Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSM-A & SP20-BSE-A & B).

Chapter 5: Inner Product Spaces

Courtesy:- Material for this lecture is selected from Kolman's book, and handouts of Virtual University, Lahore, Virtual COMSATS and Imperial College London.

Objective of this Lecture:-

(A) The prime objective of this chapter is to convert a set of linearly independent vectors (basis) into a set of orthonormal vectors (basis).

(B) To achieve this goal, first we need to define several bits and pieces like "dot product (inner product), length (norm or magnitude) of vectors, unit vector, distance between vectors, projection of vectors, orthogonal vectors and orthonormal vectors".

After studying this lecture, you are desired to do

Home Work: Do Questions 6, 8, 18 and 24 of Exercise 5.1, Do Questions 1-8 of Exercise 5.2; Questions 8,9,12,13,29,30 of Exercise 5.3, and Questions 1-4, 9-20 of Exercise 5.4.

Section 5.1:

Definition 1: (Dot or inner or scalar or projection product of vectors in terms of **vector components)**

Let $\boldsymbol{u}, \boldsymbol{v} \in R^n$ then $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ . \\ . \\ . \\ \boldsymbol{u_n} \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ . \\ . \\ . \\ \boldsymbol{v_n} \end{bmatrix}$, consider these vectors as $n \times 1$ matrices. Clearly $\boldsymbol{u}^T = [\boldsymbol{u_1}\ \ \boldsymbol{u_2}\ \ .\ .\ .\ \boldsymbol{u_n}]$.

Now the dot or inner product is a **scalar quantity** dented by $\boldsymbol{u} \bullet \boldsymbol{v}$ **or** $(\boldsymbol{u}, \boldsymbol{v})$ and is defined as matrix product

$$\boldsymbol{u}\bullet\boldsymbol{v} = (\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{u}^T\boldsymbol{v} = [u_1\ \ u_2\ \ .\ .\ .\ u_n\ ]\begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Note: Although notation $\boldsymbol{u}\bullet\boldsymbol{v}$ is more familiar but we will use this natation $(\boldsymbol{u},\boldsymbol{v})$ for dot/inner product to maintain the consistency with Kolman's book.

Example 1: Let $\boldsymbol{u} = \begin{bmatrix} \mathbf{6} \\ \mathbf{4} \\ \mathbf{2} \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{8} \end{bmatrix}$ then

$$(\boldsymbol{u},\boldsymbol{v}) = \mathbf{6}(\mathbf{4}) + \mathbf{4}(-\mathbf{1}) + \mathbf{2}(\mathbf{8}) = \mathbf{36} = (\boldsymbol{v},\boldsymbol{u})$$ (Inner product is commutative)

Definition 2: (Length or norm or magnitude of vector**)**

Let $\boldsymbol{v} \in R^n$ then $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix}$**,** then length of this vector is denoted by $\|\boldsymbol{v}\|$ (**scalar quantity**) and defined as $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v_1}^2 + \boldsymbol{v_2}^2 + .\ .\ .\ + \boldsymbol{v_n}^2}$. When $n = 2\ or\ 3$ i.e., our vector is from R^2 (2D) or R^3(3D) then

$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v_1}^2 + \boldsymbol{v_2}^2}$ And $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v_1}^2 + \boldsymbol{v_2}^2 + \boldsymbol{v_3}^2}$, respectively.

Observation: A nice relation between dot product and norm of a vector.

$$(\boldsymbol{v},\boldsymbol{v}) = \boldsymbol{v_1v_1} + \boldsymbol{v_2v_2} + \cdots + \boldsymbol{v_nv_n} = \boldsymbol{v_1}^{\mathbf{2}} + \boldsymbol{v_2}^{\mathbf{2}} + .\ .\ . + \boldsymbol{v_n}^{\mathbf{2}} = \|\boldsymbol{v}\|^{\mathbf{2}}$$

Example 2: Let $\boldsymbol{v}$ be vector in $\boldsymbol{R}^{\mathbf{3}}$, such that $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-2} \end{bmatrix}$ then $\|\boldsymbol{v}\| = \sqrt{1+4+4}$=3

Definition 3: (Unit vector and normalization**)**

A vector having length 1 is called a unit vector. We can convert any arbitrary vector $\boldsymbol{v}$ in to a unit vector $\boldsymbol{u}$ using formula

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

This way of getting $\boldsymbol{u}$ from $\boldsymbol{v}$ is called normalization of $\boldsymbol{v}$ and $\boldsymbol{u}$ is called normalized vector. Both $\boldsymbol{u}$ **and** $\boldsymbol{v}$ are in same direction.

Example 3: Let $\boldsymbol{v}$ be vector in $\boldsymbol{R}^3$, such that $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-2} \end{bmatrix}$ then $\|\boldsymbol{v}\| = \sqrt{1+4+4}$=3

Then $\boldsymbol{u} = \frac{v}{\|v\|} = \frac{1}{3}\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-2} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{1}}{3} \\ \frac{\mathbf{2}}{3} \\ \frac{\mathbf{-2}}{3} \end{bmatrix}$. Now verify

$$\|\boldsymbol{u}\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2} = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1$$

Definition 1a: (Dot or inner product of vectors in terms of **projection)**

Here we present an alternative formula for dot/inner product in terms of **projection of vectors.**

Let $\boldsymbol{u}, \boldsymbol{v} \in R^n$ then $(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos(\theta)$, Proof of this formula is simple using Law of cosines of triangle.

$(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos(\theta)$, $= \|\boldsymbol{v}\|$ (scalar projection of $\boldsymbol{u}$ on $\boldsymbol{v}$) , Where θ is angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. **(See Figure 1)**

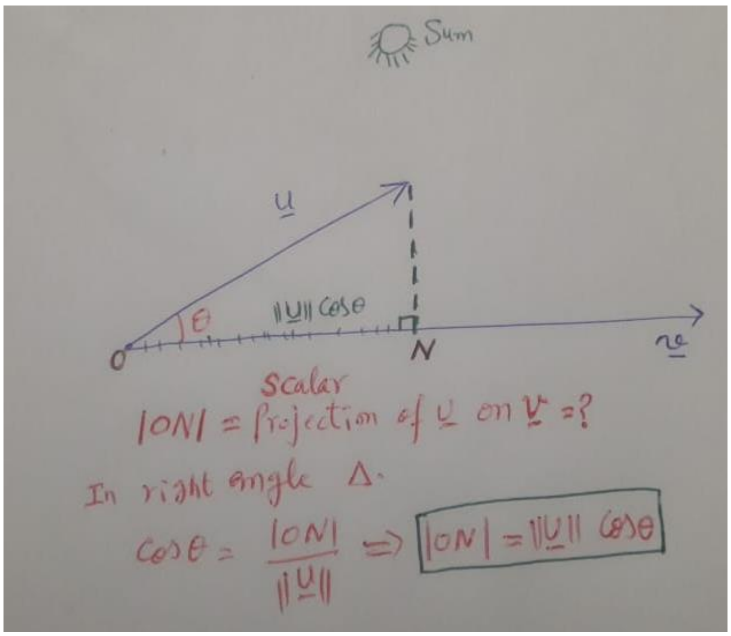


Figure 1

Observation: The formula in **Definition 1a** is excellent in establishing various **facts about** dot/inner product whereas formula in **Definition 1** is handy for calculations.

NOTE: Watch the video given in link below, only for students interested in proof, for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^2$,

$$(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u_1 v_1} + \boldsymbol{u_2 v_2} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \boldsymbol{cos}(\boldsymbol{\theta})$$

https://www.youtube.com/watch?v=LzP_9jJpR-g

Some facts about dot/inner product:

Fact 1: $(\boldsymbol{u}, \boldsymbol{v}) = 0$ **if** $\boldsymbol{\theta} = \frac{\pi}{2}$ **i.e., vectors are orthogonal to each other when their dot product is zero.**

Fact 2: $(\boldsymbol{0}, \boldsymbol{v}) = 0$ **if i.e., Zero vector is orthogonal to every vector.**

Fact 3: $(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ **if** $\boldsymbol{\theta} = 0$ **i.e., vectors are pointing in same direction.**

Fact 4: $(\boldsymbol{u}, \boldsymbol{v}) = -\|\boldsymbol{u}\|\|\boldsymbol{v}\|$ **if** $\boldsymbol{\theta} = \boldsymbol{\pi}$ **i.e., vectors are in opposite direction.**

Definition 4: (Distance between two vectors)

Let $\boldsymbol{u}, \boldsymbol{v} \in R^n$ then distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v})$ and is defined as

$$\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(\boldsymbol{u_1} - \boldsymbol{v_1})^2 + (\boldsymbol{u_2} - \boldsymbol{v_2})^2 + . . . + (\boldsymbol{u_n} - \boldsymbol{v_n})^2}.$$

Note: Next figure is only for interested readers, proof can directly be seen geometrically from law of parallelogram of vectors.

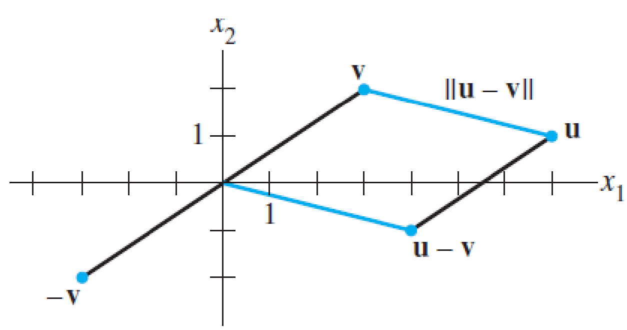


FIGURE 4 The distance between **u** and **v** is the length of **u** – **v**.

Now we present some theorems having many applications in Mathematics and proofs are beyond the scope of this course.

Do questions 6, 8, 18 and 24 of Exercise 5.1

Theorem 1: Cauchy-Schwartz Inequality:-

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are any two vectors in $\boldsymbol{R}^n$, then $|(\boldsymbol{u},\boldsymbol{v})| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$

i.e., Absolute value of dot product is less or equal to product of magnitudes of vectors.

Example 4:

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

be in the Euclidean space R^3 with the standard inner product. Then $(\mathbf{u}, \mathbf{v}) = -5$, $\|\mathbf{u}\| = \sqrt{14}$, and $\|\mathbf{v}\| = \sqrt{17}$. Therefore, $|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$. ■

i.e., $|-5| \leq \sqrt{14}\,.\sqrt{17} \;\Rightarrow 5 \;\leq 15.43$

Theorem 2: Triangle Inequality:-

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are any two vectors in $\boldsymbol{R}^n$, then $\|\boldsymbol{u}+\boldsymbol{v}\| \;\leq\; \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

Example 5: Consider again $\boldsymbol{u}$ and $\boldsymbol{v}$ given in example 4, then $\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$ and

$\|\boldsymbol{u}+\boldsymbol{v}\| = \sqrt{21}$, Now $\sqrt{21} \;\leq \sqrt{14}+\sqrt{17} \;\Rightarrow \mathbf{4.6 \leq\; 7.86}$.

Theorem 3: The Pythagorean Theorem:-

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^n$are **orthogonal** if and only if

$$(\|\boldsymbol{u}+\boldsymbol{v}\|)^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2.$$

Definition 5: (Orthogonal Set)

Let $S = \{V_1, V_2, \ldots, V_p\}$ be the set of non-zero vectors in $\boldsymbol{R}^n$, then S is said to be orthogonal set if all vectors in S are mutually **orthogonal**. Mathematically

$$(V_i, V_j) = 0 \ \ for\ all\ i \neq j$$

Theorem 4:- Let $S = \{V_1, V_2, \dots, V_p\}$ be a finite orthogonal set of vectors of a Euclidean space $\boldsymbol{R^n}$. Then S is **linearly independent.**

Definition 6: (Orthogonal Basis)

Let $S = \{V_1, V_2, \dots, V_p\}$ be a **basis** for subspace $\boldsymbol{W}$ in $\boldsymbol{R^n}$, $\boldsymbol{(dim\, W = p < n)}$, then S is said to be **orthogonal basis** if set S is an orthogonal set.

Definition 7: (Orthonormal Set)

Let $S = \{U_1, U_2, \dots, U_p\}$ be the **set** of non-zero vectors in $\boldsymbol{R^n}$**,** then S is said to be **orthonormal** set if S is orthogonal set of unit vectors. Mathematically

$$(U_i, U_j) = 0 \ \ for\ all\ i \neq j\ and\ \|U_i\| = 1\ for\ all\ U_i \in S.$$

Definition 8: (Orthonormal Basis)

Let $S = \{U_1, U_2, \dots, U_p\}$ be a **basis** for subspace $\boldsymbol{W}$ in $\boldsymbol{R^n}$**,** then S is said to be orthonormal basis if set S is an orthonormal set.

Example 6:

If $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal set (verify). The vectors

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

are unit vectors in the directions of $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. Since $\mathbf{x}_3$ is also a unit vector, we conclude that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{x}_3\}$ is an orthonormal set. ■

Now we have enough tools and finally arrived at the beginning of our core problem.

Gram-Schmidt Process to obtain orthonormal basis:-

Our life is very easy while working with orthonormal bases. We observed that to work with the natural/standard basis for R^2 and $R^3$the computations are kept to a minimum. (See my previous recording)

Recall (1) if $S = \{e_1, e_2, e_3\}$ are standard basis of $\boldsymbol{R^3}$, where $e_1 = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, e_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, e_3 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ then any vector $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} \in \boldsymbol{R^3}$ can directly be written as linear combination of basis i.e., $\boldsymbol{v} = \boldsymbol{a}e_1 + \boldsymbol{b}e_2 + \boldsymbol{c}e_3$. (Coefficients of bases vectors are coordinates of the vector $\boldsymbol{v}$)

Recall (2) if set $S = \{V_1, V_2, V_3\}$ is any basis of $\boldsymbol{R^3}$, and vector $\boldsymbol{v} \in \boldsymbol{R^3}$ then $\boldsymbol{v}$ can be written as $\boldsymbol{v} = \boldsymbol{a}V_1 + \boldsymbol{b}V_2 + \boldsymbol{c}V_3$. (**See Examle 1(LC)**), We have to solve a non-homogeneous system to obtain the coordinates of $\boldsymbol{v}$.

Following theorem shows the advantage of orthonormal basis over any basis of vector space.

Theorem 5:- (In book Theorem 5.5)

Let $S = \{V_1, V_2, \ldots, V_n\}$ be a finite orthonormal basis of finite vectors of a Euclidean space $\boldsymbol{V}$ and let $\boldsymbol{v}$ be any vector in $\boldsymbol{V}$. Then

$$\boldsymbol{v} = c_1V_1 + c_2V_2 + c_3V_3 + \cdots + c_nV_n$$

$$(\boldsymbol{v}, V_1) = c_1(V_1, V_1) = c_1||V_1||^2 = c_1$$

Where $c_i = (\boldsymbol{v}, V_i),\ i = 1,2,3, \ldots, n$

Note:- We only need to compute few dot product in presence of orthonormal basis.

Example 7:-

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Note that S is an orthonormal set. Write the vector

$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

as a linear combination of the vectors in S.

Solution:-

We have

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

Theorem 5.5 shows that c_1, c_2, and c_3 can be derived without having to solve a linear system of three equations in three unknowns. Thus

$$c_1 = (\mathbf{v}, \mathbf{u}_1) = 1, \quad c_2 = (\mathbf{v}, \mathbf{u}_2) = 0, \quad c_3 = (\mathbf{v}, \mathbf{u}_3) = 7,$$

and $\mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3$. ■

Theorem 6:- (In book Theorem 5.6)

Let $V = R^{\mathrm{n}}$ be a Euclidean space and W an m-dimensional subspace of V, then there exists an **orthonormal basis** $T = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_m}\}$.

OR equivalently, for subspace W **given** any basis set say $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_m}\}$

Obtain/convert (1) set S to an **orthogonal basis** set say $T^* = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_m}\}$ (using Gram-Schmidt process)

(2) Finally normalize this set to get **orthonormal basis**

$$T = \left\{\boldsymbol{w_1} = \frac{\boldsymbol{v_1}}{||\boldsymbol{v_1}||}, \boldsymbol{w_2}, \dots, \boldsymbol{w_m}\right\}.$$

NOTE:-PROOF OF THIS THEOREM IS BEYOND THE SCOPE OF OUR COURSE, HOWEVER I WILL GIVE GENERAL OUTLINE OF PROOF.

Proof:- Given $S = \{ \underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m \}$

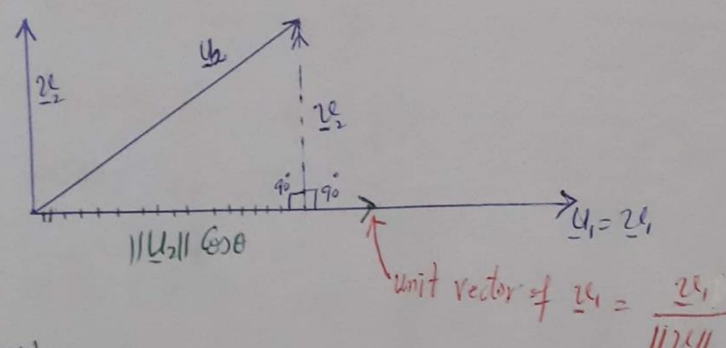
Proof is Constructive:-

Take $\boxed{\underline{u}_1 = \underline{v}_1}$

Target $\underline{v}_2$ such that $\underline{v}_2$ is orthogonal to $\underline{v}_1$

or $\boxed{(\underline{v}_2, \underline{v}_1) = 0}$

Consider



We know

$$(\underline{u}_2, \underline{v}_1) = \|\underline{u}_2\| \cdot \|\underline{v}_1\| \cos\theta$$

$$\Rightarrow \boxed{\|\underline{u}_2\| \cos\theta = \frac{(\underline{u}_2, \underline{v}_1)}{\|\underline{v}_1\|}} \quad \text{— (1)}$$

Now

$\underline{u}_2 =$ vector Projection of $\underline{u}_2$ on $\underline{v}_1 + \underline{v}_2$

$\underline{u}_2 =$ Scalar Projection of $\underline{u}_2$ X unit vector of $\underline{v}_1 + \underline{v}_2$

$$\text{''} = \|\underline{u}_2\| \cos\theta \times \frac{\underline{v}_1}{\|\underline{v}_1\|} + \underline{v}_2$$

$$\text{''} = \frac{(\underline{u}_2, \underline{v}_1)}{\|\underline{v}_1\|} \times \frac{\underline{v}_1}{\|\underline{v}_1\|} + \underline{v}_2 \qquad (\text{using } (1))$$

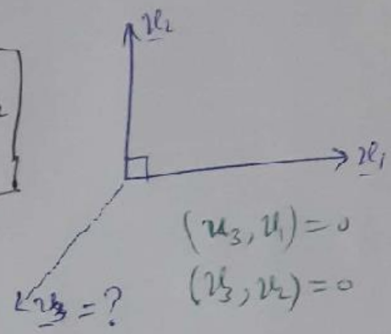
B

$$u_2 = \frac{(u_2, v_1)}{(v_1, v_1)} v_1 + v_2 \qquad \text{using } ||v_1||^2 = (v_1, v_1)$$

Now $\boxed{v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1}$ Required formula

Similarly.

$$\boxed{v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2}$$



Likewise

$$v_4 = u_4 - \frac{(u_4, v_1)}{(v_1, v_1)} v_1 - \frac{(u_4, v_2)}{(v_2, v_2)} v_2 - \frac{(u_4, v_3)}{(v_3, v_3)} v_3$$

$\vdots$

$\boldsymbol{w_1 = \frac{v_1}{||v_1||}}$; $\boldsymbol{w_2 = \frac{v_2}{||v_2||}}$; $\boldsymbol{w_3 = \frac{v_3}{||v_3||}}$; $\boldsymbol{w_4 = \frac{v_4}{||v_4||}}$

Do Example 2 on page (323) of Kolman's book. For video support of this please visit following links.

https://www.youtube.com/watch?v=KKffS_U6_34&t=104s

https://www.youtube.com/watch?v=swXcm_vTjWU

Exercise 5-4:-

Q# 2:- Use the Gram - Schmidt process to transform the basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ for the subspace W of Euclidean space $\mathbb{R}^3$ into

(a) an orthogonal basis.

(b) an orthonormal basis.

Solution:- Given vectors $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

(a) We start process by taking $\underline{v}_1 = \underline{u}_1$ and compute $\underline{v}_2$ using formula.

$$\underline{v}_2 = \underline{u}_2 - \frac{(\underline{u}_2, \underline{v}_1)}{(v_1, v_1)} \cdot \underline{v}_1$$

$$\underline{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} - \frac{(1(-2)+0(1)+1(3))}{(1(1)+0(0)+1(1))} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$'' = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2-\frac{1}{2} \\ 1-0 \\ 3-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1 \\ 5/2 \end{bmatrix}$$

$$\text{or } \underline{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$$

$\Rightarrow$ $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $\underline{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$ are orthogonal basis for W.

Check $(\underline{v}_1, \underline{v}_2) = 0$ (orthogonal)

(b) To obtain orthonormal basis simply divide each vector in orthogonal basis by their norm

i.e $\quad \underline{w}_1 = \dfrac{\underline{v}_1}{\|\underline{v}_1\|}$ & $\underline{w}_2 = \dfrac{\underline{v}_2}{\|\underline{v}_2\|}$

$$\Rightarrow \underline{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \quad \& \quad \underline{w}_2 = \frac{1}{\sqrt{54}}\begin{bmatrix}-5\\2\\5\end{bmatrix} = \begin{bmatrix}-5/\sqrt{54}\\2/\sqrt{54}\\5/\sqrt{54}\end{bmatrix}$$

Check $\|\underline{w}_1\| = 1$, $\|\underline{w}_2\| = 1$

Q# 3:- Consider the Euclidean Space R_4 & let W be the subspace that has $S = \{[1 \ 1 \ -1 \ 0], [0 \ 2 \ 0 \ 1]\}$ as a basis. Use Gram-Schmidt process to obtain orthonormal basis for W.

Solution:- We will proceed as we did in Q2 But here vectors are row vectors.

Take $\underline{u}_1 = [1 \ 1 \ -1 \ 0]$, $\underline{u}_2 = [0 \ 2 \ 0 \ 1]$

let $\underline{v}_1 = \underline{u}_1$, Now orthogonal vector to $\underline{v}_1$ can be calculated as.

$$\underline{v}_2 = \underline{u}_2 - \frac{(\underline{u}_2, \underline{v}_1)}{(\underline{v}_1, \underline{v}_1)}\underline{v}_1$$

$$'' = [0 \ 2 \ 0 \ 1] - \frac{(1(0)+1(2)-1(0)+0(1))}{(1(1)+1(1)-1(-1)+0(0))}[1 \ 1 \ -1 \ 0]$$

$$'' = [0 \ 2 \ 0 \ 1] - \frac{2}{3}[1 \ 1 \ -1 \ 0]$$

$$\underline{v}_2 = \left[0-\tfrac{2}{3} \ \ 2-\tfrac{2}{3} \ \ 0+\tfrac{2}{3} \ \ 1-0\right] = \left[-\tfrac{2}{3} \ \ \tfrac{4}{3} \ \ \tfrac{2}{3} \ \ 1\right]$$

Or $\underline{u}_2 = [-2 \quad 4 \quad 2 \quad 3]$

Therefor orthognal basis for W are

$\underline{u}_1 = [1 \quad 1 \quad -1 \quad 0]$, $\underline{u}_2 = [-2 \quad 4 \quad 2 \quad 3]$

Check $(\underline{u}_1, \underline{u}_2) = 0$

By dividing each vector by its magnitude, we get orthonormal basis.

$$\underline{w}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|} = \frac{1}{\sqrt{1+1+1+0}}[1 \quad 1 \quad -1 \quad 0] = \left[\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{-1}{\sqrt{3}} \quad 0\right]$$

$$\underline{w}_2 = \frac{\underline{u}_2}{\|\underline{u}_2\|} = \frac{1}{\sqrt{4+16+4+9}}[-2 \quad 4 \quad 2 \quad 3] = \left[\frac{-2}{\sqrt{33}} \quad \frac{4}{\sqrt{33}} \quad \frac{2}{\sqrt{33}} \quad \frac{3}{\sqrt{33}}\right]$$

Q#9:- Find an orthonormal basis for the Euclidean space $\boxed{\mathbb{R}^3}$ that contains the vectors $\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$ & $\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Solution:- Let $\underline{u}_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$, $\underline{u}_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$

Observation $(\underline{u}_1, \underline{u}_2) = 0$, $\|\underline{u}_1\| = 1$, $\|\underline{u}_2\| = 1$

hence given vectors are orthonormal and We have to find third vector (orthonormal) to be Part of basis for $\mathbb{R}^3$.

Suppose $\underline{u}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the vector to be found.

Now, $\underline{u}_3$ (being orthonormal) obey following conditions

$$\underbrace{(\underline{u}_1, \underline{u}_3) = 0, \; (\underline{u}_2, \underline{u}_3) = 0}_{\text{Orthogonality}}, \; \underbrace{(\underline{u}_3, \underline{u}_3) = 1}_{\text{Normality}}$$

$\Rightarrow$

$$\left.\begin{aligned} \tfrac{2}{3}x - \tfrac{2}{3}y + \tfrac{z}{3} = 0 &\Rightarrow 2x - 2y + z = 0 \quad -(1) \\ \tfrac{2}{3}x + \tfrac{y}{3} - \tfrac{2}{3}z = 0 &\Rightarrow 2x + y - 2z = 0 \quad -(2) \\ x^2 + y^2 + z^2 = 1 &\Rightarrow x^2 + y^2 + z^2 = 1 \quad -(3) \end{aligned}\right\} \text{System of Non-Linear equations}$$

Subtracting Eq. (2) from Eq. (1) we get $\boxed{y = z}$

Put this value in Eq (2) we get $\boxed{2x = z}$ or $\boxed{x = \frac{z}{2}}$

Put these values in (3), we get

$$\frac{z^2}{4} + z^2 + z^2 = 1 \Rightarrow \frac{9z^2}{4} = 1 \Rightarrow z^2 = \frac{4}{9}$$

$$\Rightarrow z = \pm 2/3$$

take $\boxed{z = \frac{2}{3}} \Rightarrow \boxed{x = \frac{1}{3}}$ and $\boxed{y = \frac{2}{3}}$

$$\Rightarrow \underline{u}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Hence $S = \left\{ \underline{u}_1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1/3 \end{bmatrix}, \underline{u}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \underline{u}_3 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right\}$

is set of orthonormal basis for $\mathbb{R}^3$.

Question 9: **Another solution:**

Hint: Containing (Leading 1)

Merge given vectors with standard basis; {u1, u2, e1,e2,e3}

Do yourself (I am going to use linear Algebra toolkit)

$$\boldsymbol{S} = \{u_1, u_2, u_3\}$$

Where $u_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}; u_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}; u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(1) Use Gram-Schmidt Process $v_1, v_2, v_3 = ?$

$$v_1 = u_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} - \frac{0}{(v_1, v_1)} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\frac{2}{3}}{1} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} - \frac{\frac{2}{3}}{1} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{4}{9} - \frac{4}{9} \\ 0 + \frac{4}{9} - \frac{2}{9} \\ 0 - \frac{2}{9} + \frac{4}{9} \end{bmatrix} = \begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$

$$w_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \;;\quad w_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \;;$$

$$w_3 = \frac{v_3}{||v_3||} = \frac{1}{1/3}\begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \end{bmatrix} = 3\begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$W = \{[b + 2c - d \quad b \quad c \quad d] : a = b + 2c - d\}$$

$$[b + 2c - d \quad b \quad c \quad d] = b[1 \quad 1 \quad 0 \quad 0] + c[2 \quad 0 \quad 1 \quad 0] + d[-1 \quad 0 \quad 0 \quad 1]$$

Q11 & 12:- Hint:-

Given Spanning set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

clearly $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is redundant and linearly dependent.

$\Longrightarrow \quad S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Now apply Gram - Schmidt Process $\downarrow$

Q13 $\longrightarrow$ 16 are similar and easy.

Q16 Find orthonormal basis for the subspace of $\mathbb{R}_4$ consisting of all vectors $[a \ b \ c \ d]$ such that

$$\boxed{a - b - 2c + d = 0}$$

Solution:- Substitute $a = b + 2c - d$ in the given vector

$$[b+2c-d \quad b \quad c \quad d] = b[1 \ 1 \ 0 \ 0] + c[2 \ 0 \ 1 \ 0] + d[-1 \ 0 \ 0 \ 1]$$

Hence basis for subspace of $\mathbb{R}^3$ is

$$S = \{ \underset{\underline{u}_1}{[1 \ 1 \ 0 \ 0]}, \underset{\underline{u}_2}{[2 \ 0 \ 1 \ 0]}, \underset{\underline{u}_3}{[-1 \ 0 \ 0 \ 1]} \}$$

Let $\underline{v}_1 = \underline{u}_1$

Now $\underline{v}_2 = \underline{u}_2 - \dfrac{(\underline{u}_2, \underline{v}_1)}{(\underline{v}_1, \underline{v}_1)} \underline{v}_1$

$$\underline{v}_2 = [2 \quad 0 \quad 1 \quad 0] - \frac{(1(2)+1(0)+0(1)+0(0))}{(1(1)+1(1)+0(0)+0(0))}[1 \quad 1 \quad 0 \quad 0]$$

$$= [2 \quad 0 \quad 1 \quad 0] - \frac{2}{2}[1 \quad 1 \quad 0 \quad 0]$$

$$= [2-1 \quad 0-1 \quad 1-0 \quad 0-0]$$

$$\underline{v}_2 = [1 \quad -1 \quad 1 \quad 0]$$

$$\underline{v}_3 = \underline{u}_3 - \frac{(\underline{u}_3, \underline{v}_1)}{(\underline{v}_1, \underline{v}_1)}\underline{v}_1 - \frac{(\underline{u}_3, \underline{v}_2)}{(\underline{v}_2, \underline{v}_2)}\underline{v}_2$$

$$= [-1 \quad 0 \quad 0 \quad 1] - \frac{(-1)}{2}[1 \quad 1 \quad 0 \quad 0] - \frac{(-1)}{3}[1 \quad -1 \quad 1 \quad 0]$$

$$= [-1 \quad 0 \quad 0 \quad 1] + [\tfrac{1}{2} \quad \tfrac{1}{2} \quad 0 \quad 0] + [\tfrac{1}{3} \quad -\tfrac{1}{3} \quad \tfrac{1}{3} \quad 0]$$

$$= [-1+\tfrac{1}{2}+\tfrac{1}{3} \quad 0+\tfrac{1}{2}-\tfrac{1}{3} \quad 0+0+\tfrac{1}{3} \quad 1+0+0]$$

$$\underline{v}_3 = [-\tfrac{1}{6} \quad \tfrac{1}{6} \quad \tfrac{1}{3} \quad 1]$$

or $\underline{v}_3 = [-1 \quad 1 \quad 2 \quad 6]$

$T^* = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is orthogonal basis.

Now to get orthonormal basis, we only need to divide each vector by its norm.

$T = \{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$ where

$$\underline{w}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}, \quad \underline{w}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|}, \quad \underline{w}_3 = \frac{\underline{v}_3}{\|\underline{v}_3\|}$$

$$\underline{w}_1 = \left[\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0 \quad 0\right]$$
$$\underline{w}_2 = \left[\frac{1}{\sqrt{3}} \quad \frac{-1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad 0\right]$$
$$\underline{w}_3 = \left[\frac{-1}{\sqrt{42}} \quad \frac{1}{\sqrt{42}} \quad \frac{2}{\sqrt{42}} \quad \frac{6}{\sqrt{42}}\right]$$

Q18:- Hint:- Find Solution space of given homogeneous system. Then convert the basis of solution space or Null space into orthonormal basis.

Q20 Let $S = \{[1 \ -1 \ 0], [1 \ 0 \ -1]\}$ be a basis for a subspace W of R_3.

(a) Use Gram-Schmidt Process to obtain an orthonormal basis for W.

~~Solution:-~~

(b) Using Theorem 5.5, write $\underline{u} = [5 \ -2 \ -3]$ as a linear combination of the vectors obtained in Part (a).

Solution:- At this stage, we left Part (a) for reader to Prove $\underline{w}_1 = \left[\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right]$, $\underline{w}_2 = \left[\frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{-2}{\sqrt{6}}\right]$ are orthonormal basis for W.

(b) We know $\underline{u}$ can be written $\boxed{\underline{u} = c_1 \underline{w}_1 + c_2 \underline{w}_2}$ — (1)

where $c_1 = (\underline{u}, \underline{w}_1) = 5\left(\frac{1}{\sqrt{2}}\right) - 2\left(-\frac{1}{\sqrt{2}}\right) - 3(0) = \boxed{\frac{7}{\sqrt{2}}}$

$c_2 = (\underline{u}, \underline{w}_2) = 5\left(\frac{1}{\sqrt{6}}\right) - 2\left(\frac{1}{\sqrt{6}}\right) - 3\left(\frac{-2}{\sqrt{6}}\right) = \boxed{\frac{9}{\sqrt{6}}}$

$$(1) \Rightarrow \underline{u} = \frac{7}{\sqrt{2}}\left[\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right] + \frac{9}{\sqrt{6}}\left[\frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{-2}{\sqrt{6}}\right] = [5 \quad -2 \quad -3]$$

18. Find an orthonormal basis for the solution space of the homogeneous system

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & 2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution: Solve using augmented matrix: [A|O] (Do yourself)

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4r \\ 5r \\ r \end{bmatrix} = r\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

Basis of solution (Null) space $S = \left\{\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}\right\}$

$$u_1 = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

$$v_1 = u_1 = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

$$w_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{42}}\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$