Example 21.10 **Field of a ring of charge**

A ring-shaped conductor with radius a carries a total charge Q uniformly distributed around it (Fig. 21.24). Find the electric field at a point P that lies on the axis of the ring at a distance x from its center.

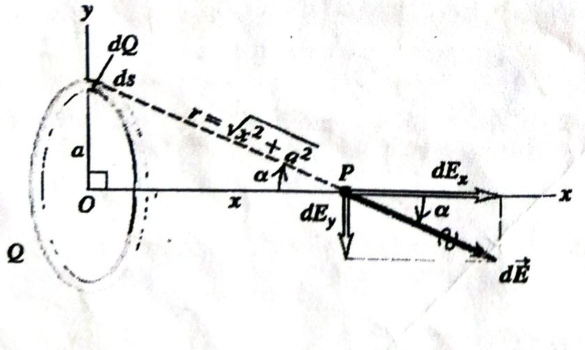
SOLUTION

IDENTIFY: This is a problem in the superposition of electric fields. The new wrinkle is that the charge is distributed continuously around the ring rather than in a number of point charges.

SET UP: The field point is an arbitrary point on the x-axis in Fig. 21.24. Our target variable is the electric field at such a point as a function of the coordinate x.

EXECUTE: As shown in Fig. 21.24, we imagine the ring divided into infinitesimal segments of length ds. Each segment has charge dQ and acts as a point-charge source of electric field. Let $d\vec{E}$ be the electric field from one such segment; the net electric field at P is then the sum of all contributions $d\vec{E}$ from all the segments that make up the ring. (This same technique works for any situation in which charge is distributed along a line or a curve.)

The calculation of $\vec{E}$ is greatly simplified because the field point P is on the symmetry axis of the ring. Consider two segments at the top and bottom of the ring: The contributions $d\vec{E}$ to the field at P from these segments have the same x-component but opposite y-components. Hence the total y-component of field due to this pair of segments is zero. When we add up the contributions from all such pairs of segments, the total field $\vec{E}$ will have only a component along the ring's symmetry axis (the x-axis), with no component perpendicular to that axis (that is, no y-component or z-component). So the field at P is described completely by its x-component E_x.



21.24 Calculating the electric field on the axis of a ring of charge. In this figure, the charge is assumed to be positive.

To calculate E_x, note that the square of the distance r from a ring segment to the point P is $r^2 = x^2 + a^2$. Hence the magnitude of this segment's contribution $d\vec{E}$ to the electric field at P is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}$$

Using $\cos\alpha = x/r = x/(x^2 + a^2)^{1/2}$, the x-component dE_x of this field is

$$dE_x = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}\frac{x}{\sqrt{x^2 + a^2}} = \frac{1}{4\pi\epsilon_0}\frac{x\,dQ}{(x^2 + a^2)^{3/2}}$$

To find the *total* x-component E_x of the field at P, we integrate this expression over all segments of the ring:

$$E_x = \int \frac{1}{4\pi\epsilon_0}\frac{x\,dQ}{(x^2 + a^2)^{3/2}}$$

Since x does not vary as we move from point to point around the ring, all the factors on the right side except dQ are constant and can be taken outside the integral. The integral of dQ is just the total charge Q, and we finally get

$$\vec{E} = E_x\hat{\imath} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{3/2}}\hat{\imath} \qquad (21.8)$$

EVALUATE: Our result for $\vec{E}$ shows that at the center of the ring ($x = 0$) the field is zero. We should expect this; charges on opposite sides of the ring would push in opposite directions on a test charge at the center, and the forces would add to zero. When the field point P is much farther from the ring than its size (that is, $x \gg a$), the denominator in Eq. (21.8) becomes approximately equal to x^3, and the expression becomes approximately

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}\hat{\imath}$$

In other words, when we are so far from the ring that its size a is negligible in comparison to the distance x, its field is the same as that of a point charge. To an observer far from the ring, the ring would appear like a point, and the electric field reflects this.

In this example we used a *symmetry argument* to conclude that $\vec{E}$ had only an x-component at a point on the ring's axis of symmetry. We'll use symmetry arguments many times in this and subsequent chapters. Keep in mind, however, that such arguments can be used only in special cases. At a point in the xy-plane that is not on the x-axis in Fig. 21.24, the symmetry argument doesn't apply, and the field has in general both x- and y-components.