

Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

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Lecture Outline

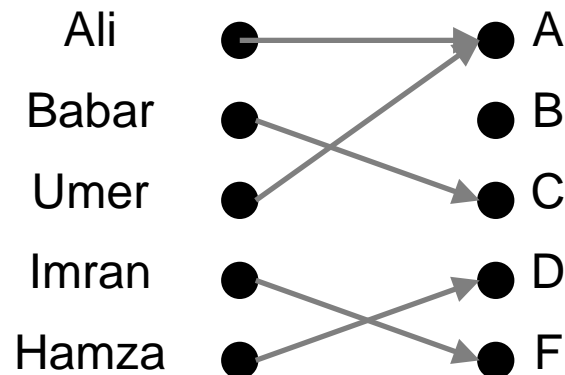
- Functions
 - What is a function?
 - Function Arithmetic
 - One-to-one, Onto and Bijective Functions
 - Identity and Inverse Function
 - Composition of Functions

Application of Functions

- Define discrete structures such as sequences and strings
- Represent the time that a computer takes to solve problems of a given size
- Represent the complexity of algorithms
- ...

Functions

- In many examples we assign to each element of a set, a particular element of a second set (which may be the same as the first).
- For example, suppose that each student in a discrete structures class is assigned a letter grade from the set $\{A, B, C, D, F\}$.
- This assignment is an example of a function.



Functions

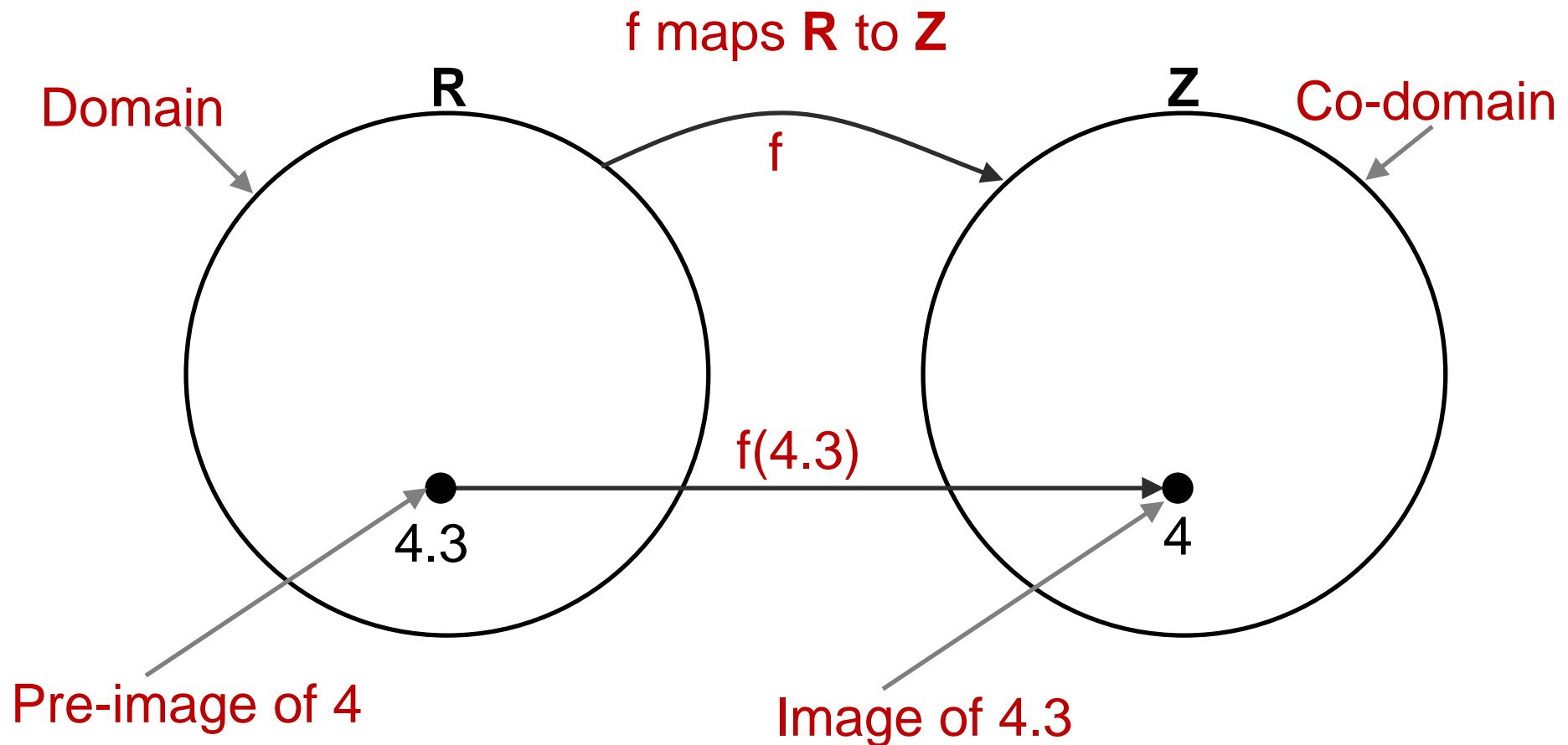
- Let **A** and **B** be nonempty sets.
- A **function** f from **A** to **B** is an assignment of exactly one element of **B** to each element of **A**.
- If **b** is the unique element of **B** assigned by the function f to the element **a** of **A**, we write $f(\mathbf{a}) = \mathbf{b}$.
- If f is a function from **A** to **B**, we write $f: \mathbf{A} \rightarrow \mathbf{B}$.

Functions

- If f is a function from A to B , we say that A is the domain of f and B is the codomain of f .
- if $f(a) = b$, we say that b is the image of a and a is the pre-image of b .
- The range of f is the set of all images of elements of A .
- Also, if f is function from A to B , we say that f maps A to B .

Functions

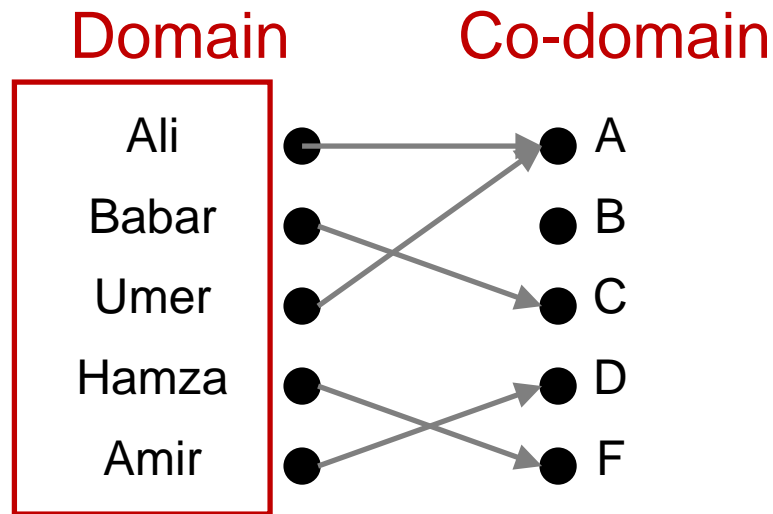
- A function takes an element from a set and maps it to a UNIQUE element in another set.



Arrow Diagram of Functions

- The definition of a function implies that the arrow diagram for a function f has the following two properties:
- Every element of \mathbf{A} has an arrow coming out of it.
- No elements of \mathbf{A} has two arrows coming out of it that point to two different elements of \mathbf{B} .

Arrow Diagram of Functions(example)



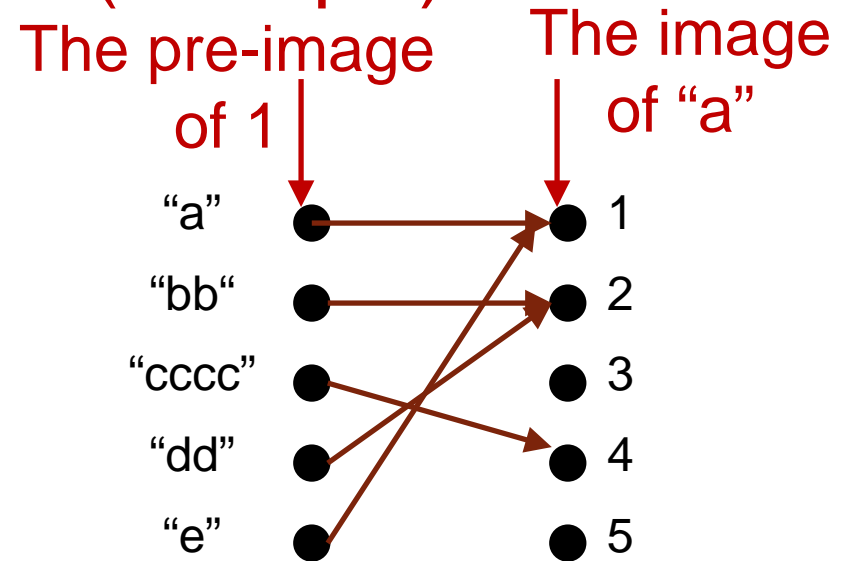
A class grade function

$$g(\text{Ali}) = A$$

$$g(\text{Babar}) = C$$

$$g(\text{Umer}) = A$$

...



A string length function

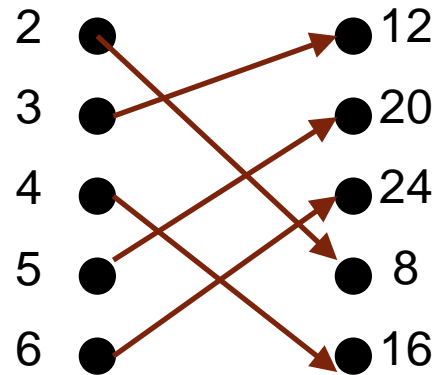
$$f(x) = \text{length } x$$

$$f(\text{"a"}) = 1$$

$$f(\text{"bb"}) = 2$$

...

Arrow Diagram of Functions(example)



$$f(x) = 4x$$

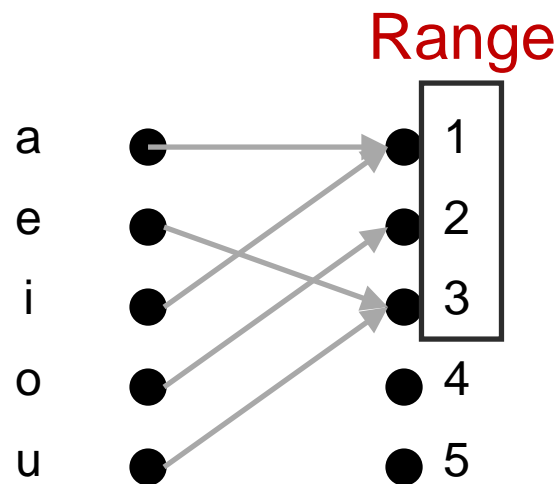
$$f(2) = 8$$

$$f(3) = 12$$

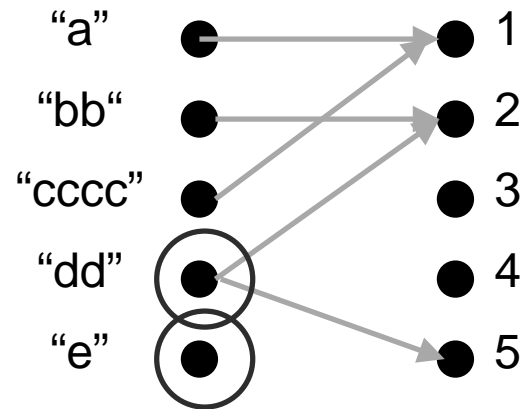
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Functions

The **range** of f is the set of all images of elements of A .



Functions



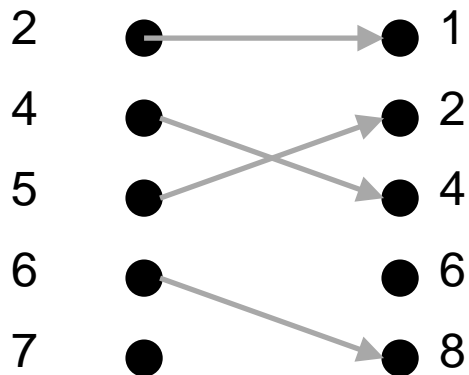
Not a valid function

Functions and Non-Functions

- Which of the arrow diagrams define functions from
- $A = \{2,4,5,6,7\}$ to $B = \{1,2,4,6,8\}$.

Domain

Co-domain



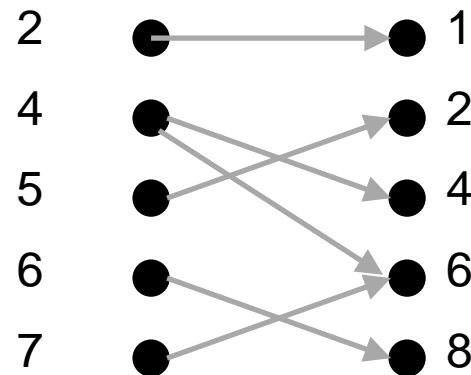
A

B

Not a Function

Domain

Co-domain



A

B

Not a Function

Example

- Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$
assign the square of an integer to this integer
- What is $f(x) = ?$
 - $f(x) = x^2$
- What is domain of f ?
 - Set of all integers
- What is codomain of f ?
 - Set of all integers
- What is the range of f ?
 - $\{0, 1, 4, 9, \dots\}$. All integers that are perfect squares

Function Arithmetic

- Just as we are able to **add (+)**, **subtract (-)**, **multiply (*)**, and **divide (÷)** two or more numbers, we are able to **+**, **-**, *****, and **÷** two or more functions.
- Let f and g be functions from \mathbf{A} to \mathbf{R} . Then $f + g$, $f - g$, $f * g$ and f/g are also functions from \mathbf{A} to \mathbf{R} defined for all $\mathbf{x} \in \mathbf{A}$ by:
 - $(f + g)(x) = f(x) + g(x)$
 - $(f - g)(x) = f(x) - g(x)$
 - $(f * g)(x) = f(x) * g(x)$
 - $(f/g)(x) = f(x)/g(x)$ given that $g(x) \neq 0$

Function Arithmetic

Let f_1 and f_2 be functions from \mathbf{R} to \mathbf{R} such that:

- $f_1(x) = 2x$
- $f_2(x) = x^2$
- Find $f_1 + f_2$ and $f_1 * f_2$.
- $f_1 + f_2 = (f_1 + f_2)(x) = f_1(x) + f_2(x) = 2x + x^2$
- $f_1 * f_2 = (f_1 * f_2)(x) = f_1(x) * f_2(x) = 2x * x^2 = 2x^3$

Function Arithmetic

- Let f and g be functions from \mathbf{R} to \mathbf{R} such that:
- $f(x) = 3x+2$ $g(x) = -2x + 1$
- What is the function $f * g$?

Function Arithmetic

- Let f and g be functions from \mathbf{R} to \mathbf{R} such that:
- $f(x) = 3x+2$ $g(x) = -2x + 1$
- What is the function $f * g$?
- $f * g = (f * g)(x) = f(x) * g(x) = (3x+2) * (-2x+1) = -6x^2 - x + 2$

Function Arithmetic

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- $f(x) = 3x+2$ $g(x) = -2x + 1$
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Let $x = -1$, What is $f(-1) * g(-1)$ and $(f * g)(-1)$?

Function Arithmetic

- Let f and g be functions from \mathbf{R} to \mathbf{R} such that:
- $f(x) = 3x+2$ $g(x) = -2x + 1$
- What is the function $f * g$?
- $f * g = (f * g)(x) = f(x) * g(x) = (3x+2) * (-2x+1) = -6x^2 - x + 2$

Let $x = -1$, What is $f(-1) * g(-1)$ and $(f * g)(-1)$?

- $f(-1) = 3(-1) + 2 = -1$
- $g(-1) = -2(-1) + 1 = 3$
- $f(-1) * g(-1) = -1 \times 3 = -3$
- $(f * g)(-1) = -6(-1)^2 - (-1) + 2 = -6 + 1 + 2 = -3$

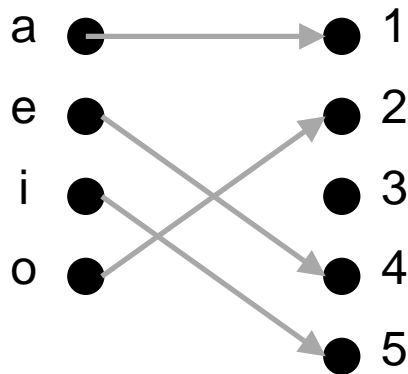
Function Arithmetic

- let $f(x)=5x+2$ and $g(x)=x^2-1$, At $x=4$, $f(4)=22$ and $g(4)=15$.

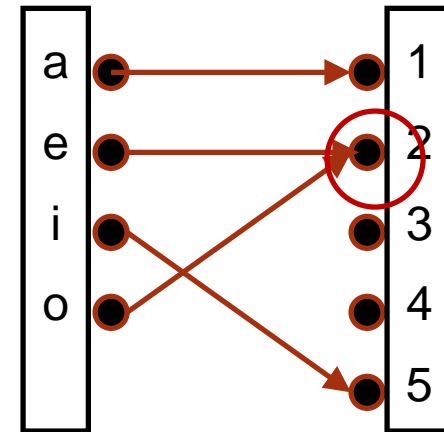
Arithmetic Expression	Combine evaluate	Evaluate, then combine
$(f+g)(x)$ $(5x+2) + (x^2-1)$ $=x^2+5x+1$	$(f+g)(4)$ $4^2+5(4)+1 =16+20+1$ $=37$	$f(4)+g(4)$ $22+15$ $=37$
$(f-g)(x)$ $(5x+2) - (x^2-1)$ $=-x^2+5x+3$	$(f-g)(4)$ $-4^2+5(4)+3$ $=-16+20+3 =7$	$f(4)-g(4)$ $22-15$ $=7$
$(f*g)(x)$ $(5x+2)*(x^2-1)$ $=5x^3+2x^2-5x-2$	$(f*g)(4)$ $5(4^3)+2(4^2)-5(4)-2$ $=5(64)+2(16)-20-2 =330$	$f(4)*g(4)$ $22*15$ $=330$
$(f/g)(x)$ $(5x+2)/(x^2-1)$	$(f/g)(4)$ $[5(4)+2]/[4^2-1]$ $=22/15$	$f(4)/g(4)$ $22/15$

One-to-One Function

- A function is one-to-one if each element in the co-domain has a unique pre-image
- Formal definition: A function f is one-to-one if $f(a) = f(b)$ implies $a = b$ for all a and b in the domain of f .



A one-to-one function



A function that is not one-to-one

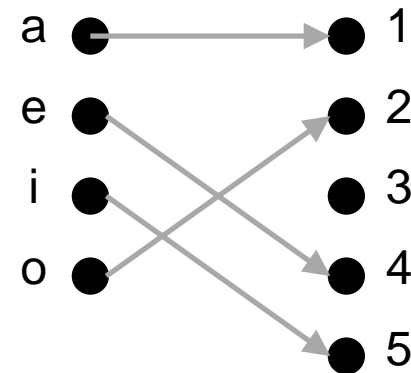
One-to-One Function

- f is one-to-one using quantifiers as
- $\forall a \forall b (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b (a \neq$

More on One-to-One Functions

- Injective is synonymous with one-to-one
 - “A function is injective”
- A function is an injection if it is one-to-one

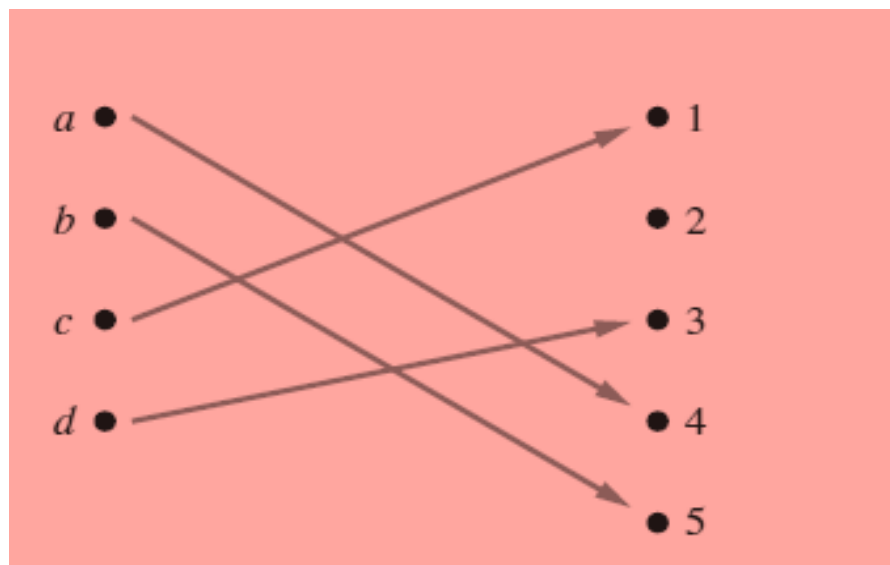
- Note that there can be un-used elements in the co-domain



A one-to-one function

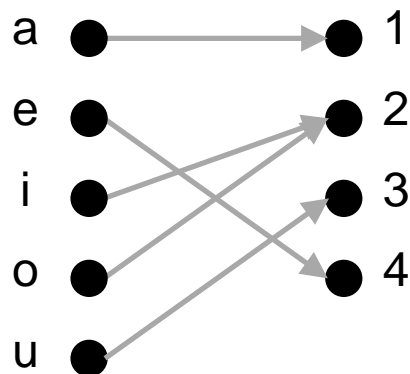
Example one-to-one function

- Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

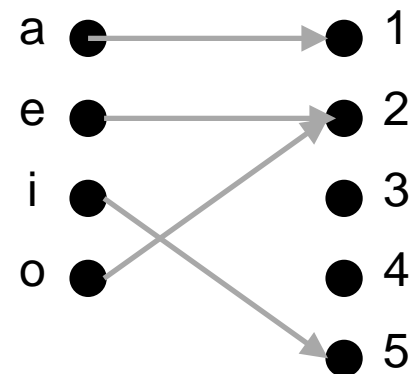


Onto Functions

- A function is onto if each element in the co-domain is an image of some pre-image
- Formal definition: A function f is onto if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.



An onto function



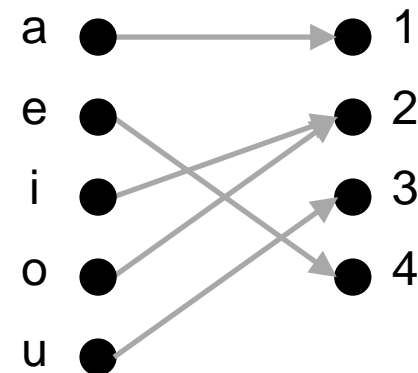
A function that
is not onto

Onto functions

- A function f is onto if $\forall b \exists a (f(a) = b)$, where the domain for a is the domain of the function and the domain for b is the codomain of the function.

More on Onto Functions

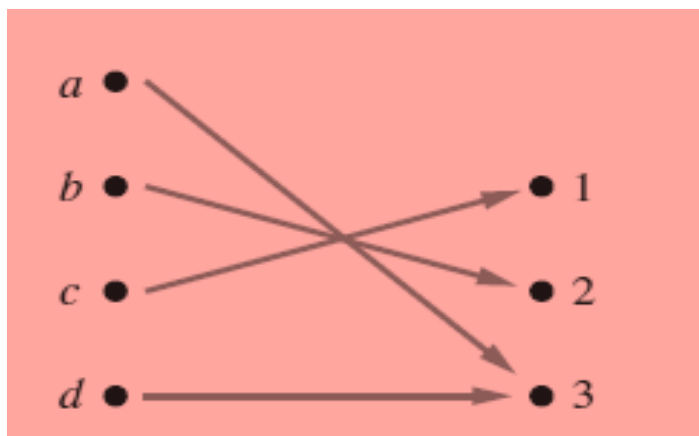
- Surjective is synonymous with onto
 - “A function is surjective”
- A function is a surjection if it is onto
- Note that there can be multiple used elements in the co-domain



An onto function

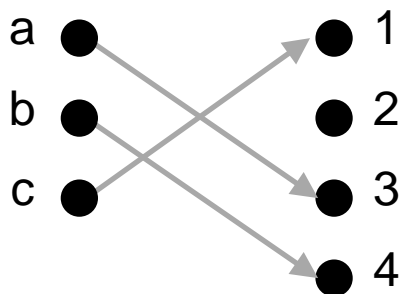
Example onto function

- Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

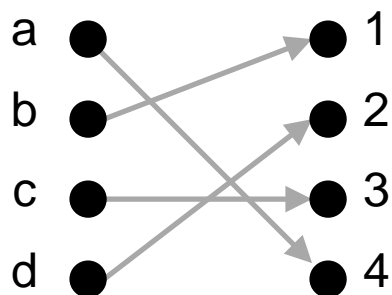


Onto vs One-to-One

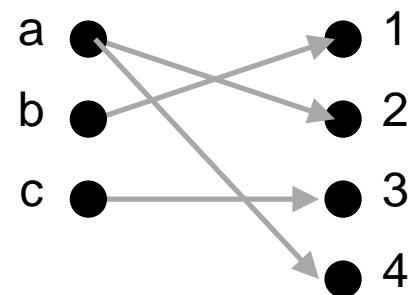
- Are the following functions onto, one-to-one, both, or neither?



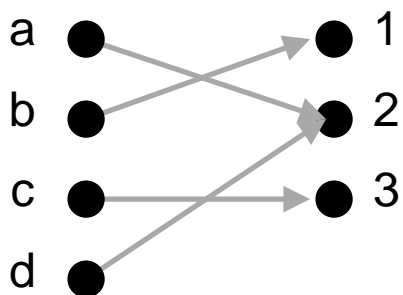
1-to-1, not onto



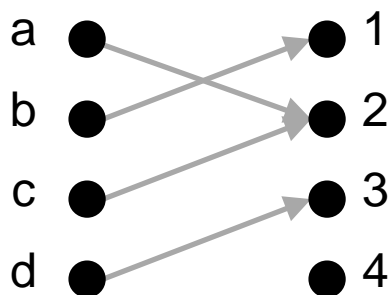
Both 1-to-1 and onto



Not a valid function



Onto, not 1-to-1



Neither 1-to-1 nor onto

Example

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

$$f: \mathbb{Z} \rightarrow \mathbb{Z}; f(x) = x^2$$

To show f is one to one, let $x_1, x_2 \in \mathbb{Z}$ and suppose

$$f(x_1) = f(x_2)$$

$$x_1^2 = x_2^2$$

$$x_1^2 - x_2^2 = 0$$

$$(x_1 + x_2)(x_1 - x_2) = 0$$

$$x_1 + x_2 = 0, \quad x_1 - x_2 = 0$$

$$x_1 = -x_2, \quad x_1 = x_2$$

$$x_1 = \pm x_2$$

Hence f is not one to one.

Example

- Determine whether the function $f(x) = x + 1$ from the set of integers to the set of integers is onto.

$$f: \mathbb{Z} \rightarrow \mathbb{Z}; f(x) = x + 1$$

Let $y \in \mathbb{Z}$. We look for an $x \in \mathbb{Z}$ such that $f(x) = y$

$$f(x) = y$$

$$x + 1 = y \quad \text{By definition of } f$$

$$x = y - 1$$

Thus for each $y \in \mathbb{Z}$, there exists $x = y - 1 \in \mathbb{Z}$

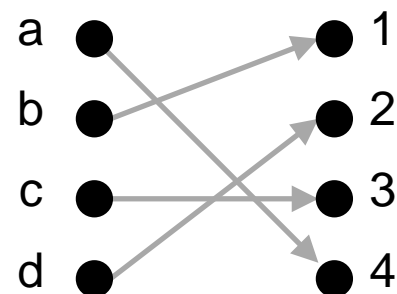
$$\text{such that } f(x) = f(y - 1)$$

$$= (y - 1) + 1 = y$$

Hence f is onto.

Bijections

- Consider a function that is both one-to-one and onto:
- Such a function is a one-to-one correspondence, or a bijection.



Example

- Determine whether the following functions are bijective or not?

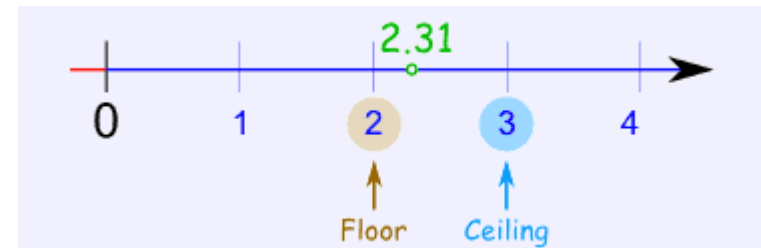
$$f: R \rightarrow R; f(x) = x^3$$

$$f: R - \{0\} \rightarrow R; f(x) = \frac{x+1}{x}$$

Floor and Ceiling

- $\text{floor}(x) = \lfloor x \rfloor$ is the largest integer that is less than or equal to x and $\text{ceiling}(x) = \lceil x \rceil$ is the smallest integer that is greater than or equal to x
- The floor and ceiling functions give you the **nearest integer** up or down.

Sample value x	Floor $\lfloor x \rfloor$	Ceiling $\lceil x \rceil$
$12/5 = 2.4$	2	3
2.7	2	3
-2.7	-3	-2
-2	-2	-2



Identity Functions

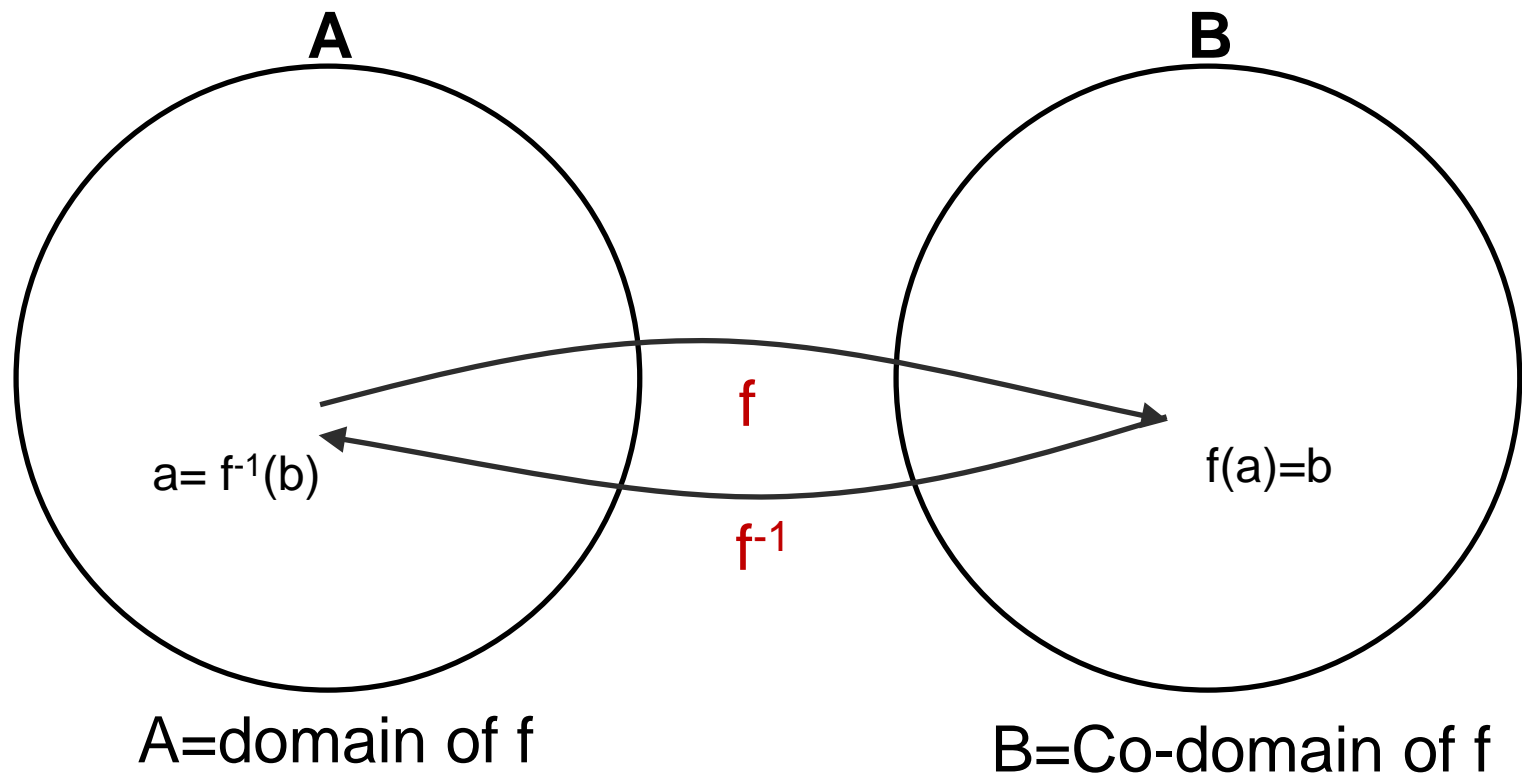
- A function such that the image and the pre-image are ALWAYS equal
- $f(x) = 1 * x$
- $f(x) = x + 0$
- The domain and the co-domain must be the same set.

Inverse Function

- Let f be a one-to-one correspondence from the set A to the set B .
- The inverse function of f is the function that assigns to an element in b belonging to B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1} .
- Hence , $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Functions

If $f(a) = b$, then $f^{-1}(b) = a$

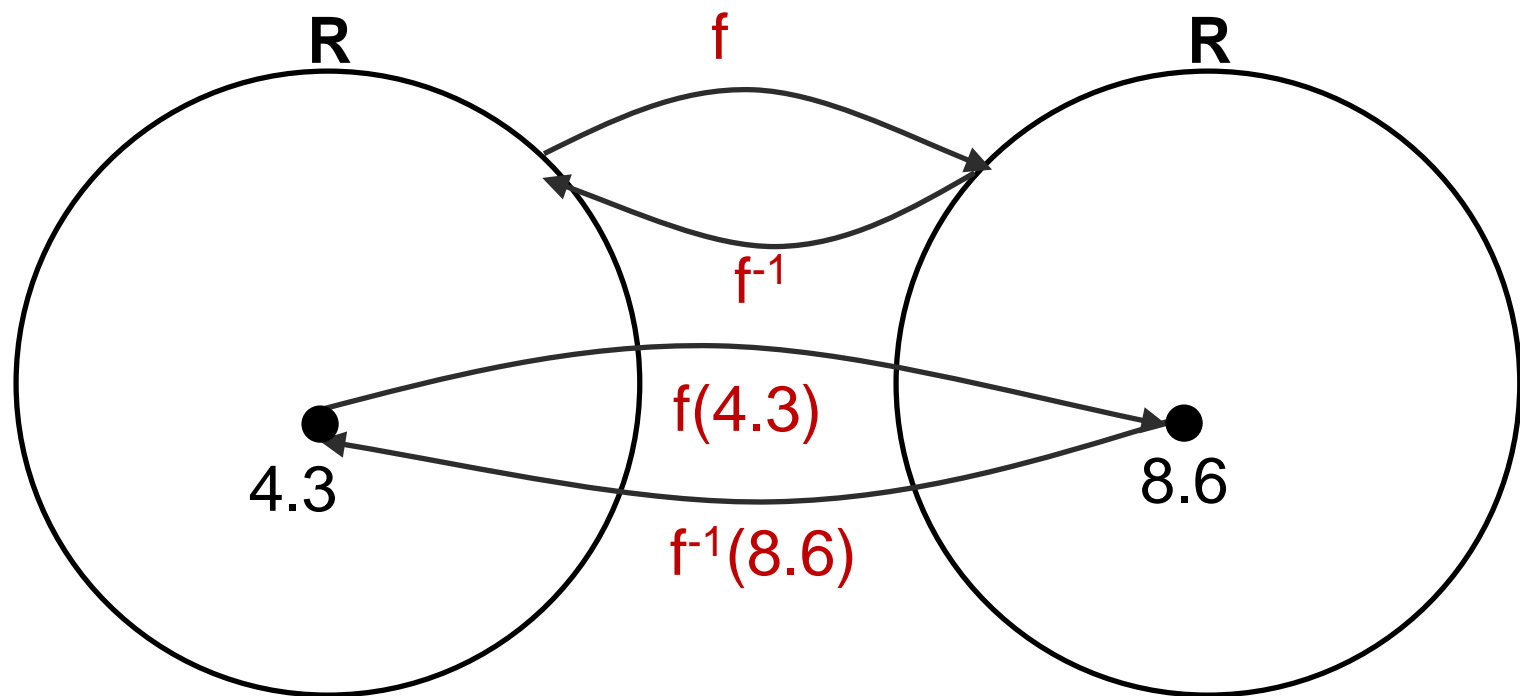


Inverse Functions

If $f(x) = y$, then $f^{-1}(y) = x$

Let $f(x) = 2 \cdot x$

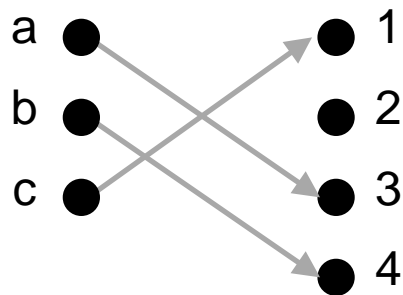
$f(x) = y$



Then $f^{-1}(y) = y/2$

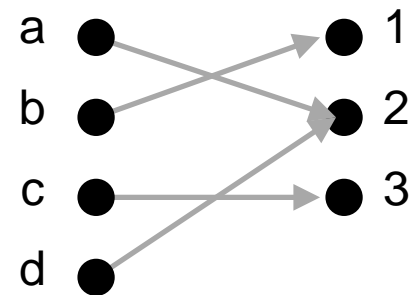
More on Inverse Functions

- Can we define the inverse of the following functions?



What is $f^{-1}(2)$?

Not onto!



What is $f^{-1}(2)$?

Not 1-to-1!

- An inverse function can ONLY be defined on a bijection

More on Inverse Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Example

- Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.
- Is f invertible, and if it is, what is its inverse?

Example

- Let ***g*** be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ such that
 $g(a) = 3$, $g(b) = 4$, $g(c) = 1$, and $g(d) = 3$.
- Is ***g*** invertible, and if it is, what is its inverse?

Working Rule to Find Inverse Function

- Let $f: X \rightarrow Y$ be a one-to-one correspondence defined by the formula $f(x) = y$.
 1. Solve the equation $f(x) = y$ for x in terms of y .
 2. $f^{-1}(y)$ equals the right hand side of the equation found in step 1.

Example

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

The given function f is defined by rule

$$f(x) = y$$

$$x + 1 = y$$

$$x = y - 1$$

$$\text{Hence } f^{-1}(y) = y - 1$$

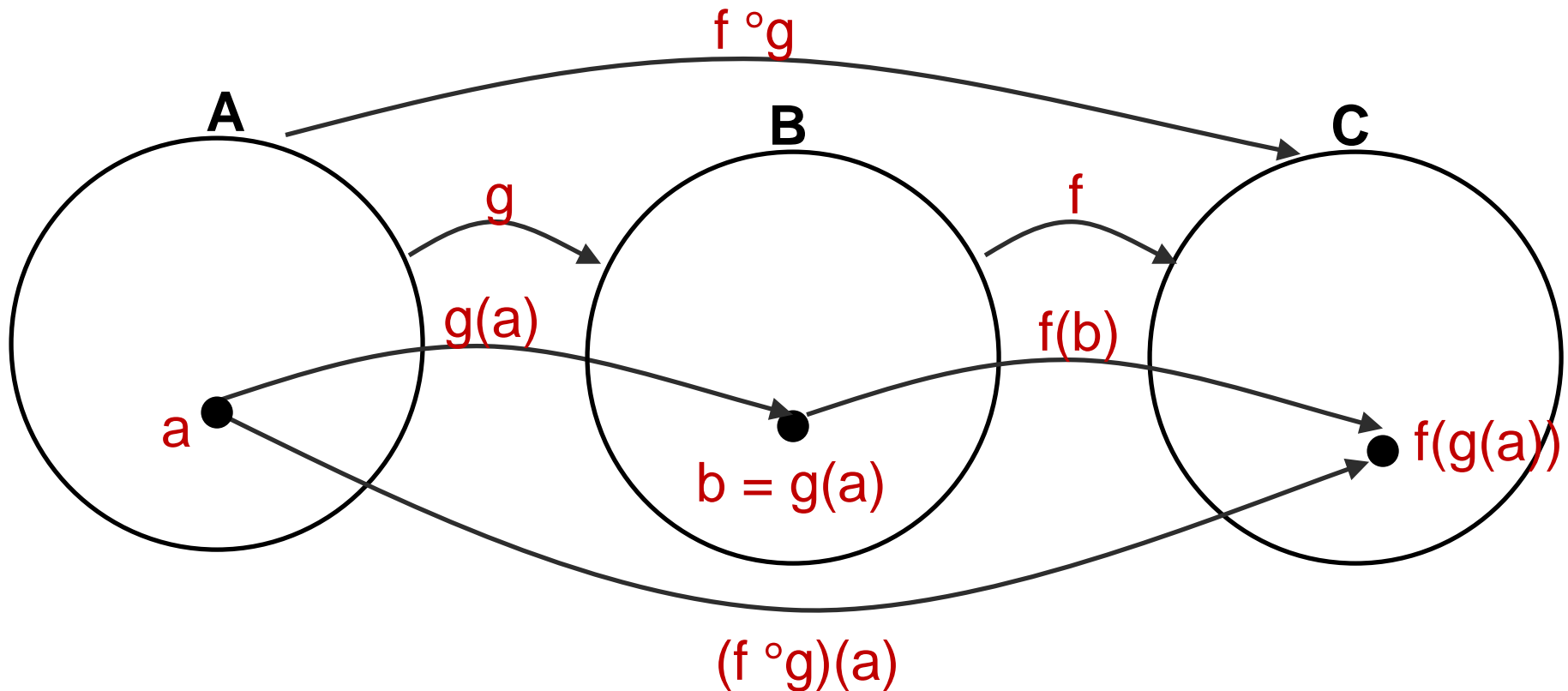
Compositions of Functions

- Let **g** be a function from the set **A** to the set **B** and let **f** be a function from the set **B** to the set **C** . the compositions of the functions f and g , denoted by **$f \circ g$** , is defined by

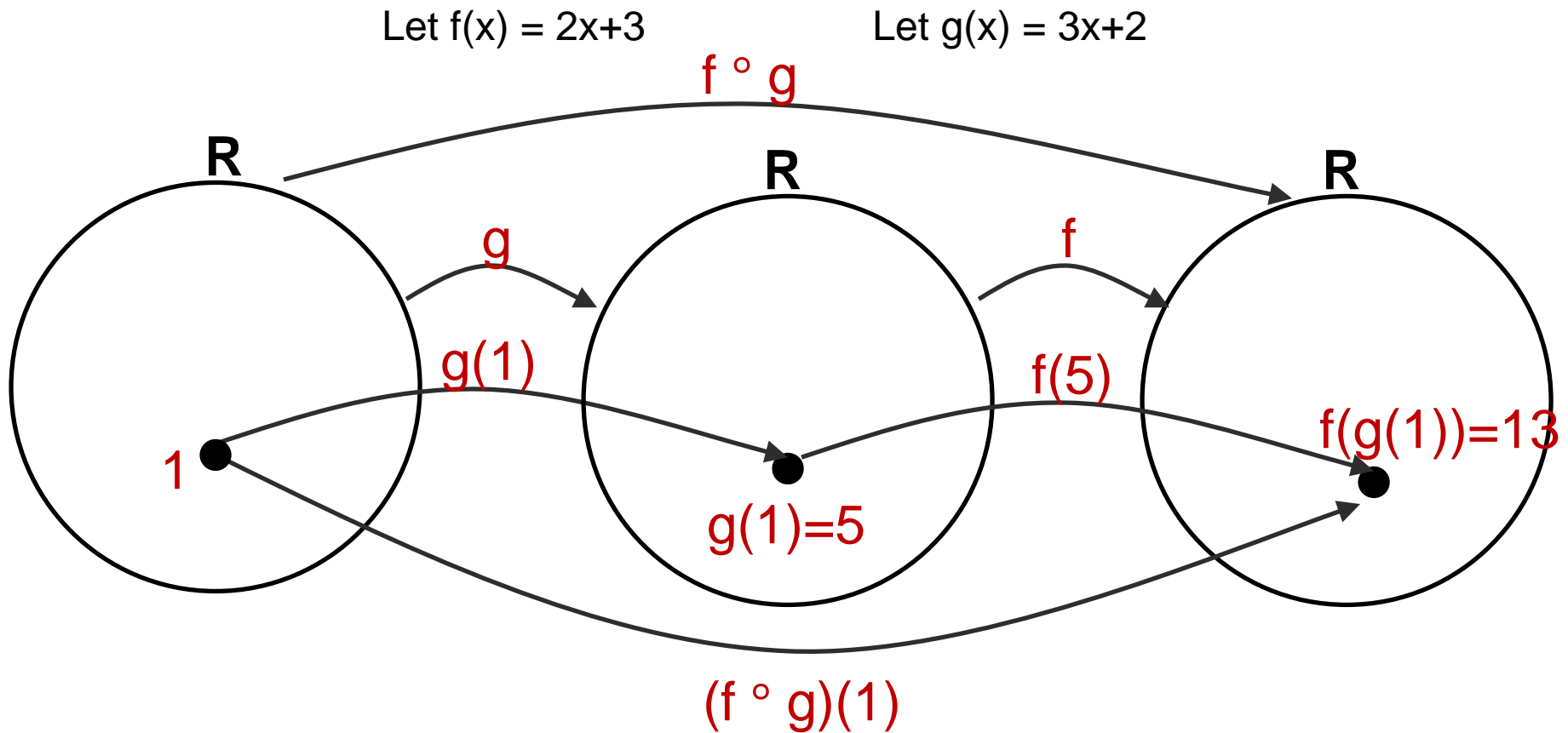
$$(f \circ g)(a) = f(g(a))$$

Compositions of Functions

$$(f \circ g)(a) = f(g(a))$$



Compositions of Functions



$$f(g(x)) = 2(3x+2)+3 = 6x+7$$

Compositions of Functions

Does $f(g(x)) = g(f(x))$?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$$f(g(x)) = 2(3x+2)+3 = 6x+7$$

$$g(f(x)) = 3(2x+3)+2 = 6x+11$$

Not equal!

Function composition is not commutative!

Compositions of Functions

- Let $A = \{1, 2, 3, 4, 5\}$

$f: A \rightarrow A$ and $g: A \rightarrow A$

$f(1) = 3, f(2) = 5, f(3) = 3, f(4) = 1, f(5) = 2$

$g(1) = 4, g(2) = 1, g(3) = 1, g(4) = 2, g(5) = 3$

Find the composition functions $f \circ g$ and $g \circ f$.

$f \circ g$	$g \circ f$
$(f \circ g)(1) = f(g(1)) = f(4) = 1$	$(g \circ f)(1) = g(f(1)) = g(3) = 1$
$(f \circ g)(2) = ?$	$(g \circ f)(2) = ?$
$(f \circ g)(3) = ?$	$(g \circ f)(3) = ?$
$(f \circ g)(4) = ?$	$(g \circ f)(4) = ?$
$(f \circ g)(5) = ?$	$(g \circ f)(5) = ?$

Compositions of Functions

- Let $A = \{1, 2, 3, 4, 5\}$

$f: A \rightarrow A$ and $g: A \rightarrow A$

$f(1) = 3, f(2) = 5, f(3) = 3, f(4) = 1, f(5) = 2$

$g(1) = 4, g(2) = 1, g(3) = 1, g(4) = 2, g(5) = 3$

Find the composition functions $f \circ g$ and $g \circ f$.

$f \circ g$	$g \circ f$
$(f \circ g)(1) = f(g(1)) = f(4) = 1$	$(g \circ f)(1) = g(f(1)) = g(3) = 1$
$(f \circ g)(2) = 3$	$(g \circ f)(2) = 3$
$(f \circ g)(3) = 3$	$(g \circ f)(3) = 1$
$(f \circ g)(4) = 5$	$(g \circ f)(4) = 4$
$(f \circ g)(5) = 3$	$(g \circ f)(5) = 1$

Compositions of Functions

Let $g : A \rightarrow A$ be the function, Set $A = \{a, b, c\}$ such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let $f : A \rightarrow B$ be the function, Set $A = \{a, b, c\}$ to the set $B = \{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g , and what is the composition of g and f ?

Compositions of Functions

Let $g : A \rightarrow A$ be the function, Set $A = \{a, b, c\}$ such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let $f : A \rightarrow B$ be the function, Set $A = \{a, b, c\}$ to the set $B = \{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g , and what is the composition of g and f ?

Solution:

The composition $f \circ g$ is defined by

$$(f \circ g)(a) = f(g(a)) = f(b) = 2,$$

$$(f \circ g)(b) = f(g(b)) = f(c) = 1,$$

$$(f \circ g)(c) = f(g(c)) = f(a) = 3.$$

$g \circ f$ is not defined, because the range of f is not a subset of the domain of g .

Exercise Questions

Chapter # 2

Topic # 2.3

Questions 1, 2, 8, 9, 10, 11, 12, 22, 23, 36, 37