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Assignment-3

Question 1:

$S = \{p_1(t), p_2(t), p_3(t)\}$ and $T = \{q_1(t), q_2(t), q_3(t)\}$

$p_1(t) = -3t^2 + 3$, $p_2(t) = -3t^2 + 2t - 1$, $p_3(t) = t^2 + 6t - 1$

$q_1(t) = -6t^2 - 6t$, $q_2(t) = -2t^2 - 6t + 4$, $q_3(t) = -2t^2 - 3t + 7$

$p(t) = \cancel{-0.8}\ -5t^2 + 8t - 5$

Solution

Given

$S = \{v_1, v_2, v_3\}$

where $v_1 = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

And, $T = \{w_1, w_2, w_3\}$

where $w_1 = \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}$, $w_3 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$

be ordered basis for $\mathbb{R}^3$

$p(t) = -5t^2 + 8t - 5$

$\Rightarrow v = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$

(1) Transition Matrix from S to T

The transition matrix $Q_{T\leftarrow S}$ is matrix defined as

$$Q_{T\leftarrow S} = \left[[\underline{v}_1]_T \ [\underline{v}_2]_T \ [\underline{v}_3]_T \right]$$

which can be achieved by solving the following three non-homogenous simultaneously.

$$\underline{v}_1 = a_1 \underline{w}_1 + a_2 \underline{w}_2 + a_3 \underline{w}_3$$
$$\underline{v}_2 = b_1 \underline{w}_1 + b_2 \underline{w}_2 + b_3 \underline{w}_3$$
$$\underline{v}_3 = c_1 \underline{w}_1 + c_2 \underline{w}_2 + c_3 \underline{w}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c|c|c} -6 & -2 & -2 & -3 & -3 & 1 \\ -6 & -6 & -3 & 0 & 2 & 6 \\ 0 & 4 & 7 & 3 & -1 & -1 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c|c|c} -6 & -2 & -2 & -3 & -3 & 1 \\ 0 & -4 & -1 & +3 & 5 & 5 \\ 0 & 4 & 7 & 3 & -1 & -1 \end{array}\right] \quad R_2 - R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c|c|c} 1 & 1/3 & 1/3 & 1/2 & 1/2 & -1/6 \\ 0 & -4 & -1 & +3 & 5 & 5 \\ 0 & 4 & 7 & 3 & -1 & -1 \end{array}\right] \quad \frac{R_1}{-6}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c|c|c} 1 & 1/3 & 1/3 & 1/2 & 1/2 & -1/6 \\ 0 & -4 & -1 & +3 & 5 & 5 \\ 0 & 0 & 6 & 6 & 4 & 4 \end{array}\right] \quad R_3 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c|c|c} 1 & 1/3 & 1/3 & 1/2 & 1/2 & -1/6 \\ 0 & 1 & +1/4 & -3/4 & -5/4 & -5/4 \\ 0 & 0 & 6 & 6 & 4 & 4 \end{array}\right] \quad \frac{R_2}{-4}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c|c|c} 1 & 0 & 1/4 & 3/4 & 11/12 & 1/4 \\ 0 & 1 & 1/4 & -3/4 & -5/4 & -5/4 \\ 0 & 0 & 6 & 6 & 4 & 4 \end{array}\right] \quad R_1 - \frac{1}{3}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c|c|c} 1 & 0 & 1/4 & 3/4 & 11/12 & 1/4 \\ 0 & 1 & 1/4 & -3/4 & -5/4 & -5/4 \\ 0 & 0 & 1 & 1 & 2/3 & 2/3 \end{array}\right] \quad R_3/6$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1/2 & 3/4 & 1/12 \\ 0 & 1 & 0 & -1 & -17/12 & -17/12 \\ 0 & 0 & 1 & 1 & -1/3 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{4}R_3 \\ R_2 - \frac{1}{4}R_3 \end{array}$$

$$\cancel{Q_{T\leftarrow S} = \begin{bmatrix} 1/4 & -1/2 & 1/6 \\ 3/4 & 1/3 & 0 \\ 1 & -1/3 & 1 \end{bmatrix}} \qquad Q_{T\leftarrow S} = \begin{bmatrix} 1/2 & 3/4 & 1/12 \\ -1 & -17/12 & -17/12 \\ 1 & 2/3 & 2/3 \end{bmatrix}$$

(2) Cordinate vector of v w.r.t basis S directly

To Find: $[V]_S = ?$

$$\underline{v} = a_1\underline{v_1} + a_2\underline{v_2} + a_3\underline{v_3}$$

$$-5t^2+8t-5 = a_1(-3t^2-3) + a_2(-3t^2+2t-1) + a_3(t^2+6t-1)$$

$$-5t^2+8t-5 = (-3a_1-3a_2+a_3)t^2 + (2a_2+6a_3)t + (-3a_1-a_2-a_3)$$

Equating co-efficients of like powers, we get

$$-3a_1 - 3a_2 + a_3 = -5$$
$$0a_1 - 2a_2 + 6a_3 = 8$$
$$-3a_1 - a_2 - a_3 = -5$$

$$\Rightarrow [A|b] = \left[\begin{array}{ccc|c} -3 & -3 & 1 & -5 \\ 0 & -2 & 6 & 8 \\ -3 & -1 & -1 & -5 \end{array}\right]$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} -3 & -3 & 1 & -5 \\ 0 & -2 & 6 & 8 \\ 0 & 2 & -2 & 0 \end{array}\right] R_3 - R_1$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -1/3 & 5/3 \\ 0 & -2 & 6 & 8 \\ 0 & 2 & -2 & 0 \end{array}\right] \frac{R_1}{-3}$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -1/3 & 5/3 \\ 0 & -2 & 6 & 8 \\ 0 & 0 & 4 & 8 \end{array}\right] R_3 + R_2$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -1/3 & 5/3 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 4 & 8 \end{array}\right] \frac{R_2}{-2}$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 8/3 & 17/3 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 4 & 8 \end{array}\right] R_1 - R_2$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 8/3 & 17/3 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] R_3/4$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 8/3\, R_3 \\ R_2 + 3R_3 \end{array}$$

$$[\underline{v}]_S = \begin{bmatrix} 1/3 \\ 2 \\ 2 \end{bmatrix} = \text{coordinate of } \underline{v} \text{ w.r.t } S \text{ basis}$$

(3) Coordinate of x w.r.t T using transition matrix

As we know $[\underline{v}]_T = Q_{T\leftarrow S} [\underline{v}]_S$

$$= \begin{bmatrix} 1/2 & 3/4 & 1/12 \\ -1 & -17/12 & -17/12 \\ 1 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 + 3/2 + 1/6 \\ -1/3 - 17/6 - 17/6 \\ 1/3 + 4/3 + 4/3 \end{bmatrix}$$

$$[\underline{v}]_T = \begin{bmatrix} 11/6 \\ -6 \\ 3 \end{bmatrix}$$

Question Q2:

(a) Find orthonormal basis for subspace W of vector space R^4 consisting of all vectors of the form $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

$W \subseteq R^4$

$$W = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} ; a, b, c \in \mathbb{R} \right\}$$

Let $w \in W$

$$W = \begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\qquad\qquad\quad \underline{u_1} \qquad\qquad \underline{u_2} \qquad\qquad \underline{u_3}$$

Let $\underline{v_1} = \underline{u_1}$

Now $\underline{v_2} = \underline{u_2} - \dfrac{(\underline{u_2}, \underline{v_1})}{(\underline{v_1}, \underline{v_1})} v_1$

$$= [-1 \ 1 \ 0 \ 1] - \frac{(-1)}{(2)} [1 \ 0 \ -1 \ 0]$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$$V_3 = U_3 - \frac{(U_3, V_1)}{(V_1, V_1)} V_1 - \frac{(U_3, V_2)}{(V_2, V_2)} V_2$$

$$V_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{-3}{10} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 0 + 1/2 - 3/10 \\ -1 + 0 + 3/5 \\ 1 - 1/2 - 3/10 \\ 0 + 0 + 3/5 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -2/5 \\ 1/5 \\ +3/5 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}$$

$T^* = \{ V_1, V_2, V_3 \}$ is orthegonal basis

Now to get orthonormal basis, we only need to divide each vector ~~basis~~ by its norm.

$T = \{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$ where

$\underline{w}_1 = \dfrac{\underline{v}_1}{\|\underline{v}_1\|}$, $\underline{w}_2 = \dfrac{\underline{v}_2}{\|\underline{v}_2\|}$, $\underline{w}_3 = \dfrac{\underline{v}_3}{\|\underline{v}_3\|}$

$$w_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} -1/\sqrt{10} \\ 2/\sqrt{10} \\ -1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} , \quad w_3 = \begin{bmatrix} 1/\sqrt{15} \\ -2/\sqrt{15} \\ 1/\sqrt{15} \\ 3/\sqrt{15} \end{bmatrix}$$

(b) write vector $v = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ as a L.C of orthonormal basis obtained in part (a)

Orthonormal Basis $T = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{10} \\ 2/\sqrt{10} \\ -1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{15} \\ -2/\sqrt{15} \\ 1/\sqrt{15} \\ 3/\sqrt{15} \end{bmatrix} \right\}$

As we know $\underline{v}$ can be written as

$$v = c_1 \underline{w}_1 + c_2 \underline{w}_2 + c_3 \underline{w}_3 \quad \text{—— (1)}$$

where $c_1 = (\underline{v}, \underline{w}_1) = (1)(1/\sqrt{2}) + (1)(1/\sqrt{2}) = 2/\sqrt{2}$

$c_2 = (\underline{v}, \underline{w}_2) = (1)(-1/\sqrt{10}) + (1)(-1/\sqrt{10}) = -2/\sqrt{10}$

$c_3 = (\underline{v}, \underline{w}_3) = (1)(1/\sqrt{15}) + (1)(1/\sqrt{15}) = 2/\sqrt{15}$

Eq ① becomes

$$\frac{2}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} - \frac{2}{\sqrt{10}}\begin{bmatrix} -1/\sqrt{10} \\ 2/\sqrt{10} \\ -1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} + \frac{2}{\sqrt{15}}\begin{bmatrix} 1/\sqrt{15} \\ -2/\sqrt{15} \\ 1/\sqrt{15} \\ 3/\sqrt{15} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$