Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BSM, SP20-BSE).

Some solved problems of 4.2

1. Let V be the set of all polynomials of (exactly) degree 2 with the definitions of addition and scalar multiplication as in Example 6.

(a) Show that V is not closed under addition.

(b) Is V closed under scalar multiplication? Explain.

Solution (1):

$$\text{(a) Given } V = P_2(t) = \{a_0 + a_1 t + a_2 t^2 : a_i \in R, a_2 \neq 0\}$$
$$= \{all\ parapolas\}$$

$$p(t) = 4t + 2t^2 \in V$$

$$q(t) = 1 + t - 2t^2 \in V$$

$$p(t) + q(t) = 1 + 5t \notin V$$

Not closed under Addition.

$$\text{(b) } c = 0 \in R,\ c.p(t) = 0.(4t + 2t^2) = 0 \notin V$$

Not closed under scalar multiplication.

2. Let V be the set of all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that the product $abcd = 0$. Let the operation $\oplus$ be standard addition of matrices and the operation $\odot$ be standard scalar multiplication of matrices.

(a) Is V closed under addition?

(b) Is V closed under scalar multiplication?

(c) What is the zero vector in the set V?

(d) Does every matrix A in V have a negative that is in V? Explain.

(e) Is V a vector space? Explain.

Solution2: (a) Given $V = M_{22} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : abcd = 0\right\}$

(a) Let $A \text{ and } B \in V$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -5 & 9 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 4 \\ -3 & 6 \end{bmatrix}$. Since the product (1)(4)(-3)(6) is not zero.

Hence $A + B$ *does not belong to* V.

(b) Let $k \in R$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ *then* $abcd = 0$

$$k.A = k.\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Now check $ka \times kb \times kc \times kd = k^4(abcd) = k^4(0) = 0$

Hence $k.A \in V$. Scalar multiplication is closed.

(c) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$ is zero matrix in our set V.

(d) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *such that* $abcd = 0$

$$-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

Now check $(-a)(-b)(-c)(-d) = abcd = 0$ *implies* $-A \in V$

(e) Clearly, V is not vector space, since it is not closed w.r.t. addition.

3. Let V be the set of all 2×2 matrices $A = \begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$.

Let the operation $\oplus$ be standard addition of matrices and the operation $\odot$ be standard scalar multiplication of matrices.

(a) Is V closed under addition?

(b) Is V closed under scalar multiplication?

(c) What is the zero vector in the set V?

(d) Does every matrix A in V have a negative that is in V? Explain.

(e) Is V a vector space? Explain.

Solution3: Do yourself. (Hint: This set forms vector space, see Question 2)

4. Let V be the set of all 2×1 matrices $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with integer entries such that $|v_1 + v_2|$ is even. For example,

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -8 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

all belong to V. Let the operation $\oplus$ be standard addition of matrices and the operation $\odot$ be standard scalar multiplication of matrices. Is V a vector space? Explain.

Solution4: (a) Given $V = M_{21} = \left\{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} : |v_1 + v_2| = even\right\}$

(1) Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ are elements of set V.

Then $|v_1 + v_2| = even$ and $|u_1 + u_2| = even$

$$u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Now check $|u_1 + v_1 + u_2 + v_2| = |u_1 + u_2 + v_1 + v_2| = |even + even| = even$

$u + v \in V$, Hence, V is closed under addition.

(b) Let $k = \frac{1}{2} \in R$ and $= \begin{bmatrix} 6 \\ 8 \end{bmatrix} \in V$. Now, calculate $k.v = \frac{1}{2}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ Clearly, 3+4 =7 and not even, So $k.v$ does not belong to V.

V is not vector space as it fails to hold scalar multiplication.

5. Prove in detail that R^n is a vector space.

Solution: See Example1

6. Show that P, the set of all polynomials, is a vector space.

Q6: Show that set of all polynomials of degree less or equal to n is vector space.

Solution: See Example3

In Exercises 7 through 11, the given set together with the given operations is not a vector space. List the properties of Definition 4.4 that fail to hold.

7. The set of all positive real numbers with the operations of $\oplus$ as ordinary addition and $\odot$ as ordinary multiplication.

8. The set of all ordered pairs of real numbers with the operations

$$(x, y) \oplus (x', y') = (x + x', y + y')$$

and

$$r \odot (x, y) = (x, ry).$$

Solution7: Given $V = set\ of\ all\ positive\ real\ numbers = \{x \in R : x > 0\}$

(1) Clearly, for any number say 2, the additive inverse -2 does not belong to V.

Show other properties that fail to hold.

Solution8:Given $V = R_2 = R^2 = \{(v_1,\ v_2): v_i \in R\} = \left\{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}: v_i \in R\right\}$

(a) $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$

(This is standard/ordinary/usual addition)

(Hint+Trick+Concept: If some standard set holds ordinary operation of either addition or scalar multiplication then it must satisfy related properties)

(b) for scalar $c \in R, v = (v_1, v_2) \in V$

$$c.v = c.(v_1, v_2) = (v_1, cv_2)$$

(property fail) (p7) Consider the property $c, d \in R, v = (v_1, v_2) \in V$

$$(c + d)\odot v = c\odot v \oplus d\odot v$$

$$L.H.S. = (c + d)\odot v = (c + d)\odot(v_1, v_2) = (v_1, (c + d)v_2)$$

$R.H.S. = c\odot v \oplus d\odot v = c\odot(v_1, v_2) \oplus d\odot(v_1, v_2) = (v_1, cv_2) \oplus (v_1, dv_2) = (2v_1, cv_2 + dv_2)$

Clearly, $L.H.S. \neq R.H.S.$

(Home work)(p8) Consider the property $c, d \in R, v = (v_1, v_2) \in V$

$c \odot (d \odot v) = (cd) \odot v$ (Do your self)

9. The set of all ordered triples of real numbers with the operations

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$$

and

$$r \odot (x, y, z) = (x, 1, z).$$

Solution9: Given $V = R^3 = \{(v_1, \ v_2, v_3): v_i \in R\} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_i \in R \right\}$

(a) $(v_1, \ v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

Given addition isstandard,hence 5 operations of addition must satisfy.

(b) for scalar $r \in R, v = (v_1, v_2, v_3) \in V$, scalar multiplication is given by

$$r.v = r.(v_1, v_2, v_3) = (v_1, 1, v_3)$$

(property fail) (p8) $1.v = v$

$$1.v = 1.(v_1, v_2, v_3) = (v_1, 1, v_3) \neq v$$

$$0.v = 0.(v_1, v_2, v_3) = (v_1, 1, v_3) \neq (0,0,0) = \mathbf{0}$$

Show other properties that fail to hold.

10. The set of all 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \leq 0$, with the usual operations in R^2.

Solution10: Given $V = R^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \leq 0 \right\}$

(a) For $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ are elements of set V.

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

(b) For $c \in R$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in V$

$$c.v = c.\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

Hint (Additive inverse) Let $v = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ Now $-v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ does not belong to V.

11. The set of all ordered pairs of real numbers with the operations $(x, y) \oplus (x', y') = (x + x', y + y')$ and $r \odot (x, y) = (0, 0)$.

Solution11: Do your self (Hint: p8 : 1.v=v)

12. Let V be the set of all positive real numbers; define $\oplus$ by $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ ($\oplus$ is ordinary multiplication) and define $\odot$ by $c \odot \mathbf{v} = \mathbf{v}^c$. Prove that V is a vector space.

Solution12: (An unusual vector space)

Given $V = set\ of\ positive\ real\ numbers = \{x \in R : x > 0\}$

(a) For $u, v \in V$ addition is given as $u \oplus v = uv \in V$ (closed under addition)

(p1) commutativity. For $u, v \in V$, then $u \oplus v = uv = vu = v \oplus u$

(p2) Assosiative law. For $u, v, w \in V$, $(u \oplus v) \oplus w = u \oplus (v \oplus w)$

$$uv \oplus w = u \oplus vw$$

$$uvw = uvw$$

(p3) Additive identity: $u \oplus 1 = u(1) = u$. Hence 1 is additive identity of V.

(p4) Additive inverse: $u \oplus \frac{1}{u} = u\left(\frac{1}{u}\right) = 1$

(b) For $c \in R$ and $v \in V$ scalar multiplication is given as $c \odot v = v^c$(set V is closed w.r.t. scalar multiplication)

$$c \odot v = v^c \in V\ if\ c\ is\ poisitive$$
$$-5 \odot v = v^{-5} = 1/v^5 \in V\ if\ c\ is\ negative$$
$$0 \odot v = v^0 = 1 \in V\ if\ c\ is\ zero$$

(p5) For $c \in R$, $and\ u, v \in V$, $c \odot (u \oplus v) = c \odot u \oplus c \odot v$

$$c \odot uv = u^c \oplus v^c$$

$$(uv)^c = u^c v^c$$

(p6)For $c, d \in R,\ and\ u \in V, (c+d) \odot u = c \odot u \oplus d \odot u$

$$u^{(c+d)} = u^c \oplus u^d$$

$$u^{(c+d)} = u^c u^d$$

$$u^{(c+d)} = u^{c+d}$$

(p7)For $c, d \in R,\ and\ u \in V, c \odot (d \odot u) = (c.d) \odot u$

$$c \odot u^d = u^{cd}$$

$$(u^d)^c = u^{cd}$$

(p8)$1 \odot u = u^1 = u$

All properties of vector space are satisfied, Hence given set V w.r.t. given operations is vector space.

Not to Do: Q13, 14,15

16. Let V be the set of all positive real numbers; define $\oplus$ by $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv} - 1$ and $\odot$ by $c \odot \mathbf{v} = \mathbf{v}$. Is V a vector space?

17. Let V be the set of all real numbers; define $\oplus$ by $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $\odot$ by $c \odot \mathbf{u} = c + \mathbf{u}$. Is V a vector space?

18. Let V be the set of all real numbers; define $\oplus$ by $\mathbf{u} \oplus \mathbf{v} = 2\mathbf{u} - \mathbf{v}$ and $\odot$ by $c \odot \mathbf{u} = c\mathbf{u}$. Is V a vector space?

What to do:Q16, 17, and 18.

Some solved problems of 4.4

1. For each of the following vector spaces, give two different spanning sets:

(a) R^3 **(b)** M_{22} **(c)** P_2

Solution: (a) Spanning sets for vector space R^3

$$S = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$T = \left\{ v_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\}$$

(b) Spanning sets for vector space M_{22}

$$S = \left\{ e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T = \left\{ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \right\}$$

(c) Spanning sets for vector space P_2.

$$All\ polynomials\ of\ degree\ less\ or\ equal\ to\ 2 = \{at^2 + bt + ct^0 : a, b, c \in R\}$$

$$S = \{e_1 = t^2, e_2 = t, e_3 = 1\}$$

Example: $-5t^2 + 5t - 7 = -5(e_1) + 5(e_2) - 7(e_3)$

$$at^2 + bt + c = a(e_1) + b(e_2) + c(e_3)$$

Span S=P_2

$$T = \{p_1(t) = -3t^2, p_2(t) = 2t, p_3(t) = 4\}$$

Span T=P_2

2. In each part, explain why the set S is not a spanning set for the vector space V.

(a) $S = \{t^3, t^2, t\}$, $V = P_3$

(b) $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $V = R^2$

(c) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, $V = M_{22}$

Solution: In each case number of vectors in set S are not sufficient to span given vector space.

3. In each part, determine whether the given vector $p(t)$ in P_2 belongs to span $\{p_1(t), p_2(t), p_3(t)\}$, where

$$p_1(t) = t^2 + 2t + 1, \quad p_2(t) = t^2 + 3,$$
$$\text{and} \quad p_3(t) = t - 1.$$

(a) $p(t) = t^2 + t + 2$

(b) $p(t) = 2t^2 + 2t + 3$

(c) $p(t) = -t^2 + t - 4$

(d) $p(t) = -2t^2 + 3t + 1$

Solution: Given $\boldsymbol{S} = \{\boldsymbol{p_1(t) = t^2 + 2t + 1, p_2(t) = t^2 + 3, p_3(t) = t - 1}\}$

Consider definition of L.C. $\boldsymbol{v = a_1v_1 + a_2v_2 + a_3v_3} \ \ldots\ldots\ldots (\mathbf{1}),$

$$\boldsymbol{p(t) = a_1p_1(t) + a_2p_2(t) + a_3p_3(t)} \ \ldots\ldots\ldots (\mathbf{2})$$

Our goal is to find scalars $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ **and** $\boldsymbol{a_3}$.

(a) $\{\boldsymbol{t^2 + t + 2 = a_1(t^2 + 2t + 1) + a_2(t^2 + 0t + 3) + a_3(0t^2 + t - 1)}$

$$\boldsymbol{t^2 + t + 2 = (a_1 + a_2)t^2 + (2a_1 + a_3)t + (a_1 + 3a_2 - a_3)}$$

$$\boldsymbol{Equating\ coefficients\ of\ like\ powers}$$

$$\boldsymbol{a_1 + a_2 = 1}$$

$$\boldsymbol{2a_1 + a_3 = 1}$$

$$\boldsymbol{a_1 + 3a_2 - a_3 = 2}\}$$

$$(\mathbf{2})\ \boldsymbol{implies}\ [A|b] = \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} & \mathbf{2} \end{array}\right] (\boldsymbol{Hint})\boldsymbol{Infinite\ many\ soln.}$$

$$\boldsymbol{p(t)} \in \boldsymbol{Span\ S}$$

(b) $[A|b] = \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} & \mathbf{3} \end{array}\right]$ $\boldsymbol{No\ soln}\ (\boldsymbol{Linear\ algebra\ toolkit})$

$$\boldsymbol{p(t)} \notin \boldsymbol{Span\ S}$$

(c) $[A|b]=\begin{bmatrix}1&1&0&|-1\\2&0&1&|\ 1\\1&3&-1&|-4\end{bmatrix}$ ***Infinite many soln***

4. In each part, determine whether the given vector A in M_{22} belongs to span $\{A_1, A_2, A_3\}$, where

$$A_1=\begin{bmatrix}1&-1\\0&3\end{bmatrix},\quad A_2=\begin{bmatrix}1&1\\0&2\end{bmatrix},$$

$$\text{and}\quad A_3=\begin{bmatrix}2&2\\-1&1\end{bmatrix}.$$

(a) $A=\begin{bmatrix}5&1\\-1&9\end{bmatrix}$ **(b)** $A=\begin{bmatrix}-3&-1\\3&2\end{bmatrix}$

(c) $A=\begin{bmatrix}3&-2\\3&2\end{bmatrix}$ **(d)** $A=\begin{bmatrix}1&0\\2&1\end{bmatrix}$

Solution: Given $S=\left\{A_1=\begin{bmatrix}1&-1\\0&3\end{bmatrix}, A_2=\begin{bmatrix}1&1\\0&2\end{bmatrix}, A_3=\begin{bmatrix}2&2\\-1&1\end{bmatrix}\right\}$

Consider definition of L.C. $v=a_1v_1+a_2v_2+a_3v_3$ ……… (1),

$$A=a_1A_1+a_2A_2+a_3A_3 \quad \ldots\ldots\ldots (2)$$

Our goal is to find scalars a_1, a_2 and a_3.

(a) $\{\begin{bmatrix}5&1\\-1&9\end{bmatrix}=a_1\begin{bmatrix}1&-1\\0&3\end{bmatrix}+a_2\begin{bmatrix}1&1\\0&2\end{bmatrix}+a_3\begin{bmatrix}2&2\\-1&1\end{bmatrix}$

$$\begin{bmatrix}5&1\\-1&9\end{bmatrix}=\begin{bmatrix}a_1&-a_1\\0&3a_1\end{bmatrix}+\begin{bmatrix}a_2&a_2\\0&2a_2\end{bmatrix}+\begin{bmatrix}2a_3&2a_3\\-1a_3&a_3\end{bmatrix}$$

$$\begin{bmatrix}5&1\\-1&9\end{bmatrix}=\begin{bmatrix}a_1+a_2+2a_3&-a_1+a_2+2a_3\\-a_3&3a_1+2a_2+a_3\end{bmatrix}$$

Equating components of equal matrices

$$a_1+a_2+2a_3=5$$

$$-a_1+a_2+2a_3=1$$

$$-a_3=-1$$

$$3a_1+2a_2+a_3=9\}$$

$$(2)\text{implies}[A|b]=\begin{bmatrix}1&1&2&|\ 5\\-1&1&2&|\ 1\\0&0&-1&|-1\\3&2&1&|\ 9\end{bmatrix}$$

$$(\boldsymbol{Toolkit}) \quad \boldsymbol{a_1 = 2, a_2 = 1\ ;\ a_3 = 1, Verify}$$

$$\boldsymbol{A \in Span\ S}$$

Home Work: Question 7 (part d); Question 9 and 10.

4.5 Linear Independence

In Section 4.4 we developed the notion of the span of a set of vectors together with spanning sets of a vector space or subspace. Spanning sets S provide vectors so that any vector in the space can be expressed as a linear combination of the members of S. We remarked that a vector space can have many different spanning sets and that spanning sets for the same space need not have the same number of vectors. We illustrate this in Example 1.

Motivation

Example 1: **Let a subspace W of R^3, where $W = \left\{\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} : a, b \in R\right\}$.**

Consider $\begin{bmatrix} a+0b \\ 0a+b \\ a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Then spanning set for W is $S_1 = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$, Here are some other spanning sets for W, e.g.,

$S_2 = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$, $S_3 = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}\right\}$, $S_4 = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

Efficient spanning set is S_1 as it is smallest and vectors are linearly independent. This gives rise to our next definition.

Linearly independent set of vectors:

Let $S = \{v_1, v_2, v_3, \dots, v_k\}$ be the set of vectors in vector space V. Then the vectors in set S are said to be "Linealy Independent (L.I.)" if

$$\boldsymbol{a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0} \ \text{ gives } \ \boldsymbol{a_1 = a_2 = \cdots = a_k} = 0$$

Some Important examples

Example 2:

Determine whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent.

We know definition of LI $\boldsymbol{a_1 v_1 + a_2 v_2 + a_3 v_3 = 0}$

Solution

Forming Equation (1),

$$a_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we obtain the homogeneous system (verify)

$$\begin{aligned} 3a_1 + a_2 - a_3 &= 0 \\ 2a_1 + 2a_2 + 2a_3 &= 0 \\ a_1 - a_3 &= 0. \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right],$$

whose reduced row echelon form is (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus there is a nontrivial solution

$$\begin{bmatrix} k \\ -2k \\ k \end{bmatrix}, \quad k \neq 0 \text{ (verify)},$$

so the vectors are linearly dependent.

Are the vectors $\mathbf{v}_1 = [1 \quad 0 \quad 1 \quad 2]$, $\mathbf{v}_2 = [0 \quad 1 \quad 1 \quad 2]$, and $\mathbf{v}_3 = [1 \quad 1 \quad 1 \quad 3]$ in R_4 linearly dependent or linearly independent?

Solution

We form Equation (1),

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0},$$

and solve for a_1, a_2, and a_3. The resulting homogeneous system is (verify)

$$\begin{aligned} a_1 \qquad + a_3 &= 0 \\ a_2 + a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \\ 2a_1 + 2a_2 + 3a_3 &= 0. \end{aligned}$$

The corresponding augmented matrix is (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right],$$

Example 3:

and its reduced row echelon form is (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the only solution is the trivial solution $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent. ■

Example 4:

Are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

in M_{22} linearly independent?

$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

Solution

We form Equation (1),

$$a_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and solve for a_1, a_2, and a_3. Performing the scalar multiplications and adding the resulting matrices gives

$$\begin{bmatrix} 2a_1 + a_2 & a_1 + 2a_2 - 3a_3 \\ a_2 - 2a_3 & a_1 + a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using the definition for equal matrices, we have the linear system

$$\begin{aligned} 2a_1 + a_2 \phantom{{}-3a_3} &= 0 \\ a_1 + 2a_2 - 3a_3 &= 0 \\ a_2 - 2a_3 &= 0 \\ a_1 \phantom{{}+2a_2} + a_3 &= 0. \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right],$$

and its reduced row echelon form is (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus there is a nontrivial solution

$$\begin{bmatrix} -k \\ 2k \\ k \end{bmatrix}, \quad k \neq 0 \text{ (verify)},$$

so the vectors are linearly dependent.

Example 5:

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

in R^3. Is $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly dependent or linearly independent?

Solution: (Hint: given number of vectors are more than dimension of space R^3,

OR number of unknowns > number of equations (infinite many solutions), hence must be dependent)

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0 - - - (1)$$

Find REF of augmented matrix obtained from (1)

$$\begin{bmatrix} 1 & 1 & -3 & 2| & 0 \\ 2 & -2 & 2 & 0| & 0 \\ -1 & 1 & -1 & 0| & 0 \end{bmatrix} \textit{Row echeln form} \sim \begin{bmatrix} 1 & 1 & -3 & 2|0 \\ 0 & 1 & -2 & 1|0 \\ 0 & 0 & 0 & 0|0 \end{bmatrix}$$

This homogeneous system has non-trivial (infinite) solution, vectors are L.D.

Extra Question! Which of these vectors are L.I.?

Answer (Easy) Select Columns with leading Ones.$\{v_1, v_2\}$

EXAMPLE 6 Are the vectors $\mathbf{v}_1 = t^2 + t + 2$, $\mathbf{v}_2 = 2t^2 + t$, and $\mathbf{v}_3 = 3t^2 + 2t + 2$ in P_2 linearly dependent or linearly independent?

Solution: Consider equation $a_1v_1 + a_2v_2 + a_3v_3 = 0 \ - - - \ (1)$

$$(1) \Rightarrow \begin{bmatrix} 1 & 2 & 3|0 \\ 1 & 1 & 2|0 \\ 2 & 0 & 2|0 \end{bmatrix} \textit{Row echeln form} \sim \begin{bmatrix} 1 & 2 & 3|0 \\ 0 & 1 & 1|0 \\ 0 & 0 & 0|0 \end{bmatrix}$$

$$a_2 + a_3 = 0; \ \ a_1 + 2a_2 + 3a_3 = 0$$

Since a_3 is arbitrary, So we have *infinite many solution*. Hence given set of vectors (polynomials) are L.D.

Home Work from exercise 4.5

Question 3, 5, 10, 11(c), 12(c), 13(c).

4.6 Basis and Dimension

In this section we continue our study of the structure of a vector space V by determining a set of vectors in V that completely describes V. Here we bring together the topics of span from Section 4.4 and linear independence from Section 4.5. In the case of vector spaces that can be completely described by a finite set of vectors, we prove further properties that reveal more details about the structure of such vector spaces.

Basis

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to form a **basis** for V if (a) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span V and (b) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Dimension

The **dimension** of a nonzero vector space V is the number of vectors in a basis for V. We often write **dim** V for the dimension of V. We also define the dimension of the trivial vector space $\{\mathbf{0}\}$ to be zero.

Let $V = R^3$. The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for R^3, called the **natural basis** or **standard basis**, for R^3. We can readily see how to generalize this to obtain the natural basis for R^n. Similarly, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is the natural basis for R_3. ■

The natural basis for R^n is denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th row;}$$

that is, $\mathbf{e}_i$ is an $n \times 1$ matrix with a 1 in the ith row and zeros elsewhere.

Example2 (Recall)

The set of vectors$\{t^2, t, 1\}$ forms a standard or natural basis for the vector space P_2(all polynomials of degree less or equal to 2).

Dimension of P_2 is 3

The set of vectors$\{t^3, t^2, t, 1\}$ forms a standard or natural basis for the vector space P_3(all polynomials of degree less or equal to 3).

Dimension of P_3 is 4

The set of vectors $\{t^n, t^{n-1}, \ldots, t, 1\}$ forms a basis for the vector space P_n called the **natural**, or **standard basis**, for P_n.

Dimension of P_n is $n+1$.

Example3 (Recall)

The set of vectors $S = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ forms a standard or natural basis for the vector space M_{22}.

Dimension of $M_{22}$is 4.

Similarly The set of vectors

$S = \left\{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\right\}$ forms a standard or natural basis for the vector space M_{23}.

Dimension of M_{23} is 6.

Some Examples helpful in exercise 4.6

Example 1:

Show that the set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2.

Solution: For this set to become basis for P_2, it must qualify two conditions.

(1) S must be L.I. (2) Span S=P_2.
First we check Linear independence, for this consider

$$\boldsymbol{a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 - - - (1)}$$

$$(1)[A|0] \rightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 2 & 0\\1 & -1 & 2 & 0\end{array}\right] Do\ yourself\ (Row\ operations) \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 2 & 0\\0 & 0 & 4 & 0\end{array}\right]$$

$$\boldsymbol{4a_3} = 0, \qquad \boldsymbol{a_2 + 2a_3} = 0, \boldsymbol{a_1} = 0$$

Hence given set **S** is L.I.

(2) **(very important) Number of elements in set S=3**

The dimension of vector space P_2=3, guarantees that Span S=P_2. Hence given set form basis for P_2.

Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 1 & -1 & 2 \end{bmatrix},$$
$$\mathbf{v}_3 = \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix},$$

Example 2: is a basis for R_4.

Solution: For this set to become basis for R_4, it must qualify two conditions.

(1) **S must be L.I. (2) Span S=R_4.**
First we check Linear independence, for this consider

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0 --- (1)$$

$$(1) \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right] (Verify) \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 1/4 & 0 \end{array}\right]$$

$$\left(\frac{1}{4}\right)a_4 = 0\,;\ a_3 - \left(\frac{1}{4}\right)a_4 = 0;\ a_2 + 2a_3 = 0;\ a_1 + a_4 = 0$$

Clearly, using backward substitution we get $a_1 = a_2 = a_3 = a_4 = 0$. Hence given set of vectors are L.I.

(2) **(very important) Number of elements in set S=4**

The dimension of vector space R_4=4, guarantees that Span S=R_4.

Hence given set form basis for R_4.

Example 3:

The set W of all 2×2 matrices with trace equal to zero is a subspace of M_{22}. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is a basis for W.

Solution:For this set to become basis for W, it must qualify two conditions.

(1) **S must be L.I. (2) Span S=W.**
First we check Linear independence, for this consider

$$a_1v_1 + a_2v_2 + a_3v_3 = 0 --- (1)$$

$$(1) \to \begin{bmatrix} 0 & 0 & 1 & |0 \\ 1 & 0 & 0 & |0 \\ 0 & 1 & 0 & |0 \\ 0 & 0 & -1 & |0 \end{bmatrix} R_{12} \sim \begin{bmatrix} 1 & 0 & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 1 & 0 & |0 \\ 0 & 0 & -1 & |0 \end{bmatrix} R_{23} \sim \begin{bmatrix} 1 & 0 & 0 & |0 \\ 0 & 1 & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 0 & -1 & |0 \end{bmatrix} R_4 + R_3 \sim \begin{bmatrix} 1 & 0 & 0|0 \\ 0 & 1 & 0|0 \\ 0 & 0 & 1|0 \\ 0 & 0 & 0|0 \end{bmatrix}$$

Clearly, we get $a_1 = a_2 = a_3 = 0$

(2) (very important) Number of elements in set S=3

The dimension of vector space M_{22}=4, But W is subspace of M_{22}. Hence dimension of W is less than four. This guarantees that Span S=W.

Hence given set form basis for W, and dimension of W is 3.

Example 4:

Let $V = R_3$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where $\mathbf{v}_1 = [1 \quad 0 \quad 1]$, $\mathbf{v}_2 = [0 \quad 1 \quad 1]$, $\mathbf{v}_3 = [1 \quad 1 \quad 2]$, $\mathbf{v}_4 = [1 \quad 2 \quad 1]$, and $\mathbf{v}_5 = [-1 \quad 1 \quad -2]$. We find that S spans R_3 (verify), and we now wish to find a subset of S that is a basis for R_3. Using the procedure just developed, we proceed as follows:

Solution: Given: vector space is $V = R_3$, and set S of vectors. What question is asked from you. (1) verify Span $S = R_3$(2) Find subset of S that is basis for R_3.

(Recall number of vectors in S are more than dimension of R_3, hence set $S$is clearly L.D.)

(2) First we find L.I. subset of S, for this consider

$$\boldsymbol{a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 = 0 - - - (1)}$$

$$(1) \to \begin{bmatrix} 1 & 0 & 1 & 1 & -1|0 \\ 0 & 1 & 1 & 2 & 1\ |0 \\ 1 & 1 & 2 & 1 & -2|0 \end{bmatrix} R_3 - R_1 \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1|0 \\ 0 & 1 & 1 & 2 & 1\ |0 \\ 0 & 1 & 1 & 0 & -1|0 \end{bmatrix}$$

$$R_3 - R_2 \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1|0 \\ 0 & 1 & 1 & 2 & 1\ |0 \\ 0 & 0 & 0 & -2 & -2|0 \end{bmatrix} \frac{R_3}{-2} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1|0 \\ 0 & 1 & 1 & 2 & 1\ |0 \\ 0 & 0 & 0 & 1 & 1\ |0 \end{bmatrix}$$

(Recall Extra Question! Which of these vectors are L.I.?

Answer (Easy) Select Columns with leading Ones.)

Thereforeset $T = \{v_1, v_2, v_4\}$is L.I. subset of S.

(1) Now, number of elements in set $T = \{v_1, v_2, v_4\}$ are 3.The dimension of vector space $R_3 = 3$, guarantees Span T =R_3.Hence set $T = \{v_1, v_2, v_4\}$form basis for R_3.

Q15:- Find all values of a for which vectors $S = \{\boldsymbol{v_1} = [a^2 \quad 0 \quad 1], \boldsymbol{v_2} = [0 \quad a \quad 2], \boldsymbol{v_3} = [1 \quad 0 \quad 1]\}$ is a basis for R_3.

Solution: To check linear independence, consider formula $\boldsymbol{a_1 v_{1+} a_2 v_2 + a_3 v_3 = 0}$

$$[A|0]=\begin{bmatrix} a^2 & 0 & 1 & | & 0 \\ 0 & a & 0 & | & 0 \\ 1 & 2 & 1 & | & 0 \end{bmatrix}$$ (To continue in this way is difficult when parameter a is in vectors)

(This is special question) Hint: Set of vectors L.I →Matrix A must have Identity form in RREF→ matrix is invertible → $|\mathbf{A}| \neq 0$.

$$|A| = \begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0 \rightarrow a(a^2-1)=0 \rightarrow \text{either a} = 0 \text{ or } a^2-1=0$$

→ $a = 0, 1, -1$given vectors are L.D.

Given set will be L.I for all values of $a \in R$ other than $0, 1, -1$. Hence form basis for R_3.

In Exercises 23 and 24, find the dimensions of the given subspaces of R_4.

24. **(a)** All vectors of the form $\begin{bmatrix} a & b & c & d \end{bmatrix}$, where $a = b$

(b) All vectors of the form

$$\begin{bmatrix} a+c & a-b & b+c & -a+b \end{bmatrix}$$

Solution24: $W = \{[a+c \quad a-b \quad b+c \quad -a+b]: a, b, c \in R\}$

Consider $w \in W$ such that

$$\boldsymbol{w} = [\boldsymbol{a}+\boldsymbol{c} \quad \boldsymbol{a}-\boldsymbol{b} \quad \boldsymbol{b}+\boldsymbol{c} \quad -\boldsymbol{a}+\boldsymbol{b}]$$
$$= \boldsymbol{a}[1 \quad 1 \quad 0 \quad -1] + \boldsymbol{b}[0 \quad -1 \quad 1 \quad 1] + \boldsymbol{c}[1 \quad 0 \quad 1 \quad 0]$$

$\boldsymbol{v_1} = [1 \quad 1 \quad 0 \quad -1]$; $\boldsymbol{v_2} = [0 \quad -1 \quad 1 \quad 1]$; $\boldsymbol{v_3} = [1 \quad 0 \quad 1 \quad 0]$

Check LI of v_1, v_2 *and* v_3.

Verify $a_1v_1 + a_2v_2 + a_3v_3 = 0$

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 1 & -1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ -1 & 1 & 0 & | & 0 \end{bmatrix}$$ Doyourself

Home Work fromexercise 4.6: Question 7, 8, 15, 24.