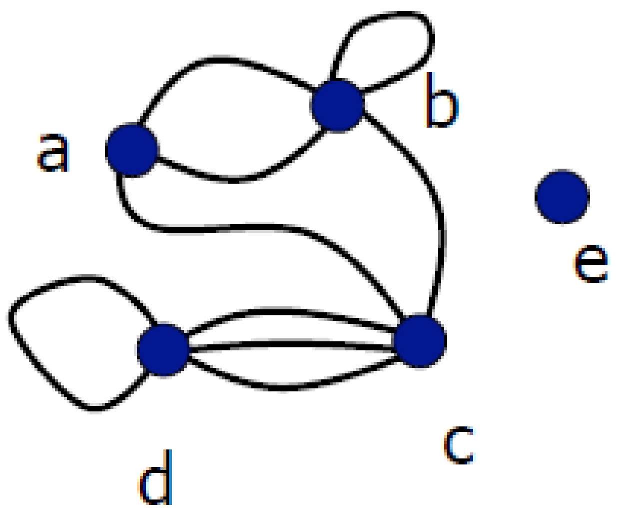
Let G be an undirected graph with edge set E. Let e ∈ E be (or map to) the pair {u,v}. Then we say:

- u, v are adjacent / neighbors / connected.
- Edge e is incident with vertices u and v.
- Edge e connects u and v.
- Vertices u and v are endpoints of edge e.



Degree of a Vertex

- Let G be an undirected graph, v ∈ V a vertex.
- The degree of v, deg(v), is its number of incident edges. (Except that any self-loops are counted twice.)



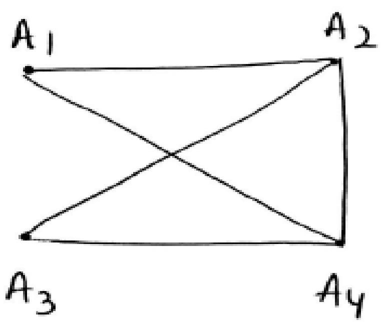
G1

deg(a) = 3
deg(b) = 5
deg(c) = 5
deg(d) = 5
deg(e) = 0 (isolated vertex)

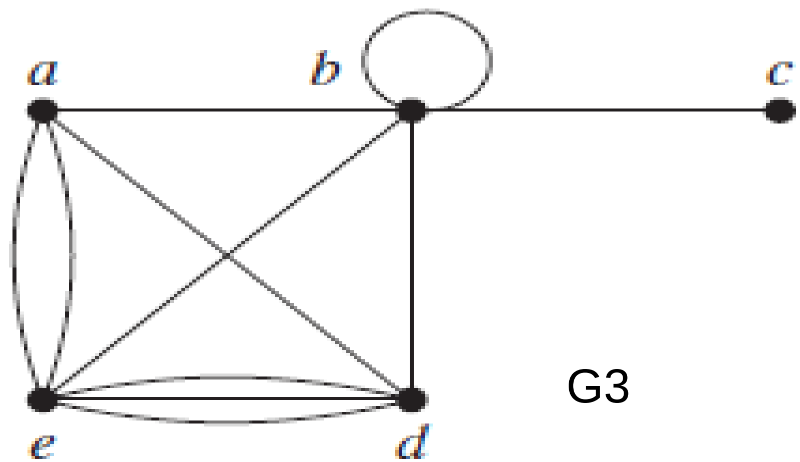
Pendant vertex = with degree 1

Example

- What is the degree of each vertex of the following graph?



G2



G3

deg(A1) = 2
deg(A2) = 3
deg(A3) = 2
deg(A4) = 3

deg(a) = 4
deg(b) = 6
deg(c) = 1
deg(d) = 5
deg(e) = 6

Handshaking Theorem

- Let G be an undirected (simple, multi-, or pseudo-) graph with vertex set V and edge set E. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

- deg(a)=3, deg(b)=5, deg(c)=5, deg(d)=5, deg(e)=0

$$\sum_{v \in \{a,b,c,d,e\}} \deg(v) = \deg(a) + \deg(b) + \deg(c) + \deg(d) + \deg(e) = 18 = 2|E|$$

Example

$\deg(A_1) = 2$
$\deg(A_2) = 3$
$\deg(A_3) = 2$
$\deg(A_4) = 3$

10

$2(5)$ — Edges

G2

$\deg(a) = 3$
$\deg(b) = 5$
$\deg(c) = 5$
$\deg(d) = 5$
$\deg(e) = 0$

18

$2(9)$ — Edges

G1

$\deg(a) = 4$
$\deg(b) = 6$
$\deg(c) = 1$
$\deg(d) = 5$
$\deg(e) = 6$

22

$2(11)$ — Edges

G3

Example

- How many edges are there in a graph with 10 vertices each of degree six?

- If a graph has 5 vertices, can each vertex have degree 3?

$10 \times 6 = 60 = 2(30) = 30$ Edges

$5 \times 3 = 15$ Not possible.

Handshaking Theorem

- Any undirected graph has an even number of vertices of odd degree.
- Let V_1 and V_2 be the set of vertices of even and odd degrees respectively, in an undirected graph, then

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

must be even since deg(v) is even for each v ∈ V1

must be even because 2m is even and the sum of degrees of vertices of even degree is even. Thus, since this is the sum of degrees of all vertices of odd degree, there must be an even number of them.

- $|V_2|$ has to be even.

Example

2 vertices with odd degree

$deg(A_1) = 2$

$deg(A_2) = 3$

$deg(A_3) = 2$

$deg(A_4) = 3$

G2

4 Vertices with odd degree

$deg(a) = 3$

$deg(b) = 5$

$deg(c) = 5$

$deg(d) = 5$

$deg(e) = 0$

G1

2 Vertices with odd degree

$deg(a) = 4$

$deg(b) = 6$

$deg(c) = 1$

$deg(d) = 5$

$deg(e) = 6$

G3

Directed Degree

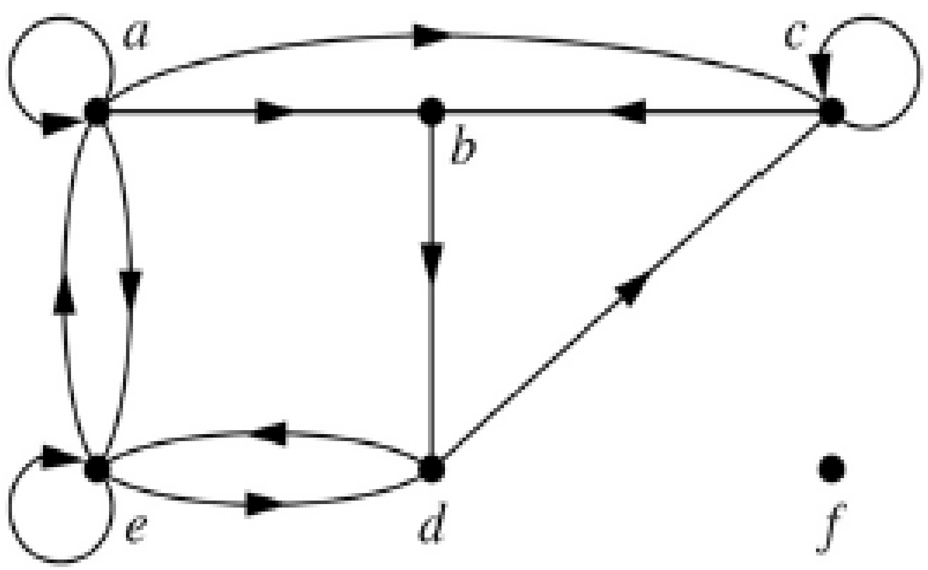
- When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v. The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v).
- The initial vertex and terminal vertex of a loop are the same.

Directed Degree

- Let G be a directed graph, v a vertex of G.
- The in-degree of v, $deg^{-}(v)$, is the number of edges with v as their terminal vertex.
- The out-degree of v, $deg^{+}(v)$, is the number of edges with v as their initial vertex.
- The degree of v, $\deg(v) :\equiv deg^{-}(v) + deg^{+}(v)$is the sum of v's in-degree and out-degree.
- Loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

Example

- Determine in/out-degree of each vertex of Graph G.

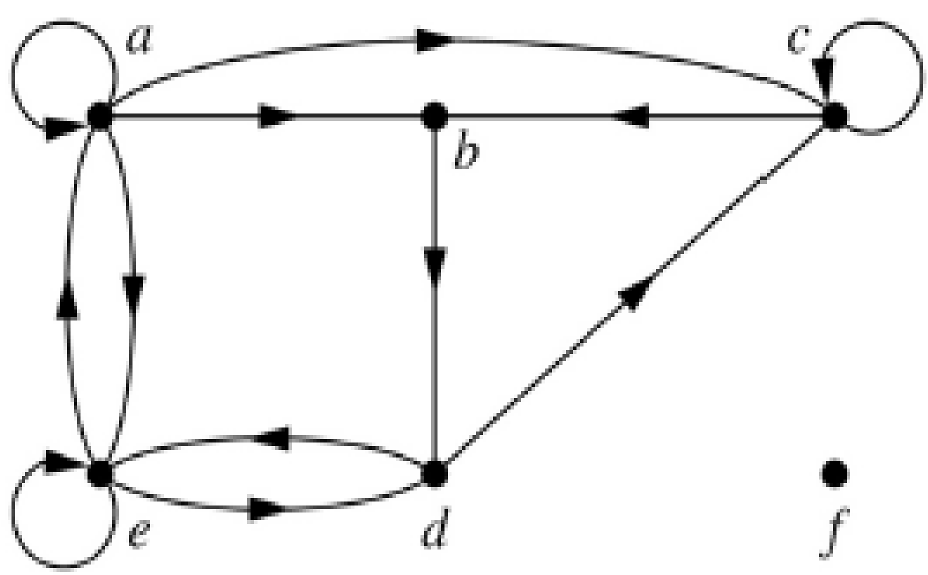


G

Example

- Determine in/out-degree of each vertex of Graph G.

in-degree	*out*-degree
$\deg^-(a) = 2$	$\deg^+(a) = 4$
$\deg^-(b) = 2$	$\deg^+(b) = 1$
$\deg^-(c) = 3$	$\deg^+(c) = 2$
$\deg^-(d) = 2$	$\deg^+(d) = 2$
$\deg^-(e) = 3$	$\deg^+(e) = 3$
$\deg^-(f) = 0$	$\deg^+(f) = 0$
$Total = 12$	$Total = 12$



G

Theorem

- Let $\mathrm{G} = (V, E)$ be a graph with directed edges then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

- The first sum counts the number of incoming edges over all vertices and the second sum counts the number of outgoing edges over all vertices Both sums must be |E|.

Chapter Exercise

- Chapter # 10
- Topic # 10.1
- Q – 1,3,4,5,6,7,8,9,13,16,19,21,22
- Topic # 10.2
- Q -1,2,3,4,7,8,9,10,21,22,23,24,27
- Topic # 10.3
- Q – 1,2,3,4,5,6,7,8,10,11,12,13,14,15,16,17,18,19,20,21, 22,23,24,26,27