(Production Planning) A manufacturer makes three different types of chemical products: A, B, and C. Each product must go through two processing machines: X and Y. The products require the following times in machines X and Y:

1. One ton of A requires 2 hours in machine X and 2 hours in machine Y.
2. One ton of B requires 3 hours in machine X and 2 hours in machine Y.
3. One ton of C requires 4 hours in machine X and 3 hours in machine Y.

Machine X is available 80 hours per week, and machine Y is available 60 hours per week. Since management does not want to keep the expensive machines X and Y idle, it would like to know how many tons of each product to make so that the machines are fully utilized. It is assumed that the manufacturer can sell as much of the products as is made.

To solve this problem, we let x_1, x_2, and x_3 denote the number of tons of products A, B, and C, respectively, to be made. The number of hours that machine X will be used is

$$2x_1 + 3x_2 + 4x_3,$$

which must equal 80. Thus we have

$$2x_1 + 3x_2 + 4x_3 = 80.$$

Similarly, the number of hours that machine Y will be used is 60, so we have

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, our problem is to find nonnegative values of x_1, x_2, and x_3 so that

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 80 \\ 2x_1 + 2x_2 + 3x_3 &= 60. \end{aligned}$$

This linear system has infinitely many solutions. Following the method of Example 4, we see that all solutions are given by

$$\begin{aligned} x_1 &= \frac{20 - x_3}{2} \\ x_2 &= 20 - x_3 \\ x_3 &= \text{any real number such that } 0 \le x_3 \le 20, \end{aligned}$$

since we must have $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$. When $x_3 = 10$, we have

$$x_1 = 5, \qquad x_2 = 10, \qquad x_3 = 10$$

while

$$x_1 = \tfrac{13}{2}, \qquad x_2 = 13, \qquad x_3 = 7$$

when $x_3 = 7$. The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

Quadratic Interpolation

Various approximation techniques in science and engineering use a parabola that passes through three given data points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, where $x_i \neq x_j$ for $i \neq j$. We call these **distinct points**, since the x-coordinates are all different. The graph of a quadratic polynomial $p(x) = ax^2+bx+c$ is a parabola, and we use the given data points to determine the coefficients a, b, and c as follows. Requiring that $p(x_i) = y_i$, $i = 1, 2, 3$, gives us three linear equations with unknowns a, b, and c:

$$\begin{aligned} p(x_1) = y_1 &\quad \text{or} \quad ax_1^2 + bx_1 + c = y_1 \\ p(x_2) = y_2 &\quad \text{or} \quad ax_2^2 + bx_2 + c = y_2 \\ p(x_3) = y_3 &\quad \text{or} \quad ax_3^2 + bx_3 + c = y_3. \end{aligned} \tag{3}$$

Let

$$A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}$$

be the coefficient matrix, $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then (3) can be written in matrix equation form as $A\mathbf{v} = \mathbf{y}$ whose augmented matrix

$$\left[A \mid \mathbf{y}\right] = \left[\begin{array}{ccc|c} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{array}\right].$$

Find the quadratic interpolant for the three distinct points $\{(1, -5), (-1, 1), (2, 7)\}$.

Solution

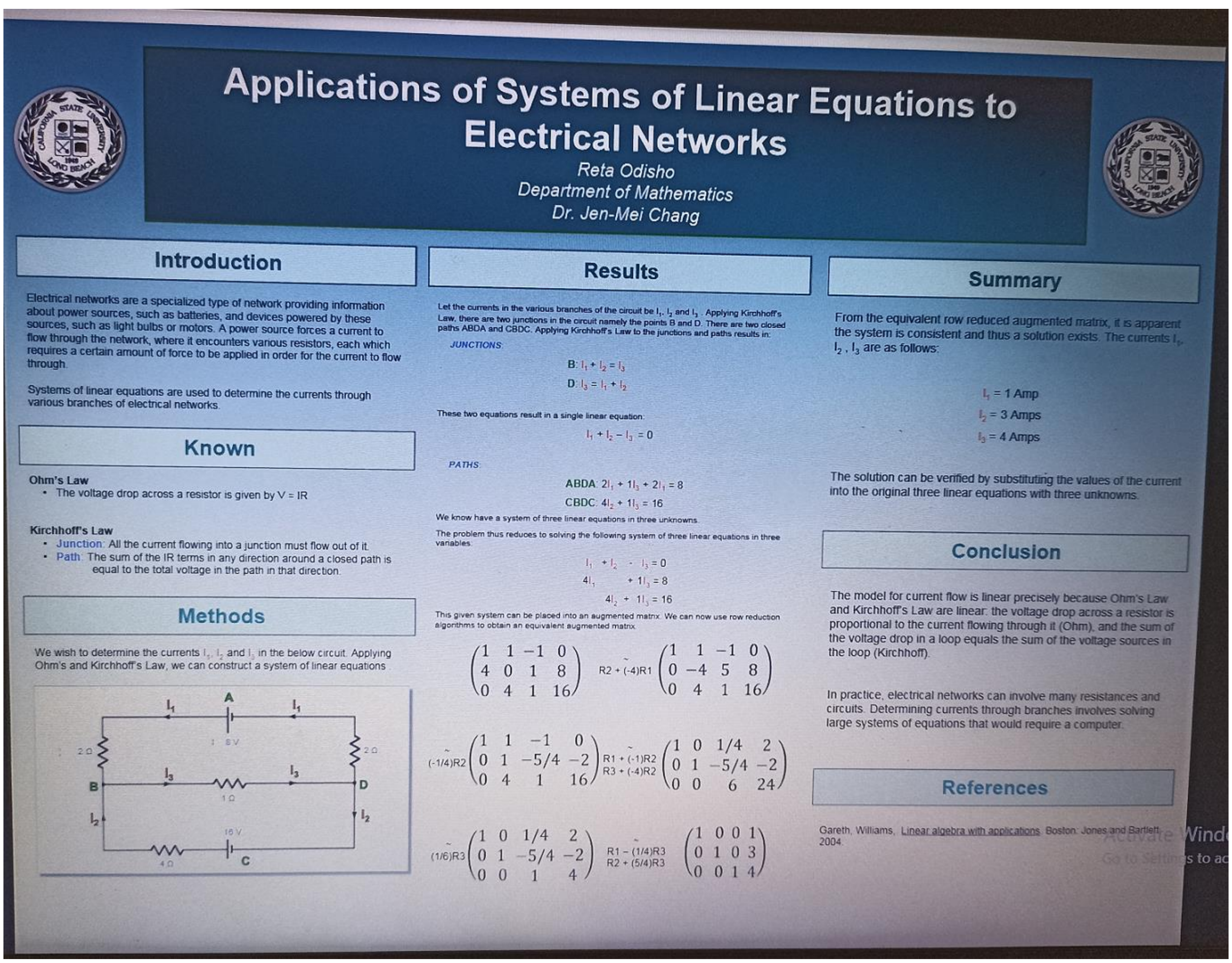
Setting up linear system (3), we find that its augmented matrix is (verify)

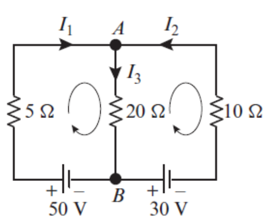
$$[A \mid \mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -5 \\ 1 & -1 & 1 & 1 \\ 4 & 2 & 1 & 7 \end{array}\right].$$

Solving this linear system, we obtain (verify)

$$a = 5, \quad b = -3, \quad c = -7.$$

Thus the quadratic interpolant is $p(x) = 5x^2 - 3x - 7$, and its graph is given in Figure 2.1. The asterisks represent the three data points. ■





▲ Figure 1.9.9

EXAMPLE 4 A Circuit with Three Closed Loops

Determine the currents I_1, I_2, and I_3 in the circuit shown in Figure 1.9.9.

Solution Using the assigned directions for the currents, Kirchhoff's current law provides one equation for each node:

Node	Current In		Current Out
A	$I_1 + I_2$	$=$	I_3
B	I_3	$=$	$I_1 + I_2$

However, these equations are really the same, since both can be expressed as

$$I_1 + I_2 - I_3 = 0 \tag{2}$$

To find unique values for the currents we will need two more equations, which we will obtain from Kirchhoff's voltage law. We can see from the network diagram that there are three closed loops, a left inner loop containing the 50 V battery, a right inner loop containing the 30 V battery, and an outer loop that contains both batteries. Thus, Kirchhoff's voltage law will actually produce three equations. With a clockwise traversal of the loops, the voltage rises and drops in these loops are as follows:

	Voltage Rises	**Voltage Drops**
Left Inside Loop	50	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	0
Outside Loop	$30 + 50 + 10I_2$	$5I_1$

These conditions can be rewritten as

$$\begin{aligned} 5I_1 \qquad\quad + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \\ 5I_1 - 10I_2 \qquad\quad &= 80 \end{aligned} \tag{3}$$

However, the last equation is superfluous, since it is the difference of the first two. Thus, if we combine (2) and the first two equations in (3), we obtain the following linear system of three equations in the three unknown currents:

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 5I_1 \qquad\quad + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \end{aligned}$$

We leave it for you to show that the solution of this system in amps is $I_1 = 6$, $I_2 = -5$, and $I_3 = 1$. The fact that I_2 is negative tells us that the direction of this current is opposite to that indicated in Figure 1.9.9. ◀

(Cryptology) Cryptology is the technique of coding and decoding messages; it goes back to the time of the ancient Greeks. A simple code is constructed by associating a different number with every letter in the alphabet. For example,

$$\begin{array}{cccccccc} A & B & C & D & \cdots & X & Y & Z \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & \cdots & 24 & 25 & 26 \end{array}$$

Suppose that Mark S. and Susan J. are two undercover agents who want to communicate with each other by using a code because they suspect that their phones are being tapped and their mail is being intercepted. In particular, Mark wants to send Susan the following message:

MEET TOMORROW

Using the substitution scheme just given, Mark sends this message:

13 5 5 20 20 15 13 15 18 18 15 23

A code of this type could be cracked without too much difficulty by a number of techniques, including the analysis of frequency of letters. To make it difficult to crack the code, the agents proceed as follows. First, when they undertook the mission, they agreed on a 3×3 nonsingular matrix, the **encoding matrix**, such as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Mark then breaks the message into four vectors in R^3. (If this cannot be done, we can add extra letters.) Thus we have the vectors

$$\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix}, \quad \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix}.$$

Mark now defines the linear transformation $L\colon R^3 \to R^3$ by $L(\mathbf{x}) = A\mathbf{x}$, so the message becomes

$$A\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 38 \\ 28 \\ 15 \end{bmatrix}, \qquad A\begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 105 \\ 70 \\ 50 \end{bmatrix},$$

$$A\begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 97 \\ 64 \\ 51 \end{bmatrix}, \qquad A\begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix} = \begin{bmatrix} 117 \\ 79 \\ 61 \end{bmatrix}.$$

Thus Mark transmits the following message:

38 28 15 105 70 50 97 64 51 117 79 61

Suppose now that Mark receives the message from Susan,

77 54 38 71 49 29 68 51 33 76 48 40 86 53 52

which he wants to decode with the same key matrix A. To decode it, Mark breaks the message into five vectors in R^3:

$$\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix}, \quad \begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix}, \quad \begin{bmatrix} 68 \\ 51 \\ 33 \end{bmatrix}, \quad \begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix}, \quad \begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix}$$