Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSE-A, FA20-BCS-A&B).

Objective of Lecture week4:- (Chapter3)

- **Determinants and their properties, Reduction to triangular form, Trace of Matrix.**
- **Adjoint of matrix, Cofactor Expansion and inverse of Matrix.**
- **Crammer's Rule to solve system of equations.**
- **Optional (Area of triangle, area of parallelogram, volume of parallelopiped)**

After studying this lecture, You are desired to do

Home Work: Do Questions 8-16 of Exercise 3.1, Questions 1-7, and 24-28 of Exercise 3.2,Questions 1-5, 7, 9-12 of Exercise 3.4,Questions 1-7 of Exercise 3.5,following link is extremely helpful in this regard.

https://www.slader.com/textbook/9780132296540-elementary-linear-algebra-with-applications-9th-edition/196/

Chapter 3:Determinants

Throughout this chapter, when we use term "Matrix", we mean "Square Matrix."

Certain important numbers (scalars) are associated with each matrix $A = [a_{ij}]_{n\times n}$ for exampleTrace of matrix and Determinant.

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- **The determinant of a 2×2 matrix, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, is the numberdet($A$)=$|A|$= $a_{11}a_{22} - a_{12}a_{21}$**
- **For a 1×1 matrix, say, A=[a_{11}], we define det (A)=a_{11}.**

- **The determinant of a 3×3 matrix, $A = \begin{bmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{bmatrix}$, is the number**

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

How formula for 3×3 determinant is attained? (very easy)

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}.$$

EXAMPLE 9 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

Evaluate $|A|$.

Solution

Substituting in (1), we find that

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = (1)(1)(2) + (2)(3)(3) + (3)(2)(1) - (1)(3)(1) - (2)(2)(2) - (3)(1)(3) = 6.$$

We could obtain the same result by using the easy method illustrated previously, as follows:

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 & 1 \end{array}$$

$$|A| = (1)(1)(2) + (2)(3)(3) + (3)(2)(1) - (3)(1)(3) - (1)(3)(1) - (2)(2)(2) = 6. \blacksquare$$

Warning The methods used for computing $\det(A)$ in Examples 7–9 do not apply for $n \geq 4$.

Exercise 3.1

Question No. 11(c) Evaluate

(c) $\det\left(\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}\right)$

Solution: $\boldsymbol{det}\left(\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{vmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{vmatrix}$ **Expand with R_1**

$$= 0\begin{vmatrix} 0 & 4 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 6 & 0 & 0 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 & 4 \\ 0 & 2 & 0 \\ 6 & 0 & 0 \end{vmatrix}$$

$= -3\begin{vmatrix} 0 & 0 & 4 \\ 0 & 2 & 0 \\ 6 & 0 & 0 \end{vmatrix}$ **Expand with R_1**

$$= -3\left(4\begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix}\right) = -3(4(-12)) = 144.$$

Question 14:

(b) $\det\left(\begin{bmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ 0 & 0 & t+1 \end{bmatrix}\right)$

16. For each of the matrices in Exercise 14, find values of t for which the determinant is 0.

Solution 14: $\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ 0 & 0 & t+1 \end{vmatrix}$ **Expand with C2**

Take care of sign with entry $a_{22} = t$

$$= t\begin{vmatrix} t-1 & 1 \\ 0 & t+1 \end{vmatrix} = t(t^2-1)$$

Solution 16: Given $\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0 \quad \Rightarrow \quad t(t^2-1) = 0$

Either $t = 0$ OR $(t^2-1) = 0 \;\Rightarrow\; t^2 = 1 \;\Rightarrow t = \pm 1.$

3.2 Properties of Determinants

Theorem 3.1 If A is a matrix, then $\det(A) = \det(A^T)$.

Theorem 3.2 If matrix B results from matrix A by interchanging two different rows (columns) of A, then $\det(B) = -\det(A)$.

Theorem 3.3 If two rows (columns) of A are equal, then $\det(A) = 0$.

Theorem 3.4 If a row (column) of A consists entirely of zeros, then $\det(A) = 0$.

Theorem 3.5 If B is obtained from A by multiplying a row (column) of A by a real number k, then $\det(B) = k\det(A)$.

An Important property (3.5.1):-

If each element of any row (or column) consists of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

$$\begin{vmatrix} a_1 + x & b_1 & c_1 \\ a_2 + y & b_2 & c_2 \\ a_3 + z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$$

Theorem 3.6 If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of the rth row (column) of A, k times the corresponding element of the sth row (column), $r \neq s$, of A, then $\det(B) = \det(A)$.

Explanation of Theorem 3.6:- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Matrix B is obtained from matrix A using row operation $R_1 + kR_3$,Then

$$B = \begin{bmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ **, Now consider**

$$det(B) = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ **Using property 3.5.1=**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{31} & ka_{32} & ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ **taking k common from R_1**

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ **Property two rows are identical.**

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k(0) = det(A)$$

Theorem 3.7 If a matrix $A = [a_{ij}]$ is upper (lower) triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$; that is, the determinant of a triangular matrix is the product of the elements on the main diagonal.

Example of Theorem 3.7:-

$$\begin{vmatrix} 3 & 2 & 1 & 9 & -6 \\ 0 & -7 & -5 & 7 & 3 \\ 0 & 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 & 10 \end{vmatrix} = 3(-7)(8)(-4)(10) = 6720$$

Theorem 3.8 If A is an $n \times n$ matrix, then A is nonsingular if and only if $\det(A) \neq 0$.

Corollary 3.1 If A is an $n \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det(A) = 0$.

Theorem 3.9 If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Corollary 3.2 If A is nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Exercise 3.2

Question No:

2. Compute the following determinants via reduction to triangular form or by citing a particular theorem or corollary:

(f) $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$

Solution: consider $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 1 & 2 & -5 & -6 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} R_2 - R_1$

$$= -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 2 & 0 & 1 & 4 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} R_{12} = -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & -4 & 11 & 16 \\ 0 & -1 & 9 & 12 \\ 0 & -14 & 51 & 72 \end{vmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 - 11R_1 \end{matrix}$$

$$= +\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & -1 & 9 & 12 \\ 0 & -4 & 11 & 16 \\ 0 & -14 & 51 & 72 \end{vmatrix} R_{23} = (-1)\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & 1 & -9 & -12 \\ 0 & -4 & 11 & 16 \\ 0 & -14 & 51 & 72 \end{vmatrix} \text{Take (-1) common from } R_2$$

$$= -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & 1 & -9 & -12 \\ 0 & 0 & -25 & -32 \\ 0 & 0 & -75 & -96 \end{vmatrix} \begin{matrix} R_3 + 4R_2 \\ R_4 + 14R_2 \end{matrix} = -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & 1 & -9 & -12 \\ 0 & 0 & -25 & -32 \\ 0 & 0 & 0 & 0 \end{vmatrix} R_4 - 3R_3$$

$= 1(1)(-25)(0) = 0$**; using property of triangular form.**

3. If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3$, find

$$\begin{vmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Solution: consider

$$\begin{vmatrix} a_1+2b_1-3c_1 & a_2+2b_2-3c_2 & a_3+2b_3-3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} 2b_1 & 2b_2 & 2b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} -3c_1 & -3c_2 & -3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ Using property 3.5.1.}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 2\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - 3\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{since two rows are identical.}$$

$$= 3 + 2(0) - 3(0) = 3.$$

5. If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4$, find

$$\begin{vmatrix} a_1 & a_2 & 4a_3-2a_2 \\ b_1 & b_2 & 4b_3-2b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3-c_2 \end{vmatrix}$$

Solution: consider

$$\begin{vmatrix} a_1 & a_2 & 4a_3-2a_2 \\ b_1 & b_2 & 4b_3-2b_2 \\ \frac{c_1}{2} & \frac{c_2}{2} & 2c_3-c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & 4a_3 \\ b_1 & b_2 & 4b_3 \\ \frac{c_1}{2} & \frac{c_2}{2} & \left(\frac{4}{2}\right)c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & -2a_2 \\ b_1 & b_2 & -2b_2 \\ \frac{c_1}{2} & \frac{c_2}{2} & -\left(\frac{2}{2}\right)c_2 \end{vmatrix} \text{ Using property 3.5.1.}$$

$$= 4\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_3}{2} \end{vmatrix} - 2\begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_2}{2} \end{vmatrix} \begin{matrix} \textit{Take 4 common from column 3} \\ \textit{Take} - 2 \textit{ common from column 3} \end{matrix}$$

$$= 4\left(\frac{1}{2}\right)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - 2\left(\frac{1}{2}\right)\begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_2 \end{vmatrix} \begin{matrix} \textit{Take } \frac{1}{2} \textit{ common from Row 3} \\ \textit{Take } \frac{1}{2} \textit{ common from Row 3} \end{matrix}$$

$$= 4\left(\frac{1}{2}\right)(4) - 2\left(\frac{1}{2}\right)(0) = 8 \text{ Since column 2 and column 3 are same.}$$

Example: **Using properties of determinants show**

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)$$

Solution: $L.H.S. = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} b^2+c^2+2bc & a^2 & bc \\ a^2+c^2+2ac & b^2 & ca \\ a^2+b^2+2ab & c^2 & ab \end{vmatrix}$

$$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ a^2+c^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix} C_1 - 2C_3 = \begin{vmatrix} b^2+c^2+a^2 & a^2 & bc \\ a^2+c^2+b^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} C_1 + C_2$$

$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$ Taking $(a^2+b^2+c^2)$ common from C_1.

(Alternate approach: You can expand above determinant using column1)

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & ca-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & -(a-b)(b+a) & c(a-b) \\ 0 & -(a-c)(c+a) & b(a-c) \end{vmatrix}$$

$= (a^2+b^2+c^2)(a-b)(a-c)\begin{vmatrix} 1 & a^2 & bc \\ 0 & -(b+a) & c \\ 0 & -(c+a) & b \end{vmatrix}$ by taking common $(a-b)$ from R_2 and $(a-c)$ from R_3.

$= (a^2+b^2+c^2)(a-b)(a-c)\begin{vmatrix} -(b+a) & c \\ -(c+a) & b \end{vmatrix}$ Expand with C_1

$= (a^2+b^2+c^2)(a-b)(a-c)[-b(b+a)+c(c+a)]$

$= (a^2+b^2+c^2)(a-b)(a-c)[-b^2-ab+c^2+ac]$

$= (a^2+b^2+c^2)(a-b)(a-c)[c^2-b^2+ac-ab]$

$= (a^2+b^2+c^2)(a-b)(a-c)[(c-b)(c+b)+a(c-b)]$

$= (a^2+b^2+c^2)(a-b)(a-c)[(c-b)(c+b+a)]$

$= (a^2+b^2+c^2)(a-b)(b-c)(c-a)[(c+b+a)] = R.H.S.$

Question 25: (b) Do yourself(Hint: Similar to Question2 or create zeros under first leading 1 and expand with Column1)

25. Use Theorem 3.8 to determine which of the following matrices are nonsingular:

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & -7 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$

Question 26:

26. Use Theorem 3.8 to determine all values of t so that the following matrices are nonsingular:

(a) $\begin{bmatrix} t & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & t \end{bmatrix}$

(c) $\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{bmatrix}$

Solution: Hint: Put det(A)=0---(1), A will be singular for all values of t found by solving (1). For all other values of t found in (1) the matrix will be nonsingular.

$$\begin{vmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{vmatrix} = t \begin{vmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} \quad \textit{Expand with Row2}$$

$$= t\left(t \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} - 0 \begin{vmatrix} t & 1 \\ 1 & t \end{vmatrix} + 1 \begin{vmatrix} 0 & t \\ 1 & 0 \end{vmatrix}\right) \quad \textit{Expand with Row1}$$

$$= t(t^3 - t) = t^2(t^2 - 1)$$

Put $|A| = 0$ $\textit{implies}\ t^2(t^2 - 1) = 0$. Hence for $t = 0, \pm 1$ matrix A

Is singular. For all values of t other than $\{0, \pm 1\}$ matrix A

Is nonsingular.

Question 27:

27. Use Corollary 3.1 to find out whether the following homogeneous system has a nontrivial solution (do *not* solve):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ **Hint: find $det(A) = |A| = ?$ If $|A| = 0$ then above homogeneous system has nontrivial solution.**

3.3 Cofactor Expansion

DEFINITION 3.3 Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. The determinant $\det(M_{ij})$ is called the **minor** of a_{ij}.

DEFINITION 3.4 Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **cofactor** A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j} \det(M_{ij})$.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}.$$

Then

$$\det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34, \quad \det(M_{23}) = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10,$$

and

$$\det(M_{31}) = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -6 - 10 = -16.$$

Also,

$$A_{12} = (-1)^{1+2} \det(M_{12}) = (-1)(-34) = 34,$$
$$A_{23} = (-1)^{2+3} \det(M_{23}) = (-1)(10) = -10,$$

and

$$A_{31} = (-1)^{3+1} \det(M_{31}) = (1)(-16) = -16. \qquad \blacksquare$$

Take entry a_{21} find minor. $det(M_{21}) = \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = -4$

Cofactor of a_{21} is $A_{21} = (-1)^{2+1} det(M_{21}) = -1(-4) = 4$

Take entry a_{33} find minor. $det(M_{33}) = \begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} = 19$

Cofactor of a_{33} is $A_{33} = (-1)^{3+3} det(M_{33}) = 19 = 19$

Theorem 3.10 Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$
[expansion of $\det(A)$ along the ith row]

$$= a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + a_{i4}A_{i4}$$

AND

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$
[expansion of $\det(A)$ along the jth column].

3.4 Inverse of a Matrix

DEFINITION 3.5 Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix adj A, called the **adjoint** of A, is the matrix whose (i, j)th entry is the cofactor A_{ji} of a_{ji}. Thus

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

Theorem 3.12 If $A = [a_{ij}]$ is an $n \times n$ matrix, then $A(\text{adj } A) = (\text{adj } A)A = \det(A)I_n$.

Corollary 3.4 If A is an $n \times n$ matrix and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}(\text{adj } A)$$

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{|A|} \begin{bmatrix} A_{11}A_{21}A_{31} \\ A_{12}A_{22}A_{32} \\ A_{13}A_{23}A_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{bmatrix}$$

Adjoint of matrix A is= matrix of cofactors.

$A_{11} = cofactor\ of\ entry\ a_{11} = (-1)^{1+1}|\ leave\ 1st\ rowand\ 1st\ column|$

$A_{32} = cofactor\ of\ entry\ a_{32}$
$= (-1)^{3+2}|\ leave\ 3rd\ row\ and\ 2nd\ column|$

$A_{23} = cofactor\ of\ entry\ a_{23}$
$= (-1)^{2+3}|\ leave\ 2nd\ row\ and\ 3rd\ column|$

EXAMPLE 2

Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. Compute adj A.

Solution

We first compute the cofactors of A. We have

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18,$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17, \quad A_{13} = (-1)^{1+3} \begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6,$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6,$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2,$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10,$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1, \quad A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28.$$

Then

$$\text{adj } A = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}.$$

EXAMPLE 3

Consider the matrix of Example 2. Then

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} = \begin{bmatrix} -94 & 0 & 0 \\ 0 & -94 & 0 \\ 0 & 0 & -94 \end{bmatrix}$$

$$= -94 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} = -94 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{vmatrix} = -94 \ then\ OK}$$

Corollary 3.4 Explanation:

$$A^{-1} = \frac{1}{|A|} adj(A)$$

Multiply both sides with A

$$AA^{-1} = \frac{1}{|A|} A.adj(A)$$

$$I = \frac{1}{|A|} A.adj(A)$$

$$|A|I = A.adj(A)$$

Or

$$|A|I = adj(A).A$$

$$A.adj(A) = det(A)I_n$$

Multiply both sides with A^{-1}

$$A^{-1}A.adj(A) = A^{-1}det(A)I_n$$

$$I_n.adj(A) = det(A)A^{-1}$$

$$adj(A) = det(A)A^{-1}$$

$$\frac{adj(A)}{det(A)} = A^{-1} \quad provided\ that \quad det(A) \neq 0.$$

2. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$.

(a) Find adj A.

(b) Compute $\det(A)$.

(c) Verify Theorem 3.12; that is, show that

$$A(\text{adj } A) = (\text{adj } A)A = \det(A)I_3.$$

3. Let $A = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Follow the directions of Exercise 2.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ -4 & 5 \end{vmatrix} = 24; \; A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 4 & 5 \end{vmatrix} = -1(-19) = 19$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 4 & -4 \end{vmatrix} = -4; \; A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 8 \\ -4 & 5 \end{vmatrix} = -1(42) = -42$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 6 & 8 \\ 4 & 5 \end{vmatrix} = -2; \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 6 & 2 \\ 4 & -4 \end{vmatrix} = -1(-32) = 32$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 8 \\ 4 & 1 \end{vmatrix} = -30; \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 6 & 8 \\ -3 & 1 \end{vmatrix} = -1(30) = -30$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 6 & 2 \\ -3 & 4 \end{vmatrix} = 30$$

$$(a) Adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix}$$

$$(b) det(A) = \begin{vmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \text{expand with R1}$$

$$= (6)(24) + (2)(19) + (8)(-4) = 144 + 38 - 32 = 150$$

expand with C2

$$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = 2(19) + 4(-2) - 4(-30) = 150$$

$$Adj(A).A = \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix} \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix}$$

$$(c) A.Adj(A) = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix}$$

$$Adj(A).A = A.Adj(A) = 150 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = det(A).I_3$$

Question3 (d) find inverse of matrix A.

$$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{150} \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix}$$

See Question from Slader.com

3.5 Other Applications of Determinants

We can use the results developed in Theorem 3.12 to obtain another method for solving a linear system of n equations in n unknowns. This method is known as **Cramer's rule**.

Limitations of Crammer's rule:

(1) Number of unknown=number of equations, i.e., Matrix A is square.
(2) Matrix A must be nonsingular, i.e., $det(A) = |A| \neq 0$

Consider a non-homogeneous system

$$a_{11}x_1 + a_{12}x_2 = c_1$$

$$a_{21}x_1 + a_{22}x_2 = c_2$$

$$AX = b --- (1)$$

Where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ **and** $det(A) \neq 0$

Multiply (1)**by** A^{-1} **on both sides, we have**

$$A^{-1}.AX = A^{-1}.b$$

$$X = A^{-1}.b$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \ Extra$$

$$A = \begin{bmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{det(A)}\begin{bmatrix} A_{11}A_{21} \\ A_{12}A_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{det(A)}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{det(A)}\begin{bmatrix} a_{22}c_1 - a_{12}c_2 \\ -a_{21}c_1 + a_{11}c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(a_{22}c_1 - a_{12}c_2)}{det(A)} \\ \dfrac{(a_{21}c_1 - a_{11}c_2)}{det(A)} \end{bmatrix}$$

Equating both sides we have

$$x_1 = \frac{(a_{22}c_1 - a_{12}c_2)}{det(A)} = \frac{\begin{vmatrix} c_1a_{12} \\ c_2a_{22} \end{vmatrix}}{det(A)}$$

$$x_2 = \frac{(a_{11}c_2 - a_{21}c_1)}{det(A)} = \frac{\begin{vmatrix} a_{11}c_1 \\ a_{21}c_2 \end{vmatrix}}{det(A)}$$

Similarly for three equations in three unknowns, solution is directly given by following formulas

Consider a non-homogeneous system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{32}x_3 = c_3$$

$$AX = b - - - (1)$$

Where $A = \begin{bmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$x_1 = \frac{\begin{vmatrix} c_1 a_{12} a_{13} \\ c_2 a_{22} a_{23} \\ c_3 a_{32} a_{33} \end{vmatrix}}{|A|}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} c_1 a_{13} \\ a_{21} c_2 a_{23} \\ a_{31} c_3 a_{33} \end{vmatrix}}{|A|}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} a_{12} c_1 \\ a_{21} a_{22} c_2 \\ a_{31} a_{32} c_3 \end{vmatrix}}{|A|}$$

EXAMPLE 1 Consider the following linear system:

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3. \end{aligned}$$

We have $|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$. Then

$$x_1 = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \frac{-4}{-2} = 2,$$

$$x_2 = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \frac{-6}{-2} = 3,$$

and

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \frac{-8}{-2} = 4.$$