

(Production Planning) A manufacturer makes three different types of chemical products: A , B , and C . Each product must go through two processing machines: X and Y . The products require the following times in machines X and Y :

1. One ton of A requires 2 hours in machine X and 2 hours in machine Y .
2. One ton of B requires 3 hours in machine X and 2 hours in machine Y .
3. One ton of C requires 4 hours in machine X and 3 hours in machine Y .

Machine X is available 80 hours per week, and machine Y is available 60 hours per week. Since management does not want to keep the expensive machines X and Y idle, it would like to know how many tons of each product to make so that the machines are fully utilized. It is assumed that the manufacturer can sell as much of the products as is made.

To solve this problem, we let x_1 , x_2 , and x_3 denote the number of tons of products A , B , and C , respectively, to be made. The number of hours that machine X will be used is

$$2x_1 + 3x_2 + 4x_3,$$

which must equal 80. Thus we have

$$2x_1 + 3x_2 + 4x_3 = 80.$$

Similarly, the number of hours that machine Y will be used is 60, so we have

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, our problem is to find nonnegative values of x_1 , x_2 , and x_3 so that

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 80 \\ 2x_1 + 2x_2 + 3x_3 &= 60. \end{aligned}$$

This linear system has infinitely many solutions. Following the method of Example 4, we see that all solutions are given by

$$\begin{aligned} x_1 &= \frac{20 - x_3}{2} \\ x_2 &= 20 - x_3 \\ x_3 &= \text{any real number such that } 0 \leq x_3 \leq 20, \end{aligned}$$

since we must have $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. When $x_3 = 10$, we have

$$x_1 = 5, \quad x_2 = 10, \quad x_3 = 10$$

while

$$x_1 = \frac{13}{2}, \quad x_2 = 13, \quad x_3 = 7$$

when $x_3 = 7$. The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

Quadratic Interpolation

Various approximation techniques in science and engineering use a parabola that passes through three given data points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, where $x_i \neq x_j$ for $i \neq j$. We call these **distinct points**, since the x -coordinates are all different. The graph of a quadratic polynomial $p(x) = ax^2 + bx + c$ is a parabola, and we use the given data points to determine the coefficients a , b , and c as follows. Requiring that $p(x_i) = y_i$, $i = 1, 2, 3$, gives us three linear equations with unknowns a , b , and c :

$$\begin{aligned} p(x_1) &= y_1 & \text{or} & & ax_1^2 + bx_1 + c &= y_1 \\ p(x_2) &= y_2 & \text{or} & & ax_2^2 + bx_2 + c &= y_2 \\ p(x_3) &= y_3 & \text{or} & & ax_3^2 + bx_3 + c &= y_3. \end{aligned} \tag{3}$$

Let

$$A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}$$

be the coefficient matrix, $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then (3) can be written in matrix equation form as $A\mathbf{v} = \mathbf{y}$ whose augmented matrix

$$[A \mid \mathbf{y}] = \left[\begin{array}{ccc|c} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{array} \right].$$

Find the quadratic interpolant for the three distinct points $\{(1, -5), (-1, 1), (2, 7)\}$.

Solution


Setting up linear system (3), we find that its augmented matrix is (verify)

$$[A | \mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -5 \\ 1 & -1 & 1 & 1 \\ 4 & 2 & 1 & 7 \end{array} \right].$$

Solving this linear system, we obtain (verify)


$$a = 5, \quad b = -3, \quad c = -7.$$

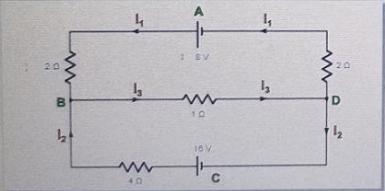
Thus the quadratic interpolant is $p(x) = 5x^2 - 3x - 7$, and its graph is given in Figure 2.1. The asterisks represent the three data points. ■

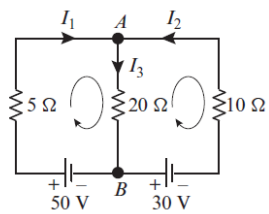


Applications of Systems of Linear Equations to Electrical Networks

Reta Odisho
Department of Mathematics
Dr. Jen-Mei Chang



Introduction	Results	Summary
<p>Electrical networks are a specialized type of network providing information about power sources, such as batteries, and devices powered by these sources, such as light bulbs or motors. A power source forces a current to flow through the network, where it encounters various resistors, each which requires a certain amount of force to be applied in order for the current to flow through.</p> <p>Systems of linear equations are used to determine the currents through various branches of electrical networks.</p>	<p>Let the currents in the various branches of the circuit be I_1, I_2 and I_3. Applying Kirchhoff's Law, there are two junctions in the circuit namely the points B and D. There are two closed paths ABDA and CBDC. Applying Kirchhoff's Law to the junctions and paths results in:</p> <p>JUNCTIONS</p> <p>B: $I_1 + I_2 = I_3$ D: $I_3 = I_1 + I_2$</p> <p>These two equations result in a single linear equation: $I_1 + I_2 - I_3 = 0$</p> <p>PATHS</p> <p>ABDA: $2I_1 + 1I_2 + 2I_3 = 8$ CBDC: $4I_2 + 1I_3 = 16$</p> <p>We know have a system of three linear equations in three unknowns. The problem thus reduces to solving the following system of three linear equations in three variables:</p> $\begin{cases} I_1 + I_2 - I_3 = 0 \\ 4I_1 + I_2 + 1I_3 = 8 \\ 4I_2 + 1I_3 = 16 \end{cases}$ <p>This given system can be placed into an augmented matrix. We can now use row reduction algorithms to obtain an equivalent augmented matrix.</p> $\left(\begin{array}{ccc c} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 8 \\ 0 & 4 & 1 & 16 \end{array} \right) \xrightarrow{R2 - 4R1, R3 - 4R2} \left(\begin{array}{ccc c} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 8 \\ 0 & 4 & 1 & 16 \end{array} \right) \xrightarrow{R2 \leftrightarrow R3} \left(\begin{array}{ccc c} 1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 16 \\ 0 & -4 & 5 & 8 \end{array} \right) \xrightarrow{R3 + R2} \left(\begin{array}{ccc c} 1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 16 \\ 0 & 0 & 6 & 24 \end{array} \right) \xrightarrow{R2 \cdot 1/4, R3 \cdot 1/6} \left(\begin{array}{ccc c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1/4 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{R2 - 1/4 R3} \left(\begin{array}{ccc c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{R1 - R2} \left(\begin{array}{ccc c} 1 & 0 & -1 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{R1 + R3} \left(\begin{array}{ccc c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right)$	<p>From the equivalent row reduced augmented matrix, it is apparent the system is consistent and thus a solution exists. The currents I_1, I_2, I_3 are as follows:</p> <p style="text-align: center;">$I_1 = 1 \text{ Amp}$ $I_2 = 3 \text{ Amps}$ $I_3 = 4 \text{ Amps}$</p> <p>The solution can be verified by substituting the values of the current into the original three linear equations with three unknowns.</p>
<p>Known</p> <p>Ohm's Law</p> <ul style="list-style-type: none"> The voltage drop across a resistor is given by $V = IR$ <p>Kirchhoff's Law</p> <ul style="list-style-type: none"> Junction: All the current flowing into a junction must flow out of it. Path: The sum of the IR terms in any direction around a closed path is equal to the total voltage in the path in that direction. 	<p>Methods</p> <p>We wish to determine the currents I_1, I_2, and I_3 in the below circuit. Applying Ohm's and Kirchhoff's Law, we can construct a system of linear equations</p> 	<p>Conclusion</p> <p>The model for current flow is linear precisely because Ohm's Law and Kirchhoff's Law are linear: the voltage drop across a resistor is proportional to the current flowing through it (Ohm), and the sum of the voltage drop in a loop equals the sum of the voltage sources in the loop (Kirchhoff).</p> <p>In practice, electrical networks can involve many resistances and circuits. Determining currents through branches involves solving large systems of equations that would require a computer.</p>
<p style="text-align: center;">References</p> <p>Gareth, Williams. <i>Linear algebra with applications</i>. Boston: Jones and Bartlett, 2004.</p>		



▲ Figure 1.9.9

► EXAMPLE 4 A Circuit with Three Closed Loops

Determine the currents I_1 , I_2 , and I_3 in the circuit shown in Figure 1.9.9.

Solution Using the assigned directions for the currents, Kirchhoff's current law provides one equation for each node:

Node	Current In		Current Out
A	$I_1 + I_2$	$=$	I_3
B	I_3	$=$	$I_1 + I_2$

However, these equations are really the same, since both can be expressed as

$$I_1 + I_2 - I_3 = 0 \quad (2)$$

To find unique values for the currents we will need two more equations, which we will obtain from Kirchhoff's voltage law. We can see from the network diagram that there are three closed loops, a left inner loop containing the 50 V battery, a right inner loop containing the 30 V battery, and an outer loop that contains both batteries. Thus, Kirchhoff's voltage law will actually produce three equations. With a clockwise traversal of the loops, the voltage rises and drops in these loops are as follows:

	Voltage Rises	Voltage Drops
Left Inside Loop	50	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	0
Outside Loop	$30 + 50 + 10I_2$	$5I_1$

These conditions can be rewritten as

$$\begin{aligned} 5I_1 + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \\ 5I_1 - 10I_2 &= 80 \end{aligned} \quad (3)$$

However, the last equation is superfluous, since it is the difference of the first two. Thus, if we combine (2) and the first two equations in (3), we obtain the following linear system of three equations in the three unknown currents:

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 5I_1 + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \end{aligned}$$

We leave it for you to show that the solution of this system in amps is $I_1 = 6$, $I_2 = -5$, and $I_3 = 1$. The fact that I_2 is negative tells us that the direction of this current is opposite to that indicated in Figure 1.9.9. ◀

(Cryptology) Cryptology is the technique of coding and decoding messages; it goes back to the time of the ancient Greeks. A simple code is constructed by associating a different number with every letter in the alphabet. For example,

A	B	C	D	...	X	Y	Z
↕	↕	↕	↕		↕	↕	↕
1	2	3	4	...	24	25	26

Suppose that Mark S. and Susan J. are two undercover agents who want to communicate with each other by using a code because they suspect that their phones are being tapped and their mail is being intercepted. In particular, Mark wants to send Susan the following message:

MEET TOMORROW

Using the substitution scheme just given, Mark sends this message:

13 5 5 20 20 15 13 15 18 18 15 23

A code of this type could be cracked without too much difficulty by a number of techniques, including the analysis of frequency of letters. To make it difficult

to crack the code, the agents proceed as follows. First, when they undertook the mission, they agreed on a 3×3 nonsingular matrix, the **encoding matrix**, such as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Mark then breaks the message into four vectors in R^3 . (If this cannot be done, we can add extra letters.) Thus we have the vectors

$$\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix}, \quad \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix}.$$

Mark now defines the linear transformation $L: R^3 \rightarrow R^3$ by $L(\mathbf{x}) = A\mathbf{x}$, so the message becomes

$$\begin{aligned} A \begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix} &= \begin{bmatrix} 38 \\ 28 \\ 15 \end{bmatrix}, & A \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix} &= \begin{bmatrix} 105 \\ 70 \\ 50 \end{bmatrix}, \\ A \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix} &= \begin{bmatrix} 97 \\ 64 \\ 51 \end{bmatrix}, & A \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix} &= \begin{bmatrix} 117 \\ 79 \\ 61 \end{bmatrix}. \end{aligned}$$

Thus Mark transmits the following message:

$$38 \quad 28 \quad 15 \quad 105 \quad 70 \quad 50 \quad 97 \quad 64 \quad 51 \quad 117 \quad 79 \quad 61$$

Suppose now that Mark receives the message from Susan,

$$77 \quad 54 \quad 38 \quad 71 \quad 49 \quad 29 \quad 68 \quad 51 \quad 33 \quad 76 \quad 48 \quad 40 \quad 86 \quad 53 \quad 52$$

which he wants to decode with the same key matrix A . To decode it, Mark breaks the message into five vectors in R^3 :

$$\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix}, \quad \begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix}, \quad \begin{bmatrix} 68 \\ 51 \\ 33 \end{bmatrix}, \quad \begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix}, \quad \begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix}$$