Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSM-A, SP20-BSE-A & B).

Note: (Courtesy) Material for this lecture is selected from Kolman book, Virtual University, Lahore, Virtual COMSATS and Housten University hand outs.

Lecture 12+13 consists of the files named as (LA-Lecture 12+13, Examples of subspaces, Vector Space-applications)

Objective of Lecture 12+13:-

(1) To check and verify that "whether given set V satisfy axioms of vector space."

(2) To check and verify that "whether given subset W of a set V satisfy axioms of subspace."

(3) Whether vector $\boldsymbol{v}$ can be written as "linear combination" of given vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$.

Linear independence of vectors, Linear span of vectors.

After studying this lecture, You are desired to do

Home Work: Do Questions 1-14 of Exercise 4.2, Questions 5-18 and 32-34 of Exercise 4.3, following link is extremely helpful in this regard.

https://www.slader.com/textbook/9780132296540-elementary-linear-algebra-with-applications-9th-edition/196/

Chapter 4: Real Vector Spaces

In this chapter, we first recall the notion of 2-vectors ($elements\ of\ R^2$) and 3-vectors ($elements\ of\ R^3$) along with their properties. As a consequence, we can extend the properties of 3-vectors to n-vectors ($elements\ of\ R^n$). Many concepts concerning vectors in R^n can be generalized to other mathematical systems (set of Matrices, set of Polynomials etc.). We can think of a vector space in general, as a collection of objects that behave as vectors do in R^n. The objects of such a set are called vectors. For applications of vector spaces see file named **"**Vector Space-applications**".** And above all, we all are living in a vector space ($3D\ or\ R^3$).

Section 4.1: Read from book on your own as it just deals with 2-vectors and 3-vectors.

Section 4.2: Definition: (Vector Space)

A **real vector space** is a set V of elements on which we have two operations $\oplus$ and $\odot$ defined with the following properties:

(a) If $\mathbf{u}$ and $\mathbf{v}$ are any elements in V, then $\mathbf{u} \oplus \mathbf{v}$ is in V. (We say that V is **closed** under the operation $\oplus$.)

(1) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}$, $\mathbf{v}$ in V.

(2) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V.

(3) There exists an element $\mathbf{0}$ in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

(4) For each $\mathbf{u}$ in V there exists an element $-\mathbf{u}$ in V such that $\mathbf{u} \oplus -\mathbf{u} = -\mathbf{u} \oplus \mathbf{u} = \mathbf{0}$.

(b) If $\mathbf{u}$ is any element in V and c is any real number, then $c \odot \mathbf{u}$ is in V (i.e., V is closed under the operation $\odot$).

(5) $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$ for any $\mathbf{u}$, $\mathbf{v}$ in V and any real number c.

(6) $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d.

(7) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d.

(8) $1 \odot \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

The elements of V are called **vectors**; the elements of the set of real numbers R are called **scalars**. The operation $\oplus$ is called **vector addition**; the operation $\odot$ is called **scalar multiplication**. The vector $\mathbf{0}$ in property (3) is called a **zero vector**. The vector $-\mathbf{u}$ in property (4) is called a **negative of u**. It can be shown (see Exercises 19 and 20) that $\mathbf{0}$ and $-\mathbf{u}$ are unique.

Examples of vector spaces: The following examples will specify a non-empty set V along with two operations: addition ($\oplus$) and scalar multiplication ($\odot$); then we shall verify that the **Ten vector space axioms** are satisfied.

Example 1 Show that the set of all ordered n-tuple $\boldsymbol{R}^n$ is a vector space under the standard operations of addition and scalar multiplication.

Solution

(i) **Closure Property:**

Suppose that $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \boldsymbol{R}^n$

Then by definition, $\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)$

$= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \in \boldsymbol{R}^n$ (By closure property)

Therefore, $\boldsymbol{R}^n$ is closed under addition.

(ii) **Commutative Property**

Suppose that $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \boldsymbol{R}^n$

Now $\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)$

$= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ (By closure property)

$= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$ (By commutative law of real numbers)

$= (v_1, v_2, \ldots, v_n) + (u_1, u_2, \ldots, u_n)$ (By closure property)

$= \boldsymbol{v} + \boldsymbol{u}$

Therefore, $\boldsymbol{R}^n$ is commutative under addition.

(iii) **Associative Property**

Suppose that $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ *and* $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \boldsymbol{R}^n$

Now $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = [(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)] + (w_1, w_2, \ldots, w_n)$

$= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) + (w_1, w_2, \ldots, w_n)$ (By closure property)

$= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n))$ (By closure property)

$= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n))$ (By associative law of real numbers)

$= (u_1, u_2, \ldots, u_n) + (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$ (By closure property)

$= (u_1, u_2, \ldots, u_n) + [(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)]$ (By closure property)

$= \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

Hence $\boldsymbol{R}^n$ is associative under addition.

(iv) **Additive Identity**

Suppose $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \boldsymbol{R}^n$. There exists $\boldsymbol{0} = (0, 0, \ldots, 0) \in \boldsymbol{R}^n$ such that
$\boldsymbol{0} + \boldsymbol{u} = (0, 0, \ldots, 0) + (u_1, u_2, \ldots, u_n)$

$= (0 + u_1, 0 + u_2, \ldots, 0 + u_n)$ (By closure property)

$= (u_1, u_2, \ldots, u_n) = \boldsymbol{u}$ (Existence of identity of real numbers)

Similarly, $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$

Hence $\boldsymbol{0} = (0, 0, \ldots, 0)$ is the additive identity for $\boldsymbol{R}^n$.

(v) **Additive Inverse**

Suppose $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \boldsymbol{R}^n$. There exists $-\boldsymbol{u} = (-u_1, -u_2, \ldots, -u_n) \in \boldsymbol{R}^n$

Such that $\boldsymbol{u} + (-\boldsymbol{u}) = (u_1, u_2, \ldots, u_n) + (-u_1, -u_2, \ldots, -u_n)$

$= (u_1 + (-u_1), u_2 + (-u_2), \ldots, u_n + (-u_n))$ (By closure property)

$= (0, 0, \ldots, 0) = \boldsymbol{0}$

Similarly, $(-\boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{0}$

Hence the inverse of each element of $\boldsymbol{R}^n$ exists in $\boldsymbol{R}^n$.

(vi) **Scalar Multiplication**

If k is any scalar and $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \boldsymbol{R}^n$.

Then by definition, $k\,\boldsymbol{u} = k\,(u_1, u_2, \ldots, u_n) = (k\,u_1, k\,u_2, \ldots, k\,u_n) \in \boldsymbol{R}^n$
(By closure property)

(vii) **Distributive Law**

Suppose k is any scalar and $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \boldsymbol{R}^n$

Now $k\,(\boldsymbol{u} + \boldsymbol{v}) = k\,[(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)]$

$= k\,(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ (By closure property)

$= (k\,(u_1 + v_1), k\,(u_2 + v_2), \ldots, k\,(u_n + v_n))$ (By scalar multiplication)

$= (k\,u_1 + k\,v_1,\, k\,u_2 + k\,v_2,\, \ldots,\, k\,u_n + k\,v_n)$ (By Distributive Law)

$= (k\,u_1,\, k\,u_2,\, \ldots,\, k\,u_n) + (k\,v_1,\, k\,v_2,\, \ldots,\, k\,v_n)$ (By closure property)

$= k\,(u_1,\, u_2,\, \ldots,\, u_n) + k\,(v_1,\, v_2,\, \ldots,\, v_n)$ (By scalar multiplication)

$= k\,\boldsymbol{u} + k\,\boldsymbol{v}$

(viii) Suppose k and l be any scalars and $\boldsymbol{u} = (u_1,\, u_2,\, \ldots,\, u_n) \in \boldsymbol{R}^n$

Then $(k + l)\,\boldsymbol{u} = (k + l)\,(u_1,\, u_2,\, \ldots,\, u_n)$

$= ((k + l)u_1,\, (k + l)u_2,\, \ldots,\, (k + l)u_n)$ (By scalar multiplication)

$= (k\,u_1 + l\,u_1,\, k\,u_2 + l\,u_2,\, \ldots,\, k\,u_n + l\,u_n)$ (By Distributive Law)

$= (k\,u_1,\, k\,u_2,\, \ldots,\, k\,u_n) + (l\,u_1,\, l\,u_2,\, \ldots,\, l\,u_n)$ (By closure property)

$= k\,(u_1,\, u_2,\, \ldots,\, u_n) + l\,(u_1,\, u_2,\, \ldots,\, u_n)$ (By scalar multiplicatio

$= k\,\boldsymbol{u} + l\,\boldsymbol{u}$

(ix) Suppose k and l be any scalars and $\boldsymbol{u} = (u_1,\, u_2,\, \ldots,\, u_n) \in \boldsymbol{R}^n$

Then $k\,(l\,\boldsymbol{u}) = k\,[l\,(u_1,\, u_2,\, \ldots,\, u_n)]$

$= k\,(l\,u_1,\, l\,u_2,\, \ldots,\, l\,u_n)$ (By scalar multiplicatio

$= (k\,(l\,u_1),\, k\,(l\,u_2),\, \ldots,\, k\,(l\,u_n))$ (By scalar multiplication)

$= ((k\,l)u_1,\, (k\,l)u_2,\, \ldots,\, (k\,l)u_n)$ (By associative law)

$= (k\,l)\,(u_1,\, u_2,\, \ldots,\, u_n)$ (By scalar multiplication)

$= (k\,l)\,\boldsymbol{u}$

(x) Suppose $\boldsymbol{u} = (u_1,\, u_2,\, \ldots,\, u_n) \in \boldsymbol{R}^n$

Then $1\,\boldsymbol{u} = 1\,(u_1,\, u_2,\, \ldots,\, u_n)$

$= (1u_1,\, 1u_2,\, \ldots,\, 1u_n)$ (By scalar multiplication)

$= (u_1,\, u_2,\, \ldots,\, u_n) = \boldsymbol{u}$ (Existence of identity in scalrs)

Hence, $\boldsymbol{R}^n$ is the real vector space with the standard operations of addition and scalar multiplication.

Note: For $n = 1, 2, 3$, we get three important vector spaces, namely, $\boldsymbol{R}$ (the real numbers), $\boldsymbol{R}^2$ (the vectors in the plane), and $\boldsymbol{R}^3$ (the vectors in 3-space), respectively.

Example 2:

Show that the set ***V*** of **all 2x2 matrices** with real entries is a vector space if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication.

$$\boldsymbol{V} = \boldsymbol{M}_{22} = \begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} \end{bmatrix} \mid \boldsymbol{a}_{ij} \in \boldsymbol{R}\Big]$$

Solution Suppose that $\boldsymbol{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \in \boldsymbol{V}$

and k and l be two any scalars.

(i) **Closure property** To prove axiom (i), we must show that $\boldsymbol{u} + \boldsymbol{v}$ is an object in $\boldsymbol{V}$: that is , we must show that $\boldsymbol{u} + \boldsymbol{v}$ is a 2x2 matrix. But this is clear from the definition of matrix addition, since $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$

(By closure property)

(ii) **Commutative property** Now it is very easy to verify the Axiom (ii)

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \quad \text{(By closure property)}$$

$$= \begin{bmatrix} v_{11} + u_{11} & v_{12} + u_{12} \\ v_{21} + u_{21} & v_{22} + u_{22} \end{bmatrix} \quad \text{(Commutative property of real numbers)}$$

$$= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \boldsymbol{v} + \boldsymbol{u}$$

(iii) **Associative property** $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$

$$= \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \quad \text{(By closure property)}$$

$$= \begin{bmatrix} (u_{11} + v_{11}) + w_{11} & (u_{12} + v_{12}) + w_{12} \\ (u_{21} + v_{21}) + w_{21} & (u_{22} + v_{22}) + w_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} + (v_{11} + w_{11}) & u_{12} + (v_{12} + w_{12}) \\ u_{21} + (v_{21} + w_{21}) & u_{22} + (v_{22} + w_{22}) \end{bmatrix} \quad \text{(By associative property of real numbers)}$$

$$=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{bmatrix}$$

$$=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\left(\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}+\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}\right)=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$$

Therefore, $\boldsymbol{V}$ is associative under '+'.

(iv) **Additive Identity** Now to prove the axiom (iv), we must find an object $\boldsymbol{0}$ in $\boldsymbol{V}$ s

that $\boldsymbol{0}$ + $\boldsymbol{v}$ = $\boldsymbol{v}$ + $\boldsymbol{0}$ = $\boldsymbol{v}$ for all $\boldsymbol{u}$ in $\boldsymbol{V}$. This can be done by defining $\boldsymbol{0}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\boldsymbol{0}+\boldsymbol{u}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} 0+u_{11} & 0+u_{12} \\ 0+u_{21} & 0+u_{22} \end{bmatrix}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\boldsymbol{u}$$

and similarly $\boldsymbol{u}$ + $\boldsymbol{0}$ = $\boldsymbol{u}$.

(v) **Additive Inverse** Now to prove the axiom (v) we must show that each object $\boldsymbol{u}$ has a negative $-\boldsymbol{u}$ such that $\boldsymbol{u}$ + $(-\boldsymbol{u})$ = **0** = $(-\boldsymbol{u})$ + **0**. Defining the negative of $\boldsymbol{u}$ to be

$$-\boldsymbol{u}=\begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}.$$

$$\boldsymbol{u}+(-\boldsymbol{u})=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+(-u_{11}) & u_{12}+(-u_{12}) \\ u_{21}+(-u_{21}) & u_{22}+(-u_{22}) \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\boldsymbol{0}$$

Similarly, $(-\boldsymbol{u})$ + $\boldsymbol{u}$ = $\boldsymbol{0}$

(vi) **Scalar Multiplication**

Axiom (vi) also holds because for any real number k we have

$$k\boldsymbol{u}=k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \qquad \text{(By closure property)}$$

so that k $\boldsymbol{u}$ is a 2x2 matrix and consequently is an object in $\boldsymbol{V}$.

(vii) **Distributive Law**

$$k(\boldsymbol{u}+\boldsymbol{v})=k\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\right)$$

$$= k\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} = \begin{bmatrix} k(u_{11}+v_{11}) & k(u_{12}+v_{12}) \\ k(u_{21}+v_{21}) & k(u_{22}+v_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11}+kv_{11} & ku_{12}+kv_{12} \\ ku_{21}+kv_{21} & ku_{22}+kv_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{bmatrix}$$

$$= k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = k\boldsymbol{u} + k\boldsymbol{v}$$

(viii) $(k+l)\boldsymbol{u} = (k+l)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} (k+l)u_{11} & (k+l)u_{12} \\ (k+l)u_{21} & (k+l)u_{22} \end{bmatrix}$

$$= \begin{bmatrix} ku_{11}+lu_{11} & ku_{12}+lu_{12} \\ ku_{21}+lu_{21} & ku_{22}+lu_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} lu_{11} & lu_{12} \\ lu_{21} & lu_{22} \end{bmatrix}$$

$$= k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + l\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\boldsymbol{u} + l\boldsymbol{u}$$

(ix) $k(l\boldsymbol{u}) = k\left(l\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\right) = k\begin{bmatrix} lu_{11} & lu_{12} \\ lu_{21} & lu_{22} \end{bmatrix}$

$$= \begin{bmatrix} k(lu_{11}) & k(lu_{12}) \\ k(lu_{21}) & k(lu_{22}) \end{bmatrix} = \begin{bmatrix} (kl)u_{11} & (kl)u_{12} \\ (kl)u_{21} & (kl)u_{22} \end{bmatrix} = (kl)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = (kl)\boldsymbol{u}$$

(x) Finally axiom (x) is a simple computation

$$1\boldsymbol{u} = 1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1u_{11} & 1u_{12} \\ 1u_{21} & 1u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \boldsymbol{u}$$

Hence the set of all 2x2 matrices with real entries is vector space under matrix addition and matrix scalar multiplication.

Note: $V = M_{m\times n} = Set\ of\ all\ matrics\ of\ order\ mn$ is vector space and $M_{2\times 2}$ is subspace of $M_{m\times n}$.

Example 3: Show that set of all polynomials of degree less or equal to n is vector space.

Solution: **We Know,**

$\boldsymbol{V = P_n(t) = \{a_n t^n + a_{n-1} t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0 : a_i \in R\}}$ **i.e. V is set of all polynomials of degree less or equal to n.**

Let $p(t), q(t),\ r(t)\ \in V$, then

$\boldsymbol{p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0}$ **and**

$\boldsymbol{q(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0}$

$\boldsymbol{r(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0}$

Property

We define $p(t) \oplus q(t)$ as

$$p(t) \oplus q(t) = (a_n + b_n)t^n + (a_{n-1} + b_{n-1})t^{n-1} + \cdots + (a_1 + b_1)t + (a_0 + b_0).$$

If c is a scalar, we also define $c \odot p(t)$ as

$$c \odot p(t) = (ca_n)t^n + (ca_{n-1})t^{n-1} + \cdots + (ca_1)t + (ca_0).$$

We now show that P_n is a vector space.

Let $p(t)$ and $q(t)$, as before, be elements of P_n; that is, they are polynomials of degree $\leq n$ or the zero polynomial. Then the previous definitions of the operations $\oplus$ and $\odot$ show that $p(t) \oplus q(t)$ and $c \odot p(t)$, for any scalar c, are polynomials of

Property1 (commutative law)

$$\boldsymbol{q(t) + p(t) = (b_n + a_n)\, t^n + (b_{n-1} + a_{n-1})\, t^{n-1} + \cdots + (b_1 + a_1) + (b_0 + a_0) = p(t) + q(t)}$$

Property2(Associative law)

Let $p(t), q(t), and\ r(t)\ \in V$, Then it is evident/obvious that

$$\boldsymbol{p(t) + \big(q(t) + r(t)\big) = \big(p(t) + q(t)\big) + r(t)}$$

Property3 (Additive identity) $\underline{0}$**=0** $= 0t^n + 0t^{n-1} + 0t^{n-2} + \cdots + 0t + 0$

Property4 (Additive inverse) $s(t) = 2t^n - 5t^{n-1} + t^{n-2} + \cdots - 6t + \sqrt{7}$

$$-s(t) = -2t^n + 5t^{n-1} - t^{n-2} + \cdots + 6t - \sqrt{7}$$

(b) Properties of scalar Multiplication.

(5) for scalar $c \in R$ and $p(t), q(t) \in V$

$$c\,.\big(p(t) + q(t)\big) = c\,.p(t) + c\,.q(t) - - - (5)$$

$$L.H.S = c\,.\big(p(t) + q(t)\big)$$

$$= c\,.\Big((\boldsymbol{a_n} + \boldsymbol{b_n})\,\boldsymbol{t^n} + (\boldsymbol{a_{n-1}} + \boldsymbol{b_{n-1}})\,\boldsymbol{t^{n-1}} + \cdots + (\boldsymbol{a_1} + \boldsymbol{b_1})\boldsymbol{t} + (\boldsymbol{a_0} + \boldsymbol{b_0})\Big)$$

$$= c\,.(\boldsymbol{a_n} + \boldsymbol{b_n})\,\boldsymbol{t^n} + c\,.(\boldsymbol{a_{n-1}} + \boldsymbol{b_{n-1}})\,\boldsymbol{t^{n-1}} + \cdots + c\,.(\boldsymbol{a_1} + \boldsymbol{b_1})\,\boldsymbol{t} + c\,.(\boldsymbol{a_0} + \boldsymbol{b_0})$$

$$= (c\,\boldsymbol{a_n} + c\,\boldsymbol{b_n})\,\boldsymbol{t^n} + (\boldsymbol{c}.\boldsymbol{a_{n-1}} + \boldsymbol{c}.\boldsymbol{b_{n-1}})\,\boldsymbol{t^{n-1}} + \cdots + (\boldsymbol{c}.\boldsymbol{a_1} + \boldsymbol{c}.\boldsymbol{b_1})\,\boldsymbol{t} + (\boldsymbol{c}.\boldsymbol{a_0} + \boldsymbol{c}.\boldsymbol{b_0})$$

$$= (c\,\boldsymbol{a_n}\boldsymbol{t^n} + \boldsymbol{c}.\boldsymbol{a_{n-1}}\boldsymbol{t^{n-1}} + \cdots + c\,\boldsymbol{a_1}\boldsymbol{t} + c\,\boldsymbol{a_0}) + (c\,\boldsymbol{b_n}\boldsymbol{t^n} + \boldsymbol{c}.\boldsymbol{b_{n-1}}\boldsymbol{t^{n-1}} + \cdots + c\,\boldsymbol{b_1}\boldsymbol{t} + c\,\boldsymbol{b_0})$$

$$= c.(\,\boldsymbol{a_n}\boldsymbol{t^n} + \boldsymbol{a_{n-1}}\boldsymbol{t^{n-1}} + \cdots + \boldsymbol{a_1}\boldsymbol{t} + \boldsymbol{a_0}) + \boldsymbol{c}.(\,\boldsymbol{b_n}\boldsymbol{t^n} + \boldsymbol{b_{n-1}}\boldsymbol{t^{n-1}} + \cdots + \boldsymbol{b_1}\boldsymbol{t} + \boldsymbol{b_0})$$

$$= c.P(t) + c.q(t) = R.H.S$$

degree $\leq n$ or the zero polynomial. That is, $p(t) \oplus q(t)$ and $c \odot p(t)$ are in P_n so that (a) and (b) in Definition 4.4 hold. To verify property (1), we observe that

$$q(t) \oplus p(t) = (b_n + a_n)t^n + (b_{n-1} + a_{n-1})t^{n-1} + \cdots + (b_1 + a_1)t + (a_0 + b_0),$$

and since $a_i + b_i = b_i + a_i$ holds for the real numbers, we conclude that $p(t) \oplus q(t) = q(t) \oplus p(t)$. Similarly, we verify property (2). The zero polynomial is the element $\mathbf{0}$ needed in property (3). If $p(t)$ is as given previously, then its negative, $-p(t)$, is

$$-a_n t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0.$$

We shall now verify property (6) and will leave the verification of the remaining properties to the reader. Thus

$$\begin{aligned}(c+d) \odot p(t) &= (c+d)a_n t^n + (c+d)a_{n-1}t^{n-1} + \cdots + (c+d)a_1 t \\ &\quad + (c+d)a_0 \\ &= ca_n t^n + da_n t^n + ca_{n-1}t^{n-1} + da_{n-1}t^{n-1} + \cdots + ca_1 t \\ &\quad + da_1 t + ca_0 + da_0 \\ &= c(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0) \\ &\quad + d(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0) \\ &= c \odot p(t) \oplus d \odot p(t). \qquad \blacksquare\end{aligned}$$

Remark We show later that the vector space P_n behaves algebraically in exactly the same manner as R^{n+1}.

$$V = P_n(t) = \{a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 : a_i \in R\}$$

$$V = P_n(t) = \{(a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0) : a_i \in R\} = R^{n+1}$$

Note: In Example 3, For $n = 1, 2, 3$, we get special cases $P(t) = \{a_0 + a_1 t : a_i \in R\}$ (all linear functions or straight lines in 2D), $P_2(t) = \{a_0 + a_1 t + a_2 t^2 : a_i \in R\}$ (all parabolas and straight lines) and $P_3(t) = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_i \in R\}$ (all cubic functions, all parabolas and straight lines), respectively. Since $P_n(t)$ is a vector space so are the $P(t)$, $P_2(t)$ and $P_3(t)$.

Exercise 4.2 (Important)

See separate file named (some solved problems of 4.2)

Section 4.3: Definition (Subspace)

Let $\boldsymbol{V}$ be a vector space and $\boldsymbol{W}$ a nonempty subset of $\boldsymbol{V}$. Then $\boldsymbol{W}$ is said to be a subspace of $\boldsymbol{V}$ if following three properties hold in $\boldsymbol{W}$

(1) The zero vector of $\boldsymbol{V}$ is in $\boldsymbol{W}$. (optional)
(2) For each vector $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{W}$ we have $\boldsymbol{u} \oplus \boldsymbol{v}$ **in** $\boldsymbol{W}$.
(3) For each $\boldsymbol{u} \in \boldsymbol{W}$ and each scalar $\boldsymbol{c} \in \boldsymbol{R}$, we get $\boldsymbol{c} \odot \boldsymbol{u} \in \boldsymbol{W}$.

Examples of Subspaces:

See uploaded file with name "examples of subspaces."

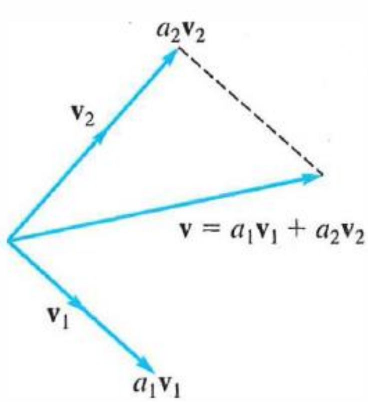
Definition (Linear combination of vectors)

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V. A vector $\mathbf{v}$ in V is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \sum_{j=1}^{k} a_j\mathbf{v}_j$$

for some real numbers $a_1, a_2, \ldots, a_k$.

See figure below to understand geometric interpretation of linear combination of two vectors.



Examle 1(LC) : Let $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ be vectors in $\boldsymbol{R}^3$, such that $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} \mathbf{6} \\ \mathbf{4} \\ \mathbf{2} \end{bmatrix}$, $\boldsymbol{v_3} = \begin{bmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{8} \end{bmatrix}$. Show that $\boldsymbol{v} = \begin{bmatrix} \mathbf{9} \\ \mathbf{2} \\ \mathbf{7} \end{bmatrix}$ is a linear combination of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$.

Solution: Consider definition of L.C. $v = a_1v_1 + a_2v_2 + a_3v_3$ ... ... (1), Our goal is to find scalars a_1, a_2 and a_3.

$$\left\{(1) \Rightarrow \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix},\right.$$

$$\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_1 \\ -a_1 \end{bmatrix} + \begin{bmatrix} 6a_2 \\ 4a_2 \\ 2a_2 \end{bmatrix} + \begin{bmatrix} 4a_3 \\ -a_3 \\ 8a_3 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} a_1 + 6a_2 + 4a_3 \\ 2a_1 + 4a_2 - a_3 \\ -a_1 + 2a_2 + 8a_3 \end{bmatrix}$$

Equating both sides we get,

$$a_1 + 6a_2 + 4a_3 = 9 \;;$$

$$2a_1 + 4a_2 - a_3 = 2 \;;$$

$$\left.-a_1 + 2a_2 + 8a_3 = 7\right\}$$

Observe: The problem of **linear combination** boils down to a problem of non-homogeneous system of linear equations. I believe you can find scalars using Gauss-Elimination method (Echelon form). Here we go.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 2 & 4 & -1 & 2 \\ -1 & 2 & 8 & 7 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 0 & -8 & -9 & -16 \\ 0 & 8 & 12 & 16 \end{array}\right]$$

$$R_3 + R_2 \sim \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 0 & -8 & -9 & -16 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{matrix} \dfrac{R_2}{-8} \\ \dfrac{R_3}{3} \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 0 & 1 & 9/8 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now rewrite this equivalent simple system,

$$a_3 = 0; \quad a_2 + \frac{9}{8}a_3 = 2\,; \quad a_1 + 6a_2 + 4a_3 = 9$$

Using backward substitution, we get $a_3 = 0\,;\ a_2 = 2\,;\ a_1 = -3$

Hence $\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$.

DEFINITION 4.8 Let S be a set of vectors in a vector space V. If every vector in V is a linear combination of the vectors in S, then the set S is said to **span** V, or V is spanned by the set S; that is, span $S = V$.

Examle 2 (Spanning set): Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be vectors in $\boldsymbol{R}^3$, such that $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$, $\boldsymbol{v_3} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$. Whether these vectors span $\boldsymbol{R}^3$?

Solution: The set of vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ spans $\boldsymbol{R}^3$ if **every vector** in $\boldsymbol{R}^3$ can be written as a linear combination (L.C.) of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$.

Take any arbitrary vector say $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix}$ from $\boldsymbol{R}^3$.

Consider $\boldsymbol{v} = \boldsymbol{a_1 v_1} + \boldsymbol{a_2 v_2} + \boldsymbol{a_3 v_3}$ ……… $(\mathbf{1})$, Our goal is to find scalars $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$ in terms **of** $\boldsymbol{a}, \boldsymbol{b}$ **and** $\boldsymbol{c}$.

$$(\mathbf{1}) \Rightarrow \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} = \boldsymbol{a_1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \boldsymbol{a_2}\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} + \boldsymbol{a_3}\begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} \\ 2\boldsymbol{a_1} \\ -\boldsymbol{a_1} \end{bmatrix} + \begin{bmatrix} 6\boldsymbol{a_2} \\ 4\boldsymbol{a_2} \\ 2\boldsymbol{a_2} \end{bmatrix} + \begin{bmatrix} 4\boldsymbol{a_3} \\ -\boldsymbol{a_3} \\ 8\boldsymbol{a_3} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} + 6\boldsymbol{a_2} + 4\boldsymbol{a_3} \\ 2\boldsymbol{a_1} + 4\boldsymbol{a_2} - \boldsymbol{a_3} \\ -\boldsymbol{a_1} + 2\boldsymbol{a_2} + 8\boldsymbol{a_3} \end{bmatrix}$$

Equating both sides we get,

$$\boldsymbol{a_1} + 6\boldsymbol{a_2} + 4\boldsymbol{a_3} = \boldsymbol{a} \;\; ; \; 2\boldsymbol{a_1} + 4\boldsymbol{a_2} - \boldsymbol{a_3} = \boldsymbol{b} \; ; \; -\boldsymbol{a_1} + 2\boldsymbol{a_2} + 8\boldsymbol{a_3} = \boldsymbol{c}$$

Observe: The problem of **spanning set for vector space** boils down to a problem of non-homogeneous system of linear equations. I believe you can find scalars using Gauss-Elimination method. Here we go.

$$[A|b] = \begin{bmatrix} 1 & 6 & 4 & | & a \\ 2 & 4 & -1 & | & b \\ -1 & 2 & 8 & | & c \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 6 & 4 & | & a \\ 0 & -8 & -9 & | & b-2a \\ 0 & 8 & 12 & | & c+a \end{bmatrix}$$

$$R_3 + R_2 \sim \begin{bmatrix} 1 & 6 & 4 & | & a \\ 0 & -8 & -9 & | & b-2a \\ 0 & 0 & 3 & | & b+c-a \end{bmatrix} \begin{matrix} \frac{R_2}{-8} \\ \frac{R_3}{3} \end{matrix} \sim \begin{bmatrix} 1 & 6 & 4 & | & a \\ 0 & 1 & 9/8 & | & (2a-b)/8 \\ 0 & 0 & 1 & | & (b+c-a)/3 \end{bmatrix}$$

Now, rewrite this equivalent simple system,

$$a_3 = \frac{b+c-a}{3}; \quad a_2 + \frac{9}{8}a_3 = \frac{2a-b}{8}\ ; \ a_1 + 6a_2 + 4a_3 = a$$

Using backward substitution, we get

$$a_2 + \frac{9}{8}\left(\frac{b+c-a}{3}\right) = \frac{2a-b}{8}$$

$$a_2 = \frac{2a-b}{8} - \frac{9}{8}\left(\frac{b+c-a}{3}\right) = \left(\frac{6a-3b-9b-9c+9a}{24}\right) = \left(\frac{15a-12b-9c}{24}\right)$$
$$= \left(\frac{5a-4b-3c}{8}\right)$$

$$a_1 + 6a_2 + 4a_3 = a$$

$$\rightarrow a_1 = a - 6a_2 - 4a_3 = a - 6\left(\frac{5a-4b-3c}{8}\right) - 4\left(\frac{b+c-a}{3}\right) = \frac{20b+11c-17a}{12}$$

$$a_3 = \frac{b+c-a}{3}\ ; \ a_2 = \frac{5a-4b-3c}{8}\ ; \ a_1 = \frac{20b+11c-17a}{12}$$

Hence given set of vectors v_1, v_2 and v_3 spans R^3.

Verification of answer: a=9, b=2, c=7, Then $a_1 = \frac{40+77-153}{12} = -3$

$$a_2 = \frac{45-8-21}{8} = 2; \ a_3 = \frac{2+7-9}{3} = 0$$

$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

$$a_1e_1 + a_2e_2 + a_3e_3 = 0$$

$$a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{a_1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{a_2} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{a_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \boldsymbol{a_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

DEFINITION 4.9 The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly dependent** if there exist constants $a_1, a_2, \ldots, a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j \mathbf{v}_j = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}. \qquad (1)$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if, whenever $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$,

$$a_1 = a_2 = \cdots = a_k = 0.$$

Examle 3: (Linear Independent): Let $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ be vectors in $\boldsymbol{R^3}$, such that $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} \mathbf{6} \\ \mathbf{4} \\ \mathbf{2} \end{bmatrix}$, $\boldsymbol{v_3} = \begin{bmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{8} \end{bmatrix}$. Are these vectors linearly independent?

Solution: Consider $\mathbf{0} = \boldsymbol{a_1 v_1} + \boldsymbol{a_2 v_2} + \boldsymbol{a_3 v_3}$ ... ... $(\mathbf{1})$, Our goal is to find scalars $\boldsymbol{a_1}$, $\boldsymbol{a_2}$**and** $\boldsymbol{a_3}$.

$$(\mathbf{1}) \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{a_1} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix} + \boldsymbol{a_2} \begin{bmatrix} \mathbf{6} \\ \mathbf{4} \\ \mathbf{2} \end{bmatrix} + \boldsymbol{a_3} \begin{bmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{8} \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{2a_1} \\ \boldsymbol{-a_1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{6a_2} \\ \boldsymbol{4a_2} \\ \boldsymbol{2a_2} \end{bmatrix} + \begin{bmatrix} \boldsymbol{4a_3} \\ \boldsymbol{-a_3} \\ \boldsymbol{8a_3} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1 + 6a_2 + 4a_3} \\ \boldsymbol{2a_1 + 4a_2 - a_3} \\ \boldsymbol{-a_1 + 2a_2 + 8a_3} \end{bmatrix}$$

Equating both sides we get,

$$\boldsymbol{a_1 + 6a_2 + 4a_3} = 0 \quad ;$$

$$\boldsymbol{2a_1 + 4a_2 - a_3} = 0 \quad ;$$

$$\boldsymbol{-a_1 + 2a_2 + 8a_3} = 0$$

Observe: The problem of **linear independence** boils down to a problem of **homogeneous** system of linear equations. I believe you can find scalars using Gauss-Elimination method. Here we go.

$$[\boldsymbol{A}|\boldsymbol{b}] = \begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{4} & | & 0 \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} & | & 0 \\ \mathbf{-1} & \mathbf{2} & \mathbf{8} & | & 0 \end{bmatrix} \begin{matrix} \boldsymbol{R_2 - 2R_1} \\ \boldsymbol{R_3 + R_1} \end{matrix} \sim \begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{4} & | & 0 \\ \mathbf{0} & \mathbf{-8} & \mathbf{-9} & | & 0 \\ \mathbf{0} & \mathbf{8} & \mathbf{12} & | & 0 \end{bmatrix}$$

$$\boldsymbol{R_3 + R_2} \sim \begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{4} & | & 0 \\ \mathbf{0} & \mathbf{-8} & \mathbf{-9} & | & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{3} & | & 0 \end{bmatrix} \begin{matrix} \dfrac{\boldsymbol{R_2}}{\mathbf{-8}} \\ \dfrac{\boldsymbol{R_3}}{\mathbf{3}} \end{matrix} \sim \begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{4} & | & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{9/8} & | & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & | & 0 \end{bmatrix}$$

Now rewrite this equivalent simple system,

$$\boldsymbol{a_3} = 0; \;\; \boldsymbol{a_2} + \frac{\mathbf{9}}{\mathbf{8}}\boldsymbol{a_3} = 0\,; \;\; \boldsymbol{a_1} + \mathbf{6}\boldsymbol{a_2} + \mathbf{4}\boldsymbol{a_3} = 0$$

Using backward substitution, we get $\boldsymbol{a_3} = 0\,;\;\; \boldsymbol{a_2} = 0\,;\;\; \boldsymbol{a_1} = 0$

Hence given set of vectors are linearly independent.

Examle 2:

In R^3 let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ if we can find real numbers a_1, a_2, and a_3 so that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{v}.$$

Substituting for $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we have

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}.$$

Equating corresponding entries leads to the linear system (verify)

$$\begin{aligned} a_1 + a_2 + a_3 &= 2 \\ 2a_1 + a_3 &= 1 \\ a_1 + 2a_2 &= 5. \end{aligned}$$

Solving this linear system by the methods of Chapter 2 gives (verify) $a_1 = 1$, $a_2 = 2$, and $a_3 = -1$, which means that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Thus

$$\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3.$$

Figure 4.21 shows $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ■

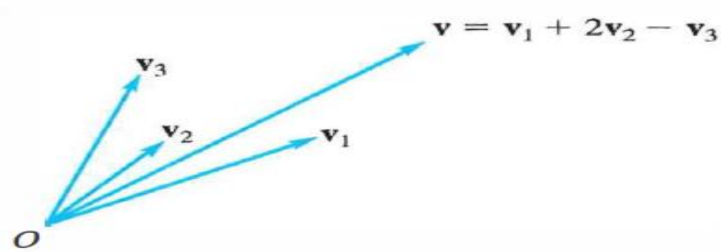


FIGURE 4.21

Figure 4.21 is just for explanation, not to draw in exam.

Examle 2: The set $\boldsymbol{W}$ of all 2×3 matrices of the form $\begin{bmatrix} a & b & c \\ a & 0 & 0 \end{bmatrix}$, where $c = a + b$, is a subspace of $M_{2\times 3}$. Show that every vector in $\boldsymbol{W}$ is a **linear combination** of $w_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution: $\boldsymbol{W} = \left\{ \begin{bmatrix} a & b & c \\ a & 0 & 0 \end{bmatrix} : c = a + b; a, b, c \in R \right\}$

Consider $w \in \boldsymbol{W}$, Then

$$w = \begin{bmatrix} a & b & c \\ a & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & a+b \\ a & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & a \\ a & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & b \\ 0 & 0 & 0 \end{bmatrix}$$
$$= a\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = aw_1 + bw_2$$

Hence every vector in $\boldsymbol{W}$ can be written as linear combination of w_1 and w_2.