Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSE-A&B, FA20-BSE-A).

Objective of Lecture week3:-

- **Chapter2: Row Echelon Form (REF) OR Gauss-Elimination Method,**
  Row Reduced Echelon form (RREF) OR Gauss-Jordan Elimination Method
- **Row operations (Allowed).**
- **Optional (Quadratic interpolation, cubic interpolation, Global Positioning system).**

After studying this lecture, You are desired to do

Home Work: Do Questions 1-8 of Exercise 2.1, Questions 1-23, and 26, 27, 28 of Exercise 2.2, Questions 1-21 of Exercise 2.3, following link is extremely helpful in this regard.

https://www.slader.com/textbook/9780132296540-elementary-linear-algebra-with-applications-9th-edition/196/

Chapter 2: Solving Linear System

DEFINITION 2.1

An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following properties:

(a) All zero rows, if there are any, appear at the bottom of the matrix.

(b) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.

(c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

(d) If a column contains a leading one, then all other entries in that column are zero.

An $m \times n$ matrix satisfying properties (a), (b), and (c) is said to be in **row echelon form**. In Definition 2.1, there may be no zero rows.

EXAMPLE 1

The following are matrices in reduced row echelon form, since they satisfy properties (a), (b), (c), and (d):

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices that follow are not in reduced row echelon form. (Why not?)

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

■

EXAMPLE 2

The following are matrices in row echelon form:

$$H = \begin{bmatrix} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 & 1 & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

DEFINITION 2.2

An **elementary row (column) operation** on a matrix A is any one of the following operations:

(a) **Type I:** Interchange any two rows (columns).
(b) **Type II:** Multiply a row (column) by a nonzero number.
(c) **Type III:** Add a multiple of one row (column) to another.

DEFINITION 2.3 An $m \times n$ matrix B is said to be **row (column) equivalent** to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations to A.

Exercise 2.1.

5. Find the reduced row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 9 \\ 3 & 2 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 7 & -5 \end{bmatrix}$

Solution (b):-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 7 & -5 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_4 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 7 & -3 \end{bmatrix} R_{23} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 7 & -3 \end{bmatrix}$$

$$\begin{matrix} R_3 - 2R_2 \\ R_4 - 7R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \end{bmatrix} (-1)R_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} R_4 + 3R_3 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 2.2

5. Consider the linear system

$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + z &= -5 \\ 3x + y + z &= 3. \end{aligned}$$

(a) Find all solutions, if any exist, by using the Gaussian elimination method.

(b) Find all solutions, if any exist, by using the Gauss–Jordan reduction method.

6. Repeat Exercise 5 for each of the following linear systems:

(a)
$$\begin{aligned} x + y + 2z + 3w &= 13 \\ x - 2y + z + w &= 8 \\ 3x + y + z - w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 1 \\ x + y - 2z &= 3 \\ 2x + y + z &= 2 \end{aligned}$$

(c)
$$\begin{aligned} 2x + y + z - 2w &= 1 \\ 3x - 2y + z - 6w &= -2 \\ x + y - z - w &= -1 \\ 6x + z - 9w &= -2 \\ 5x - y + 2z - 8w &= 3 \end{aligned}$$

Solution 6(c): (a) Row Echelon Form (Gauss Elimination method)

Given system can be written in compact form as

$$AX = b$$

Where $A = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 3 & -2 & 1 & -6 \\ 1 & 1 & -1 & -1 \\ 6 & 0 & 1 & -9 \\ 5 & -1 & 2 & -8 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}; b = \begin{bmatrix} 1 \\ -2 \\ -1 \\ -2 \\ 3 \end{bmatrix}$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & 1 \\ 3 & -2 & 1 & -6 & -2 \\ 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right] R_{13} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 3 & -2 & 1 & -6 & -2 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$$

$$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 6R_1 \\ R_5 - 5R_1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & -6 & 7 & -3 & 8 \end{array}\right]$$

$$R_5 - R_4 \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

Read Row 4 and write as $0x + 0y + 0z + 0w = 4 \quad \to 0 = 4$ (*No Solution*).

A variation of Question 6(c):

(a) Find all solutions, if any exist, by using the Gaussian elimination method.

$$2x + y + z - 2w = 1$$

$$3x - 2y + z - 6w = -2$$

$$x + y - z - w = -1$$

$$6x + z - 9w = -2$$

$$5x - y + 2z - 8w = -1$$

Given system can be written in compact form as

$$AX = b$$

Where $A = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 3 & -2 & 1 & -6 \\ 1 & 1 & -1 & -1 \\ 6 & 0 & 1 & -9 \\ 5 & -1 & 2 & -8 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ -2 \\ -1 \\ -2 \\ -1 \end{bmatrix}$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & 1 \\ 3 & -2 & 1 & -6 & -2 \\ 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & -1 \end{array}\right] R_{13} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 3 & -2 & 1 & -6 & -2 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & -1 \end{array}\right]$$

$$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 6R_1 \\ R_5 - 5R_1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & -6 & 7 & -3 & 4 \end{array}\right]$$

$$R_5 - R_4 \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_{23} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$(-1)R_2 \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} R_3 + 5R_2 \\ R_4 + 6R_2 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & -11 & -3 & -14 \\ 0 & 0 & -11 & -3 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_4 - R_3 \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & -11 & -3 & -14 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \frac{R_3}{-11} \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 3/11 & 14/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We arrived at ROW ECHELN FORM (REF) and will find solution by backward substitution.

$$z + \frac{3}{11}w = \frac{14}{11} \quad \ldots\ldots\ldots (1)$$

$$y - 3z = -3 \ldots\ldots\ldots (2)$$

$$x + y - z - w = -1 \ldots\ldots\ldots (3)$$

3 equations and 4 unknowns (Unknown > Equatins) implies infinite many solutions

$$(1) \Rightarrow z = \frac{14}{11} - \frac{3}{11}w$$

Let $w = r \in R$, Then $z = \frac{14}{11} - \frac{3}{11}r$.

Put value of z in (2) we get

$$y = -3 + 3z = -3 + \frac{42}{11} - \frac{9}{11}r \Rightarrow y = \frac{9}{11} - \frac{9}{11}r$$

Put value of y, z and w in (3) we get

$$x + y - z - w = -1 \Rightarrow x = -y + z + w - 1 \Rightarrow x = -\frac{9}{11} + \frac{9}{11}r + \frac{14}{11} - \frac{3}{11}r + r - 1$$

$$x = \frac{-6}{11} + \frac{17}{11}r$$

Additional Solution Set: $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{-6}{11} + \frac{17}{11}r \\ \frac{9}{11} - \frac{9}{11}r \\ \frac{14}{11} - \frac{3}{11}r \\ 0 + r \end{bmatrix} = \begin{bmatrix} \frac{-6}{11} \\ \frac{9}{11} \\ \frac{14}{11} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{17}{11}r \\ \frac{-9}{11}r \\ \frac{-3}{11}r \\ r \end{bmatrix} = \begin{bmatrix} \frac{-6}{11} \\ \frac{9}{11} \\ \frac{14}{11} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{17}{11}r \\ \frac{-9}{11}r \\ \frac{-3}{11}r \\ \frac{11r}{11} \end{bmatrix}$

$$= \frac{1}{11}\begin{bmatrix} -6 \\ 9 \\ 14 \\ 0 \end{bmatrix} + \frac{r}{11}\begin{bmatrix} 17 \\ -9 \\ -3 \\ 11 \end{bmatrix}$$

A variation of Question 6(c): CONTINUED

(b) Find all solutions, if any exist, by using the Gauss–Jordan reduction method.

Now we will work for Reduced Row Echelon Form (**RREF**). Proceed part (a) as follows.

$$[A|b] \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 3/11 & 14/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_1 - R_2 \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 3/11 & 14/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -17/11 & -6/11 \\ 0 & 1 & 0 & 9/11 & 9/11 \\ 0 & 0 & 1 & 3/11 & 14/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$z + \frac{3}{11}w = \frac{14}{11} \Rightarrow z = \frac{14}{11} - \frac{3}{11}r, where\ w = r \in R$$

$$y + \frac{9}{11}w = \frac{9}{11} \Rightarrow y = \frac{9}{11} - \frac{9}{11}r$$

$$x-\frac{17}{11}w=\frac{-6}{11}\Rightarrow x=\frac{-6}{11}+\frac{17}{11}r$$

Another variation of Question 6(c):

$$2x+y+z-w=1$$

$$3x-2y+z-6w=-2$$

$$x+y-z-w=-1$$

$$6x+z-9w=-2$$

$$5x-y+2z-8w=-1$$

(a) Find all solutions, if any exist, by using the Gaussian elimination method.

OR

(b) Find all solutions, if any exist, by using the Gauss–Jordan reduction method.

(DO YOURSELF, IMPORTANT)

Solution is given here.(Unique solution)

$$X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-6/11\\9/11\\14/11\\0\end{bmatrix}$$

12. Find a 3×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x}=3\mathbf{x},\quad \text{where } A=\begin{bmatrix}1&2&-1\\1&0&1\\4&-4&5\end{bmatrix}.$$

Solution: $AX=3X$ **where** $X=\begin{bmatrix}x\\y\\z\end{bmatrix}=?$

Consider $AX=3X\ \ gives\ \ AX-3X=O_{3\times1}$ **implies** $AX-3X=O_{3\times1}$

$$(A - 3I)X = O_{3\times 1}$$

$$\left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - - - (1) \quad AX = O$$

We will work on augmented matrix to find solution of above homogeneous system

$$[A|O] = \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right] R_{12} \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ -2 & 2 & -1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right]$$

$$\begin{matrix} R_2 + 2R_1 \\ R_3 - 4R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 8 & -2 & 0 \end{array}\right] \frac{R_2}{-4} \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 8 & -2 & 0 \end{array}\right]$$

$$R_3 - 8R_2 \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 + 3R_2 \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - -(2)$$

Now read rows of last matrix (RREF) and write equivalent system as

$$0x + 0y + 0z = 0$$

$y - \frac{1}{4}z = 0 \rightarrow y = \frac{z}{4}$; $x + \frac{1}{4}z = 0 \rightarrow x = \frac{-z}{4}$**; where** $z = r \in R$ **is an arbitrary or free variable.**

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-r}{4} \\ \frac{r}{4} \\ r \end{bmatrix} = \begin{bmatrix} \frac{-r}{4} \\ \frac{r}{4} \\ \frac{4r}{4} \end{bmatrix} = \frac{r}{4}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \text{ Non trivial solution.}$$

14. In the following linear system, determine all values of a for which the resulting linear system has

(a) no solution;

(b) a unique solution;

(c) infinitely many solutions:

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

Solution: $[A|b] = \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 1 & | & 3 \\ 1 & 1 & a^2-5 & | & a \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & a^2-4 & | & a-2 \end{bmatrix}$

Case1: If $a^2 - 4 = 0$ and $a - 2 \neq 0$ implies No solution;

Now $a^2 = 4 \rightarrow a = \pm 2$; when $a = -2$, then

$$\begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & | & -2-2 \end{bmatrix}$$

$0x + 0y + 0z = -4$, not acceptable. Hence No solution at $a = -2$

Now take $a = 2$, then

$$\begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$, (This expression signals about "Infinite many solutions")

Case2: Infinite many solution:

$$\begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & a^2-4=0 & | & a-2=0 \end{bmatrix}$$

Implies $a^2 - 4 = 0$ and $a - 2 = 0$ both should be zero at the same time.

For $a = 2$ given system has Infinite many solution.

Case3: Unique Solution: For all values of $a \in R$ other than ± 2 system has Unique solution.

16. Repeat Exercise 14 for the linear system

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2-5)z &= a. \end{aligned}$$

Solution: $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \begin{array}{c} R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-6 & a-2 \end{array}\right]$

Case1: If $a^2 - 6 = 0$ and $a - 2 \neq 0$ implies No solution; $a^2 = 6 \rightarrow a = \pm\sqrt{6}$

When $a = -\sqrt{6}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -\sqrt{6}-2 \end{array}\right]$$

$0x + 0y + 0z = \sqrt{6} - 2$ Invalid, No solution.

When $a = \sqrt{6}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \sqrt{6}-2 \end{array}\right]$$

$0x + 0y + 0z = \sqrt{6} - 2$, Invalid, No solution.
for $a = \pm\sqrt{6}$ we have NO SOLUTION.

Case2: Infinite many solution: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-6=0 & a-2=0 \end{array}\right]$

Implies $a^2 - 6 = 0$ and $a - 2 = 0$ both should be zero at the same time.

There will be no value of a for which given system has Infinite many solution.

Case3: Unique Solution: For all values of $a \in R$ other than $\pm\sqrt{6}$ system has Unique solution.

21. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

Solution: $\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \quad \to AX = b$

$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ -3 & -2 & -1 & 2 \\ -2 & 0 & 2 & 4 \end{array}\right]$ **(I am going to give its solution via linear algebra toolkit) VERIFY IT**

$z = r \in R, y = -2r + 2, x = r - 2$

22. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Solution: $[A|b]$ =[A|b]= $\left[\begin{array}{ccc|c} 4 & 1 & 3 & a \\ 2 & -1 & 3 & b \\ 2 & 2 & 0 & c \end{array}\right] R_{13} \sim \left[\begin{array}{ccc|c} 2 & 2 & 0 & c \\ 2 & -1 & 3 & b \\ 4 & 1 & 3 & a \end{array}\right]$

$\frac{R_1}{2} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c/2 \\ 2 & -1 & 3 & b \\ 4 & 1 & 3 & a \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c/2 \\ 0 & -3 & 3 & b - c \\ 0 & -3 & 3 & a - 2c \end{array}\right]$

$R_3 - R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c \\ 0 & -3 & 3 & b - c \\ 0 & 0 & 0 & a - 2c - (b - c) \end{array}\right]$

$$= \begin{bmatrix} 1 & -3 & 2 & | & c \\ 0 & -3 & 3 & | & b-3c \\ 0 & 0 & 0 & | & a-b-c \end{bmatrix} \frac{R_2}{-3} \sim \begin{bmatrix} 1 & -3 & 2 & | & c \\ 0 & 1 & -1 & | & (3c-b)/3 \\ 0 & 0 & 0 & | & a-b-c=0 \end{bmatrix}$$

For system to be consistent (either unique solution OR infinite many solutions), our expression $a-b-c$ appeared in Row Echelon Form must be zero, i.e.

$$a-b-c=0$$

27. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} 2x + 2y + 3z &= a \\ 3x - y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

Solution: $[A|b] = \begin{bmatrix} 2 & 2 & 3 & | & a \\ 3 & -1 & 5 & | & b \\ 1 & -3 & 2 & | & c \end{bmatrix} R_{13} \sim \begin{bmatrix} 1 & -3 & 2 & | & c \\ 3 & -1 & 5 & | & b \\ 2 & 2 & 3 & | & a \end{bmatrix}$

$$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -3 & 2 & | & c \\ 0 & 8 & -1 & | & b-3c \\ 0 & 8 & -1 & | & a-2c \end{bmatrix}$$

$$R_3 - R_2 \sim \begin{bmatrix} 1 & -3 & 2 & | & c \\ 0 & 8 & -1 & | & b-3c \\ 0 & 0 & 0 & | & a-2c-(b-3c) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 & | & c \\ 0 & 8 & -1 & | & b-3c \\ 0 & 0 & 0 & | & a-b+c \end{bmatrix}$$

$$\frac{R_2}{8} \sim \begin{bmatrix} 1 & -3 & 2 & | & c \\ 0 & 1 & -1/8 & | & (b-3c)/8 \\ 0 & 0 & 0 & | & a-b+c=0 \end{bmatrix}$$

For system to be consistent (either unique solution OR infinite many solutions), our expression $a-b+c$ appeared in Row Echelon Form must be zero, i.e. $a-b+c=0$

For instance, take $a=5,\ b=4, c=-1$; Also when $a=9,\ b=3, c=-6$

2.3 Elementary Matrices; Finding A^{-1}

Recall, we defined three elementary row operations on a matrix A:

1. Interchange two rows.

2. Multiply a row by a nonzero constant c.

3. Add a constant c times one row to another.

DEFINITION 1 Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Our next goal is to show how matrix multiplication can be used to carry out an elementary row operation.

DEFINITION 2 A matrix E is called an ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

▶ EXAMPLE 1 Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1. ◀

Example: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $R_1 - 4R_2 \sim \begin{bmatrix} -15 & -10 \\ 4 & 3 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R_1 - 4R_2$ $E = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

$$EA = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -15 & -10 \\ 4 & 3 \end{bmatrix}$$

$E_k E_{k-1} \ldots E_3 E_2 E_1 A = I$ (if A is non-singular)

$$E_k^{-1} E_k E_{k-1} \ldots E_3 E_2 E_1 A = E_k^{-1} I$$

$$E_{k-1}^{-1} E_{k-1} \ldots E_3 E_2 E_1 A = E_{k-1}^{-1} E_k^{-1} I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_{k-1}^{-1} E_k^{-1} I \quad --- (2.3.1)\ \textit{Next Class}$$

Question: Write a matrix A as product of elementary matrices.

> **THEOREM 1.5.2** *Every elementary matrix is invertible, and the inverse is also an elementary matrix.*

Result: Every matrix A can be written as a product of elementary matrices (Important question). **Use** $(AB)^{-1} = (B)^{-1}(A)^{-1}$

$$A^{-1} = \left(E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_{k-1}^{-1} E_k^{-1} I\right)^{-1} = I E_k E_{k-1} \ldots E_3 E_2 E_1$$
$$= E_k E_{k-1} \ldots E_3 E_2 E_1 I --- (2.3.2)\ \textit{Today's class}$$

2.3.2 gives rise to idea of inverse of matrix using row operations.

Table 1

Row Operation on I That Produces E	**Row Operation on E That Reproduces I**
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c time row i to row j	Add $-c$ times row i to row j

Corollary 2.2 A is nonsingular if and only if A is row equivalent to I_n. (That is, the reduced row echelon form of A is I_n.)

Theorem 2.9 The homogeneous system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if A is singular. (That is, the reduced row echelon form of $A \neq I_n$.) ■

Note that at this point we have shown that the following statements are equivalent for an $n \times n$ matrix A:

1. A is nonsingular.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. A is row (column) equivalent to I_n. (The reduced row echelon form of A is I_n.)
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. A is a product of elementary matrices.

Inverse of Matrix using row operations.

$$[A \mid I] \textit{ Row operations to get reduced echelonform of } A$$

$$[I \mid A^{-1}]$$

Exercise 2.3

Question 11(b): Find the inverse of matrix A, using row operations.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_3+2R_2 \\ R_4-2R_2 \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 & -3 & 2 & 1 & 0 \\ 0 & 0 & 6 & -1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_4+2R_3 \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 & -3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -5 & 2 & 2 & 1 \end{array}\right]$$

$$\begin{matrix} \frac{R_3}{-3} \\ \frac{R_4}{3} \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & 1 & -2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/3 & 2/3 & 2/3 & 1/3 \end{array}\right]$$

$$R_1-R_2 \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & 1 & -2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/3 & 2/3 & 2/3 & 1/3 \end{array}\right]$$

$$\begin{matrix} R_1-3R_3 \\ R_2+2R_3 \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1/3 & 1 & -1/3 & -2/3 & 0 \\ 0 & 0 & 1 & -2/3 & 1 & -2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/3 & 2/3 & 2/3 & 1/3 \end{array}\right]$$

$$\begin{matrix} R_1-2R_4 \\ R_2+\frac{1}{3}R_4 \\ R_3+\frac{2}{3}R_4 \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 7/3 & -1/3 & -1/3 & -2/3 \\ 0 & 1 & 0 & 0 & 4/9 & -1/9 & -4/9 & 1/9 \\ 0 & 0 & 1 & 0 & -1/9 & -2/9 & 1/9 & 2/9 \\ 0 & 0 & 0 & 1 & -5/3 & 2/3 & 2/3 & 1/3 \end{array}\right] = [I \mid A^{-1}]$$

In Exercises 13 and 14, prove that each given matrix A is nonsingular and write it as a product of elementary matrices. (Hint: First, write the inverse as a product of elementary matrices; then use Theorem 2.7.)

13. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

Solution: $[A|I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$ $R_2 - 3R_1 \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$

$$\frac{R_2}{-2} \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$R_1 - 2R_2 \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] = [I \mid A^{-1}]$$

Matrix A is non singular.

$$R_2 - 3R_1 \textit{ on } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textit{ gives } E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad \textit{then } {E_1}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\frac{R_2}{-2} \textit{ on } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textit{ gives } E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \textit{ then } {E_2}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$R_1 - 2R_2 \textit{ on } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textit{ gives } E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \textit{ then } {E_3}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Verify $A = {E_1}^{-1}{E_2}^{-1}{E_3}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Extra work

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = E_3E_2E_1I = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$