An Introductory Lecture

Scalars and Vectors

By
Muhammad Kaleem Ullah
Lecturer
Department of Physics
CUI, Lahore Campus.

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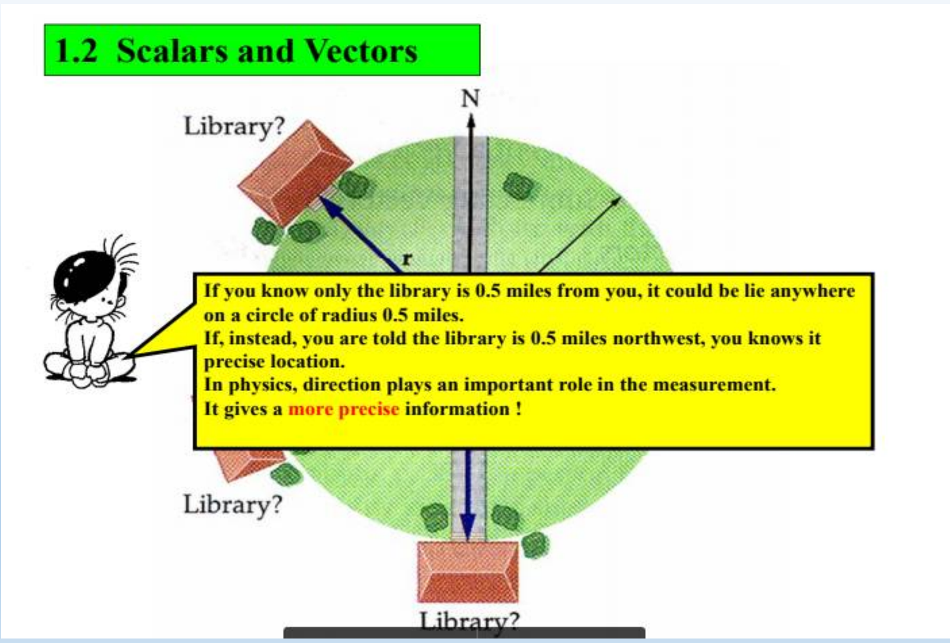

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Scalar Quantity

- Quantity which has only magnitude.
- Example: mass, distance, speed, work, pressure, current, temperature.

Vector Quantity

- Quantity which has both magnitude and direction.
- Example: displacement, velocity, force, momentum, impulse, electric field, magnetic field.

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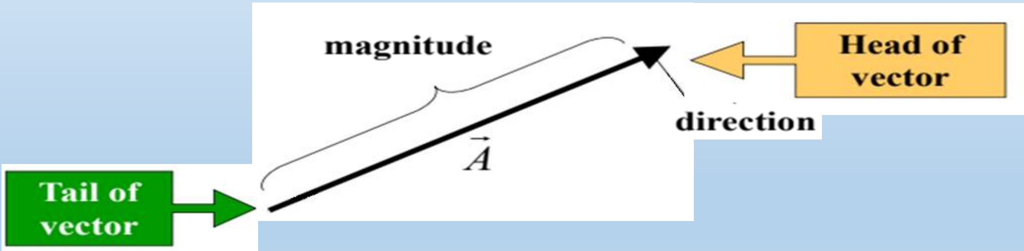
Representing vectors

- Symbols for vectors : **A** or $\vec{A}$
- A vector $\vec{A}$ can be represented by an arrow.
- The length of the arrow indicates its magnitude
- Arrow head shows the direction .



- Magnitude of the vector $\vec{A}$ is written as |A|

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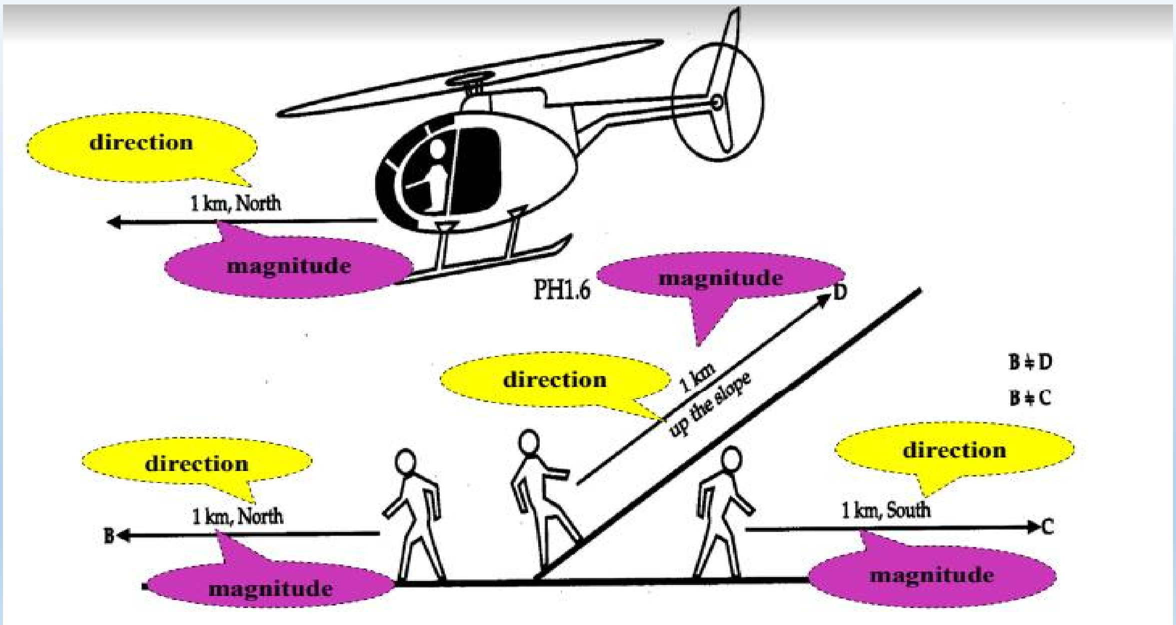

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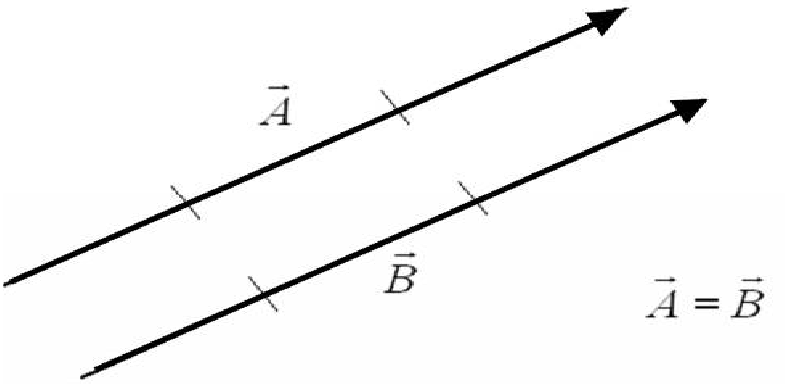
Equality of two vectors

- 2 vectors $\vec{A}$ & $\vec{B}$ are equal if they have the same magnitude and point in the same direction.



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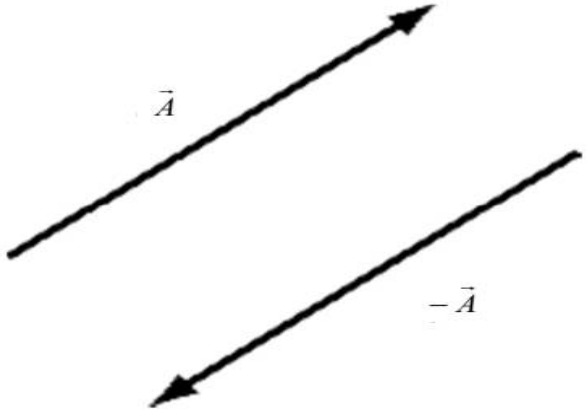
Negative Vector

Negative vector $-\vec{A}$ is a vector with the **same magnitude** as $\vec{A}$ but **points in opposite direction**.



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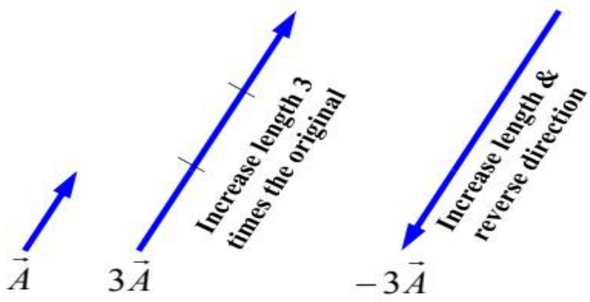
Multiplying a vector by a scalar quantity, *k*

Multiplying a vector by a scalar quantity +k, will get a vector with the same direction, but different magnitude, as the original.
If scalar, k is negative, then the direction of vector is reversed by scalar multiplication.



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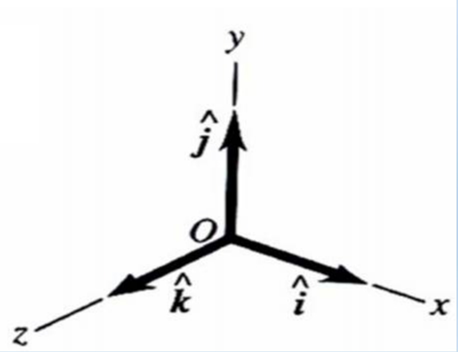
Unit vectors

A unit vector is a vector that has a magnitude of 1 with no units.

Are use to specify a given direction in space.

$\hat{i}$, $\hat{j}$ & $\hat{k}$ is used to represent unit vectors pointing in the positive x, y & z directions.

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$



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Vector Addition & Subtraction

Addition

- The addition of 2 vector, $\vec{A}$ and $\vec{B}$ result in a third vector called $\vec{R}$ resultant vector.
- Resultant vector is a single vector that will have the same effect as 2 or more vectors.
- 2 methods of vector addition:

(1) Drawing / Graphical method - tail to head & Parallelogram

(2) Mathematic Calculation – unit vector & trigonometry

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Adding Parallel Vectors

(1) vectors in the same directions

$\vec{A} = 3N$ $\quad$ $\vec{B} = 7N$

Resultant, $\vec{R} = 10\ N$

$\vec{R} = (+3\,N) + (+7\,N) = +10\,N$ **To the right**

(2) vectors in the opposite directions

$\vec{A} = 3N$ $\quad$ $\vec{B} = 7N$

Resultant, $\vec{R} = 4N$

The direction of resultant vector **R** is in the direction of the bigger vector

$\vec{R} = (+7\,N) + (-3\,N) = +4\,N$ **To the right**

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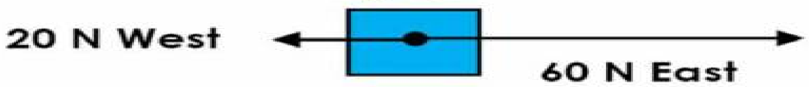
Resultant , R= 9 N to the East



Resultant, R = 40 N to the East

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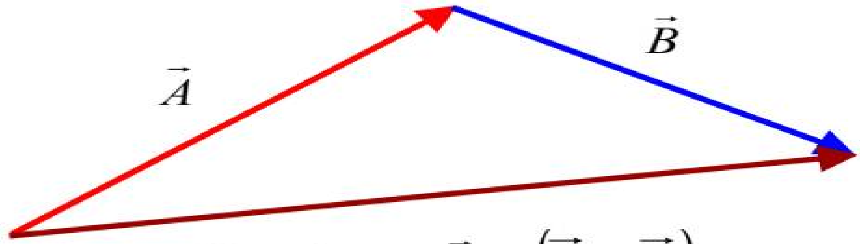
Two equivalent ways to add vectors graphically: the tail-to-head method and the parallelogram method.

(a) Tail to head method

Placing the tail of B so that it meets the head of A



Resultant, $\vec{R} = \left(\vec{A} + \vec{B}\right)$

The Resultant, $\vec{R} = (\vec{A} + \vec{B})$ is the vector from the tail of A to the head of B

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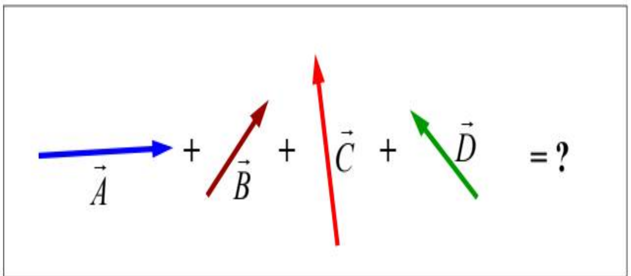
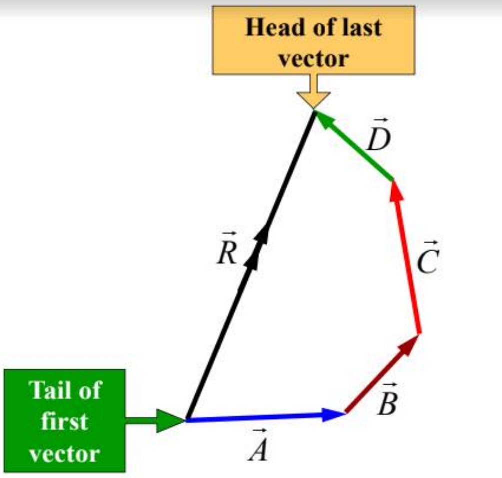
How to add vector A, B, C and D ?



Placing the tail of each successive arrow at the head of the previous one. The resultant vector is the arrow drawn from the tail of the first vector to the head of the last vector.



$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

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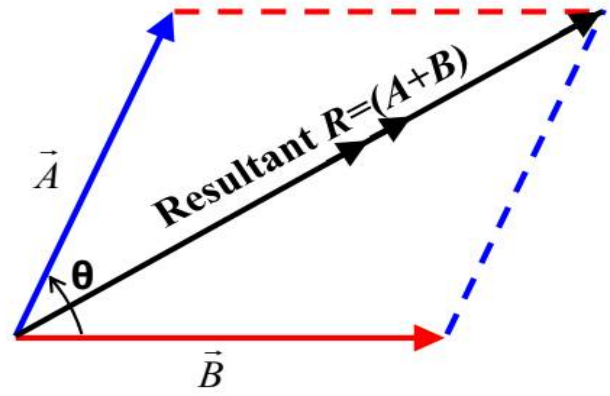
(b) Parallelogram method



Resultant vector, $\vec{R}$ diagonal of a parallelogram formed with $\vec{A}$ & $\vec{B}$ as two of its 4 sides.

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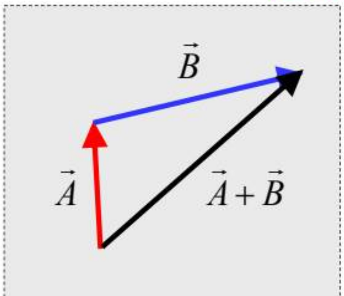
Vector Addition Rules

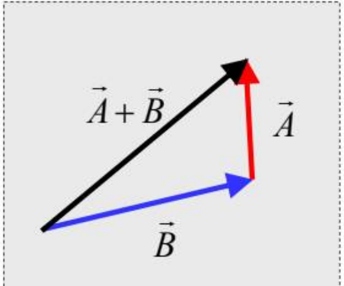
1. Commutative law of Addition

   When 2 vectors are added, the sum is independent of the order of addition.

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$



(i)



(ii)

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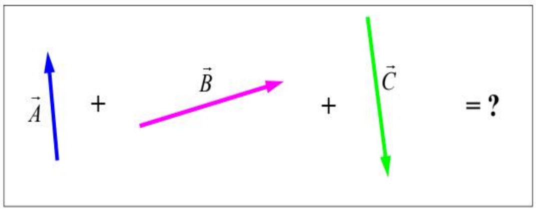
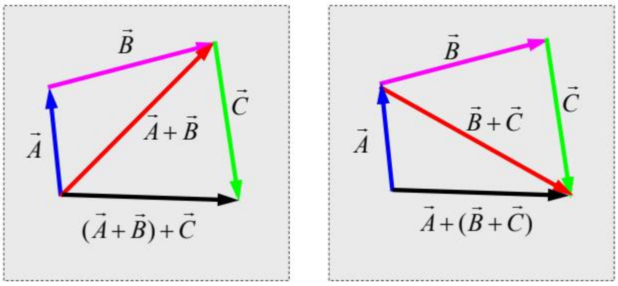
2. Associative law of Addition

When 3 or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together.





$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$$

The order makes no difference !

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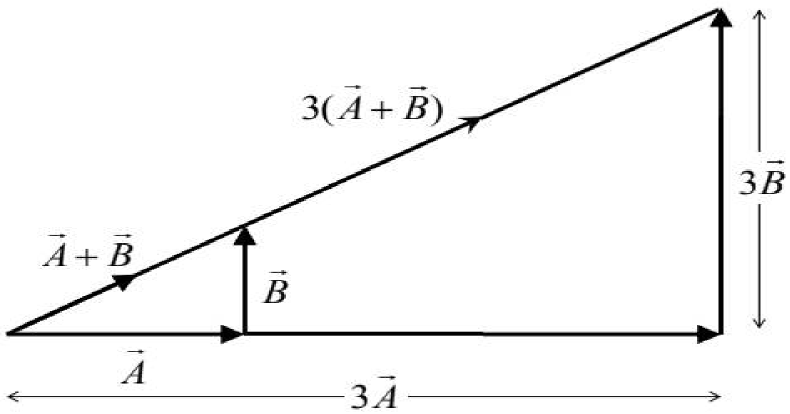
3. Distributive Law of Addition

$$m(\vec{A}+\vec{B})=m\vec{A}+m\vec{B}$$

where m - scalar quantity



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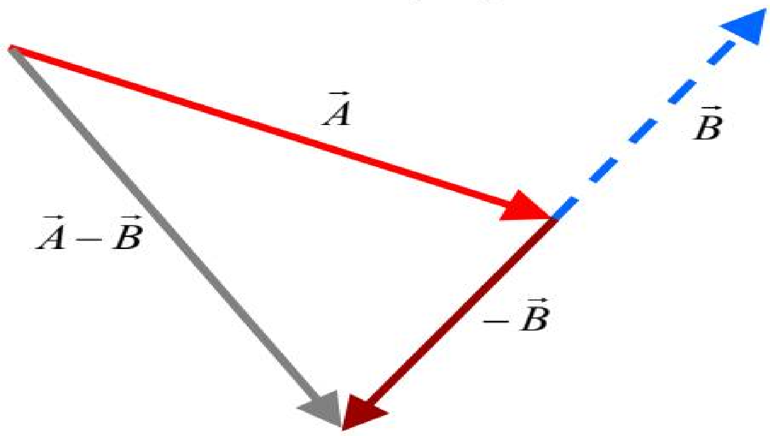
Vectors Subtraction

– is done by adding negative vector.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



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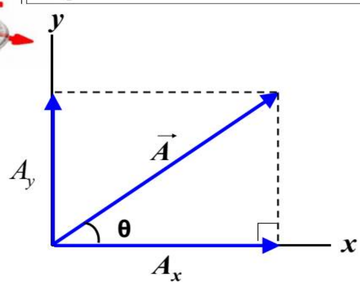
Resolving vector into 2 perpendicular components (2D)

A vector may be expressed in terms of its components.



with the aid of trigonometry:

$$\cos\theta = \frac{A_x}{\vec{A}} \mapsto A_x = \vec{A}\cos\theta$$

$$\sin\theta = \frac{A_y}{\vec{A}} \mapsto A_y = \vec{A}\sin\theta$$

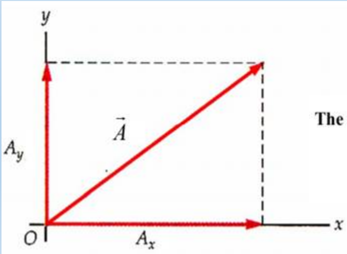
Magnitude of vector A :

$$|A| = \sqrt{A_x^2 + A_y^2}$$

Direction of vector A :

$$\tan\theta = \frac{A_y}{A_x}$$

* θ is always measured from +x axis.



The vector $\vec{A}$ can also written in unit vector form:

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

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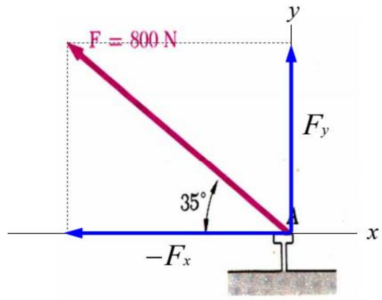
Example 5

A force of 800 N is exerted on a bolt A as shown in Fig. below. Determine the horizontal and vertical components of the force.



Solution

with the aid of trigonometry:

Horizontal component of force

$$F_x = -F\cos\theta$$
$$= -800\cos 35°$$
$$\boxed{F_x = -655\,N}$$

Vertical component of force

$$F_y = F\sin\theta$$
$$= 800\sin 35°$$
$$\boxed{F_y = 459\,N}$$

We may write $\vec{F}$ **in the unit vector form**

$$\vec{F} = -(655N)\hat{i} + (459N)\hat{j}$$

OR USING TABLE:

Vector	Component x	Component y
F	$-F\cos\theta$ $-800\cos 35°$	$+F\sin\theta$ $+800\sin 35°$
R	$\sum F_x = -655N$	$\sum F_y = +459N$

Horizontal component of the force, $\boxed{\sum F_x = -655N}$

Vertical component of the force, $\boxed{\sum F_y = +459N}$

We may write $\vec{F}$ **in the unit vector form**

$$\vec{F} = -(655N)\hat{i} + (459N)\hat{j}$$

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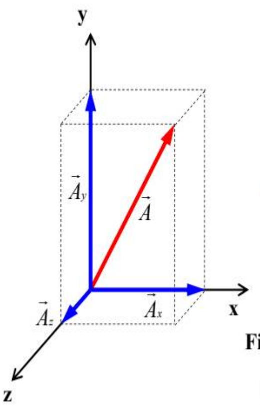
Resolving vector into 3 perpendicular components (3D)



In 3D space, vector $\vec{A}$ can be written in unit vector as :

$$\boxed{\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}$$

Magnitude of vector A :

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

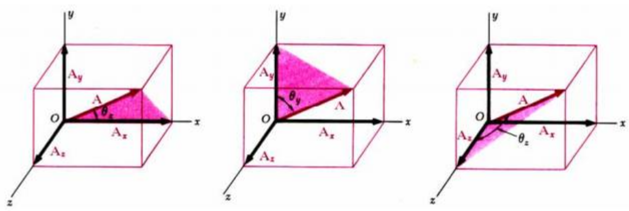
Example :

Given vector $\vec{A} = -2\hat{i} + 5\hat{j} + 8\hat{k}$

Find the magnitude of vector, A?

$$|A| = \sqrt{(-2)^2 + (5)^2 + (8)^2} = 9.64$$

vector can be resolved into 3 components : A_x, A_y & A_z



$$\boxed{A_x = A\cos\theta_x} \quad \boxed{A_y = A\cos\theta_y} \quad \boxed{A_z = A\cos\theta_z}$$

where θ_x, θ_y and θ_z are the angles that vector A forms with x, y & z axes respectively

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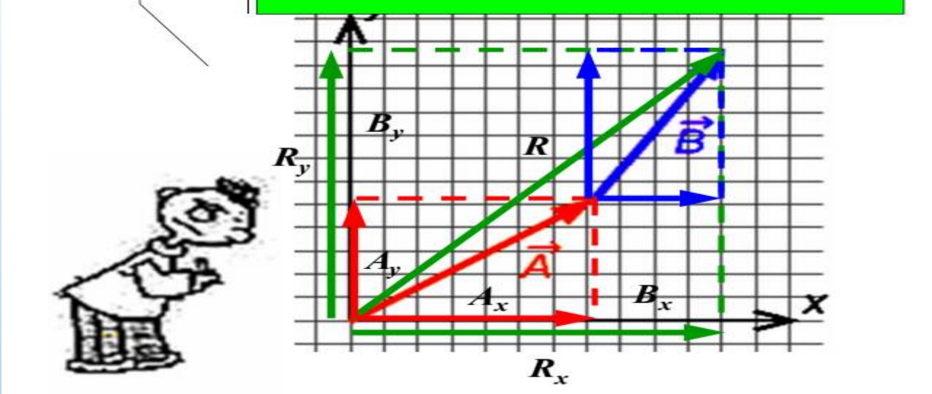
Addition of

We use components to calculate the vector sum (resultant) of two or more vectors.



$$\mathbf{A} + \mathbf{B} = (\mathbf{A}_x + \mathbf{B}_x) + (\mathbf{A}_y + \mathbf{B}_y)$$

$$\mathbf{R} = \mathbf{R}_x + \mathbf{R}_y$$

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Steps of adding vectors using components

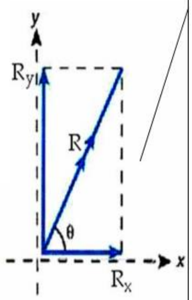
1. Resolve each vector into its x and y components.
   Pay careful attention to signs:
   any component that points along the negative x or y axis get a − sign.
2. Add all the x components together to get the x component of resultant.

   $$R_x = A_x + B_x + \text{any other}$$

   Ditto for y:

   $$R_y = A_y + B_y + \text{any other}$$

   * do not add x components to y components



3. The magnitude of the resultant vector, R is given by:

   $$|R| = \sqrt{R_x^{\,2} + R_y^{\,2}}$$

   Direction of the resultant vector :

   $$\tan\theta = \frac{R_y}{R_x}$$

   * θ is measured from x – axis.

   * vector diagram drawn help to obtain the correct position of the angle θ

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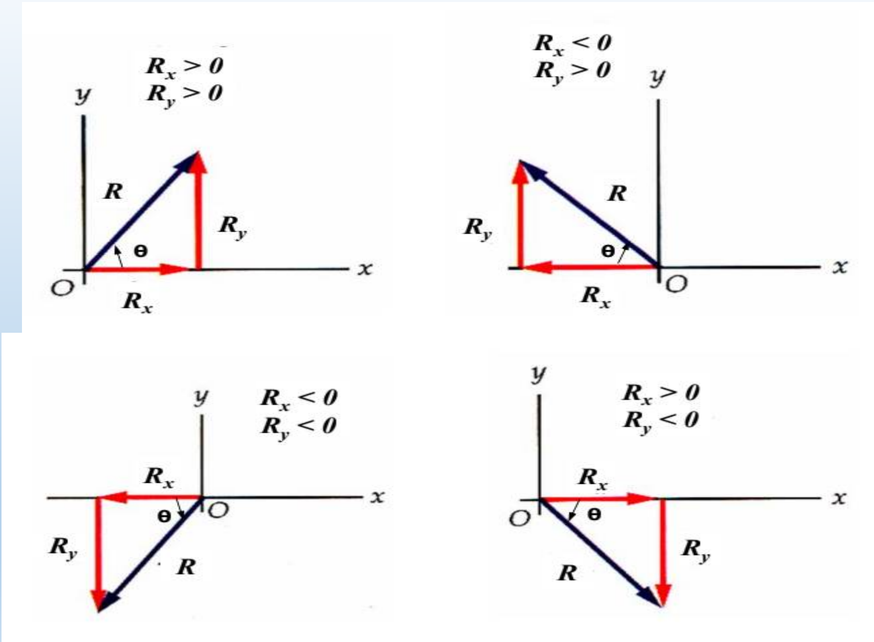

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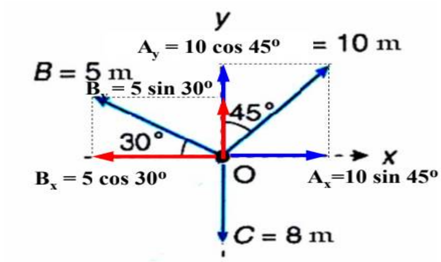
Example 6

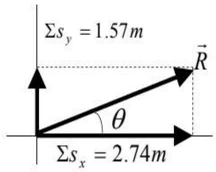
The magnitudes of the 3 displacement vectors shown in drawing. Determine the resultant value when these vectors are added together.



Vector	Component x	Component y
A	+10 sin 45° = +7.07m	+10 cos 45° = +7.07m
B	–5 cos 30 = –4.33m	+5 sin 30 =+ 2.50m
C	0m	– 8m
R	$\boxed{\sum s_x = +2.74m}$	$\boxed{\sum s_y = +1.57m}$

Magnitude of resultant vector

$$\vec{R} = \sqrt{\Sigma s_x^{\,2} + \Sigma s_y^{\,2}}$$
$$= \sqrt{(2.74)^2 + (1.57)^2}$$
$$= 3.16\ m$$



Direction of resultant vector

$$Tan\ \theta = \frac{\Sigma s_y}{\Sigma s_x} = \frac{1.57}{2.74} = 0.573$$

$$\theta = 29.81° \text{ above positive x - axis}$$

Resultant vector , R = 3.16 m at 29.81° above positive x-axis

Or can write in unit vector form

$$\vec{R} = +(2.74m)\hat{i} + (1.57m)\hat{j}$$

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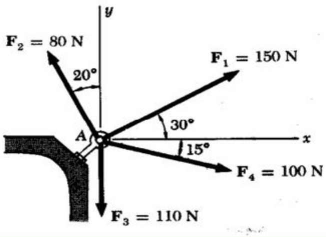
Follow Up Exercise 3

Four forces act on bolt A shown. Determine the resultant of the forces on the bolt .



Answer :
R = (199.1N)i + (14.3N)j
or R = 199.6 N at 4.1° above positive x-axis.

Answer Follow Up Exercise 3

Vector	Component -x	Component –y
F_1	$= +150 \cos 30°$ $= +129.90\ N$	$= +150 \sin 30°$ $= +75.00\ N$
F_2	$= -80 \sin 20°$ $= -27.36\ N$	$= +80 \cos 20°$ $= +75.18\ N$
F_3	$= 0\ N$	$= -110.00\ N$
F_4	$= +100 \cos 15°$ $= +96.59\ N$	$= -10 \sin 15°$ $= -25.88\ N$
ΣF	$\Sigma F_x = +199.13\ N$	$\Sigma F_y = +14.30\ N$

Magnitude of resultant vector

$$F_{nett} = \sqrt{\Sigma F_x{}^2 + \Sigma F_y{}^2}$$
$$= \sqrt{(199.13)^2 + (14.30)^2}$$
$$= 199.60\ N$$

Direction of resultant vector

$$Tan\ \theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{14.30}{199.13} \rightarrow \theta = 4.1°$$

Resultant Force, ΣF=199.6N at θ=4.1° above positive x-axis

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Example 7

Let : $\vec{a} = 2\hat{i} + 5\hat{j}$

$\vec{b} = 5\hat{i} - 3\hat{j}$

Find : (a) $\vec{a} + \vec{b}$

(b) $2\vec{a} - 3\vec{b}$

(c) $|2\vec{a}|$

Solution

(a) $\vec{a} + \vec{b} = (2\hat{i} + 5\hat{j}) + (5\hat{i} - 3\hat{j})$

$= 7\hat{i} + 2\hat{j}$

(b) $2\vec{a} - 3\vec{b} = 2(2\hat{i} + 5\hat{j}) - 3(5\hat{i} - 3\hat{j})$

$= 4\hat{i} + 10\hat{j} - 15\hat{i} + 9\hat{j}$

$= -11\hat{i} + 19\hat{j}$

(c) To find the magnitude of $|2\vec{a}|$, **1st we have to calculate** $2\vec{a}$

$2\vec{a} = 2(2\hat{i} + 5\hat{j}) = 4\hat{i} + 10\hat{j}$

$|2\vec{a}| = \sqrt{4^2 + 10^2}$

$= 10.77$

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Follow Up Exercise 4

1. Find the sum of two vectors A and B in unit vector:

$\vec{A} = (2.0\hat{i} + 2.0\hat{j})\,\text{m}$ and $\vec{B} = (2.0\hat{i} - 4.0\hat{j})\,\text{m}$

2. A particle undergoes three consecutive displacements:

$\vec{d}_1 = (15\hat{i} + 30\hat{j} + 12\hat{k})\,\text{cm}$

$\vec{d}_2 = (23\hat{i} - 14\hat{j} - 5\hat{k})\,\text{cm}$ and $\vec{d}_3 = (-13\hat{i} + 15\hat{j})\,\text{cm}$

Find the components of the resultant displacement and its magnitude.

answer : (1) $\vec{R} = (4.0\hat{i} - 2.0\hat{j})\,\text{m}$
(2) R_x = 25 cm; R_y = 31 cm ; R_z = 7.0 cm; R = 40.44 cm

Answer Follow Up Exercise 4

1. Find the sum of two vectors A and B in unit vector:

Solution:

$$\vec{A} + \vec{B} = (2.0\hat{i} + 2.0\hat{j})\,\text{m} + (2.0\hat{i} + 4.0\hat{j})\text{m}$$
$$= (4.0\hat{i} - 2.0\hat{j})\,\text{m}$$

2. A particle undergoes three consecutive displacements:

$\vec{d}_1 = (15\hat{i} + 30\hat{j} + 12\hat{k})$ cm, $\vec{d}_2 = (23\hat{i} - 14\hat{j} - 5\hat{k})$ cm and $\vec{d}_3 = (-13\hat{i} + 15\hat{j})$ cm

Find the components of the resultant displacement and its magnitude.

vector	Component-x	Component-y	Component-z
d_1	+15cm	+30cm	+12cm
d_2	+23cm	-14cm	-5cm
d_3	-13cm	+15cm	0cm
$\sum R$	$\sum d_x = +25cm$	$\sum d_y = +31cm$	$\sum d_z = +7cm$

$$\vec{R} = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(25)^2 + (31)^2 + (7)^2} = 40.44\ cm$$

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Multiplying a vector by a vector

Dot (scalar) product	$\vec{A} \bullet \vec{B}$
Cross (vector) product	$\vec{A} \times \vec{B}$

Dot Product ($\vec{A} \bullet \vec{B}$)

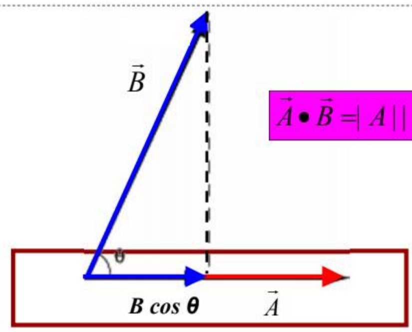
$$\vec{A} \bullet \vec{B} = |A||B|\cos\theta$$

where |A| : magnitude of vector $\vec{A}$
|B|: magnitude of vector $\vec{B}$
θ : angle between $\vec{A}$ & $\vec{B}$

$0° \le \theta \le 180°$

Physical Meaning of $\vec{A} \bullet \vec{B}$

$\vec{A} \bullet \vec{B}$ is the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ parallel to $\vec{A}$.



$$\vec{A} \bullet \vec{B} = |A||B|\cos\theta$$

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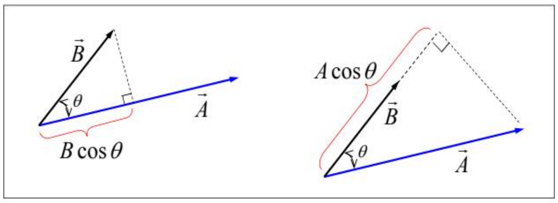
Physical Meaning of $\vec{A} \bullet \vec{B}$

$\vec{A} \bullet \vec{B}$ is the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ <u>parallel</u> to $\vec{A}$.

or

$\vec{A} \bullet \vec{B}$ is the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ <u>parallel</u> to $\vec{B}$.



$\vec{A} \bullet \vec{B}$ = **zero** when **θ = 90°** because cos 90° = 0

$\vec{A} \bullet \vec{B}$ = **maximum** value when **θ = 0°** because cos 0° = 1

Commutative law applied to dot product :

$$\boxed{\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}}$$

Example of physical quantity : $W = \vec{F} \bullet \vec{s}$

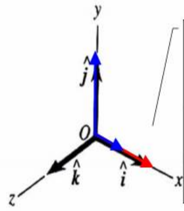
<u>Dot product Calculation</u>

Given 2 vector :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

How to perform $\vec{A} \bullet \vec{B}$?

$$\vec{A} \bullet \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \bullet (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$



Remember :

$$\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = (1)(1)\cos 0° = 1$$
$$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = (1)(1)\cos 90° = 0$$

$$\vec{A} \bullet \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

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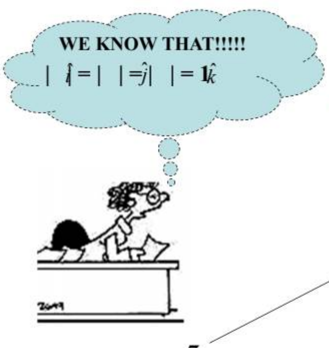
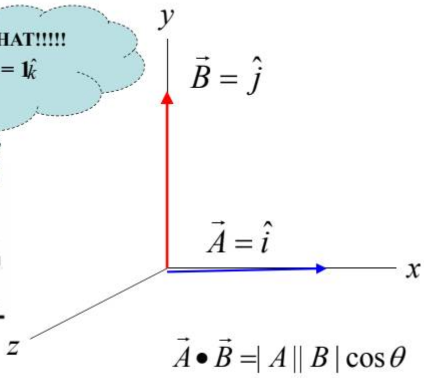
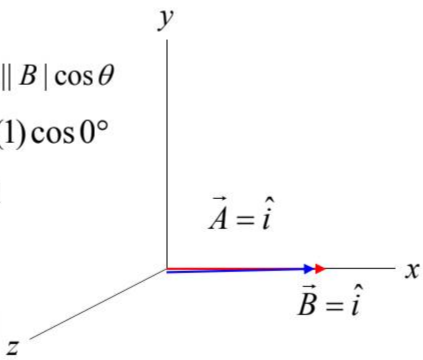




$$\vec{A} \bullet \vec{B} = |A||B|\cos\theta$$
$$= 1(1)\cos 90°$$
$$= 0$$

$$\vec{A} \bullet \vec{B} = |A||B|\cos\theta$$
$$= 1(1)\cos 0°$$
$$= 1$$



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Example 8

Given 2 vectors :

$$\vec{A} = (3i + 2j - 4k)$$
$$\vec{B} = (-5i + 8j - 2k)$$

Calculate
(a) the value of $\vec{A} \bullet \vec{B}$
(b) the angle θ between 2 vectors

Solution

(a) $\vec{A} \bullet \vec{B} = (3i + 2j - 4k) \bullet (-5i + 8j - 2k)$

$= (3)(-5)$

$\vec{A} \bullet \vec{B} = \boxed{9}$ ⇐ **produces a scalar**

(b) from : $\vec{A} \bullet \vec{B} = |A||B|\cos\theta$

$$|A| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$
$$= \sqrt{(3)^2 + (2)^2 + (-4)^2} = 5.39$$
$$|B| = \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}$$
$$= \sqrt{(-5)^2 + (8)^2 + (-2)^2} = 9.64$$

$$\cos\theta = \frac{\vec{A} \bullet \vec{B}}{|A||B|} = \frac{9}{(5.39)(9.64)} \Rightarrow \boxed{\theta = 80.03^\circ}$$

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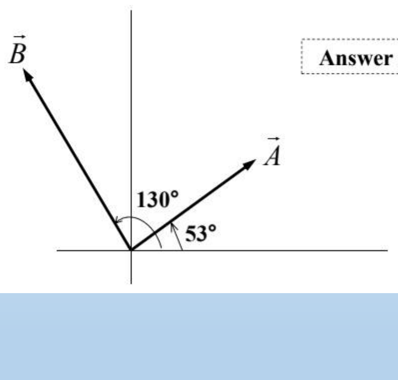
Follow Up Exersice 5

Find the scalar product $\vec{A} \bullet \vec{B}$ **of the two vectors in figure. The magnitude of the vectors are A = 4.0 N and B = 5.0 m**



Answer : 4.50 Nm

Answer Follow Up Exersice 5

Solution :

Given : A = 4.0 N and B = 5.0 m

$$\theta = (130 - 53)^\circ = 77^\circ$$

$$\vec{A} \bullet \vec{B} = |A||B|\cos\theta$$
$$= (4)(5)\cos 77^\circ$$
$$= 4.50\,Nm$$

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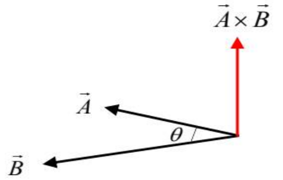
Cross Product ($\vec{A}\times\vec{B}$)

- Also called **vector product.**
- produce a **third vector**, which is **perpendicular** to both of the original vectors.

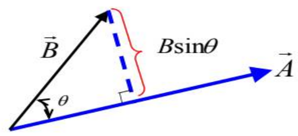


- The **magnitude of the cross product** is given by:

$$\boxed{|\vec{A}\times\vec{B}| = |A||B|\sin\theta} \qquad 0° \le \theta \le 180°$$

Physical Meaning of $\vec{A}\times\vec{B}$

$|\vec{A}\times\vec{B}|$ is equals the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ perpendicular to $\vec{A}$.



-- if $\vec{A}\,\&\,\vec{B}$ is parallel @ anti parallel (θ=0° @ 180°) → $|\vec{A}\times\vec{B}| = 0$

$$|\vec{A}\times\vec{B}| = |A||B|\cancel{\sin 0°}^{\,0} = 0$$

-- if $\vec{A}\,\&\,\vec{B}$ is 90° → $|\vec{A}\times\vec{B}| = \max$

$$|\vec{A}\times\vec{B}| = |A||B|\cancel{\sin 90°}^{\,1} = AB$$

Example of physical quantity :

1) Force acting on a charge moving in magnetic field

$$F_m = q\vec{v}\times\vec{B}$$

2) Torque, $\tau = \vec{r}\times\vec{F}$

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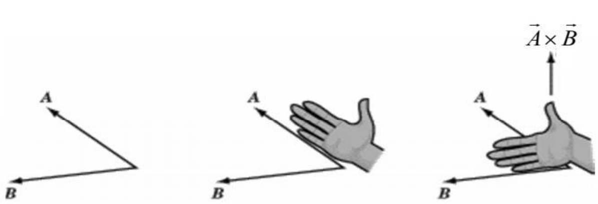
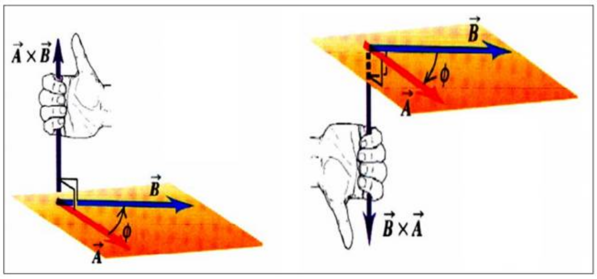
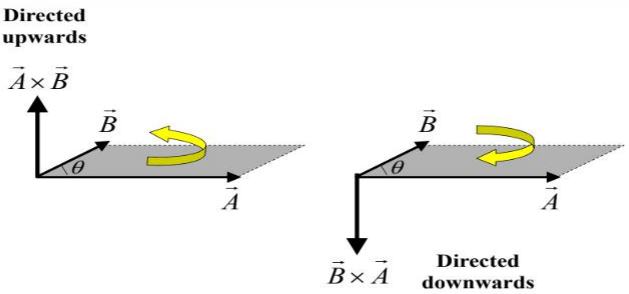
- the **direction of new vector** ($\vec{A}\times\vec{B}$) is **normal** to the plane that contain vector $\vec{A}$ & $\vec{B}$ → given by **Right Hand Rule**





$$\boxed{\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})}$$



$$\boxed{\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})}$$

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Example 9

Given 2 vector :

$$\vec{A} = (3i + 2j)$$
$$\vec{B} = (-5i + 8j)$$

Calculate magnitude and direction of $\vec{A} \times \vec{B}$ if the vectors are perpendicular to each other.

Solution

$$|\vec{A} \times \vec{B}| = |A||B|\sin\theta$$
$$|A| = \sqrt{(3)^2 + (2)^2} = 3.606$$
$$|B| = \sqrt{(-5)^2 + (8)^2} = 9.434$$

vector perpendicular to each other, $\theta = 90^\circ$

$$|A \text{ x } B| = (3.606)(9.434)\sin 90^\circ$$
$$\boxed{|A \text{ x } B| = 34.02}$$

By using Right Hand Rule: the direction $\vec{A} \times \vec{B}$ is out of paper.

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Example 10

If vector has a magnitude of 3 unit and vector has a magnitude of 4 unit, find the angle between and if

(a) $\vec{C} \text{ x } \vec{D} = 0$

(b) $|\vec{C} \text{ x } \vec{D}| = 12$

Solution

$$|\vec{C} \text{ x } \vec{D}| = CD\sin\theta$$

(a) $0 = CD\sin\theta$

$\therefore \theta = 0^0$

(b) $12 = (3)(4)\sin\theta$

$1 = \sin\theta$

$\therefore \theta = 90^0$

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Follow Up Exercise 6

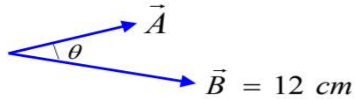
1. A force $\vec{F} = (\hat{i} - 5\hat{j})$ is acting on an object. The displacement of the object is given by $\vec{x} = (10\hat{i} + \hat{j})$
   Find
   (a) the work done by this force
   (b) the angle between the force & the displacement.

2. Given 2 vector as below :
   $\vec{A} = 3\hat{i} + 3\hat{j}$ $\quad \vec{B} = 5\hat{i} + 2\hat{j}$

Find the cross product of the two vector
State its magnitude & draw the vector diagram to shows the direction of the new vector ($\vec{A} \times \vec{B}$

3. Given 2 vectors A = 2i –5j and B = 5i +7j, and the angle between these two vectors is 35°
   (i) Find the magnitudes of A and B
   (ii) The scalar product of these vectors
   (iii) The vector product for these vectors

4. Given two vectors A and B as in diagram below.



(a) Determine the direction of a new vector (B x A)

(b) If the magnitude of (B x A) = 25 cm, and $\theta = 24.6^\circ$
Find the magnitude of vector A

Answer Follow Up Exercise 6

(i) $Use\ |A| = \sqrt{(A_x)^2 + (A_y)^2}$

$$|A| = \sqrt{(2)^2 + (-5)^2} = 5.385$$

$$|B| = \sqrt{(5)^2 + (7)^2} = 8.602$$

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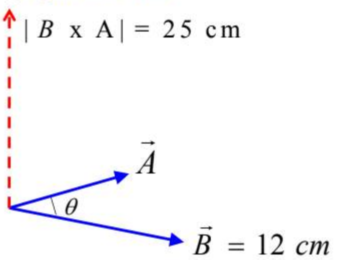
(ii) $$\vec{A} \bullet \vec{B} = (2i - 5j) \bullet (5i + 7j)$$
$$= [(2)(5) + (-5)(7)]$$
$$= -25$$

(iii) $$|\vec{A} \times \vec{B}| = |A||B| \sin\theta$$
$$= (5.385)(8.602) \sin 35^\circ$$
$$= 26.57$$

Solution

Using right hand rule:



(a) The new vector, (B x A) directed UPWARD

(b) $|B \text{ x A}| = \text{BA} \sin\theta$

$$25 = (12)(A) \sin 24.6^\circ$$
$$\text{A} = 5.0 \text{ cm}$$

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