Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

Muhammad Zuhair Qadir

Lecture Outline

- Predicate Logic
  - Predicate
  - Quantifier
  - Translation of Quantified Statements

Predicate Logic

- Proposition, YES or NO?
  - $3 + 2 = 5$ Yes
  - $X + 2 = 5$ No
  - $X + 4 = 5$ for some X in $\{1, 2, 3\}$ Yes
  - Computer X is under attack by an intruder No

Why Predicate Logic?

- **Propositional Logic is not expressive enough**
  - It cannot adequately express the meaning of statements in mathematics and in natural language

Example 1:

“Every computer connected to the university network is functioning properly.”

- No rules of propositional logic allow us to conclude the truth of the statement.

Why Predicate Logic?

Example 2:

- **“There is a computer on the university network that is under attack by an intruder.”**

Predicate Logic is more expressive and powerful

Propositional Functions(Example)

- “x is greater than 3” or $(x > 3)$
  - The variable x: subject of the statement
  - “is greater than 3”: predicate
  - $P(x)$: propositional function P at x

- Let $P(x) = x > 3$
  - $P(x)$ has no truth values (x is not given a value)
  - $P(10)$ is true: The proposition $10 > 3$ is true.
  - $P(1)$ is false: The proposition $1 > 3$ is false.
  - $P(x)$ will create a proposition when given a value

Propositional Functions(Example)

- Let $A(x)$ = "Computer x is under attack by an intruder."
- Suppose computers on campus, only CS2 and MATH1 are currently under attack by intruders.
- What are truth values of A(CS1), A(CS2), and A(MATH1)?

- The statement A(CS1) by setting x = CS1 in the statement "Computer x is under attack by an intruder."
- CS1 is not on the list of computers currently under attack, A(CS1) is false.
- CS2 and MATH1 are on the list of computers under attack, A(CS2) and A(MATH1) are true.

Propositional Functions

- Functions with multiple variables:
  - $P(x,y) = x + y == 0$
    - $P(1,2)$ is false, $P(1,-1)$ is true
  - $P(x,y,z) = x + y == z$
    - $P(3,4,5)$ is false, $P(1,2,3)$ is true
  - $P(x_1,x_2,x_3 \dots x_n) = \dots$
- Anatomy of a propositional function
- $$P(\underset{\text{variable}}{x}) = \underbrace{x + 5 > x}_{\text{predicate}}$$

Predicates

- A predicate is a declarative statement with at least one variable (i.e. unknown value).
- A predicate, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Predicates

- Suppose Q(x,y) = "x > y"

Proposition, YES or NO?

Q(x,y)	No
Q(3,4)	Yes
Q(x,9)	No

Predicate, YES or NO?

Q(x,y)	Yes
Q(3,4)	No
Q(x,9)	Yes

Quantification

- Quantification expresses the extent to which a predicate is true over a range of elements.
- In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.
- The area of logic that deals with predicates and quantifiers is called the **predicate calculus.**

Types of Quantifiers

- A quantifier is “an operator that limits the variables of a proposition”.

- Two types:
  - Universal
  - Existential

Universal Quantifiers

- Represented by an upside-down A: $\forall$
  - It means "for all"
  - Let $P(x) = x+1 > x$
- We can state the following:
  - $\forall x\, P(x)$
  - English translation: "for all values of x, $P(x)$ is true"
  - English translation: "for all values of x, $x+1>x$ is true"

Besides "for all", universal quantification can be expressed in many other ways: "for every", "all of", "for each", "given any", "for arbitrary", "for each" and "for any"

Universal Quantifiers

- You need to specify the **universe of quantification**!
  - What values x can represent
  - Called the "domain of discourse" or "universe of discourse"
  - Or just "domain" or "universe"

- The meaning of the universal quantification of *P(x)* changes when we change the domain. The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

Universal Quantifiers

- Let the universe of discourse be the real numbers.
- Let P(x) = x/2 < x
  - Not true for the negative numbers!
  - Thus, $\forall x\, P(x)$ is false, When the domain is all the real numbers

- In order to prove that a universal quantification is true, it must be shown for ALL cases

- In order to prove that a universal quantification is false, it must be shown to be false for only ONE case

Universal Quantifiers

- Let *P(x)* is "$x^2 > 0$." To show that the statement ∀*xP(x)* is false where the universe of discourse consists of all integers, we give a counterexample.
- *x* = 0 is a counterexample because $x^2 = 0$ when *x* = 0, so that x^2 is not greater than 0 when *x* = 0.

Universal Quantification

- Given some propositional function P(x) And values in the universe $x_1 .. x_n$
- The universal quantification $\forall x$ P(x) implies:
- $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

Question

- What is the truth value of $\forall x\, P\,(x)$, where $P(x)$ is the statement $\mathrm{x}^2 < 10$ and the domain consists of the positive integers not exceeding 4?

Solution:

- The statement $\forall x P(x)$ is the same as the conjunction

$P(1) \land P(2) \land P(3) \land\ P(4)$,

- Because $\boldsymbol{P}(\mathbf{4}) \equiv \mathbf{4}^{\mathbf{2}} < \mathbf{10}$**, is false**, it follows that $\forall \boldsymbol{x}\, \boldsymbol{P}(\boldsymbol{x})$ **is false**.

Existential Quantification

- Represented by an backwards E: ∃
  - It means "there exists", there is", "for some", etc.
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\exists x\ P(x)$
  - English translation: "there exists (a value of) x such that P(x) is true"
  - English translation: "for at least one value of x, x+1>x is true"
  - English translation: "for some x, P(x)"

Existential Quantification

- Let $P(x) = x+1 > x$
  - There is a numerical value for which $x+1>x$
  - In fact, it's true for all of the values of x. Thus, $\exists x\ P(x)$ is true

- In order to show an existential quantification is true, you only have to find ONE value

- In order to show an existential quantification is false, you have to show it's false for ALL values

Existential Quantification

- **Example:** Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

- **Solution:** Because "$x > 3$" is sometimes true—for instance, when $x = 4$ the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Existential Quantification

- **Example:** Let Q(x) denote the statement “x ==x + 1.”What is the truth value of the quantification ∃xQ(x), where the domain consists of all real numbers?

- **Solution:** Because Q(x) is false for every real number x, the existential quantification of Q(x), which is ∃xQ(x), is false.

Existential Quantification

- Given some propositional function $P(x)$ And values in the universe $x_1 .. x_n$
- The existential quantification $\exists x\, P(x)$ implies:
- $P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n)$

Summary

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\ P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Quantifiers with Restricted Domain

- An abbreviated notation is often used to restrict the domain of a quantifier.
- In this notation, a condition a variable must satisfy is included after the quantifier.

- $\forall x < 0(x^2 > 0)$ where domain is real numbers

Quantifiers with Restricted Domain

- $\forall x < 0(x^2 > 0) \equiv \forall x((x < 0) \rightarrow (x^2 > 0))$
- The restriction of a universal quantification is the same as the universal quantification of a conditional statement.

Quantifiers with Restricted Domain

- $\forall y \neq 0 (y^3 \neq 0) \equiv \forall y (y \neq 0 \to y^3 \neq 0)$

Quantifiers with Restricted Domain

- $\exists z > 0\ (z^2 = 2) \equiv \exists z(z > 0 \wedge z^2 = 2)$
- The restriction of an existential quantification is the same as the existential quantification of a conjunction.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators from propositional calculus.
- e.g $\forall x$ P(x) $\vee$ Q(x) is the disjunction of $\forall x$ P(x) and Q(x).

Binding Variables

- When a quantifier is used on a variable *x*, we say that this occurrence of the variable is **bound**.
- An occurrence of a variable that is not bound by a quantifier or not set equal to a particular value is said to be **free**.
- The part of a logical expression to which a quantifier is applied is called the **scope** of the quantifier.
- All the variables that occur in a logical expression must be bound or set equal to a particular value to turn into a proposition.

Binding Variables

- Examples:

- $P(x)$ x is free
- $P(5)$ x is bound to 5
- $\forall x\, P(x)$ x is bound by quantifier

Binding Variables

- $\exists x\, (P(x) \wedge Q(x)) \vee (\forall x R(x))$
  - All variables are bound.
- The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement.
- Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$.
- That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$.

Binding Variables

- $\exists x\ (x + y = 1)$
  - x is bound by $\exists x$ and y is free; thus not a proposition
- $(\exists x\ P(x)) \vee Q(x)$
  - The x in Q(x) is not bound; thus not a proposition
- $(\exists x\ P(x)) \vee (\forall x\ Q(x))$
  - Both x values are bound; thus it is a proposition
- $\exists x\ (P(x) \wedge Q(x)) \vee (\forall y\ R(y))$
  - All variables are bound; thus it is a proposition
- $(\exists x\ P(x) \wedge Q(y)) \vee (\forall y\ R(y))$
  - The y in Q(y) is not bound; thus not a proposition

A note on quantifiers

- Recall that P(x) is a propositional function
  - Let P(x) be “x == 0”
- Recall that a proposition is a statement that is either true or false
  - P(x) is not a proposition
- There are two ways to make a propositional function into a proposition:
  - Supply it with a value
    - For example, P(5) is false, P(0) is true
  - Provide a quantification
    - For example, $\forall x\, P(x)$ is false and $\exists x\, P(x)$ is true
    - Let the universe of discourse be the real numbers

Translating From English to Logical Expressions

- Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

- **Solution:**
- Assume domain is students in the class

"For every student in this class, that student has studied calculus."

"For every student x in this class, x has studied calculus."

$C(x)$ = "x has studied calculus."

$\forall x C(x)$

Translating From English to Logical Expressions

- Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
- *Let* $C(x)$ = "x has studied calculus."

  $S(x)$ = "person x is student in this class."

  The domain for x consists of all people.

- "For every person x, if person x is a student in this class then x has studied calculus."

- The statement can be expressed as $\forall x(S(x) \rightarrow C(x))$.

Negating Quantified Expressions

- Consider the statement

  "Every student in this class has studied calculus."
- This statement is a universal quantification, namely, $\forall x C(x)$,
  - $C(x)$ is the statement "x has studied calculus"
  - Domain consists of the students in the class.
- The negation of this statement is
  - "It is not the case that every student in this class has studied calculus."
  - This is equivalent to "There is a student in this class who has not studied calculus."
- This is simply the existential quantification of the negation of the original propositional function, namely, $\exists x \neg C(x)$.

Negating Quantified Expressions

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Negation	Equivalent Statement	When is Negation True?	When False?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negating Quantified Expressions

- **Example:**

What is the negation of the statement "There is an honest politician"?

Solution:

- Let $H(x)$ denote "x is honest."
- The statement is represented by $\exists x H(x)$, where the domain consists of all politicians.
- The negation of this statement is $\neg \exists x H(x)$, which is equivalent to $\forall x \neg H(x)$.
- This negation can be expressed as "Every politician is dishonest." or Not all politicians are honest."

Negating Quantified Expressions

Example:

What is the negation of the statement "All Americans eat cheeseburgers"?

Solution:

- $C(x)$ denote "x eats cheeseburgers."
- The statementis represented by $\forall x C(x)$, where the domain consists of all Americans.
- The negation of this statement is $\neg\forall x C(x)$, which is equivalent to $\exists x \neg C(x)$.
- This negation can be expressed as "Some American does not eat cheeseburgers" and "There is an American who does not eat cheeseburgers."

Negating Quantified Expressions

- What are the negations of the statements $\forall x(x^2 > x)$ *and* $\exists x(x^2 = x)$?

$$\forall x(x^2 > x)$$
$$\equiv \neg\forall x(x^2 > x)$$
$$\equiv \exists x \neg(x^2 > x)$$
$$\equiv \exists x(x^2 \leq x)$$

$$\exists x(x^2 = x)$$
$$\equiv \neg\exists x(x^2 = x)$$
$$\equiv \forall x \neg(x^2 = x)$$
$$\equiv \forall x(x^2 \neq x)$$

De Morgan's Laws for Quantifiers

- $\forall x\ P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- $\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots . \vee P(x_n)$

- $\neg\exists x P(x) \equiv \neg(P(x_1) \vee P(x_2) \vee \ldots . \vee P(x_n))$

  $\equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots . \wedge \neg(P(x_n))$

  $\equiv \forall x \neg P(x)$

- $\neg\forall x P(x) \equiv \neg(P(x_1) \wedge P(x_2) \wedge \ldots . \wedge P(x_n))$

  $\equiv \neg P(x_1) \vee \neg P(x_2) \vee \ldots . \vee \neg(P(x_n))$

  $\equiv \exists x \neg P(x)$

Translating From English to Logical Expressions

- Let *R(x)* = "*x* can speak Russian"

  C(x) = "*x* knows the computer language C++."

Express each of these sentences in terms of *R(x)*, *C(x)*, quantifiers, and logical connectives.

The domain for quantifiers consists of all students at your school.

- There is a student at your school who can speak Russian and who knows C++.

$$\exists \boldsymbol{x}\big(\boldsymbol{R}(\boldsymbol{x}) \wedge \boldsymbol{C}(\boldsymbol{x})\big)$$

Translating From English to Logical Expressions

- Let $R(x)$ = "x can speak Russian"

  $C(x)$ = "x knows the computer language C++."

- There is a student at your school who can speak Russian but who doesn't know C++.

$$\exists \boldsymbol{x}\left(\boldsymbol{R}(\boldsymbol{x}) \wedge \neg \boldsymbol{C}(\boldsymbol{x})\right)$$

Translating From English to Logical Expressions

- Let $R(x)$ = "x can speak Russian"

  $C(x)$ = "x knows the computer language C++."

- Every student at your school either can speak Russian or knows C++.

$$\forall \boldsymbol{x}\left(\boldsymbol{R}(\boldsymbol{x}) \vee \boldsymbol{C}(\boldsymbol{x})\right)$$

Translating From English to Logical Expressions

- Let *R(x)* = "*x* can speak Russian"
  C(x) = "*x* knows the computer language C++."

- No student at your school can speak Russian or knows C++.

$$\forall \boldsymbol{x} \neg\left(\boldsymbol{R}(\boldsymbol{x}) \vee \boldsymbol{C}(\boldsymbol{x})\right) / \neg \exists \boldsymbol{x}\left(\boldsymbol{R}(\boldsymbol{x}) \vee \boldsymbol{C}(\boldsymbol{x})\right)$$

The Four Aristotelian Forms

1. All A's are B's
2. Some A's are B's
3. No A's are B's
4. Some A's are not B's

- These are four of the most common quantificational sentences used in quantificational reasoning.

The First Aristotelian Form

- The Form: *All A's are B's*
- Example: All comedian are funny.
  - Rephrase: For every x, if x is a comedian then x is funny
  - Translation: $\forall x\ (Comedian(x) \rightarrow Funny(x))$
  - This translation has the form: $\forall x\ (A(x) \rightarrow B(x))$
- General Fact
  - All A's are B's translates as $\forall x\ (A(x) \rightarrow B(x))$

The Second Aristotelian Form

- The Form: *Some A's are B's*
- Example: Some comedian are funny
  - Rephrase: Some thing x is both comedian and funny
  - Translation: $\exists x\ (\text{Comedian}(x) \wedge \text{Funny}(x))$
  - This translation has the form: $\exists x\ (A(x) \wedge B(x))$
- General Fact
  - Some A's are B's translates as $\exists x\ (A(x) \wedge B(x))$

The Third Aristotelian Form

- The Form: *No A's are B's*
- Example: No students are failed
  - Rephrase: For every x, if x is a student then x is not failed
  - Translation: $\forall x\ (\text{Student}(x) \rightarrow \neg\text{Failed}(x))$
  - This translation has the form: $\forall x\ (A(x) \rightarrow \neg B(x))$
- General Fact
  - No A's are B's translates as $\forall x\ (A(x) \rightarrow \neg B(x))$

The Fourth Aristotelian Form

- The Form: *Some A's are not B's*
- Example: Some excuses are not believable
  - Rephrase: For some x, x is an excuse and x is not believable
  - Translation: $\exists x\ (Excuse(x) \wedge \neg Believable(x))$
  - This translation has the form: $\exists x\ (A(x) \wedge \neg B(x))$
- General Fact
  - Some A's are not B's translates as $\exists x\ (A(x) \wedge \neg B(x))$

Summary

- The Aristotelian Forms and Their Translations
  - All A's are B's $\forall x\ (A(x) \rightarrow B(x))$
  - Some A's are B's $\exists x\ (A(x) \land B(x))$
  - No A's are B's $\forall x\ (A(x) \rightarrow \neg B(x))$
  - Some A's are not B's $\exists x\ (A(x) \land \neg B(x))$

Predicates - Examples

L(x) = “x is a lion.”
F(x) = “x is fierce.”
C(x) = “x drinks coffee.”
Assuming that the domain consists of all creatures.

- All lions are fierce.

  $\forall x\ (L(x) \rightarrow F(x))$
- Some lions don’t drink coffee.

  $\exists x\ (L(x) \land \neg C(x))$
- Some fierce creatures don’t drink coffee.

  $\exists x\ (F(x) \land \neg C(x))$

Predicates - Examples

B(x) = “x is a hummingbird.”
L(x) = “x is a large bird.”
H(x) = “x lives on honey.”
R(x) = “x is richly colored.”
Assuming that the domain consists of all birds.

- All hummingbirds are richly colored.

  $\forall x\ (B(x) \rightarrow R(x))$

- No large birds live on honey.

  $\forall x\ (L(x) \rightarrow \neg H(x))$

- Birds that do not live on honey are dully colored.

  $\forall x\ (\neg H(x) \rightarrow \neg R(x))$

- Hummingbirds are small.

  $\forall x\ (B(x) \rightarrow \neg L(x))$

Example

- Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English as well.

a) Some old dogs can learn new tricks.

b) No rabbit knows calculus.

c) Every bird can fly.

d) There is no dog that can talk.

e) There is no one in this class who knows French and Russian.

Chapter Reading

- ***Chapter 1***, Kenneth H. Rosen, Discrete Mathematics and Its Applications, Section 1.4

Chapter Exercise (For Practice)

- Question # 1, 2, 5, 6, 7, 8, 10, 11, 12, 14, 17, 18, 35, 36, 59(a, b, c), 60(a, b, c), 61(a, b, c, d)