Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

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Lecture Outline

- Rules of Inference
  - Rules of inference with quantifiers
  - Proof using rules of Inference

Rules of Inference for the Universal Quantifier

- Assume that we know that $\forall x\, P(x)$ is true
  - Then we can conclude that P(c) is true
  - Here c is particular value in domain
  - This is called “universal instantiation”
  - Example: All women are wise therefore Lisa is wise.

- Assume that we know that P(c) is true for all value of c in domain
  - Then we can conclude that $\forall x\, P(x)$ is true
  - This is called “universal generalization”
  - Example: Student **s** has taken calculus therefore All students has taken calculus.

Rules of Inference for the Existential Quantifier

- Assume that we know that $\exists x\ P(x)$ is true
  - Then we can conclude that P(c) is true for some value of c
  - This is called “existential instantiation”.
  - Example: There is a fish in a pool therefore Some fish **a** is in pool.

- Assume that we know that P(c) is true for some value of c
  - Then we can conclude that $\exists x\ P(x)$ is true
  - This is called “existential generalization”.
  - Example: Ali is in the store therefore There is a person in store.

Rules of Inference for Quantified Statements

Rule of Inference	***Name***
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Example

$$\begin{array}{l} \forall x(D(x) \to C(x)) \\ \underline{D(Ali) \qquad\qquad\qquad} \\ \therefore C(Ali) \end{array}$$

- Given the hypothesis:

"Everyone in this discrete mathematics class has taken a course in computer science" and "Ali is in this discrete mathematics class" imply the conclusion "Ali has taken a course in computer science."

- Let D(x) = "x is in this discrete mathematics class,"
- Let C(x) = "x has taken a course in computer science."

- The hypothesis are $\forall x(D(x) \to C(x))$ and D(Ali).
- The conclusion is C(Ali).

Example

$$\forall x(D(x) \to C(x))$$
$$\underline{D(Ali)}$$
$$\therefore C(Ali)$$

- The following steps can be used to establish the conclusion from the hypothesis.

- **Step** **Reason**

Step	Reason
1. $\forall x(D(x) \to C(x))$	Hypothesis
2. $D(Ali) \to C(Ali)$	Universal instantiation from 1
3. $D(Ali)$	Hypothesis
4. $C(Ali)$	Modus ponens from 2 and 3

Example

- Given the hypothesis

  "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

- Let C(x) = "x is in this class,"
- Let B(x) = "x has read the book,"
- Let P(x) = "x passed the first exam."

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \underline{\forall x(C(x) \rightarrow P(x))} \\ \therefore \exists x(P(x) \land \neg B(x)) \end{array}$$

- The hypothesis are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$.
- The conclusion is $\exists x(P(x) \land \neg B(x))$.

Example Proof

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore \exists x(P(x) \land \neg B(x)) \end{array}$$

1.	$\exists x(C(x) \land \neg B(x))$	Hypothesis
2.	$C(a) \land \neg B(a)$	Existential instantiation from 1
3.	$C(a)$	Simplification from 2
4.	$\forall x(C(x) \rightarrow P(x))$	Hypothesis
5.	$C(a) \rightarrow P(a)$	Universal instantiation from 4
6.	$P(a)$	Modus ponens from 3 and 5
7.	$\neg B(a)$	Simplification from 2
8.	$P(a) \land \neg B(a)$	Conjunction from 6 and 7
9.	$\exists x(P(x) \land \neg B(x))$	Existential generalization from 8

Universal Modus Ponens

- If $\forall x(P(x) \rightarrow Q(x))$ is true, and if P(a) is true for a particular element ‘a’ in the domain of the universal quantifier, then Q(a) must also be true.

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ \underline{P(a), \text{ where a is a particular element in the domain}} \\ \therefore Q(a) \end{array}$$

Universal Modus Tollens

- If $\forall x(P(x) \rightarrow Q(x))$ is true, and if Q(a) is false for a particular element ‘a’ in the domain of the universal quantifier, then P(a) must also be false.

$\forall x(P(x) \rightarrow Q(x))$

$\neg Q(a)$, where a is a particular element in the domain

$\therefore \neg P(a)$

Example of Proof

- Given the hypotheses:
  - “Maria is a student in this class, owns a red convertible.”
  - “Everybody who owns a red convertible has gotten a speeding ticket”
- Can you conclude: “Somebody in this class has gotten a speeding ticket”?

- Let C(x) = “x is student in class”.
- Let R(x) = “x has owns a red convertible”.
- Let T(x) = “x has gotten speeding ticket”.

- The hypothesis are $\forall x\ (R(x) \rightarrow T(x))$, C(Maria) and R(Maria).
- The conclusion is $\exists x\ (C(x) \wedge T(x))$.

Example of Proof

1.	$\forall x\,(R(x) \rightarrow T(x))$	Hypothesis
2.	$R(Maria) \rightarrow T(Maria)$	Universal instantiation using 1
3.	$R(Maria)$	Hypothesis
4.	$T(Maria)$	Modes ponens using 2 and 3
5.	$C(Maria)$	Hypothesis
6.	$C(Maria) \wedge T(Maria)$	Conjunction using 5 and 4
7.	$\exists x\,(C(x) \wedge T(x))$	Existential generalization using 6

How do you know which one to use?

- Experience!
- In general, use quantifiers with statements like “for all” or “there exists”

Chapter Reading

- ***Chapter 1***, Kenneth H. Rosen, Discrete Mathematics and Its Applications, Section 1.6

Exercise Questions

Q – 1, 2, 3, 4, 5, 6,13-b,13-d, 23, 24