**Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSM-A, SP20-BSE-A & B).**

## Change of basis and Transition Matrix:

**Exercise 4.8:- Question 1-27; Question 1-15(very simple) ; Question 15 to 19 (similar)**

**Question 20 to 23 (similar); Question 24, 25, 26, 27 (similar).**

### ■ Coordinates

If $V$ is an $n$-dimensional vector space, we know that $V$ has a basis $S$ with $n$ vectors in it; thus far we have not paid much attention to the order of the vectors in $S$. However, in the discussion of this section we speak of an **ordered basis** $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $V$; thus $S_1 = \{\mathbf{v}_2, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a different ordered basis for $V$.

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered basis for the $n$-dimensional vector space $V$, then by Theorem 4.8 every vector $\mathbf{v}$ in $V$ can be uniquely expressed in the form

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n,$$

where $a_1, a_2, \ldots, a_n$ are real numbers. We shall refer to

$$[\mathbf{v}]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

as the **coordinate vector of v with respect to the ordered basis** $S$. The entries of $[\mathbf{v}]_S$ are called the **coordinates of v with respect to** $S$.

**Example 1: Given $S = \left\{e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is set of standard basis for $R^3$. Find coordinates of $v = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$ with respect to $S$. OR $[v]_S = ?$**

**Solution:-** $v = a_1e_1 + a_2e_2 + a_3e_3$

$$\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

**Coordinates of $v$ w.r.t. $S$**=$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$

**Next example shows the cost we have to bear in absence of standard basis**

**Examle 2(LC) : Given** $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} \right\}$ *be ordered basis.*

**Find coordinates of** $v = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$ **with respect to** $T$. **OR** $[v]_T = ?$

**Solution:** Consider $v = a_1 v_1 + a_2 v_2 + a_3 v_3$ ... ... ... (1), Our goal is to find scalars $a_1, a_2$ **and** $a_3$.

$$(1) \Rightarrow \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_1 \\ -a_1 \end{bmatrix} + \begin{bmatrix} 6a_2 \\ 4a_2 \\ 2a_2 \end{bmatrix} + \begin{bmatrix} 4a_3 \\ -a_3 \\ 8a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} a_1 + 6a_2 + 4a_3 \\ 2a_1 + 4a_2 - a_3 \\ -a_1 + 2a_2 + 8a_3 \end{bmatrix}$$

Equating both sides we get,

$$a_1 + 6a_2 + 4a_3 = 9 \;\; ; \; 2a_1 + 4a_2 - a_3 = 2 \;\; ; \;\; -a_1 + 2a_2 + 8a_3 = 7$$

**Observe:** The problem of **linear combination** boils down to a problem of non-homogeneous system of linear equations. I believe you can find scalars using Gauss-Elimination method. Here we go.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 2 & 4 & -1 & 2 \\ -1 & 2 & 8 & 7 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 0 & -8 & -9 & -16 \\ 0 & 8 & 12 & 16 \end{array}\right]$$

$$R_3 + R_2 \sim \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 0 & -8 & -9 & -16 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{matrix} \dfrac{R_2}{-8} \\ \dfrac{R_3}{3} \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 6 & 4 & 9 \\ 0 & 1 & 9/8 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now rewrite this equivalent simple system,

$$a_3 = 0; \;\; a_2 + \frac{9}{8}a_3 = 2\,; \; a_1 + 6a_2 + 4a_3 = 9$$

Using backward substitution, we get $a_3 = 0\,;\;\; a_2 = 2\,;\;\; a_1 = -3$

Hence $v = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$.

$$[v]_T = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}_T = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

**Example 3: (a)** $S = \{e_1 = t, e_2 = 1\}$ **standard basis for vector space** $P_1(t)$.

**Find coordinates of** $v = 5t - 2$**. OR** $[v]_S = ?$

**Solution:**

$$v = 5(t) - 2(1) = a_1 e_1 + a_2 e_2$$

**Coordinates of** $v$ **w.r.t.** $S$=$[v]_S = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

**(b)** $T = \{v_1 = t + 1, v_2 = t - 1\}$ **ordered basis for vector space** $P_1(t)$.

$v = 5t - 2$ **; Find coordinates of** $v$**, OR** $[v]_T = ?$

$$v = a_1 v_1 + a_2 v_2 - - - - (1)$$

$$5t - 2 = a_1(t + 1) + a_2(t - 1)$$

$$5t - 2 = (a_1 + a_2)t + (a_1 - a_2)$$

**Equating coefficients of like powers**

$$a_1 + a_2 = 5 \quad ; \quad a_1 - a_2 = -2$$

**Coordinates of** $v$ **w.r.t.** $T$=$[v]_T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \end{bmatrix}$

## ■ Transition Matrices

We now look at the relationship between two coordinate vectors for the same vector $\mathbf{v}$ with respect to different bases. Thus, let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be two ordered bases for the $n$-dimensional vector space $V$. If $\mathbf{v}$ is any vector in $V$, then

$S = \{v_1, v_2, v_3\}$ **;** $T = \{w_1, w_2, w_3\}$

$v = c_1 w_1 + c_2 w_2 + c_3 w_3 \quad - - - (1)$; $[v]_T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

**Write (1) w.r.t. S basis;** $[v]_S = [c_1 w_1 + c_2 w_2 + c_3 w_3]_S$

$$[v]_S = [c_1 w_1]_S + [c_2 w_2]_S + [c_3 w_3]_S$$

$$[v]_S = c_1 [w_1]_S + c_2 [w_2]_S + c_3 [w_3]_S$$

$$[v]_S = [[w_1]_S \quad [w_2]_S \quad [w_3]_S]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [?]\ [v]_T$$

$$[v]_S = P_{S\leftarrow T}[v]_T$$

**i.e.** $[?] = P_{S\leftarrow T} = [[w_1]_S \quad [w_2]_S \quad [w_3]_S]$

**Recall,** $[v]_S = ? = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ **Implies** $a_1v_1 + a_2v_2 + a_3v_3 = v ----(*)$

$[w_1]_S = ? = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ **Implies** $a_1v_1 + a_2v_2 + a_3v_3 = w_1 \quad ----(1)$

$[w_2]_S = ? = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ **Implies** $b_1v_1 + b_2v_2 + b_3v_3 = w_2 ----(2)$

$[w_3]_S = ? = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ **Implies** $d_1v_1 + d_2v_2 + d_3v_3 = w_3 ----(3)$

$$[?] = P_{S\leftarrow T} = [[w_1]_S \quad [w_2]_S \quad [w_3]_S]$$

$$[?] = P_{S\leftarrow T} = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

$$P_{S\leftarrow T} = [v_1 \quad v_2 \quad v_3 \quad | \quad w_1 \quad | \quad w_2 \quad | \quad w_3]$$

**Formula (1)** $[v]_S = P_{S\leftarrow T}[v]_T$ **(Transition from T to S basis.)**

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**Formula (2)** $[v]_T = Q_{T\leftarrow S}[v]_S$ **(Transition from S to T basis.)**

$S = \{v_1, v_2, v_3\}\ ;\ T = \{w_1, w_2, w_3\}$

$v = c_1v_1 + c_2v_2 + c_3v_3 \quad --- (1);$ $[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

**Write (1) w.r.t. T basis;** $[v]_T = [c_1v_1 + c_2v_2 + c_3v_3]_T$

$$[v]_T = [c_1v_1]_T + [c_2v_2]_T + [c_3v_3]_T$$

$$[v]_T = c_1[v_1]_T + c_2[v_2]_T + c_3[v_3]_T$$

$$[v]_T = [[v_1]_T \quad [v_2]_T \quad [v_3]_T]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [?]\,[v]_S$$

$$[v]_T = Q_{T\leftarrow S}[v]_S$$

i.e. $[?] = Q_{T\leftarrow S} = [[v_1]_T \quad [v_2]_T \quad [v_3]_T]$

$[v_1]_T = ? = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ Implies $v_1 = a_1 w_1 + a_2 w_2 + a_3 w_3 \quad ----(1)$

$[v_2]_T = ? = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Implies $v_2 = b_1 w_1 + b_2 w_2 + b_3 w_3 ----(2)$

$[v_3]_T = ? = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ Implies $v_3 = d_1 w_1 + d_2 w_2 + d_3 w_3 ----(3)$

$Q_{T\leftarrow S} = [w_1 \quad w_2 \quad w_3 \quad | \quad v_1 \quad | \quad v_2 \quad | \quad v_3]$

**Alternate formula of Formula (2):- We know**

$$[v]_S = P_{S\leftarrow T}[v]_T$$

$$P^{-1}{}_{S\leftarrow T}[v]_S = P^{-1}{}_{S\leftarrow T}P_{S\leftarrow T}[v]_T$$

$$P^{-1}{}_{S\leftarrow T}[v]_S = [v]_T \text{ (Transition from S to T basis.)}$$

$$[v]_T = Q_{T\leftarrow S}[v]_S$$

$$Q_{T\leftarrow S} = P^{-1}{}_{S\leftarrow T}$$

**How to find transition matrix , We explain through example**

**Example 1:-** $S = \{v_1, v_2, v_3\}$ and $T = \{w_1, w_2, w_3\}$; $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$.

$w_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, w_3 = \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}$ **Find transition matrices $P_{S\leftarrow T}$ and $Q_{T\leftarrow S}$ .**

**Solution:** $[v_1 \quad v_2 \quad v_3 \quad | \quad w_1 \quad | \quad w_2 \quad | \quad w_3] \to \left[\begin{array}{ccc|c|c|c} 1 & 6 & 4 & 3 & 3 & -2 \\ 2 & 4 & 0 & -2 & 1 & 4 \\ -1 & 2 & 8 & 1 & 2 & 10 \end{array}\right]$

$RREF \sim \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -5 & -3/4 & 12 \\ 0 & 1 & 0 & 2 & 5/8 & -5 \\ 0 & 0 & 1 & -1 & 0 & 4 \end{array}\right]$

$$\Rightarrow \quad P_{S\leftarrow T} = [[w_1]_S \quad [w_2]_S \quad [w_3]_S] = \begin{bmatrix} -5 & -3/4 & 12 \\ 2 & 5/8 & -5 \\ -1 & 0 & 4 \end{bmatrix}$$

**Now to find $Q_{T\leftarrow S}$ we proceed as**

$$[v_1 \quad v_2 \quad v_3 \quad | \quad w_1 \quad | \quad w_2 \quad | \quad w_3] \to \left[\begin{array}{ccc|c|c|c} 3 & 3 & -2 & 1 & 6 & 4 \\ -2 & 1 & 4 & 2 & 4 & -1 \\ 1 & 2 & 10 & -1 & 2 & 8 \end{array}\right]$$

$$RREF \sim \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -10/11 & -12/11 & 15/11 \\ 0 & 1 & 0 & 12/11 & 32/11 & 4/11 \\ 0 & 0 & 1 & -5/22 & -3/11 & 13/22 \end{array}\right]$$

$$Q_{T\leftarrow S} = [[v_1]_T \quad [v_2]_T \quad [v_3]_T] = \begin{bmatrix} -10/11 & -12/11 & 15/11 \\ 12/11 & 32/11 & 4/11 \\ -5/22 & -3/11 & 13/22 \end{bmatrix}$$

**Example2:- Let** $Q_{T\leftarrow S} = \begin{bmatrix} -3 = a_1 & 2 = b_1 \\ 2 = a_2 & -1 = b_2 \end{bmatrix} = [[v_1]_T \quad [v_2]_T]$

$T = \{w_1 = t - 1, w_2 = t + 1\}$. **Find** $S\{v_1, v_2\}$.

**Solution:** $v_1 = a_1 w_1 + a_2 w_2$ **then** $[v_1]_T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$v_1 = -3w_1 + 2w_2 = -3(t-1) + 2(t+1) = -t + 5$$

$v_2 = b_1 w_1 + b_2 w_2$ **then** $[v_2]_T = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$v_2 = 2w_1 - 1w_2 = 2(t-1) - 1(t+1) = t - 3$$

**16.** Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

and

$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

be ordered bases for $R^3$. Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -1 \\ 8 \\ -2 \end{bmatrix}.$$

Follow the directions for (a) through (f) in Exercise 15.

**17.** Let $S = \{t^2 + 1, t - 2, t + 3\}$ and $T = \{2t^2 + t, t^2 + 3, t\}$ be ordered bases for $P_2$. Let $\mathbf{v} = 8t^2 - 4t + 6$ and $\mathbf{w} = 7t^2 - t + 9$. Follow the directions for (a) through (f) in Exercise 15.

**(a)** Find the coordinate vectors of **v** and **w** with respect to the basis $T$.

**(b)** What is the transition matrix $P_{S \leftarrow T}$ from the $T$- to the $S$-basis?

**(c)** Find the coordinate vectors of **v** and **w** with respect to $S$, using $P_{S \leftarrow T}$.

**(d)** Find the coordinate vectors of **v** and **w** with respect to $S$ directly.

**(e)** Find the transition matrix $Q_{T \leftarrow S}$ from the $S$- to the $T$-basis.

**(f)** Find the coordinate vectors of **v** and **w** with respect to $T$, using $Q_{T \leftarrow S}$. Compare the answers with those of (a).

Exercise 4.8

Q17 (1)

$S = \{t^2+1, t-2, t+3\}$

$T = \{2t^2+t, t^2+3, t\}$ ordered basis for $P_2$

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$\underline{v}_1 = t^2+1$, $\underline{v}_2 = t-2$, $\underline{v}_3 = t+3$

$\underline{w}_1 = 2t^2+t$, $\underline{w}_2 = t^2+3$, $\underline{w}_3 = t$

We know $P_2(t) = \{a_2t^2 + a_1t + a_0 : a_i \in \mathbb{R}\}$

= All Polynomials of degree less or equal to 2

We know $P_2(t) \cong \mathbb{R}^3$ using isomorphism

$f: P_2 \to \mathbb{R}^3$; $f(a_2t^2+a_1t+a_0) = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$ Applicable in whole Linear Algebra Course

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Solution 1 $\Rightarrow$ $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$\underline{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\underline{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\underline{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now Completely Proceed like Q16

Q17 $S = \{ t^2+1, t-2, t+3 \}$, $T = \{ 2t^2+t, t^2+3, t \}$

$$\underset{\underline{v}_1}{t^2+1},\ \underset{\underline{v}_2}{t-2},\ \underset{\underline{v}_3}{t+3} \qquad \underset{\underline{w}_1}{2t^2+t},\ \underset{\underline{w}_2}{t^2+3},\ \underset{\underline{w}_3}{t}$$

(a) given $\underline{v} = 8t^2-4t+6$, $\underline{w} = 7t^2-t+9$

$[\underline{v}]_T$, $[\underline{w}]_T = ?$

$$a_1\underline{w}_1 + a_2\underline{w}_2 + a_3\underline{w}_3 = \underline{v}$$

$$a_1(2t^2+t) + a_2(t^2+3) + a_3 t = 8t^2-4t+6$$

$$(2a_1+a_2)t^2 + (a_1+a_3)t + 3a_2 = 8t^2-4t+6$$

Equating co-efficients of like powers, we get

$$\left.\begin{aligned} 2a_1 + a_2 &= 8 \quad \text{--- (1)} \\ a_1 + a_3 &= -4 \quad \text{--- (2)} \\ 3a_2 &= 6 \quad \text{--- (3)} \end{aligned}\right\} \text{Simple Non-homogeneous System.}$$

(3) $\Rightarrow$ $\boxed{a_2 = 2}$ put in (1) $\boxed{a_1 = 3}$ put in (2) $\boxed{a_3 = -7}$

$$[\underline{v}]_T = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix} = \text{Coordinate vector of } \underline{v} \text{ w.r.t. } T \text{ basis.}$$

$\rightarrow$ Prove yourself $[\underline{w}]_T = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$

(b) The transition matrix $P_{S\leftarrow T}$ from $T$ to $S$ is matrix with columns as follows

$$P_{S\leftarrow T} = \left[ [\underline{w}_1]_S \ [\underline{w}_2]_S \ [\underline{w}_3]_S \right], \text{ where}$$

$$\begin{aligned} \underline{w}_1 &= a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 \\ \underline{w}_2 &= b_1\underline{v}_1 + b_2\underline{v}_2 + b_3\underline{v}_3 \\ \underline{w}_3 &= c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 \end{aligned} \Rightarrow \begin{array}{cccccc} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \end{array}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & -2 & 3 & 0 & 3 & 0 \end{array}\right]$$

$$\begin{array}{cccccc} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 & \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \end{array}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & -2 & 3 & 0 & 3 & 0 \end{array}\right]$$

$$\underset{\sim}{RREF} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2/5 & 3/5 \\ 0 & 0 & 1 & 0 & 2/5 & 2/5 \end{array}\right]$$

$$P_{S\leftarrow T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2/5 & 3/5 \\ 0 & 2/5 & 2/5 \end{bmatrix}$$

(c) Find $[\underline{v}]_S$ & $[\underline{w}]_S$ using $P_{S\leftarrow T}$

We know $[\underline{v}]_S = P_{S\leftarrow T}[\underline{v}]_T$ ($[\underline{v}]_T$ is computed in Part (a))

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2/5 & 3/5 \\ 0 & 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}$$

likewise $[\underline{w}]_S = P_{S\leftarrow T}[\underline{w}]_T$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2/5 & 3/5 \\ 0 & 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

(d) $[\underline{v}]_S$ , $[\underline{w}]_S = ?$ Directly

$\underline{v} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3$

$8t^2 - 4t + 6 = a_1(t^2+1) + a_2(t-2) + a_3(t+3)$

$8t^2 - 4t + 6 = a_1 t^2 + (a_2 + a_3)t + (a_1 - 2a_2 + 3a_3)$

Equating Co-efficients of like Powers, we get

$\boxed{a_1 = 8}$ — ①  $\boxed{a_2 + a_3 = -4}$ — ②  $\boxed{a_1 - 2a_2 + 3a_3 = 6}$ — ③

This Non-homogeneous system is simple and can easily be solved.

Put value of $a_1$ in ③ we have

$$-2a_2 + 3a_3 = -2 \quad \text{— ④}$$

Solve ② & ④

$$2a_2 + 2a_3 = -8$$
$$-2a_2 + 3a_3 = -2$$

$$5a_3 = -10 \Rightarrow \boxed{a_3 = -2}$$

and $\boxed{a_2 = -2}$

$[\underline{v}]_S = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}$ Same as in Part ©

Similarly Verify yourself

$[\underline{w}]_S = ? \Rightarrow \boxed{a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3 = \underline{w}}$

$$\downarrow$$

$[\underline{w}]_S = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$

(e) The transition matrix $Q_{T\leftarrow S}$ is matrix defined as $Q_{T\leftarrow S} = \left[ [\underline{v_1}]_T \; [\underline{v_2}]_T \; [\underline{v_3}] \right]$

which can be achieved by solving following three non-homogeneous simultaneously.

$$\underline{v_1} = a_1 \underline{w_1} + a_2 \underline{w_2} + a_3 \underline{w_3}$$

$$\underline{v_2} = b_1 \underline{w_1} + b_2 \underline{w_2} + b_3 \underline{w_3}$$

$$\underline{v_3} = c_1 \underline{w_1} + c_2 \underline{w_2} + c_3 \underline{w_3}$$

$$\Rightarrow \begin{array}{c} \begin{array}{cccccc} \underline{w_1} & \underline{w_2} & \underline{w_3} & \underline{v_1} & \underline{v_2} & \underline{v_3} \end{array} \\ \left[\begin{array}{ccc|c|c|c} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 1 & -2 & 3 \end{array}\right] \end{array}$$

$$\underline{\text{RREF}} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{-2}{3} & 1 \\ 0 & 0 & 1 & \frac{-1}{3} & 2/3 & \frac{3}{2} \end{array}\right]$$

$$Q_{T\leftarrow S} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & \frac{-2}{3} & 1 \\ \frac{-1}{3} & 2/3 & 3/2 \end{pmatrix}$$

(5)

(f) Find $[\underline{v}]_T = ?$ $[\underline{w}]_T = ?$
using $Q_{T\leftarrow S}$

We know $[\underline{v}]_T = Q_{T\leftarrow S}[\underline{v}]_S$

$$[\underline{v}]_T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{2}{3} & 1 \\ -\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}$$

Similarly

$$[\underline{w}]_T = Q_{T\leftarrow S}[\underline{w}]_S$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{2}{3} & 1 \\ -\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

**19.** Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

and

$$T = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

be ordered bases for $M_{22}$. Let

$$\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Follow the directions for (a) through (f) in Exercise 15.

**Solution Part(b):**

$$[v_1 \quad v_2 \quad v_3 \quad v_4 \mid w_1 \mid w_2 \mid w_3 \mid w_4] = \left[\begin{array}{cccc|c|c|c|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$RREF\sim\left[\begin{array}{cccc|c|c|c|c}1&0&0&0&1&0&0&1\\0&1&0&0&1/3&2/3&-2/3&0\\0&0&1&0&1/3&-1/3&1/3&0\\0&0&0&1&-1/3&1/3&2/3&0\end{array}\right]$$

$$P_{S\leftarrow T}=\begin{bmatrix}1&0&0&1\\1/3&2/3&-2/3&0\\1/3&-1/3&1/3&0\\-1/3&1/3&2/3&0\end{bmatrix}$$

Part (d): $\left[\begin{array}{cccc|c|c|c|c}1&0&0&1&1&0&0&0\\1&0&0&0&0&1&2&0\\0&1&0&0&0&1&0&1\\0&0&1&0&0&0&1&1\end{array}\right]$

$$RREF\sim\left[\begin{array}{cccc|c|c|c|c}1&0&0&0&0&1&2&0\\0&1&0&0&0&1&0&1\\0&0&1&0&0&0&1&1\\0&0&0&1&1&-1&-2&0\end{array}\right]$$

$$Q_{T\leftarrow S}=\begin{bmatrix}0&1&2&0\\0&1&0&1\\0&0&1&1\\1&-1&-2&0\end{bmatrix}$$

**HOME WORK part (Complete question 19)**

**22.** Let

$$S=\{[-1\quad 2\quad 1],[0\quad 1\quad 1],[-2\quad 2\quad 1]\}$$

and

$$T=\{[-1\quad 1\quad 0],[0\quad 1\quad 0],[0\quad 1\quad 1]\}$$

be ordered bases for $R_3$. If $\mathbf{v}$ is in $R_3$ and

$$[\mathbf{v}]_S=\begin{bmatrix}2\\0\\1\end{bmatrix},$$

determine $[\mathbf{v}]_T$.

**Solution: (Alternate solution )** Since $R_3\cong R^3$; $S=\left\{v_1=\begin{bmatrix}-1\\2\\1\end{bmatrix},v_2=\begin{bmatrix}0\\1\\1\end{bmatrix},v_3=\begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}$;

$$T=\left\{w_1=\begin{bmatrix}-1\\1\\0\end{bmatrix},w_2=\begin{bmatrix}0\\1\\0\end{bmatrix},w_3=\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

$$[v]_S=\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$$

We know coordinates of $v$ w.r.t. S basis can be written as

$$a_1v_1 + a_2v_2 + a_3v_3 = v$$

$$2\begin{bmatrix}-1\\2\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\1\end{bmatrix} + 1\begin{bmatrix}-2\\2\\1\end{bmatrix} = v$$

$$\begin{bmatrix}-4\\6\\3\end{bmatrix} = v$$

$$[v]_T = ? = \text{coordinates of } v \text{ w.r.t. T basis}$$

$$b_1w_1 + b_2w_2 + b_3w_3 = v \quad ----(1)$$

$$[A|b] = \left[\begin{array}{ccc|c}-1 & 0 & 0 & -4\\1 & 1 & 1 & 6\\0 & 0 & 1 & 3\end{array}\right] \; RREF \approx \left[\begin{array}{ccc|c}1 & 0 & 0 & 4\\0 & 1 & 0 & -1\\0 & 0 & 1 & 3\end{array}\right]$$

RREF .... $[v]_T = \begin{bmatrix}4\\-1\\3\end{bmatrix}$

**24.** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be ordered bases for $R^3$, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Suppose that the transition matrix from $T$ to $S$ is

$$P_{S\leftarrow T} = \begin{bmatrix}1 & 1 & 2\\2 & 1 & 1\\-1 & -1 & 1\end{bmatrix}.$$

Determine $T$.

**Solution:** $P_{S\leftarrow T} = [[w_1]_S \quad [w_2]_S \quad [w_3]_S] = \begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\2 & 1 & 1\\-1 & -1 & 1\end{bmatrix}$

$$a_1v_1 + a_2v_2 + a_3v_3 = w_1 \Rightarrow 1\begin{bmatrix}1\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\0\end{bmatrix} - 1\begin{bmatrix}0\\0\\1\end{bmatrix} = w_1 \Rightarrow \begin{bmatrix}3\\2\\0\end{bmatrix} = w_1$$

$$b_1v_1 + b_2v_2 + b_3v_3 = w_2 \Rightarrow 1\begin{bmatrix}1\\0\\1\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} - 1\begin{bmatrix}0\\0\\1\end{bmatrix} = w_2 \Rightarrow \begin{bmatrix}2\\1\\0\end{bmatrix} = w_2$$

$$c_1v_1 + c_2v_2 + c_3v_3 = w_3 \Rightarrow 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix} = w_3 \Rightarrow \begin{bmatrix}3\\1\\3\end{bmatrix} = w_3.$$

**27.** Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be ordered bases for $P_1$, where

$$\mathbf{w}_1 = t - 1, \quad \mathbf{w}_2 = t + 1.$$

If the transition matrix from $T$ to $S$ is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, determine $S$.

**Solution: Recall $P_n(t) \cong R^{n+1}$, implies $P_1(t) \cong R^2$**

**Given $T = \left\{w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ and $P_{S\leftarrow T} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.**

**To determine basis set $S = \{v_1, v_2\}$ we need $Q_{T\leftarrow S}$.**

**We know $Q_{T\leftarrow S} = P_{S\leftarrow T}^{-1}$.**

$$P_{S\leftarrow T}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$Q_{T\leftarrow S} = [[v_1]_T \quad [v_2]_T\,] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$a_1 w_1 + a_2 w_2 = v_1 \Rightarrow \ -3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 \Rightarrow \begin{bmatrix} -1 \\ 5 \end{bmatrix} = v_1 = -t + 5$$

$$b_1 w_1 + b_2 w_2 = v_2 \Rightarrow \ 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 \Rightarrow \begin{bmatrix} 1 \\ -3 \end{bmatrix} = v_1 = t - 3$$