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Date:

Differential Equations

Cauchy Euler Differential Equations:

$$a_n x^n \frac{d^n}{dx^n} y + a_{n-1} x^{n-1} \frac{d^{n-1}}{dx^{n-1}} y \ldots\ldots + a_1 x \frac{d}{dx} y + a_0 y = 0$$

$$\overbrace{x^2 \frac{d^2}{dx^2}}^{\text{same power}} y + \overbrace{x^1 \frac{d^1}{dx}}^{\text{same power}} y + y = 0$$

Solution:

["t" can be replaced by any other variable name]

let $x = e^t$

Taking log

$$\ln x = \ln(e^t)$$

$$\ln x = t$$

$$\frac{d}{dx} = D$$

$$x.D = m$$

$$x^2.D^2 = m(m-1)$$

$$x^3.D^3 = m(m-1)(m-2)$$

$$x^4.D^4 = m(m-1)(m-2)(m-3)$$

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$$x^2\frac{d^2}{dx^2}y + 7x\frac{d}{dx}y + 5y = x^5$$

let $x = e^t$

$\ln x = t$

$\frac{d}{dx} = D$

$x \cdot D = m$

$x^2 \cdot D^2 = m(m-1)$

$$(x^2D^2 + 7xD + 5)y = (e^t)^5$$

$$(m(m-1) + 7m + 5)y = e^{5t}$$

$$(m^2 - m + 7m + 5)y = e^{5t}$$

$$(m^2 + 6m + 5)y = e^{5t} \quad \rightarrow \text{equ (A)}$$

(m^2+6m+5) is an Auxilary Equation, so

$$m^2 + 6m + 5 = 0$$

$$m^2 + m + 5m + 5 = 0$$

$$m(m+1) + 5(m+1) = 0$$

$$(m+1)(m+5) = 0$$

$m = -1$, $m = -5$ [Roots are Real & Distinct]

So, $y_c = C_1e^{-t} + C_2e^{-5t}$

as equ(A) is

$$(m^2+6m+5)y = e^{5t}$$

So $y_p = \frac{1}{m^2+6m+5}(e^{5t})$

$m = 5$
This will for only exponents like x^5

$$y_p = \frac{e^{5t}}{5^2 + 6(5) + 5} = \frac{e^{5t}}{60}$$

$$y = y_c + y_p$$

$$y = C_1e^{-t} + C_2e^{-5t} + \frac{e^{5t}}{60}$$

Re-substitution

$$y = C_1(e^t)^{-1} + C_2(e^t)^{-5} + \frac{(e^t)^5}{60}$$

$$y = C_1x^{-1} + C_2x^{-5} + \frac{x^5}{60}$$

$$y = \frac{C_1}{x^1} + \frac{C_2}{x^5} + \frac{x^5}{60}$$

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② $x^2 \frac{d^2}{dx^2} y - 2x \frac{d}{dx} y - 4y = x^4$

let $x = e^t$
$\ln x = t$
$\frac{d}{dx} = D$
$x.D = m$
$x^2.D^2 = m(m-1)$

$(x^2.D^2 - 2x.D - 4)y = x^4$

$[m(m-1) - 2m - 4]y = e^{4t}$

$[m^2 - m - 2m - 4]\, y = e^{4t}$

$[m^2 - 3m - 4]y = e^{4t} \rightarrow$ equ(1)

$(m^2 - 3m - 4)$ is an auxilary Equation, So

$m^2 - 3m - 4 = 0$

$m^2 + m - 4m - 4 = 0$

$m(m+1) - 4(m+1) = 0$

$(m+1)(m-4) = 0$

$m = -1$, $m = 4$ [Roots are real & Distinct]

So, $y_c = C_1 e^{4t} + C_2 e^{-t}$

As equ(1) is

$[m^2 - 3m - 4]\, y = e^{4t}$

$y_p = \frac{1}{m^2 - 3m - 4}(e^{4t})$ $\therefore m = 4$

$y_p = \frac{e^{4t}}{4^2 - 3(4) - 4}$

$y_p = \frac{e^{4t}}{16-16} = \infty$

$y_p =$ undefined.

As $y_p = \infty$ after putting value of m in equation, so

$y_p = \frac{(e^{4t})(t)}{2m-3}$ ⟶ [Multiply with "t"]
⟶ [take derivative of Equation]

$y_p = \frac{t.e^{4t}}{2(4)-3} = \frac{t.e^{4t}}{5}$ $\therefore m = 4$

General Solution

$y = y_c + y_p$

$y = C_1 e^{4t} + C_2 e^{-t} + \frac{t.e^{4t}}{5}$

$y = c_1 (e^t)^4 + C_2 (e^t)^{-1} + \frac{t.(e^t)^4}{5}$

$y = c_1 x^4 + \frac{c_2}{x} + \frac{\ln x\,(x^4)}{5}$

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(3) $x^2\frac{d^2}{dx^2}y + x\frac{d}{dx}y + 4y = \sin(\log x^2)$

let $x = e^t$

$\log x = t$

$\frac{d}{dx} = D$

$x.D = m$

$x^2.D^2 = m(m-1)$

$[x^2D^2 + xD + 4]y = \sin(2\log x)$

$[m(m-1) + m + 4]y = \sin(2t)$

$[m^2 - m + m + 4]y = \sin[2t]$

$[m^2+4]y = \sin[2t] \longrightarrow$ equ(1)

$m^2+4 = 0$ [Auxilary Equation]

$m^2 = -4 = 4i^2$

$m = 0 \pm 2i$

$\alpha \quad \beta$

$y_c = e^{\alpha x}[C_1\cos\beta t + C_2\sin\beta t]$

$y_c = e^{0x}[C_1\cos 2t + C_2\sin 2t]$

$y_c = C_1\cos 2t + C_2\sin 2t$

As equ(1) is

$[m^2+4]y = \sin(2t)$

$y_p = \frac{\sin(2t)}{m^2+4}$

$y_p = \frac{\sin 2t}{m^2+4}$ (α)

As $m^2 = -\alpha^2$

~~So, $m^2 = -4$~~

we can only put value of m^2 in equ like $y_p = \frac{\sin 2t}{-4+4}$

$y_p = \infty$

As $y_p = \infty$ after putting value of "m^2"

So, $y_p = \frac{t\sin 2t}{2m}$

(Method 1)

$y_p = \left(\frac{t}{2}\right)\left(\frac{1}{m}\right)\sin 2t$

$y_p = \frac{t}{2}\left(\int \sin 2t\right) \quad \because m = D = \frac{d}{dx}, \ \frac{1}{m} = \int$

$y_p = \left(\frac{t}{2}\right)\left(\frac{-\cos 2t}{2}\right)$

$y_p = \frac{-t.\cos 2t}{4}$

Method (2)

$y_p = \left(\frac{t}{2}\right)\left(\frac{1}{m}\right)\sin 2t$

$y_p = \left(\frac{t}{2}\right)\left(\frac{1}{m^2}\right)m\sin 2t$

$y_p = \left(\frac{t}{2}\right)\left(\frac{1}{-4}\right)\frac{d}{dx}\sin 2t$

$y_p = -\frac{t}{8}\cos 2t\,(2)$

$y_p = -t/4\,(\cos 2t)$

General Solution

$y = y_c + y_p$

$y = C_1\cos 2t + C_2\sin 2t - \frac{t\cos 2t}{4}$

$y = C_1\cos 2(\log x) + C_2\sin 2(\log x) - \frac{\log x[\cos 2(\log x)]}{4}$

$y = C_1\cos(\log x^2) + C_2\sin(\log x^2) - \frac{\log x[\cos(\log x^2)]}{4}$

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(4) $x^3 \frac{d^3}{dx^3} y + 2x^2 \frac{d^2}{dx^2} y + 2y = 10x + \frac{10}{x}$

let $x = e^t$

$\ln x = t$

$d/dx = D$

$x.D = m$

$x^2.D^2 = m(m-1) = m^2 - m$

$x^3.D^3 = m(m-1)(m-2)$

$= (m^2-m)(m-2)$

$= m^3 - 2m^2 - m^2 + 2m$

$= m^3 - 3m^2 + 2m$

$[x^3.D^3 + 2x^2.D^2 + 2]y = 10e^t + 10e^{-t}$

$[m(m-1)(m-2) + 2m(m-1) + 2]y = 10e^t + 10e^{-t}$

$[m^3 - 3m^2 + \cancel{2m} + 2m^2 - \cancel{2m} + 2]y = 10e^t + 10e^{-t}$

$[m^3 - m^2 + 2]y = 10e^t + 10e^{-t} \rightarrow$ equ(1)

$m^3 - m^2 + 2 = 0$ [Auxilary Equation]

$m = -1, \ 1 \pm i$ [Roots are real & imagenery]

$y_c = c_1 e^{-t} + e^t[c_2 \cos t + c_3 \sin t]$

As equ(1)

$[m^3 - m^2 + 2]y = 10e^t + 10e^{-t}$

$y_p = \frac{10e^t + 10e^{-t}}{m^3 - m^2 + 2}$

$y_p = \frac{10e^t}{m^3 - m^2 + 2} + \frac{10e^{-t}}{m^3 - m^2 + 2}$

$y_p = \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{(10e^{-t})t}{3m^2 - 2m}$

This part give ∞ after putting value of "m" so, we take ~~derivative of equation~~

$y_p = \frac{10}{2}e^t + \frac{10t.e^{-t}}{3(-1)^2 - 2(-1)}$

$y_p = 5e^t + 2te^{-t}$

General solution

$y = y_c + y_p$

$y = c_1 e^{-t} + e^t[c_2 \cos t + c_3 \sin t] + 5e^t + 2te^{-t}$

$y = \frac{c_1}{e^t} + e^t[c_2 \cos t + c_3 \sin t] + 5e^t + \frac{2t}{e^t}$ $\qquad \because \ln x = t$

$y = \frac{c_1}{x} + x[c_2 \cos(\ln x) + c_3 \sin(\ln x)] + 5x + \frac{2 \ln x}{x}$

$y = \frac{c_1}{x} + c_2 x.\cos(\ln x) + c_3 x \sin(\ln x) + 5x + \frac{\ln x^2}{x}$

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