Example 21.11 Field of a line of charge

Positive electric charge Q is distributed uniformly along a line with length $2a$, lying along the y-axis between $y = -a$ and $y = +a$. (This might represent one of the charged rods in Fig. 21.1.) Find the electric field at point P on the x-axis at a distance x from the origin.

SOLUTION

IDENTIFY: As in Example 21.10, our target variable is the electric field due to a continuous distribution of charge.

SET UP: Figure 21.25 shows the situation. We need to find the electric field at P as a function of the coordinate x. The x-axis is the perpendicular bisector of the charged line, so as in Example 21.10 we will be able to make use of a symmetry argument.

EXECUTE: We divide the line charge into infinitesimal segments, each of which acts as a point charge; let the length of a typical segment at height y be dy. If the charge is distributed uniformly, the linear charge density λ at any point on the line is equal to $Q/2a$ (the total charge divided by the total length). Hence the charge dQ in a segment of length dy is

$$dQ = \lambda dy = \frac{Q dy}{2a}$$

The distance r from this segment to P is $(x^2 + y^2)^{1/2}$, so the magnitude of field dE at P due to this segment is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2} = \frac{Q}{4\pi\epsilon_0}\frac{dy}{2a(x^2 + y^2)}$$

We represent this field in terms of its x- and y-components:

$$dE_x = dE\cos\alpha \qquad dE_y = -dE\sin\alpha$$

We note that $\sin\alpha = y/(x^2 + y^2)^{1/2}$ and $\cos\alpha = x/(x^2 + y^2)^{1/2}$; combining these with the expression for dE, we find

$$dE_x = \frac{Q}{4\pi\epsilon_0}\frac{x\,dy}{2a(x^2 + y^2)^{3/2}}$$

$$dE_y = -\frac{Q}{4\pi\epsilon_0}\frac{y\,dy}{2a(x^2 + y^2)^{3/2}}$$

To find the total field components E_x and E_y, we integrate these expressions, noting that to include all of Q, we must integrate from $y = -a$ to $y = +a$. We invite you to work out the details of the integration; an integral table is helpful. The final results are

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Qx}{2a}\int_{-a}^{a}\frac{dy}{(x^2 + y^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{x\sqrt{x^2 + a^2}}$$

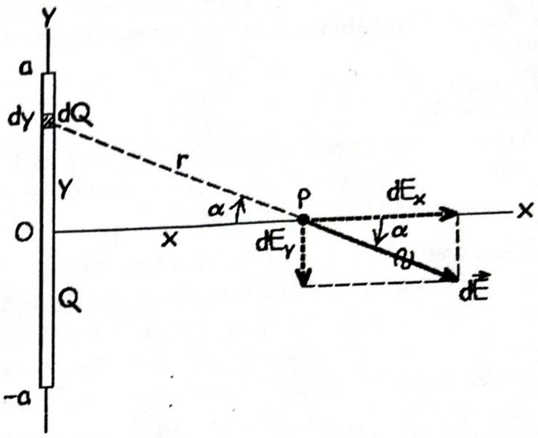
$$E_y = -\frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{a}\frac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$$

or, in vector form,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x\sqrt{x^2 + a^2}}\hat{\imath} \qquad (21.9)$$

EVALUATE: Using a symmetry argument as in Example 21.10, we could have guessed that E_y would be zero; if we place a positive test charge at P, the upper half of the line of charge pushes downward on it, and the lower half pushes up with equal magnitude.

21.25 Our sketch for this problem.



To explore our result, let's first see what happens in the limit that x is much larger than a. Then we can neglect a in the denominator of Eq. (21.9), and our result becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}\hat{\imath}$$

This means that if point P is very far from the line charge in comparison to the length of the line, the field at P is the same as that of a point charge. We found a similar result for the charged ring in Example 21.10.

To further explore our exact result for $\vec{E}$, Eq. (21.9), let's express it in terms of the linear charge density $\lambda = Q/2a$. Substituting $Q = 2a\lambda$ into Eq. (21.9) and simplifying, we get

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{x\sqrt{(x^2/a^2) + 1}}\hat{\imath} \qquad (21.10)$$

Now we can answer the question: What is $\vec{E}$ at a distance x from a *very* long line of charge? To find the answer we take the *limit* of Eq. (21.10) as a becomes very large. In this limit, the term x^2/a^2 in the denominator becomes much smaller than unity and can be thrown away. We are left with

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 x}\hat{\imath}$$

The field magnitude depends only on the distance of point P from the line of charge. So at any point P at a perpendicular distance r from the line in any direction, $\vec{E}$ has magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \qquad \text{(infinite line of charge)}$$

Thus the electric field due to an infinitely long line of charge is proportional to $1/r$ rather than to $1/r^2$ as for a point charge. The direction of $\vec{E}$ is radially outward from the line if λ is positive and radially inward if λ is negative.

There's really no such thing in nature as an infinite line of charge. But when the field point is close enough to the line, there's very little difference between the result for an infinite line and the real-life finite case. For example, if the distance r of the field point from the center of the line is 1% of the length of the line, the value of E differs from the infinite-length value by less than 0.02%.

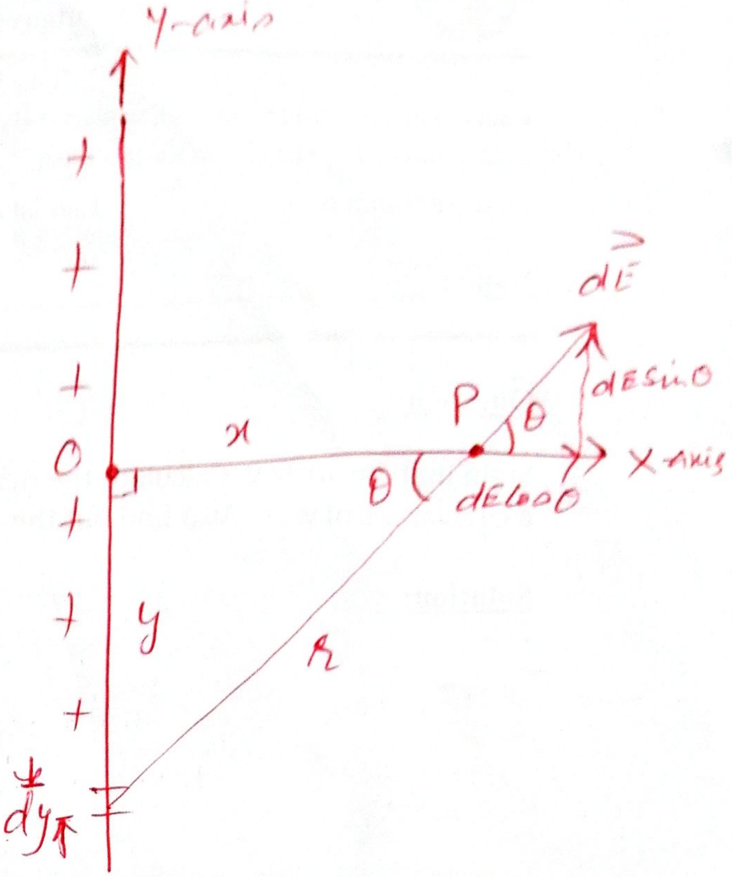
Charge $dq = \lambda dy$

linear density $= \lambda = \dfrac{dq}{dy}$

$$dE = \frac{1}{4\pi\epsilon_0}\,\frac{dq}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda dy}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda dy}{x^2+y^2}$$



$$dE_x = dE\cos\theta$$

$$dE_y = dE\sin\theta$$

$$E_y = \int_{-\infty}^{+\infty} dE_y = 0$$

So

$$E = \int_{-\infty}^{+\infty} dE_x$$

$$= \int_{-\infty}^{+\infty} dE\cos\theta = \int_{-\infty}^{+\infty} \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda dy}{(x^2+y^2)}\cdot\cos\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dy}{(x^2+y^2)}\cos\theta$$

$$= \frac{\lambda}{2\pi\epsilon_0}\int_0^{\infty}\frac{\cos\theta}{x^2+y^2}\,dy$$

let $y = x\tan\theta \Rightarrow dy = x\sec^2\theta\, d\theta$

$y\to 0,\ \theta\to 0,\quad y\to\infty,\ \theta = \pi/2$

$$E = \frac{\lambda}{2\pi\epsilon_0}\int_0^{\pi/2}\frac{x\sec^2\theta\, d\theta\cos\theta}{x^2+x^2\tan^2\theta} = \frac{\lambda}{2\pi\epsilon_0 x}\int_0^{\pi/2}\cos\theta\, d\theta$$

$$E = \frac{\lambda}{2\pi\epsilon_0 x}\left|\sin\theta\right|_0^{\pi/2} = \boxed{\frac{\lambda}{2\pi\epsilon_0 x} = E}$$