**Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSM-A, SP20- A & B).**

# Chapter 7 (Eigenvalues and Eigenvectors)

**Exercise 7.1: 5-8,17,18,30 ; Exercise 7.2: 1-3,5-20.**

**Objective of Lecture:-**

(1) To find eigenvalues and eigenvectors of a matrix.

(2) Cayley-Hamilton Theorem.

(3) Diagonalization of Matrix.

**Skills Needed:- Determinant (Ch-3), Homogeneous system (Ex-4.7), Inverse of matrix [Ch-2 (Row operations) and Ch-3 (Adjoint method)], matrix product. (I believe, you have hands on experience on these skills already)**

Let $A$ be a **square** matrix, if $\lambda \in R$ *and* $\boldsymbol{X} \neq \boldsymbol{0}$ such that $A\boldsymbol{X} = \lambda\boldsymbol{X}$, then $\lambda$ is called eigenvalue (proper, characteristic, latent) and $\boldsymbol{X}$ is called eigenvector of matrix $A$.

Question:- How to find eigenvalue $\lambda$ and corresponding eigenvector $\boldsymbol{X}$?

Consider $A\boldsymbol{X} = \lambda\boldsymbol{X} \quad - - - (1)$
We can write it as $A\boldsymbol{X} - \lambda\boldsymbol{X} = \boldsymbol{0}$
Or $A\boldsymbol{X} - \lambda I\boldsymbol{X} = \boldsymbol{0}$
Or $(A - \lambda I)\boldsymbol{X} = \boldsymbol{0} \quad - - - (2)(\boldsymbol{Homogeneous\ sytem})$

**(Recall:** Homogeneous system $AX = 0$ have trivial solution or non-trivial (infinite many solution).
We know homogenous system has trivial solution if $|A| = det(A) \neq 0$ We are not interested in trivial solution (useless). Moreover, homogenous system has non-trivial solution if $|A| = det(A) = 0$.)

$Homogeneous\ sytem$ (2) have non-trivial solution only if $|A - \lambda I| = det(A - \lambda I) = 0 - (3)$
Equation (3) is called "characteristic equation" and gives "characteristic polynomial".
Roots of this "characteristic polynomial" are called eigenvalues, and corresponding eigenvectors can be found by putting these eigenvalues in (2).

**A Tiny Example:** Find eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$

**Solution:-** Consider $|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 3-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0$

$$[\,(3-\lambda)(-3-\lambda) + 5] = 0 \quad \rightarrow \quad \lambda^2 - 4 = 0 \quad \rightarrow \lambda = -2, 2$$

Case1: when $\lambda = -2$, Solve $AX = \lambda X$ $ie$ $(A - \lambda I)X = O$ where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A - \lambda I = A + 2I = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 1 & -1 \end{bmatrix}$$

$$[A-\lambda I|O]=\begin{bmatrix}5 & -5 & | & 0\\ 1 & -1 & | & 0\end{bmatrix}\ RREF\ \begin{bmatrix}1 & -1 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

$$x_1-x_2=0\ ;\ x_1=x_2=r\ \ and\ \ X=\begin{bmatrix}r\\ r\end{bmatrix}=r\begin{bmatrix}1\\ 1\end{bmatrix}$$

Verify: $AX=\begin{bmatrix}3 & -5\\ 1 & -3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}-2\\ -2\end{bmatrix}=-2\begin{bmatrix}1\\ 1\end{bmatrix}=\lambda X$

Case2: when $\lambda=2$, Solve $AX=\lambda X\ \ ie\ (A-\lambda I)X=O$

$$A-\lambda I=A-2I=\begin{bmatrix}3 & -5\\ 1 & -3\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -5\\ 1 & -5\end{bmatrix}$$

$$[A-\lambda I|O]=\begin{bmatrix}1 & -5 & | & 0\\ 1 & -5 & | & 0\end{bmatrix}\ RREF\ \begin{bmatrix}1 & -5 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

$$x_1-5x_2=0\ ;\ x_1=5x_2;\ x_2=s\ then\ x_1=5s\ \ \ and\ \ X=\begin{bmatrix}5s\\ s\end{bmatrix}=s\begin{bmatrix}5\\ 1\end{bmatrix}$$

Verify: $AX=\begin{bmatrix}3 & -5\\ 1 & -3\end{bmatrix}\begin{bmatrix}5\\ 1\end{bmatrix}=\begin{bmatrix}10\\ 2\end{bmatrix}=2\begin{bmatrix}5\\ 1\end{bmatrix}=\lambda X$

$$AX=\begin{bmatrix}2 & 0\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}-1\\ -1\end{bmatrix}=\begin{bmatrix}-2\\ -0.5\end{bmatrix}$$

## Exercise 7.1:- (1)

Q6 (d) Find the characteristic Polynomial, the eigenvalues and associated eigenvectors of each of following matrices.

**Solution:-**

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

We know $A\underline{x} = \lambda \underline{x}$ where $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow (A - \lambda I)\underline{x} = \underline{0} \quad \text{—— (1)}$$

(1) is homogeneous system in three unknowns $x_1, x_2$ & $x_3$.

Now (1) has non-trivial (Infinite) solution only if

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 2 & 2-\lambda & -2 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

Expand (Open) this determinant using first row.

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ 3 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 2-\lambda \\ 3 & 1 \end{vmatrix} = 0$$

$$(2-\lambda)\left[(2-\lambda)(1-\lambda)+2\right] - 1\left[2(1-\lambda)+6\right] + 2\left[2-3(2-\lambda)\right] = 0$$

$$(2-\lambda)\left[2-2\lambda-\lambda+\lambda^2+2\right] - 1\left[2-2\lambda+6\right] + 2\left[2-6+3\lambda\right] = 0$$

$$(2-\lambda)[\lambda^2-3\lambda+4]-[8-2\lambda]+2[3\lambda-4]=0$$

$$\Rightarrow 2\lambda^2-6\lambda+8-\lambda^3+3\lambda^2-4\lambda-8+2\lambda+6\lambda-8=0$$

$$\Rightarrow \boxed{-\lambda^3+5\lambda^2-2\lambda-8=0}$$ Characteristic Polynomial, $P(\lambda)$

Either use calculator CASIO 991ES'

Mode $\rightarrow$ 5 (Eqn) $\rightarrow$ 4 (Cubic)

Press $\boxed{-1}$ $\boxed{=}$ $\boxed{5}$ $\boxed{=}$ $\boxed{-2}$ $\boxed{=}$ $\boxed{-8}$ $\boxed{=}$

Now press $\boxed{=}$ $x_1=-1$

$\boxed{=}$ $x_2=2$

$\boxed{=}$ $x_3=4$

OR Hit & trial then synthetic division:-

$\rightarrow$ Possible integer solution/root of $P(\lambda)$ are $\pm1, \pm2, \pm3, \ldots$

Check

$P(1) = -1+5-2-8 = 10$ ✗

$P(-1) = 1+5+2-8 = 0$ ✓

$\boxed{\lambda=-1}$ is root of $P(\lambda)$

$$\begin{array}{r|rrrr} -1 & -1 & 5 & -2 & -8 \\ & \downarrow & 1 & -6 & 8 \\ \hline & -1 & 6 & -8 & \boxed{0} \end{array}$$

$(-\lambda^2+6\lambda-8)$ is other factor.

$$\Rightarrow (\lambda+1)(-\lambda^2+6\lambda-8)=0$$

$$-(\lambda+1)(\lambda^2-6\lambda+8)=0$$

$$(\lambda+1)(\lambda-2)(\lambda-4)=0$$

$\lambda=-1, 2, 4$ are eigenvalues of $A$.

When $\lambda = -1$ and $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Then ① becomes

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|\underline{0}] = \left[\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 2 & 3 & -2 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right]$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 2 & 3 & -2 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right] \frac{R_1}{3}$$

$$R \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & 2/3 & 0 \\ 0 & 7/3 & -10/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & \frac{-10}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \times \frac{3}{7}$$

$$x_2 - \frac{10}{7} x_3 = 0 \quad \& \quad x_1 + \frac{x_2}{3} + \frac{2}{3} x_3 = 0$$

Let $\boxed{x_3 = r}$ then $\boxed{x_2 = \frac{10}{7} x_3 = \frac{10}{7} r}$

$$x_1 = \frac{-x_2}{3} - \frac{2}{3} x_3 = \frac{-1}{3} \times \frac{10r}{7} - \frac{2}{3} r$$

$$x_1 = \frac{-10}{9} r - \frac{2}{7} r = \cancel{\frac{-16}{9}} = \frac{-88}{63} r$$

$$x_1 = -\frac{x_2}{3} - \frac{2}{3}x_3$$

$$x_1 = -\frac{1}{3}\left(\frac{10}{7}r\right) - \frac{2}{3}r = -\frac{10r}{21} - \frac{2}{3}r$$

$$x_1 = \frac{-10r - 14r}{21} = -\frac{24r}{21}$$

$$\boxed{x_1 = -\frac{8}{7}r}$$

$$\underline{x} = \begin{bmatrix} -\frac{8}{7}r \\ \frac{10}{7}r \\ r \end{bmatrix} = r\begin{bmatrix} -8/7 \\ 10/7 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to eigenvalue $\boxed{\lambda = -1}$ is $\begin{bmatrix} -8/7 \\ 10/7 \\ 1 \end{bmatrix}$.

Case 2: When $\lambda = 2$ Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector then ① becomes.

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & -2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow \quad \boxed{x_1 = x_3 = r} \;\&\; \boxed{x_2 = -2x_3 = -2r}$$

$$\Rightarrow \underline{x} = \begin{bmatrix} r \\ -2r \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Case 3 When $\lambda = 4$, let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be eigenvector

Then (1) becomes

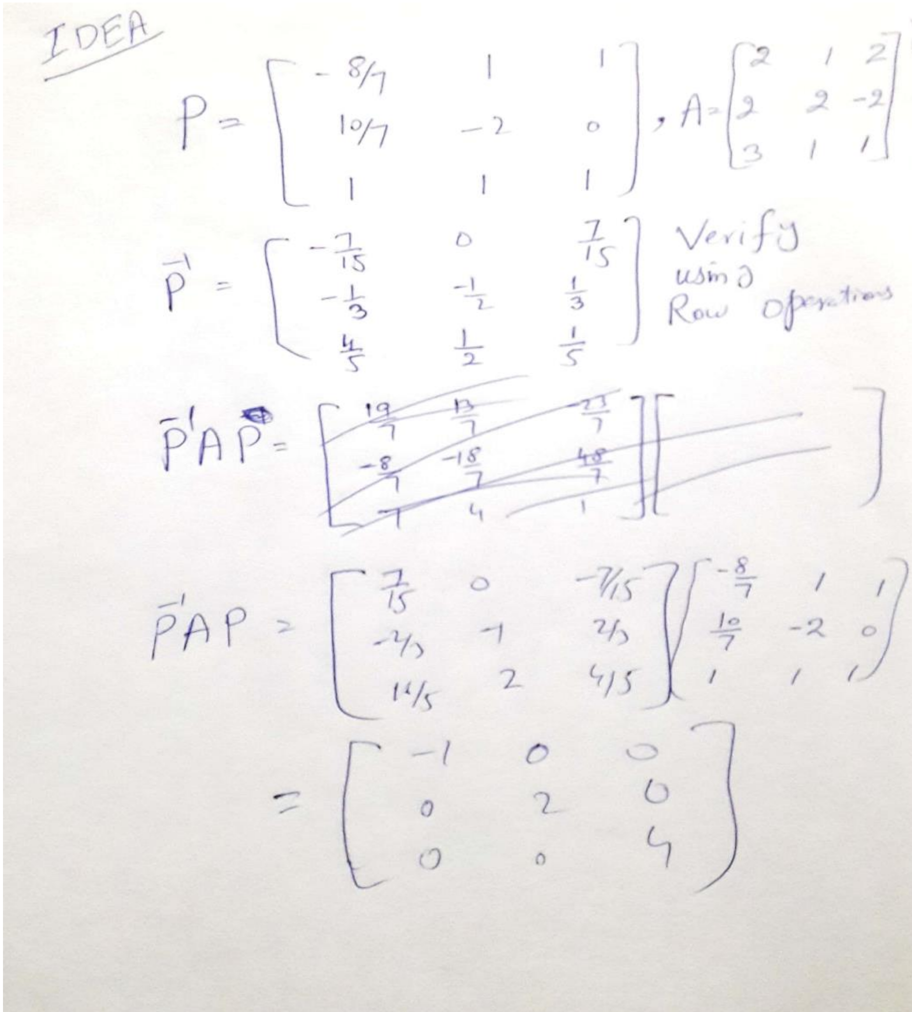
$$\begin{bmatrix} -2 & 1 & 2 \\ 2 & -2 & -2 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ Homogeneous System

$$\downarrow \quad \boxed{x_1 = x_3 = r} \quad \& \quad \boxed{x_2 = 0}$$

$$\underline{x} = \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Conclusion :-

| Eigen Values | Eigenvectors |
|---|---|
| $\lambda = -1$ | $\begin{bmatrix} -8/7 \\ 10/7 \\ 1 \end{bmatrix}$ |
| $\lambda = 2$ | $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ |
| $\lambda = 4$ | $\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$ |



IDEA

$$P = \begin{bmatrix} -8/7 & 1 & 1 \\ 10/7 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{7}{15} & 0 & \frac{7}{15} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{1}{3} \\ \frac{4}{5} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$$ Verify using Row operations

$$\cancel{P^{-1}AP = \begin{bmatrix} \frac{19}{7} & \frac{13}{7} & -\frac{23}{7} \\ -\frac{8}{7} & -\frac{18}{7} & \frac{48}{7} \\ 7 & 4 & 1 \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}$$

$$P^{-1}AP = \begin{bmatrix} 7/15 & 0 & -7/15 \\ -2/3 & -1 & 2/3 \\ 14/5 & 2 & 4/5 \end{bmatrix} \begin{bmatrix} -8/7 & 1 & 1 \\ 10/7 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

**Example 1: (Repeated roots)** $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

$$-\lambda^3 + 3\lambda^2 + 0\lambda - 4 = 0 \quad (characteristic\ Polynomial)$$

$$\lambda = -1, 2, 2$$

Case 1: For $\lambda = -1; (A - \lambda I)X = \mathbf{0}$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 1 & | & 0 \\ 2 & 2 & -1 & | & 0 \\ 0 & -1 & 2 & | & 0 \end{bmatrix}$ (Do yourself, here I am using LAT) $X = \begin{bmatrix} -3/2r \\ 2r \\ r \end{bmatrix} = r \begin{bmatrix} -3/2 \\ 2 \\ 1 \end{bmatrix}$.

Case 1: For $\lambda = 2$ $(A - \lambda I)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 2 & -1 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \end{bmatrix}$$ (Do yourself, here I am using LAT) $X = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

**Example 2: (Repeated roots)**

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution: consider $|A - \lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$

$$Expand\ with\ Row\ 1 \rightarrow -\lambda \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$-\lambda[(2-\lambda)(3-\lambda) - 0] - 2[0 - (2-\lambda)] = 0 \rightarrow (2-\lambda)[-\lambda(3-\lambda) + 2] = 0$$

Either $2 - \lambda = 0$ or $[-\lambda(3-\lambda) + 2] = 0.$

Gives $\lambda = 2$ and $\lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda = 1, 2$

Case 1: For $\lambda = 1$, Solve homogeneous system $AX = \lambda X$ Or $(A - \lambda I)X = O$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|O] = \begin{bmatrix} -1 & 0 & -2 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 1 & 0 & 2 & | & 0 \end{bmatrix} RREF \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = s\,; \qquad x_2 = x_3 = s$$

$$x_1 = -2x_3 = -2s$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Case 2: For $\lambda = 2$, Solve homogeneous system $AX = \lambda X$ Or $(A - \lambda I)X = O$

$$(A - 2I)X = O$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|O] = \begin{bmatrix} -2 & 0 & -2 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{bmatrix} RREF \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = r \quad ; \quad x_2 = t\,; \quad \boldsymbol{x_1 + x_3 = 0} \ \ implies \ \ x_1 = -x_3 = -r$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r \\ t \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Observation: If for 3X3 matrix you have repeated eigenvalues then either you have less than 3 eigenvectors or 3 eigenvectors.

**30.** The **Cayley*–Hamilton**† **theorem** states that a matrix satisfies its characteristic equation; that is, if $A$ is an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n,$$

then

$$A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n = O.$$

The proof and applications of this result, unfortunately, lie beyond the scope of this book. Verify the Cayley–Hamilton theorem for the following matrices:

**(a)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

Solution:- **Hint** (a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$; consider $|A - \lambda I| = 0$ $(characteristic\ Equation)$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 5 \\ 3 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 - 24\lambda - 36 = 0 \quad (characteristic\ Polynomial)$$

No need to find eigenvalues and eigenvectors. Only show, $A^3 - A^2 - 24A - 36I = O$

$$A^3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

$$24A = 24\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

$$36I = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

## 7.2 Diagonalization and Similar Matrices

**Similar Matrices:**

If $A$ and $B$ are $n \times n$ matrices, we say that $B$ is **similar** to $A$ if there is a nonsingular matrix $P$ such that $B = P^{-1}AP$.

Note: Similar matrices have same determinant; have same eigenvalues, have same properties.

**Diagonalizable Matrix:**

A square matrix $A$ is said to be "diagonalizable" if $A$ is similar to a "diagonal matrix" that is

$$\boldsymbol{D = P^{-1}AP}$$

**Note: Columns of $\boldsymbol{P}$ are eigenvectors of $\boldsymbol{A}$ and diagonal entries of $\boldsymbol{D}$ are eigenvalues of $\boldsymbol{A}$.**

Example:- Diagonalize the following matrix, **if possible.**

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution: consider $|A - \lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$

$Expand\ with\ Row\ 1 \rightarrow -\lambda \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix} = 0$

$$-\lambda[(2-\lambda)(3-\lambda) - 0] - 2[0 - (2-\lambda)] = 0 \ \rightarrow (2-\lambda)[-\lambda(3-\lambda) + 2] = 0$$

Either $2 - \lambda = 0$ or $[-\lambda(3-\lambda) + 2] = 0.$

Gives $\lambda = 2$ and $\lambda^2 - 3\lambda + 2 = 0 \ \rightarrow \ \lambda = 1, 2$

Case 1: For $\lambda = 1$, Solve homogeneous system $AX = \lambda X$ Or $(A - \lambda I)X = O$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|O] = \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] RREF \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = s\ ; \qquad x_2 = x_3 = s$$

$$x_1 = -2x_3 = -2s$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Case 2: For $\lambda = 2$, Solve homogeneous system $AX = \lambda X$ Or $(A - \lambda I)X = O$

$$(A - 2I)X = O$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|O] = \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] RREF \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = r \quad ; \quad x_2 = t\,; \quad \boldsymbol{x_1 + x_3 = 0} \;\; implies \;\; x_1 = -x_3 = -r$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r \\ t \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

**Verify (Do yourself) using row operations or Adjoint method**

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## The Diagonalization Theorem

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

The following theorem provides a *sufficient* condition for a matrix to be diagonalizable.

An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

Example:-

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution:-

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$. However, it is easy to verify that each eigenspace is only one-dimensional:

$$\text{Basis for } \lambda = 1: \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Question No 8 (Exercise 7.2) ( Question 9 Home work )

Target $A = ?$

Given $\lambda = 2,\ X = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and when $\lambda = -3,\ X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution: We know $P^{-1}AP = D$

$$PP^{-1}AP = PD$$

$$APP^{-1} = PDP^{-1}$$

This implies $A = PDP^{-1}$

Where $P = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ And $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

$$P^{-1} = \frac{-1}{3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$A = PDP^{-1} = \frac{-1}{3}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= \frac{-1}{3}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 6 & 3 \end{bmatrix}$$

$$= \frac{-1}{3}\begin{bmatrix} 4 & 5 \\ 10 & -1 \end{bmatrix} = \begin{bmatrix} -4/3 & -5/3 \\ -10/3 & 1/3 \end{bmatrix}$$

**Hint for Question 9:-**

Given $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Find $P^{-1}$ using Row operations

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Compute $A = PDP^{-1}$ and you are done

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Idea for taking higher powers of matrix A

(Application of diagonalization: We know $P^{-1}AP = D$

$$PP^{-1}AP = PD \text{ Multiplying both sides on left with } P.$$

$$APP^{-1} = PDP^{-1} \text{ Multiplying both sides on right with } P^{-1}.$$

$$A = PDP^{-1}$$

$$A^9 = (PDP^{-1})^9 \text{ Taking power 9 of both sides.}$$

$$A^9 = PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}$$

$$A^9 = PDIDIDIDIDIDIDIDIDP^{-1}$$

$$A^9 = PDDDDDDDDDP^{-1} = PD^9P^{-1}$$

Similarly, in general $A^n = PD^nP^{-1}$)

Question19 (Ex 7.2) $\quad A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$ $then\ A^9 = ?$

We use formula $\quad A^9 = PD^9P^{-1}$

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 3-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$$[\ (3-\lambda)(-3-\lambda) + 5] = 0 \ \rightarrow \lambda^2 - 4 = 0 \ \ and\ \lambda = -2, 2$$

Case1: when $\lambda = -2$, Solve $AX = \lambda X\ \ ie\ (A - \lambda I)X = O$

$$[A|O] = \left[\begin{array}{cc|c} 5 & -5 & 0 \\ 1 & -1 & 0 \end{array}\right] \ \ RREF \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 - x_2 = 0\ ;\ x_1 = x_2 = r\ \ and\ \ X = \begin{bmatrix} r \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Case2: when $\lambda = 2$, Solve $AX = \lambda X\ \ ie\ (A - \lambda I)X = O$

$$[A|O] = \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 1 & -5 & 0 \end{array}\right] \quad RREF \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 5x_2 = 0\ ;\ x_1 = 5x_2;\ x_2 = s\ then\ x_1 = 5s\quad and\ \ X = \begin{bmatrix} 5s \\ s \end{bmatrix} = s\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

We know $D = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$

$$P^{-1} = \frac{1}{-4}\begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix}$$

$$A^9 = PD^9P^{-1} = \frac{-1}{4}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-2)^9 & 0 \\ 0 & 2^9 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -512 & 0 \\ 0 & 512 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix}$$

$$A^9 = \frac{-512}{4}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix} = -128\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ -1 & 1 \end{bmatrix} = \mathbf{-128}\begin{bmatrix} \mathbf{-6} & \mathbf{10} \\ \mathbf{-2} & \mathbf{6} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{-768} & \mathbf{1280} \\ \mathbf{-256} & \mathbf{768} \end{bmatrix}$$

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(1) Eigenvalues of a matrix A (3x3) are 1, 2, -3 then Is A diagonalizable? (T/F)

(2) Eigenvalues of a matrix A are 2, 2, -3 then Is A diagonalizable? (T/F)

(3) Eigenvalues of a matrix A are 1, 2, -3 then eigenvalues of $A^2$are 1, 4 and 9. (T/F)

(4) If A is 3x3 matrix then it has three distinct eigenvalues. (T/F)

(5) If A is 3x3 real symmetric matrix then it has three distinct eigenvalues. (T/F)