Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

Mahwish Waqas

Lecture Outline

- Relations
  - What is a Relation?
  - Binary Relation
  - Representation of Relation
  - Relation Properties
  - Combining Relations

Relations

- The connections between people and things.

- Between people, family relation
  - 'to be brothers' x is a brother of y
  - 'to be older' x is older than y
  - 'to be parents' x and y are parents of z
- Between things, numerical relations
  - 'to be greater than' $x > y$ on the real numbers
  - 'to be divisible by' x is divisible by y on the set of integers
- Between things and people, legal relations
  - 'to be an owner' x is an owner of y

Cartesian Product

- The Cartesian product of sets A and B, denoted by A × B, is the set of all ordered pairs of elements from A and B.
- $A \times B = \{(a, b) | a \in A \wedge b \in B\}$

- The elements of the Cartesian product are ordered pairs. In particular $(a, b) = (c, d) \leftrightarrow (a = c) \wedge (b = d)$

- $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$
- $\{Mon, Tue\} \times \{Jan, Feb\} = \{(Mon, Jan), (Mon, Feb), (Tue, Jan), (Tue, Feb)\}$

Cartesian Product of More Than Two Sets

- Instead of ordered pairs we may consider ordered triples, or, more general, k-tuples
- (a, b), an ordered pair
- (a, b, c), an ordered triple
- (a, b, c, d), an ordered quadruple
- $(a_1, a_2, \ldots, a_k)$, a k-tuple

- Pairs, triples, quadruples, and k-tuples are elements of Cartesian products of 2, 3, 4, and k sets, respectively

Relations

- Relationship between elements of sets are represented using the structure called a relation, which is just a subset of the cartesian product of sets.

Binary Relation

- Let A and B be sets. A binary relation from set A to set B is any subset of $A \times B$.
- Binary relations represent relationship between two sets.

Example

- $A = \{Ali, Junaid, Hassan\}$
- $B = \{CSC102,\ CSC222,\ CSC106\}$
- R: "*relation of students enrolled in courses*"
- $A \times B = \{(Ali, CSC102), (Ali, CSC222), (Ali, CSC106),$
  $(\mathrm{Juanid}, \mathrm{CSC102}), (\mathrm{Juanid}, \mathrm{CSC222}), (\mathrm{Juanid}, \mathrm{CSC106}),$
  $(\mathrm{Hassan}, \mathrm{CSC102}),\ (\mathrm{Hassan}, \mathrm{CSC222}), (\mathrm{Hassan}, \mathrm{CSC106})\}$
- $R = \{(Junaid, CSC102),\ (Hassan, CSC222)\}$

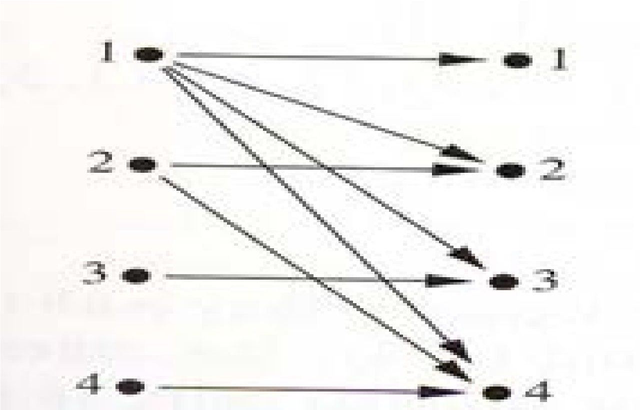
- $(Junaid, \mathrm{CSC102}) \in R$
- $(Ali, \mathrm{CSC102}) \notin R$
- $(Junaid, CSC222) \notin R$
- $(Ali, CSC106) \notin R$

More Relations

- Relations can be generalized to subsets of cartesian products of more than two sets
- Any subset of the Cartesian product of 3 sets is called a ternary relation
  - 'x and y are parents of z' $\subseteq$ People $\times$ People $\times$ People
- Any subset of the Cartesian product of k sets is called a k-ary relation

Functions as Relations

- A function f:A→B is a relation from A to B
- A relation from A to B is not always a function f:A→B
- Relations are generalizations of functions!



Relations on a Set

- A (binary) relation from a set *A* to itself is called a relation on the set *A*.

$$A = \{1, 2, 3, 4\}$$

$$R_1 = \{(a, b) \mid a \text{ divides } b\}$$

$$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4),$$
$$(3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

$$R_1 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

$$R_1 \subseteq A \times A$$

Relations on a Set

- A (binary) relation from a set *A* to itself is called a relation on the set *A*.

$$A = \{1, 2, 3, 4\}$$

$$R_2 = \{(a,b) \mid a > b\}$$

$$R_2 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

$$R_2 \subseteq A \times A$$

$$R_3 = \{(a,b) \mid a + b = 4\}$$

$$R_3 = \{(1,3), (2,2), (3,1)\}$$

$$R_3 \subseteq A \times A$$

Describing Binary Relations

- A binary relation can be described using a list of pairs.

$$A = \{1, 2, 3, 4\}$$

$$R_1 = \{(a, b) \mid a \text{ divides } b\}$$

$$R_1 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

Describing Binary Relations

- A binary relation can also be described using a matrix.
  - Rows are labeled with elements of A
  - Columns are labeled with elements of B.
- We write 1 in row a, column b if and only if $(a, b) \in R$; otherwise we write 0.

Describing Binary Relations

$$A = \{1, 2, 3, 4\}$$

$$R_1 = \{(a,b) \mid a \text{ divides } b\}$$

$$R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Describing Binary Relations

$R_2 = \{(a,b) \mid a > b\}$

$R_2 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$$M_{R_2} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Describing Binary Relations

- A binary relation can also be described using a directed graph.
- A graph of a relation $R \in A \times B$ consists of two sets of vertices labeled by elements of A and B.
- A vertex **a** is connected to a vertex **b** with an edge (arc) if and only if $(a, b) \in R$.

Reflexivity

- Note: from now on we will consider only binary relations on A.
- That is such relations are subsets of $A \times A$.
- Reflexivity: A binary relation $\mathrm{R} \subseteq A \times A$ is said to be reflexive if $\forall a \in A, (a, a) \in R$
- $R = \{(a, b) | (a, b) \in Z \times Z, a \leq b\}$ is reflexive.

	a	b	c	d	e	...
a	1	.	.	.	.	.
b	.	1	.	.	.	.
c	.	.	1	.	.	.
d	.	.	.	1	.	.
e	.	.	.	.	1	.
...	.	.	.	.	.	1

Reflexivity

$\forall a \in A \quad (a,a) \in R$

$A = \{1, 2, 3, 4\}$

$(1,1), (2,2), (3,3), (4,4) \in R$

If Diagonal contains all 1's
then
Reflexive
else
ir-reflexive

$R_1 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$R_2 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$$M_{R_1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

ir-reflexive because Diagonal contains zero's.

Symmetricity

- Symmetricity: A binary relation $\mathrm{R} \subseteq A \times A$ said to be symmetric if, $\forall a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$
- $R = \{(a, b) | (a, b) \in people \times people, a \ and \ b \ are \ siblings\}$ is symmetric, because if a is a sibling of b, then b is a sibling of a

	a	b	c	d	e	...
a	0	1	0	0	0	.
b	1	0	0	1	0	.
c	0	0	0	0	1	.
d	0	1	0	0	0	.
e	0	0	1	0	0	.
...	.	.	.	.	.	.

Symmetricity

$\forall a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$

$R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

$R_2 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

if $M_R = M_R^t$

then symmetric

else

not-symmetric

$M_{R_1} \neq M_{R_1}^t$ Not Symmetric

$M_{R_2} \neq M_{R_2}^t$ Not symmetric

Transitivity:

- Transitivity: A binary relation $\mathrm{R} \subseteq A \times A$ said to be transitive if, $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$
- $R = \{(a, b) | (a, b) \in Z \times Z, a \leq b\}$ is transitive, because if, $a \leq b$ and $b \leq c$ then $a \leq c$.

$\forall a, b, c \in A,$	**If** $(a, b) \in R \wedge (b, c) \in R$ $\boldsymbol{then}$ $(a, c) \in R$		**Transitive**
Case 1	True	True	True
Case 2	True	False	False
Case 3	False		True

Transitivity:

$R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R_1} * M_{R_1} = M_{R_1}^2$$

M_R	M_R^2	Transitive
Non-Zero	Non-Zero	True
Non-Zero	Zero	True
Zero	Non-Zero	False

Anti-Symmetricity

- Anti-Symmetricity: A binary relation $\mathrm{R} \subseteq A \times A$ said to be anti-symmetric, if $\forall a, b \in A, (a, b) \in R \wedge (b, a) \in R \rightarrow a = b$
- The relation Motherhood ('x is the mother of y') is anti-symmetric, because if x is a mother of y, then y is not the mother of x.

	a	b	c	d	e	...
a	0	0	0	0	0	.
b	1	0	0	0	0	.
c	0	0	1	0	0	.
d	0	0	0	0	0	.
e	0	0	1	0	0	.
...	.	.	.	.	.	.

Anti-Symmetricity

$\forall a, b \in A$	$if\ (a,b) \in R \land (b,a) \in R\ then\ a = b$		Anti-Symmetricity
Case 1	True	True	True
Case 2	True	False	False
Case 3	False		True

$$M_{R_1} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Antisymmetric

$R_3 = \{(a,b) \mid a+b=4\}$.

$R_3 = \{(1,3),(2,2),(3,1)\}$

$(1,3) \in R_3 \land (3,1) \in R_3 \rightarrow 1 \neq 3$

Not Antisymmetric

Equivalence Relations

- A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example

- Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack
- { (0,0), (1,1), (2,2), (3,3) }
  - Has all the properties, thus, is an equivalence relation
- { (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
  - Not reflexive: (1,1) is missing
  - Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)
- { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
  - Has all the properties, thus, is an equivalence relation
- { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
  - Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)
- { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
  - Not symmetric: (1,2) is present, but not (2,1)
  - Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

Combining Relations

- Relations from A to B are subsets of $\mathrm{A} \times \mathrm{B}$, two relations from A to B can be combined in any way two sets can be combined.
- Let $A = \{\ 1, 2, 3\}\ and\ B = \{1, 2, 3, 4\}$
- The relations $R_1 = \{(1,1), (2,2), (3,3)\}\ and$

$$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$$

- $R_1 \cup R_2$
- $R_1 \cap R_2$
- $R_1 - R_2$
- $R_2 - R_1$

Combining Relations

- Let $A = \{ 1, 2, 3\}\ and\ B = \{1, 2, 3, 4\}$
- The relations $R_1 = \{(1,1), (2,2), (3,3)\}\ and$
  $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$
- $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$
- $R_1 \cap R_2 = \{(1,1)\}$
- $R_1 - R_2 = \{(2,2), (3,3)\}$
- $R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Combining Relations

- Let R be a relation from a set A to a set B and S be a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $\mathrm{a} \in A, c \in C$, and for which their exists an element $\mathrm{b} \in B$ such that $(\mathrm{a}, \mathrm{b}) \in R$ and $(\mathrm{b}, \mathrm{c}) \in S$. We denote the composite of R and S by $S{\circ}R$.
- What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with

$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and

S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with

$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Combining Relations

$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$
$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$

Find $S^\circ R$.

If $(a,b) \in R$ & $(b,c) \in S$ then $(a,c) \in S^\circ R$

$(1,1) \in R \wedge (1,0) \in S$ then $(1,0) \in S^\circ R$

$(1,4) \in R \wedge (4,1) \in S$ then $(1,1) \in S^\circ R$

$(2,3) \in R \wedge (3,1) \in S$ then $(2,1) \in S^\circ R$

$(2,3) \in R \wedge (3,2) \in S$ then $(2,2) \in S^\circ R$

$(3,1) \in R \wedge (1,0) \in S$ then $(3,0) \in S^\circ R$

$(3,4) \in R \wedge (4,1) \in S$ then $(3,1) \in S^\circ R$.

Exercise

1. Find $R∘R$. Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$
2. Find $S∘R$ using matrix representing of the relations.
   Let R is the relation from $\{1,2,3\}\ to\ \{1,2,3,4\}$ with
   $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$ and
   S is the relation from $\{1,2,3,4\}\ to\ \{0,1,2\}$ with
   $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$.

Exercise

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Examples

- Parenthood ('x is the parent of y')
- Brotherhood ('x is the brother of y')
- Neighborhood ('x is the neighbor of y')
- Ownership ('x is the owner of y‘)
- 'x divides y'
- $x \leq y$
- $x = y$
- $x < y$

Chapter Reading and Exercise

Chapter # 9

Topic # 9.1

Q-1,2,3,10,11,18,30,32

Topic # 9.3

Q-1,2,3,4,5,6,7,8,13-c,14,23-28