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ASSIGNMENT-2

QUESTION 1: Show that

$$\begin{vmatrix} 3x+2y & 2y & 2z \\ 4x+3y & 3y & 3z \\ 5x+5y & 4y & 5z \end{vmatrix} = xyz$$

$$L.H.S = \begin{vmatrix} 3x+2y & 2y & 2z \\ 4x+3y & 3y & 3z \\ 5x+5y & 4y & 5z \end{vmatrix}$$

$$= \begin{vmatrix} 3x & 2y & 2z \\ 4x & 3y & 3z \\ 5x & 4y & 5z \end{vmatrix} + \begin{vmatrix} 2y & 2y & 2y \\ 3y & 3y & 3y \\ 5y & 4y & 5y \end{vmatrix}$$ Using splitting property

$$= xyz\begin{vmatrix} 3 & 2 & 2 \\ 4 & 3 & 3 \\ 5 & 4 & 5 \end{vmatrix} + y^2z\begin{vmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 5 & 4 & 5 \end{vmatrix}$$

Taking x, y, z common from C_1, C_2 and C_3 respectively. Taking y, z common from C_1, C_2 and C_3

$$= xyz\left[3\begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & 3 \\ 5 & 5 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 \\ 5 & 4 \end{vmatrix}\right] + y^2z(0) \quad \because C_1 = C_3$$

Expanded from R_1

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$= xyz[3(3) - 2(5) + 2(1)]$

$= xyz(9 - 10 + 1)$

$= xyz$

$=$ RHS

Hence proved.

Q2: $W = \left\{ \begin{bmatrix} 2x+3y \\ -x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

(i) Is W a subspace of $V = \mathbb{R}^3$

(a) let $\underline{u}, \underline{v} \in W$

$$\underline{u} = \begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} 2v_1 + 3v_2 \\ -v_1 \\ v_2 \end{bmatrix}$$

$$\underline{u} \oplus \underline{v} = \begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} 2v_1 + 3v_2 \\ -v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(u_1 + v_1) + 3(u_2 + v_2) \\ -(u_1 + v_1) \\ u_2 + v_2 \end{bmatrix} \in W$$

Hence W is closed w.r.t addition

(b) Let $c \in \mathbb{R}$

$$c \odot \underline{u} = c \begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2(cu_1) + 3(cu_2) \\ -cu_1 \\ cu_2 \end{bmatrix} \in W$$

Hence W is closed with respect to multiplication

(ii) If yes, find the basis of subspace W

Let $w \in W$

So, $w = \begin{bmatrix} 2x+3y \\ -x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -x \\ 0 \end{bmatrix} + \begin{bmatrix} 3y \\ 0 \\ y \end{bmatrix}$

$= x\underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{V_1} + y\underbrace{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{V_2}$

Therefore, the basis of subspace W is

$S = \left\{ u_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, u_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

QUESTION 3: Determine whether $S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \right\}$ form a basis of M_{22}

The given set S form the basis for M_{22}, if:

(1) S is Linearly Independent

(2) Span $S = M_{22}$

So,

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, V_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, V_3 \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, V_4 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$

(1) Checking LI of S.

Consider:

$$a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 = \underline{0}$$

Therefore,

$$[A|0] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_4 - 2R_2 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_2 + R_3 \\ R_4 - 2R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] R_1 - R_4$$

Hence, $a_1 = a_2 = a_3 = a_4 = 0$

So it is Linearly Independent.

(ii) No. of elements in set $S = 4$, the dimension of vector space $M_{22} = 4$, gaurentees that $S = M_{22}$

If yes the Express the vector $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ as a linear combination of the vectors in S.

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = \underline{v}$$

So,

$$[A \mid \underline{b}] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 3 & 2 & 2 & 0 \end{array}\right]$$

$$R \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \Rightarrow R \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \end{array}\right] R_4 - 2R_2$$

$$R \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right] R_4 - 2R_3 \Rightarrow R \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] -R_4$$

So, $\boxed{a_4 = 1}$

$a_3 + a_4 = 1 \Rightarrow \boxed{a_3 = 0}$

$a_2 - a_3 = -1 \Rightarrow \boxed{a_2 = 1}$

$a_1 + a_2 + 2a_3 + a_4 = 1 \Rightarrow \boxed{a_1 = 1}$

Hence
$$1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

L.H.S = R.H.S

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Question 4 : For which real values of a do the polynomial form basis for vector space $P_2(t)$?

$P_1(t) = at^2 - \frac{1}{2}t - \frac{1}{2}$; $P_2(t) = -\frac{1}{2}t^2 + at - \frac{1}{2}$;
$P_3(t) = -\frac{1}{2}t^2 - \frac{1}{2}t + a$

$$V_1 = \begin{bmatrix} a \\ -\frac{1}{2} \\ -1/2 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1/2 \\ a \\ -1/2 \end{bmatrix}, \quad V_3 \begin{bmatrix} -1/2 \\ -1/2 \\ a \end{bmatrix}$$

To find the basis for vector space $P_2(t)$

Consider : $a_1V_1 + a_2V_2 + a_3V_3 = \underline{0}$

$$[A/0] = \left[\begin{array}{ccc|c} a & -1/2 & -1/2 & 0 \\ -1/2 & a & -1/2 & 0 \\ -1/2 & -1/2 & a & 0 \end{array}\right]$$

This method would become difficult to find values of a so, we use another way.

As we know, $AX = \underline{0}$, where $|A| \neq 0$ is Linearly independent. So

$$|A| = \begin{vmatrix} a & -1/2 & -1/2 \\ -1/2 & a & -1/2 \\ -1/2 & -1/2 & a \end{vmatrix} \Rightarrow \begin{vmatrix} a-\frac{1}{2} & -1/2 & -1/2 \\ a-\frac{1}{2} & a & -1/2 \\ \cancel{a}-1 & -1/2 & a \end{vmatrix} \quad C_1 + C_2$$

$$|A| = \begin{vmatrix} a-1 & -1/2 & -1/2 \\ a-1 & a & -1/2 \\ a-1 & -1/2 & a \end{vmatrix} \quad C_1 + C_3$$

$$= (a-1)\begin{vmatrix} 1 & -\frac{1}{2} & -1/2 \\ 1 & a & -1/2 \\ 1 & -1/2 & a \end{vmatrix} \Rightarrow (a-1)\begin{vmatrix} 1 & -1/2 & -1/2 \\ 0 & \frac{2a+1}{2} & 0 \\ 0 & 0 & \frac{2a+1}{2} \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Taking $(a-1)$ common from C_1

Expanding from C_1

$= (a-1)\left[1\left((a+1/2)^2 - 0\right) + 0 + 0\right] = (a-1)\left(a+\frac{1}{2}\right)^2$

if $|A| = 0$ then A is L.D

So $(a-1)\left(a+\frac{1}{2}\right)^2 = 0$

$\boxed{a=1}$; $\left(a+\frac{1}{2}\right)^2 = 0 \Rightarrow \boxed{a = -\frac{1}{2}}$

So, given set will be L.I for all the values of $a \in \mathbb{R}$ other than $-1/2$ & 1

(iii) Number of elements in set $= 3$

Dimension of vector space $= 3$

This gaurantees that span $S = P_2(t)$

Hence, given vectors form the basis for $P_2(t)$ or $\mathbb{R}^3$

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