Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSM-A, SP20-BSE-A & B).

4.7 Homogeneous Systems

The procedure for finding a basis for the solution space of a homogeneous system $A\mathbf{x} = \mathbf{0}$, or the null space of A, where A is $m \times n$, is as follows:

Step 1. Solve the given homogeneous system by Gauss–Jordan reduction. If the solution contains no arbitrary constants, then the solution space is $\{\mathbf{0}\}$, which has no basis; the dimension of the solution space is zero.

Step 2. If the solution $\mathbf{x}$ contains arbitrary constants, write $\mathbf{x}$ as a linear combination of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ with $s_1, s_2, \ldots, s_p$ as coefficients:

$$\mathbf{x} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_p\mathbf{x}_p.$$

Step 3. The set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ is a basis for the solution space of $A\mathbf{x} = \mathbf{0}$; the dimension of the solution space is p.

If A is an $m \times n$ matrix, we refer to the dimension of the null space of A as the **nullity** of A, denoted by nullity A.

Example 1:-

Find a basis for and the dimension of the solution space W of the homogeneous system

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution

Step 1. To solve the given system by the Gauss–Jordan reduction method, we transform the augmented matrix to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$x_4 + 2x_5 = 0 - - - (1)$$

$$x_2 + 2x_3 - x_5 = 0 - - - (2)$$

$$x_1 + 2x_3 + x_5 = 0 - - - (3)$$

Say $x_5 = s \in R$ be an arbitrary variable.

From equation (1) $x_4 = -2s$

Say $x_3 = t \in R$ be an arbitrary variable.

From equation (2) $x_2 = -2t + s$

From equation (3) $x_1 = -2t - s$

Step 2:- $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t - s \\ -2t + s \\ t + 0s \\ 0t - 2s \\ 0t + s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Step 3:- Basis of Null space (W) or solution space (W) is = $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Nullity=dimension of W=2

Question 10: (Exercise 4.7):- In exercise 3 through 1, find basis for and dimension of the solution space (Null Space) of the given homogeneous system.

10. $$\begin{bmatrix} 1 & 2 & -3 & -2 & 1 & 3 \\ 1 & 2 & -4 & 3 & 3 & 4 \\ -2 & -4 & 6 & 4 & -3 & 2 \\ 0 & 0 & -1 & 5 & 1 & 9 \\ 1 & 2 & -3 & -2 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: $[A|O] = \left[\begin{array}{cccccc|c} 1 & 2 & -3 & -2 & 1 & 3 & 0 \\ 1 & 2 & -4 & 3 & 3 & 4 & 0 \\ -2 & -4 & 6 & 4 & -3 & 2 & 0 \\ 0 & 0 & -1 & 5 & 1 & 9 & 0 \\ 1 & 2 & -3 & -2 & 0 & 7 & 0 \end{array}\right]$

I got Reduced Row Echelon Form using linear algebra toolkit (Do yourself: important and it is practice of row operations)

$$\sim \left[\begin{array}{cccccc|c} 1 & 2 & 0 & -17 & 0 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reading from echelon form $\boldsymbol{x_6 = 0}$; $\boldsymbol{x_5 = 0}$

$x_3 - 5x_4 = 0 \Rightarrow x_3 = 5x_4$ $Take$ $\boldsymbol{x_4 = s \in R} \Rightarrow \boldsymbol{x_3 = 5s}$

$x_1 + 2x_2 - 17x_4 = 0 \Rightarrow x_1 = -2x_2 + 17x_4$ $Take$ $\boldsymbol{x_2 = t} \in R \Rightarrow \boldsymbol{x_1 = -2t + 17s}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2t + 17s \\ t \\ 5s \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2t + 17s \\ t + 0s \\ 0t + 5s \\ 0t + s \\ 0t + 0s \\ 0t + 0s \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 17 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Basis of Null space (W) or solution space (W) is =basis of W= $\left\{ \begin{bmatrix} 17 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Nullity of W=dimension of W=2.

Relationship between Nonhomogeneous Linear Systems and Homogeneous Systems

Consider the linear system

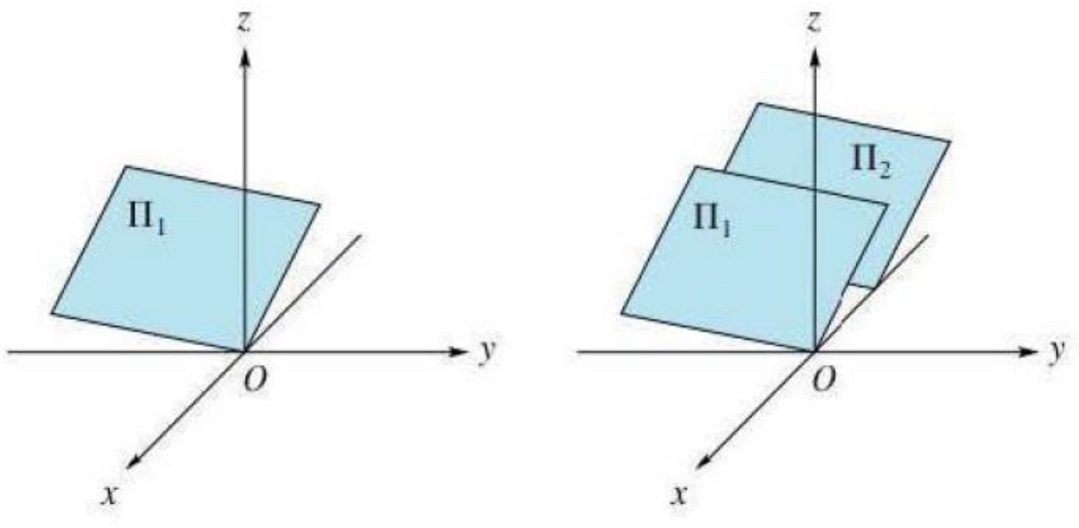
$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

The set of all solutions to this linear system consists of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 2 - 2r + 3s \\ r \\ s \end{bmatrix}$$

(verify), which can be written as

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$



Π_1 is the solution space to $A\mathbf{x} = \mathbf{0}$. Π_2 is the set of all solutions to $A\mathbf{x} = \mathbf{b}$.

$$\mathbf{x}_p = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_h = r \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

4.9 Rank of a Matrix

In this section we obtain another effective method for finding a basis for a vector space V spanned by a given set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. In Section 4.6 we developed a technique for choosing a basis for V that is a subset of S (Theorem 4.9). The method to be developed in this section produces a basis for V that is not guaranteed to be a subset of S. We shall also attach a unique number to a matrix A that we later show gives us information about the dimension of the solution space of a homogeneous system with coefficient matrix A.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix. The rows of A, considered as vectors in R_n, span a subspace of R_n called the **row space** of A. Similarly, the columns of A, considered as vectors in R^m, span a subspace of R^m called the **column space** of A.

EXAMPLE 1

Find a basis for the subspace V of R_5 that is spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 & 2 & 8 & 1 & 4 \end{bmatrix},$$
$$\mathbf{v}_3 = \begin{bmatrix} 2 & 3 & 7 & 2 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -1 & 2 & 0 & 4 & -3 \end{bmatrix}.$$

Solution

Note that V is the row space of the matrix A whose rows are the given vectors.

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}.$$

Using elementary row operations, we find that A is row equivalent to the matrix (verify)

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in reduced row echelon form. The row spaces of A and B are identical, and a basis for the row space of B consists of

$$\mathbf{w}_1 = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$
$$\text{and} \quad \mathbf{w}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Example 2:

Find a basis for the row space of the matrix A defined in the solution of Example 1 that contains only row vectors from A. Also, compute the row rank of A.

Solution

Using the procedure in the alternative proof of Theorem 4.9, we form the equation

$$a_1\begin{bmatrix}1 & -2 & 0 & 3 & -4\end{bmatrix} + a_2\begin{bmatrix}3 & 2 & 8 & 1 & 4\end{bmatrix} + a_3\begin{bmatrix}2 & 3 & 7 & 2 & 3\end{bmatrix} + a_4\begin{bmatrix}-1 & 2 & 0 & 4 & -3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\left[\begin{array}{rrrr|r} 1 & 3 & 2 & -1 & 0 \\ -2 & 2 & 3 & 2 & 0 \\ 0 & 8 & 7 & 0 & 0 \\ 3 & 1 & 2 & 4 & 0 \\ -4 & 4 & 3 & -3 & 0 \end{array}\right] = \left[A^T \mid \mathbf{0}\right]; \tag{1}$$

that is, the coefficient matrix is A^T. Transforming the augmented matrix $\left[A^T \mid \mathbf{0}\right]$ in (1) to reduced row echelon form, we obtain (verify)

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{11}{24} & 0 & 0 \\ 0 & 1 & 0 & -\frac{49}{24} & 0 \\ 0 & 0 & 1 & \frac{7}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2}$$

Since the leading 1's in (2) occur in columns 1, 2, and 3, we conclude that the first three rows of A form a basis for the row space of A. That is,

$$\left\{\begin{bmatrix}1 & -2 & 0 & 3 & -4\end{bmatrix}, \begin{bmatrix}3 & 2 & 8 & 1 & 4\end{bmatrix}, \begin{bmatrix}2 & 3 & 7 & 2 & 3\end{bmatrix}\right\}$$

is a basis for the row space of A. The row rank of A is 3. ■

Find a basis for the column space of the matrix A defined in the solution of Example 1, and compute the column rank of A.

Solution 1

Writing the columns of A as row vectors, we obtain the matrix A^T, which when transformed to reduced row echelon form is (as we saw in Example 4)

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & -\frac{49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the vectors $\begin{bmatrix}1 & 0 & 0 & \frac{11}{24}\end{bmatrix}$, $\begin{bmatrix}0 & 1 & 0 & -\frac{49}{24}\end{bmatrix}$, and $\begin{bmatrix}0 & 0 & 1 & \frac{7}{3}\end{bmatrix}$ form a basis for the row space of A^T. Hence the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{11}{24} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{49}{24} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{7}{3} \end{bmatrix}$$

form a basis for the column space of A, and we conclude that the column rank of A is 3.

Solution 2

If we want to find a basis for the column space of A that contains only the column vectors from A, we follow the procedure developed in the proof of Theorem 4.9, forming the equation

$$a_1 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 8 \\ 7 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} + a_5 \begin{bmatrix} -4 \\ 4 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

whose augmented matrix is $\left[A \mid \mathbf{0} \right]$. Transforming this matrix to reduced row echelon form, we obtain (as in Example 1)

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since the leading 1's occur in columns 1, 2, and 4, we conclude that the first, second, and fourth columns of A form a basis for the column space of A. That is,

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$$

is a basis for the column space of A. The column rank of A is 3. ■

DEFINITION 4.15 The dimension of the row (column) space of A is called the **row** (**column**) **rank** of A.

Theorem 4.18 The row rank and column rank of the $m \times n$ matrix $A = \left[a_{ij} \right]$ are equal.

Theorem 4.19 If A is an $m \times n$ matrix, then rank A + nullity $A = n$.

Question:- For given matrix, Find

(a) **Basis for row space that are not rows (vectors) of A**
(b) **Basis for row space that are rows (vectors) of A**
(c) **Basis for column space that are not columns (vectors) of A**
(d) **Basis for column space that are columns (vectors) of A**
(e) **Prove that $\boldsymbol{rank\ A + nullity\ A = number\ of\ columns}$ for following matrix.**

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} RREF \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Answer (a) **Basis for row space that are not rows (vectors) of A**

$$w_1 = [1 \quad 0 \quad 2 \quad 0 \quad 1]; w_2 = [0 \quad 1 \quad 2 \quad 0 \quad -1]; w_3 = [0 \quad 0 \quad 0 \quad 1 \quad 2]$$

(d) Basis for column space that are columns (vectors) of A

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}; v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}; v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A^T} = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 0 & 6 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix} RREF \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) **Basis for row space that are rows (vectors) of A**

$$w_1 = [1 \quad 1 \quad 4 \quad 1 \quad 2]; w_2 = [0 \quad 1 \quad 2 \quad 1 \quad 1]; w_3 = [0 \quad 0 \quad 0 \quad 1 \quad 2]$$

(c) **Basis for column space that are not columns (vectors) of A**

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}; v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -1 \end{bmatrix}; v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Row rank=number of non-zero rows of A in RREF=3

Column rank= number of leading ones=3

Rank of matrix A = Row rank= Column rank=**3**

Now to find nullity (dimension of null or solution space) augment $RREF$ with null vector. i.e.,

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 + 2x_5 = 0;\ x_2 + 2x_3 - x_5 = 0;\ x_1 + 2x_3 + x_5 = 0$

$x_4 = -2x_5;\ x_2 = -2x_3 + x_5;\ x_1 = -2x_3 - x_5$

Suppose $x_5 = r$ then $x_4 = -2r;$

Suppose $x_3 = s$ then $x_2 = -2s + r;\ x_1 = -2s - r$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - r \\ -2s + r \\ s \\ -2r \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Basis for null/solution space=$\left\{\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)\right\}$

Nullity of A=2

$$rank\ A + nullity\ A = number\ of\ columns$$

$$3 + 2 = 5$$