Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (FA20-BCS-A&B, FA20-BSE-A).

Objective of Lecture 1+2:-

- **Chapter1: Non-Homogeneous Linear system of equations**
- **Unknowns**
- **Unique solution**
- **No solution**
- **Infinitely many solutions**
- **Consistent system (unique soln or infinitely many solns.)**
- **Inconsistent system (No soln)**
- **Homogeneous system**
- **Trivial solution (all unknowns are 0)**
- **Nontrivial solution (Infinitely many solutions)**
- **Equivalent systems**
- **Method of elimination (studied in class 12^{th})**
- **Matrices their handling and properties.**
- **Chapter2: Row Echelon Form (REF) OR Gauss-Elimination Method,**
  Row Reduced Echelon form (RREF) OR Gauss-Jordan Elimination Method
- **Row operations.**

After studying this lecture, You are desired to do

Home Work: Do Questions 1-23 of Exercise 1.1, Questions 1-12 of Exercise 1.2, Questions 1-27 of Exercise 1.3, Questions 1-5 and 8-19 of Exercise 1.4 following link is extremely helpful in this regard.

https://www.slader.com/textbook/9780132296540-elementary-linear-algebra-with-applications-9th-edition/196/

Chapter 1+2: Linear Equations and Matrices,

Solving Linear System

$$2x_1 + x_1x_2 + x_3 = 5\ (product\ of\ unknowns\ is\ not\ allowed)$$

$$2x_1 + \sqrt{x_2} + x_3 = 1\ (any\ root\ is\ not\ allowed)$$

$$2x_1 + (x_2)^n + x_3 = 1\ (any\ power\ of\ unknown\ is\ not\ allowed)$$

$$2x_1 + \sqrt{3}\,x_2 + x_3 = (5)^{\frac{1}{3}}\ (is\ it\ linear\ or\ not?)\ \boldsymbol{Linear}$$

(1) Tiny System of linear eqns (Non-homogenious)

$$x_1 - 3\,x_2 = -7$$

$$2x_1 + \,x_2 = 7$$

Solution: $x_1 = 2$; $x_2 = 3$ (Unique solution)

(2) Tiny System of linear eqns (Non-homogenious)

$$x_1 - 3\,x_2 = -7$$

$$3x_1 - 9\,x_2 = -21$$

Solution: 0=0 (Important, signaling some equation is redundant or overlapping) (infinite many solution)

$$x_1 - 3\,x_2 = -7 \quad \rightarrow x_1 = -7 + 3\,x_2 \; and \; x_2 \in R$$

x_2 is free variable or arbitrary variable.

Soln1 $x_2 = 3 \; then \; x_1 = 2$

Soln2 $x_2 = 50 \; then \; x_1 = 143$ (Infinite many solution)

(3) Tiny System of linear eqns (Non-homogenious)

$$x_1 - 3\,x_2 = -7$$

$$x_1 - 3\,x_2 = 7$$

Solution: (No solution) 0 = -14

(1) Tiny System of linear eqns (Homogenious)

$$x_1 - 3\,x_2 = 0$$

$$2x_1 + \,x_2 = 0$$

Solution: $x_1 = 0$; $x_2 = 0$ (Unique solution) (Trivial soln)

(2) Tiny System of linear eqns (homogenious)

$$x_1 - 3\,x_2 = 0$$

$$3x_1 - 9\,x_2 = 0$$

Solution: $0 = 0 \quad \to x_1 - 3\,x_2 = 0 \;\to x_1 = 3\,x_2\; and\; x_2 \;\in R$

Soln1: $x_2 = 1\; then\; x_1 = 3$

Soln2: $x_2 = 5\; then\; x_1 = 15$ (Infinite many solution) (Non-Trivial soln)

(3) Tiny System of linear eqns (homogenious)

$$x_1 - 3\,x_2 = 0$$

$$x_1 - 3\,x_2 = 0$$

Solution: (No solution 0 = -14) But here 0 = 0

$$x_1 - 3\,x_2 = 0 \;\to\; x_1 = 3\,x_2\; and\;\; x_2 \;\in R$$

Soln1 $x_2 = 1\; then\; x_1 = 3$

Soln2 $x_2 = 5\; then\; x_1 = 15$ (Non-Trivial soln)

(1) Equivalent System of linear eqns (Non-homogenious)

$$x_1 - 3\,x_2 = -7$$

$$2x_1 + \; x_2 = 7$$

Solution: $x_1 = 2;\; x_2 = 3$

(2) Other System of linear eqns (Non-homogenious)

$$8x_1 - 3\,x_2 = 7$$

$$3x_1 - 2\,x_2 = 0$$

$$10x_1 - 2\,x_2 = 14$$

Solution: $x_1 = 2;\; x_2 = 3$

Question 16: (Exercise 1.1)

16. Given the linear system

$$3x + 4y = s$$
$$6x + 8y = t,$$

(a) Determine particular values for s and t so that the system is consistent.

(b) Determine particular values for s and t so that the system is inconsistent.

(c) What relationship between the values of s and t will guarantee that the system is consistent?

3x+4y=s-----(1)

6x+8y=t ------(2)

2*(1) - (2)

6x+8y=2s-----(3)

-6x-8y=-t ------(4)

0=2s-t or **t=2s-----(5)**

(a) **Consistent**

When s=1 then t=2

When s=3 then t=6

When s=10 then t=20

(b) **Inconsistent**

Let s=1 and $t \neq 2$; for s=3 take $t \neq 6$

if s=10 take $t \neq 20$

(c) **Consistent**

Relation t=2s.

17. Given the linear system

$$\begin{aligned} x + \quad 2y &= 10 \\ 3x + (6+t)y &= 30, \end{aligned}$$

(a) Determine a particular value of t so that the system has infinitely many solutions.

(b) Determine a particular value of t so that the system has a unique solution.

(c) How many different values of t can be selected in part (b)?

Question 17: (Exercise 1.1)

x+2y=10-----(1)

3x+(6+t)y=30 ------(2)

3*(1)-(2)

3x+6y=30-----(3)

-3x-(6+t)y=-30 ------(4)

6y-(6+t)y=0

6y-6y-ty=0

ty=0----(5)

(a) Infinite many soln, t=?,

when t=0 then (5) implies 0=0 (important, overlapping or redundancy)

x+2y=10-----(1)

3x+6y=30 ------(2)

x+2y=10 implies x=10-2y where y is any real number
Soln1: y=0, x=10 ; Soln2 y=10, x=-10

(b) When t=any number other than 0, say t=n then n*y=0 implies y=0 then x=10 (Unique solution)

(c) Infinite values of t.

MATRICES

See pages (11, 12, 13) of Book.

Properties of Matrices that deviate from normality: (**See page 39 of book**)

(1) Commutative property

A, B are matrices. Then **AB may or may not equal to BA.**

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times \mathbf{2}}$; $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}_{\mathbf{2}\times 3}$

$$AB = \begin{bmatrix} 5 & 8 & -1 \\ 11 & 18 & -1 \end{bmatrix}_{2\times 3}$$

Whereas BA is even not defined.

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}_{2\times \mathbf{3}} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathbf{2}\times 2}$$

(2) Cancellation Law (for real numbers ac=ad implies c=d)

For Matrices

$AB = AC$ does not imply that B=C

$If\ A$ is invertible (non-singular, i.e., det(A) is not zero) then we can cancel A and B=C .

$If\ A$ is not invertible (singular, i.e., det(A) is zero) then we cannot cancel A and $B \neq C$.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

(3) Zero product with non-Zero Matrices. (for real numbers ab=0 implies either a=0 or b=0)

Example1: $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$: $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Example2: $A = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$; $B = \begin{bmatrix} -b & 2b \\ a & -2a \end{bmatrix}$; $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Example3: $A^2 = A.A = I$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Example4: $A^2 = O$; $\quad B^2 = O$

$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Question11: Find two unequal 2×2 matrices A and B **such that** $AB = I$

Solution: $$A = \begin{bmatrix} a & a+1 \\ a-1 & a \end{bmatrix}$$

$$B = \begin{bmatrix} a & -a-1 \\ 1-a & a \end{bmatrix}$$

(We can solve example2 and question 11 using chapter 2)

Two important formulae: (1) $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\,\boldsymbol{A}^{-1}$

(2) $(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{B}^{\boldsymbol{T}}\,\boldsymbol{A}^{\boldsymbol{T}}$

Dot product and matrix multiplication

$x \in R$; $\begin{bmatrix} x \\ y \end{bmatrix} \in R^2$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3$;

Similarly in general we have, $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \in R^n$ Let $\boldsymbol{u}, \boldsymbol{v} \in R^n$, Then $= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$; $\boldsymbol{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$. **Dot product of two vectors is defined as**

$$\boldsymbol{u} \bullet \boldsymbol{v} = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

Now this dot product provide basis for matrix multiplication. Consider

$\boldsymbol{u} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}_{\boldsymbol{n}\times\mathbf{1}}$ and $\boldsymbol{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}_{\boldsymbol{n}\times\mathbf{1}}$ as matrices. Then

$$\boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{v} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}_{\mathbf{1}\times\boldsymbol{n}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}_{\boldsymbol{n}\times\mathbf{1}}$$

$$= a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n$$

$$\begin{array}{c} \\ \\ \\ \text{row}_i(A) \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{array}{c} \begin{array}{cccccc} & & & \text{col}_j(B) & & \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$(\text{row}_i(A))^T \cdot \text{col}_j(B) = \sum_{k=1}^{p} a_{ik}b_{kj} = c_{ij}$$

Matrix Product Explanation:

(*Manufacturing Costs*) A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{c} \begin{array}{cc} \text{Assembly process} & \text{Finishing process} \end{array} \\ \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Chair} \\ \text{Table} \end{array}.$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$B = \begin{array}{c} \begin{array}{cc} \text{Salt Lake City} & \text{Chicago} \end{array} \\ \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Assembly process} \\ \text{Finishing process} \end{array}.$$

What do the entries in the matrix product AB tell the manufacturer?

Solution:- $\boldsymbol{AB} = \begin{bmatrix} \mathbf{38} & \mathbf{44} \\ \mathbf{67} & \mathbf{78} \end{bmatrix}$

$$\boldsymbol{AB_{11} = 38 = cost\ of\ Chairs\ in\ Salt\ lake\ city}$$

$$\boldsymbol{AB_{22} = 78 = cost\ of\ tables\ in\ Chicago}$$

$$\boldsymbol{AB_{12} = 44 = cost\ of\ Chairs\ in\ Chicago}$$

$$\boldsymbol{AB_{21} = 67 = cost\ of\ tables\ in\ Salt\ lake\ city}$$

Question 5: (Exercise 1.2)

If $\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix}$ find a, b, c and d.

Solution: Two matrices are equal if and only if their corresponding entries are equal. i.e.

$$a + 2b = 4 \ldots..(1)$$

$$2a - b = -2 \ldots..(2)$$

$$2c + d = 4 \ldots..(3)$$

$$c - 2d = -3 \;......(4)$$

We got **simple** non-homogeneous linear system of equations in four unknowns and can easily be solved by elimination method.

Cancelling b from (1) and (2), we get $a = 0,\ b = 2$

Now cancelling d from (3) and (4), we get $c = 1,\ d = 2$

Question 5: (Exercise 1.3) Determine values of $\boldsymbol{x}$ and $\boldsymbol{y}$ so that

$\boldsymbol{v}\bullet \boldsymbol{w} = \mathbf{0}$ **and** $\boldsymbol{v}\bullet \boldsymbol{u} = \mathbf{0}$**,** where $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{x} \\ \mathbf{1} \\ \boldsymbol{y} \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} \mathbf{2} \\ -\mathbf{2} \\ \mathbf{1} \end{bmatrix}, and\ \boldsymbol{u} = \begin{bmatrix} \mathbf{1} \\ \mathbf{8} \\ \mathbf{2} \end{bmatrix}$

Solution: $\boldsymbol{v}\bullet \boldsymbol{w} = \mathbf{0}$ implies $2x - 2 + y = 0 \;.....(1)$

Similarly, $\boldsymbol{v}\bullet \boldsymbol{u} = \mathbf{0}$ gives $x + 8 + 2y = 0 \;.....(2)$

Solving (1) and (2) $x = 4$ and $y = -6$.

Chapter 2: Solving Linear Systems

DEFINITION 2.1

An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following properties:

(a) All zero rows, if there are any, appear at the bottom of the matrix.

(b) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.

(c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

(d) If a column contains a leading one, then all other entries in that column are zero.

An $m \times n$ matrix satisfying properties (a), (b), and (c) is said to be in **row echelon form**. In Definition 2.1, there may be no zero rows.

EXAMPLE 1

The following are matrices in reduced row echelon form, since they satisfy properties (a), (b), (c), and (d):

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices that follow are not in reduced row echelon form. (Why not?)

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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EXAMPLE 2

The following are matrices in row echelon form:

$$H = \begin{bmatrix} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 & 1 & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

DEFINITION 2.2

An **elementary row (column) operation** on a matrix A is any one of the following operations:

(a) **Type I:** Interchange any two rows (columns).
(b) **Type II:** Multiply a row (column) by a nonzero number.
(c) **Type III:** Add a multiple of one row (column) to another.

DEFINITION 2.3 An $m \times n$ matrix B is said to be **row (column) equivalent** to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations to A.

5. Find the reduced row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 9 \\ 3 & 2 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 7 & -5 \end{bmatrix}$

Solution:-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 7 & -5 \end{bmatrix} \underset{R_4 + 2R_1}{\overset{R_2 + R_1}{\sim}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 7 & -3 \end{bmatrix} \overset{R_{23}}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 7 & -3 \end{bmatrix}$$

$$\underset{R_4 - 7R_2}{\overset{R_3 - 2R_2}{\sim}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \end{bmatrix} \overset{R_4 - 3R_3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 + R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{(-1)R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$