Example 21.12 Field of a uniformly charged disk

Find the electric field caused by a disk of radius R with a uniform positive surface charge density (charge per unit area) σ, at a point along the axis of the disk a distance x from its center. Assume that x is positive.

SOLUTION

IDENTIFY: This example is similar to Examples 21.10 and 21.11 in that our target variable is the electric field along a symmetry axis of a continuous charge distribution.

SET UP: Figure 21.26 shows the situation. We can represent the charge distribution as a collection of concentric rings of charge dQ, as shown in Fig. 21.26. From Example 21.10 we know the field of a single ring on its axis of symmetry, so all we have to do is add the contributions of the rings.

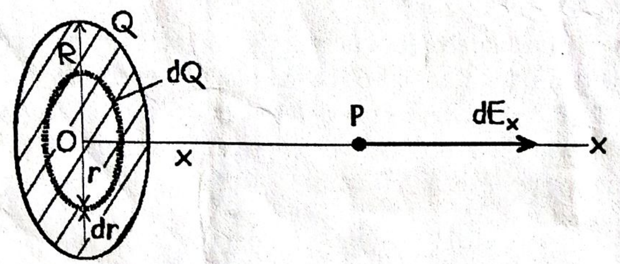
EXECUTE: A typical ring has charge dQ, inner radius r, and outer radius $r + dr$ (Fig. 21.26). Its area dA is approximately equal to its width dr times its circumference $2\pi r$, or $dA = 2\pi r\,dr$. The charge per unit area is $\sigma = dQ/dA$, so the charge of the ring is $dQ = \sigma\,dA = \sigma\,(2\pi r\,dr)$, or

$$dQ = 2\pi\sigma r\,dr$$

We use this in place of Q in the expression for the field due to a ring found in Example 21.10, Eq. (21.8), and also replace the ring radius a with r. The field component dE_x at point P due to charge dQ is

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{(2\pi\sigma r\,dr)x}{(x^2+r^2)^{3/2}}$$

21.26 Our sketch for this problem.



To find the total field due to all the rings, we integrate dE_x over r from $r = 0$ to $r = R$ (*not* from $-R$ to R):

$$E_x = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{(2\pi\sigma r\,dr)x}{(x^2+r^2)^{3/2}} = \frac{\sigma x}{2\epsilon_0}\int_0^R \frac{r\,dr}{(x^2+r^2)^{3/2}}$$

Remember that x is a constant during the integration and that the integration variable is r. The integral can be evaluated by use of the substitution $z = x^2 + r^2$. We'll let you work out the details; the result is

$$E_x = \frac{\sigma x}{2\epsilon_0}\left[-\frac{1}{\sqrt{x^2+R^2}} + \frac{1}{x}\right] = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{(R^2/x^2)+1}}\right] \qquad (21.11)$$

The electric field due to the ring has no components perpendicular to the axis. Hence at point P in Fig. 21.26, $dE_y = dE_z = 0$ for each ring, and the total field has $E_y = E_z = 0$.

EVALUATE Suppose we keep increasing the radius R of the disk, simultaneously adding charge so that the surface charge density σ (charge per unit area) is constant. In the limit that R is much larger than the distance x of the field point from the disk, the term $1/\sqrt{(R^2/x^2)+1}$ in Eq. (21.11) becomes negligibly small, and we get

$$E = \frac{\sigma}{2\epsilon_0} \qquad (21.12)$$

Our final result does not contain the distance x from the plane. Hence the electric field produced by an *infinite* plane sheet of charge is *independent of the distance from the sheet*. The field direction is everywhere perpendicular to the sheet, away from it. There is no such thing as an infinite sheet of charge, but if the dimensions of the sheet are much larger than the distance x of the field point P from the sheet, the field is very nearly given by Eq. (21.11).

If P is to the *left* of the plane $(x < 0)$, the result is the same except that the direction of $\vec{E}$ is to the left instead of the right. If the surface charge density is negative, the directions of the fields on both sides of the plane are toward it rather than away from it.