Lecture Notes: Compiled by Maqsood Ahmad (A.P. Maths.) for students of CUI, Lahore. (SP21-BSE-A, B & C).

Chapter 6: Linear Transformations (function) and Matrices

Exercise 6.1 : 1-16, 20-23; Exercise 6.2 : 1-11,16,17; Exercise 6.3 : 1-8,13,14,22.

DEFINITION 6.1

Let V and W be vector spaces. A function $L: V \to W$ is called a **linear transformation** of V into W if

(a) $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for every $\mathbf{u}$ and $\mathbf{v}$ in V.

(b) $L(c\mathbf{u}) = cL(\mathbf{u})$ for any $\mathbf{u}$ in V, and c any real number.

Important Note:(1) some books combine these two conditions into one.

$$L(a\boldsymbol{u} + b\boldsymbol{v}) = aL(\boldsymbol{u}) + bL(\boldsymbol{v}) : a, b \text{ are reals}$$

(2)$L(0_V) = 0_W$ i.e., Zero vector of vector space V always transform on zero of vector space W.

(3) Every Linear Transformation have the form $L(\boldsymbol{u}) = \boldsymbol{Au}$

(4) (a) No product of components (e.g. u_1u_2) appear in LT.

(b) No power of component is allowed in LT. (e.g.$u_1{}^2$)

(c) No constant will be added into any component in LT (e.g. $u_1 + 2$)

Examples of LT:-

(1) $Reflection\ abt\ \boldsymbol{Y}\ axis$: L: $R^2 \to R^2$ defined by

$$L(x,y) = (-x,y)\ OR\ L\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\y\end{bmatrix}\ OR\ \ L(\boldsymbol{u}) = L\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}-u_1\\u_2\end{bmatrix}$$

(2) $Reflection\ abt\ \boldsymbol{y} = \boldsymbol{x}\ line$: L: $R^2 \to R^2$defined by

$$L(x,y) = (y,x)\ OR\ L\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}\ OR\ \ L(\boldsymbol{u}) = L\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}u_2\\u_1\end{bmatrix}$$

(3) $Reflection\ abt\ XY\ plane$; L: $R^3 \to R^3$;

$$L(x,y,z) = (x,y,-z), \qquad OR\ L\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\y\\-z\end{bmatrix};\ L(\boldsymbol{u}) = L\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix}u_1\\u_2\\-u_3\end{bmatrix}$$

$$(4) Dilation\ L(u) = ru\ , r\ > 1\ \ OR\ L(\boldsymbol{u}) = L\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = r\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

(5) $Contraction\ L(u) = ru\ , 0\ < r\ < 1\quad OR\ L(\boldsymbol{u}) =\ L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = r\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

(1) $Reflection\ abt\ X\ axis: L:\ R^2 \to R^2$

$$L(\boldsymbol{u}) = L\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

$$L(e_1) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -0 \end{bmatrix}$$

$$L(e_2) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{A} = [L(e_1)\quad L(e_2)]$$

$$L(\boldsymbol{u}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boldsymbol{Au} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

(2) $Projection\ of\ vector\ in\ XY - plane; L:\ \boldsymbol{R^3} \to R^2$

$$L(\boldsymbol{u}) =\ L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

$$L(e_1) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A} = [\boldsymbol{L(e_1)}\quad \boldsymbol{L(e_2)L(e_3)}]$$

$$L(\boldsymbol{u}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \boldsymbol{Au}$$

(3) $Reflection\ abt\ XY\ plane; L:\ R^3 \to R^3;\ L(\boldsymbol{u}) =\ L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ -u_3 \end{bmatrix}$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -0 \end{bmatrix};\ L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -0 \end{bmatrix};L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A = [L(e_1)\quad L(e_2)\quad L(e_3)]$$

$$L(\boldsymbol{u}) = L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \boldsymbol{Au} = \begin{bmatrix} u_1 \\ u_2 \\ -u_3 \end{bmatrix}$$

EXAMPLE 2 Let $L: R^3 \to R^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \\ u_3 \end{bmatrix}.$$

To determine whether L is a linear transformation, let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then

$$\begin{aligned} L(\mathbf{u} + \mathbf{v}) &= L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} (u_1 + v_1) + 1 \\ 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$L(\mathbf{u}) + L(\mathbf{v}) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 + 1 \\ 2v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1) + 2 \\ 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}.$$

Letting $u_1 = 1$, $u_2 = 3$, $u_3 = -2$, $v_1 = 2$, $v_2 = 4$, and $v_3 = 1$, we see that $L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$. Hence we conclude that the function L is not a linear transformation. ■

EXAMPLE 3 Let $L: R_2 \to R_2$ be defined by

$$L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1^2 & 2u_2 \end{bmatrix}.$$

Is L a linear transformation?

Solution

Let

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}.$$

Then

$$\begin{aligned} L(\mathbf{u} + \mathbf{v}) &= L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix} + \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right) \\ &= L\left(\begin{bmatrix} u_1 + v_1 & u_2 + v_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} (u_1 + v_1)^2 & 2(u_2 + v_2) \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} L(\mathbf{u}) + L(\mathbf{v}) &= \begin{bmatrix} u_1^2 & 2u_2 \end{bmatrix} + \begin{bmatrix} v_1^2 & 2v_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 + v_1^2 & 2(u_2 + v_2) \end{bmatrix}. \end{aligned}$$

Since there are some choices of u and v such that $L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$, we conclude that L is not a linear transformation. ■

EXAMPLE 10

Let $L: R_4 \to R_2$ be a linear transformation and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for R_4, where $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & -1 & 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$. Suppose that

$$L(\mathbf{v}_1) = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad L(\mathbf{v}_2) = \begin{bmatrix} 0 & 3 \end{bmatrix},$$
$$L(\mathbf{v}_3) = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{and} \quad L(\mathbf{v}_4) = \begin{bmatrix} 2 & 0 \end{bmatrix}.$$

Let

$$\mathbf{v} = \begin{bmatrix} 3 & -5 & -5 & 0 \end{bmatrix}.$$

Find $L(\mathbf{v})$.

Solution

We first write $\mathbf{v}$ as a linear combination of the vectors in S, obtaining (verify)

$$\mathbf{v} = \begin{bmatrix} 3 & -5 & -5 & 0 \end{bmatrix} = 2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 + \mathbf{v}_4.$$

It then follows by Theorem 6.2 that

$$\begin{aligned} L(\mathbf{v}) &= L(2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 + \mathbf{v}_4) \\ &= 2L(\mathbf{v}_1) + L(\mathbf{v}_2) - 3L(\mathbf{v}_3) + L(\mathbf{v}_4) = \begin{bmatrix} 4 & 7 \end{bmatrix}. \end{aligned}$$

■

$$v = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 \;---\; (1)\ NHS$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 0 & -5 \\ 1 & -1 & 2 & 0 & -5 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right] \; RREF \; \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$v = 2v_1 + v_2 - 3v_3 + v_4$$

Now apply Linear transformation on Both Sides

$$L(v) = L(2v_1 + v_2 - 3v_3 + v_4)$$

$$part\ (a) of\ LT: L(u+v) = L(u) + L(v)$$

$$L(v) = L(2v_1) + L(v_2) + L(-3v_3) + L(v_4)$$

$$part\ (b) of\ LT: L(cu) = cL(u)$$

$$L(v) = 2L(v_1) + L(v_2) - 3L(v_3) + L(v_4)$$

$$L(v) = 2[1 \quad 2] + [0 \quad 3] - 3[0 \quad 0] + [2 \quad 0] = [4 \quad 7]$$

Exercise 6.1

14. Let $L: R_2 \to R_2$ be a linear transformation for which we know that

$$L\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \end{bmatrix},$$
$$L\left(\begin{bmatrix} -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \end{bmatrix}.$$

(a) What is $L\left(\begin{bmatrix} -1 & 5 \end{bmatrix}\right)$?

(b) What is $L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right)$?

Solution: Method 1: (a) Let $v_1 = [1 \quad 1]$and $v_2 = [-1 \quad 1]$, $v = [-1 \quad 5], w = [u_1 u_2]$

$$v = \boldsymbol{c_1} v_1 + \boldsymbol{c_2} v_2$$

$$[-1 \quad 5] = \boldsymbol{c_1}[1 \quad 1] + \boldsymbol{c_2}[-1 \quad 1]$$

$$[-1 \quad 5] = [\boldsymbol{c_1 c_1}] + [-\boldsymbol{c_2 c_2}]$$

$$[-1 \quad 5] = [\boldsymbol{c_1 - c_2 c_1 + c_2}]$$

Equating components of equal vectors

$$c_1 - c_2 = -1$$

$$c_1 + c_2 = 5$$

$$2c_1 = 4 \; gives \; \boldsymbol{c_1 = 2 \; then \; c_2 = 3}$$

$$[-1 \quad 5] = 2[1 \quad 1] + 3\,[-1 \quad 1]$$

Now apply transformation on Both Sides

$$L[-1 \quad 5] = L(2[1 \quad 1] + 3\,[-1 \quad 1])$$

$$L[-1 \quad 5] = L(2[1 \quad 1]) + L(3\,[-1 \quad 1]); \;\; part\ (a) of\ LT: L(u+v) \; = L(u) + L(v)$$

$$L[-1 \quad 5] = 2L([1 \quad 1]) + 3L(\,[-1 \quad 1]); \;\; part\ (b) of\ LT: L(\boldsymbol{c}u) = \boldsymbol{c}L(u)$$

$$L[-1 \quad 5] = 2[1 \quad -2] + 3[2 \quad 3] = [2+6 \quad -4+9] = [8 \quad 5]$$

(b)$w = c_1 v_1 + c_2 v_2$

$$[u_1 u_2] = c_1[1 \quad 1] + c_2[-1 \quad 1]$$

$$[u_1 u_2] = [c_1 c_1] + [-c_2 c_2]$$

$$[u_1 u_2] = [c_1 - c_2 c_1 + c_2]$$

$$c_1 - c_2 = u_1$$

$$c_1 + c_2 = u_2$$

Adding both equations $2c_1 = u_1 + u_2$

$$[u_1 u_2] = \left(\frac{u_1 + u_2}{2}\right)[1 \quad 1] + \left(\frac{u_2 - u_1}{2}\right)[-1 \quad 1]$$

Now apply transformation on BS;

$$L[u_1 u_2] = L\left(\left(\frac{u_1 + u_2}{2}\right)[1 \quad 1] + \left(\frac{u_2 - u_1}{2}\right)[-1 \quad 1]\right)$$

$$L[u_1 u_2] = L\left(\left(\frac{u_1 + u_2}{2}\right)[1 \quad 1]\right) + L\left(\left(\frac{u_2 - u_1}{2}\right)[-1 \quad 1]\right);$$

$$part\ (a) of\ LT: L(u+v) \ = L(u)+L(v)$$

$$L[u_1 u_2] = \left(\frac{u_1+u_2}{2}\right) L[1 \quad 1] + \left(\frac{u_2-u_1}{2}\right) L\,[-1 \quad 1];\ part\ (b) of\ LT: L(ku) = kL(u)$$

$$L[u_1 u_2] = \left(\frac{u_1+u_2}{2}\right)[1 \quad -2] + \left(\frac{u_2-u_1}{2}\right)[2 \quad 3]$$

$$L[u_1 u_2] = \left[\left(\frac{u_1+u_2}{2}\right) \quad -2\left(\frac{u_1+u_2}{2}\right)\right] + \left[2\left(\frac{u_2-u_1}{2}\right) \quad 3\left(\frac{u_2-u_1}{2}\right)\right]$$

$$L[u_1 u_2] = \left[\frac{u_1+u_2}{2} + u_2 - u_1 \quad -(u_1+u_2) + 3\left(\frac{u_2-u_1}{2}\right)\right]$$

$$L[u_1 u_2] = \left[-\frac{1}{2}u_1 + \frac{3}{2}u_2 \quad -\frac{5}{2}u_1 + \frac{1}{2}u_2\right]$$

***Method* 2: See Slader.com**

6.2 Kernel and Range of a Linear Transformation

Definition1: Let $L: V \to W$ be a linear transformation of a vector space V into a vector space W. The **kernel** of L, ker L, is the subset of V consisting of all elements $\mathbf{v}$ of V such that $L(\mathbf{v}) = \mathbf{0}_W$.

Note: Problem of finding $Ker\ L$ is same as solving homogeneous system (Null or solution space).

Definition2: A linear transformation $L: V \to W$ is called **one-to-one** if it is a one-to-one function; that is, if $\mathbf{v}_1 \neq \mathbf{v}_2$ implies that $L(\mathbf{v}_1) \neq L(\mathbf{v}_2)$. An equivalent statement is that L is one-to-one if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$. (See Figure A.2 in Appendix A.)

Theorem1 : Let $L: V \to W$ be a linear transformation of a vector space V into a vector space W. Then

(a) ker L is a subspace of V.

(b) L is one-to-one if and only if ker $L = \{\mathbf{0}_V\}$.

Definition3: If $L: V \to W$ is a linear transformation of a vector space V into a vector space W, then the **range** of L or **image** of V under L, denoted by range L, consists of all those vectors in W that are images under L of vectors in V. Thus $\mathbf{w}$ is in range L if there exists some vector $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{w}$. The linear transformation L is called **onto** if range $L = W$.

Note: Problem of finding $Range\ L$(onto) is same as of finding spanning set of vector space.

Theorem 6.6: $dim(Ker(L)) + dim(Range(L)) = dim(V)$

Example:

Let $L: R^3 \to R^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

(a) Is L onto?
(b) Find a basis for range L.
(c) Find ker L.
(d) Is L one-to-one?

Solution (a):- $L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + 0u_2 + u_3 \\ u_1 + u_2 + 2u_3 \\ 2u_1 + u_2 + 3u_3 \end{bmatrix}$

Extra $L(e_1) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$; $L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$; $L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(a) Is L is onto $\rightarrow$ To find Range(L)= R^3

$$L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{bmatrix} 101 \\ 112 \\ 213 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} --(1)$$

$$\begin{bmatrix} 101|a \\ 112|b \\ 213|c \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 101| & a \\ 011| & b-a \\ 011| & c-2a \end{bmatrix} R_3 - R_2 \begin{bmatrix} 101| & a \\ 011| & b-a \\ 000| & c-a-b \end{bmatrix}$$

No Solution to NHS (1) therefore L is not onto(Range(L)$\neq R^3$).

(b) Find a basis for Range of L

Consider $L(\boldsymbol{u}) = L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 101 \\ 112 \\ 213 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + 0u_2 + u_3 \\ u_1 + u_2 + 2u_3 \\ 2u_1 + u_2 + 3u_3 \end{bmatrix} = u_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + u_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + u_3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is spanning set for Range of L.

Take LI vectors from S.

$$\begin{bmatrix} 101|0 \\ 112|0 \\ 213|0 \end{bmatrix} REF \begin{bmatrix} 101|0 \\ 011|0 \\ 000|0 \end{bmatrix}$$

Basis for Range of L = $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) To find Ker(L); $L(u) = 0 \rightarrow L \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 101 \\ 112 \\ 213 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 101|0 \\ 112|0 \\ 213|0 \end{bmatrix} \quad RREF \quad \begin{bmatrix} 101|0 \\ 011|0 \\ 000|0 \end{bmatrix}$$

$$\boldsymbol{u_1} + \boldsymbol{u_3} = 0, \boldsymbol{u_2} + \boldsymbol{u_3} = 0\,;\ \boldsymbol{u_3} = a\ \boldsymbol{then}\ \boldsymbol{u_2} = -\boldsymbol{u_3} = -a;\ \boldsymbol{u_1} = -\boldsymbol{u_3} = -a$$

$$\boldsymbol{Ker}\ (\boldsymbol{L}) = \left\{ \begin{bmatrix} -a \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} :\ a\ \in \boldsymbol{R} \right\}$$

Verify $\begin{bmatrix} 101 \\ 112 \\ 213 \end{bmatrix} \begin{bmatrix} -a \\ -a \\ a \end{bmatrix} = \begin{bmatrix} -a + 0 + a \\ -a - a + 2a \\ -2a - a + 3a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) Since $\boldsymbol{Ker}\ (\boldsymbol{L}) \neq \{\boldsymbol{0}\}$ **hence** $\boldsymbol{L}$ **is not one to one.**

Note: Additional question: find basis for $Ker\ L$and $dim(Ker\ L)$;$dim(Range\ L)$ and prove$dim(Ker(L)) + dim(Range(L)) = dim(R^3)$

Basis for $Ker\ L = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $dim(Ker\ L) = Number\ of\ vectors\ in\ basis\ of\ Ker\ L = 1$

$$dim(Range\ L) = Number\ of\ vectors\ in\ basis\ of\ Range\ L = 2$$

$$1 + 2 = 3$$

Exercise 6.2

3. Let $L : R_4 \rightarrow R_2$ be the linear transformation defined by

$$L([u_1 \quad u_2 \quad u_3 \quad u_4]) = [u_1 + u_3 \quad u_2 + u_4].$$

(a) Is $[2 \quad 3 \quad -2 \quad 3]$ in ker L?

(b) Is $[4 \quad -2 \quad -4 \quad 2]$ in ker L?

(c) Is $[1 \quad 2]$ in range L?

(d) Is $[0 \quad 0]$ in range L?

(e) Find $\ker L$.

(f) Find a set of vectors spanning range L.

Solution:(a)$L([2 \quad 3 -2 \quad 3]) = [0 \quad 6] \Longrightarrow [2 \quad 3 -2 \quad 3] \notin Ker\ L$

(b) $L([4 \quad -2 -4 \quad 2]) = [0 \quad 0] \Longrightarrow [4 \quad -2 -4 \quad 2] \in Ker\ L$

(c) $[\boldsymbol{u_1} + \boldsymbol{u_3} \quad \boldsymbol{u_2} + \boldsymbol{u_4}] = [\mathbf{1} \quad \mathbf{2}] \Longrightarrow \boldsymbol{u_1} + \boldsymbol{u_3} = \mathbf{1}\ \& \boldsymbol{u_2} + \boldsymbol{u_4} = \mathbf{2}.$

We have 2 equations and 4 unknowns and have infinite many solution.

One solution may be, $If\ u_1 = 1\ then\ \ u_3 = 0\ \&\ if\ \ u_2 = 1\ then\ \ u_4 = 1$

$$L([1 \quad 10 \quad 1]) = [1 \quad 2] \in Range\ L$$

(d) $[u_1 + u_3 \quad u_2 + u_4] = [0 \quad 0] \ \Longrightarrow\ u_1 + u_3 = 0\ \& u_2 + u_4 = 0.$

We have 2 equations and 4 unknowns and have infinite many solution.

One solution may be, $If\ u_1 = 1\ then\ \ u_3 = -1\ \&\ if\ \ u_2 = 1\ then\ \ u_4 = -1$

$$L([1 \quad 1 -1 \quad -1]) = [0 \quad 0] \in Range\ L$$

(e) To obtain$Ker\ L$,

find$[u_1 \quad u_2 u_3 \quad u_4] \in\ R_4$such that$L(\boldsymbol{u}) = \mathbf{0}\ \ implies\ \ L([u_1 \quad u_2 u_3 \quad u_4]) = \mathbf{0}_{\boldsymbol{R_2}}$

$$\Longrightarrow\ [u_1 + u_3 \quad u_2 + u_4] = [0 \quad 0] \Longrightarrow u_1 + u_3 = 0\ \& u_2 + u_4 = 0\ (HS)$$

$\Longrightarrow u_1 = -u_3 \& u_2 = -u_4;$

We have 2 equations and 4 unknowns and have infinite many solution. Suppose

$u_3 = r\ then\ u_1 = -r \& u_4 = s\ then\ u_2 = -s$

$$Ker\ L\{[u_1 \quad u_2 u_3 \quad u_4] = [-r \quad -sr \quad s] : r, s\ \in R\}$$

(f) To find spanning set for $Range\ L$, consider the element of range

$$L(\boldsymbol{u}) = L([u_1 \quad u_2 u_3 \quad u_4]) = [u_1 + u_3 \quad u_2 + u_4]$$
$$= u_1[1 \quad 0] + u_2[0 \quad 1] + u_3[1 \quad 0] + u_4[0 \quad 1]$$

Spanning set for $Range\ L = \{[1 \quad 0], [0 \quad 1]\}$

Note: Additional question: find basis for $Ker\ L$ and $dim(Ker\ L)$.

Answer: consider$[-r \quad -sr \quad s] = r[-1 \quad 01 \quad 0] + s[0 \quad -10 \quad 1]$

Hence **basis for** $Ker\ L = \{[-1 \quad 0 1 \quad 0], [0 \quad -1 0 \quad 1]\}$.

$$dim(Ker\ L) = 2$$

Note: Additional question: find basis for $Range\ L$ and $dim(Range\ L)$.

Basis for $Range\ L = \{[1 \quad 0], [0 \quad 1]\}$

$$dim(Range\ L) = 2$$

$$*** \ dim(Ker(L)) + dim(Range(L)) = dim(R_4)$$

$$2 + 2 = 4$$

6. Let $L: P_2 \to P_3$ be the linear transformation defined by $L(p(t)) = t^2 p'(t)$.

(a) Find a basis for and the dimension of ker L.

(b) Find a basis for and the dimension of range L.

Solution:(a) **To obtain** $\boldsymbol{Ker\ L}$**, find** $\boldsymbol{p(t) = at^2 + bt + c \in P_2}$ **such that**

$$L(p(t)) = 0_{P_3}$$

$$\Longrightarrow \ \boldsymbol{t^2 p'(t)} = 0t^3 + 0t^2 + 0t + 0$$

$$\Longrightarrow \ \boldsymbol{t^2(2at + b)} = 0t^3 + 0t^2 + 0t + 0 \Longrightarrow \boldsymbol{2at^3 + bt^2} = 0t^3 + 0t^2 + 0t + 0$$

Equating coefficients of like powers

$2a = 0, b = 0, \boldsymbol{c} \ \in \boldsymbol{R} \quad \Longrightarrow \quad \boldsymbol{p(t) = c}$

$\boldsymbol{Ker\ L} = \{at^2 + bt + c : a = 0, b = 0\} = \{\boldsymbol{c}\} = \{\boldsymbol{c}(1)\}$

Basis of $\boldsymbol{KerL} = \{1\}$ **and** $\boldsymbol{dim}\ (\boldsymbol{Ker\ L}) = 1$

(b) To obtain $\boldsymbol{Range\ L}$**, Let** $\boldsymbol{p(t) = at^2 + bt + c \in P_2}$

Consider $L(p(t)) = \boldsymbol{t^2 p'(t) = t^2(2at + b) = 2a(t^3) + b(t^2)}$

$\boldsymbol{Spanning\ set\ for\ Range\ L} = \{v_1 = \boldsymbol{t^3}, v_2 = \boldsymbol{t^2}\}$

Basis of $\boldsymbol{RangeL} = \{\boldsymbol{t^2, t^3}\}$ **and** $\boldsymbol{dim\ (Range\ L) = 2}$

$$*** \ dim(Ker(L)) + dim(Range(L)) = dim(P_2)$$

$$1 + 2 = 3$$

11. Let $L: M_{22} \to M_{22}$ be the linear operator defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}.$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

Solution: $Generally \quad L: V \to W$; $\quad Here\ L: M_{22} \to M_{22}$

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$$

(a) $\quad L(v) = 0_W, \quad L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0_{M_{22}}$

$$\begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a+b=0\,; \quad b+c=0 \quad ; \quad a+d=0\,; \quad b+d=0$$

$$AX = 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad , |A| \neq 0; \textit{unique solution, trivial soln}$$

(Homogeneous System of 4 linear eqs in 4 unknowns)

$$a = b = c = d = 0$$

$Ker\ L = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$(Transformation is one to one)

$Basis\ of\ Ker\ L = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\};\quad \dim Ker\ (L) = 0$

(b) For Range L Consider;

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$Span(Range\ L) = \left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$$

$$***** \ dim(Ker(L)) + \dim(\text{Range}(L)) = \dim(V)$$

$$dim(Ker(L)) + \dim(\text{Range}(L)) = \dim(M_{22})$$

$$0 + \dim(\text{Range}(L)) = 4$$

$$Basis\ of\ Range\ L = \left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$$

6.3 Matrix of a Linear Transformation

Working in Exercise 6.1, 6.2:

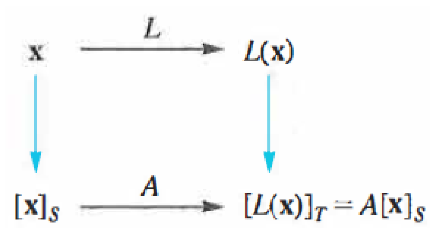
Theorem: $: V \to W$; $Standard\ Basis\ of\ V = S = \{e_1, e_2, e_3\}$

$$Standard\ Basis\ of\ W = T = \{E_1, E_2, E_3\}$$

Then $L[\boldsymbol{X}] = A\,\boldsymbol{X}$ $\quad for\ all\ X \in V$

$$\boldsymbol{A} = [\boldsymbol{L}(\boldsymbol{e_1}) \quad \boldsymbol{L}(\boldsymbol{e_2}) \quad \boldsymbol{L}(\boldsymbol{e_3})]$$

Note: But Now to deal with "Ordered Basis"



Theorem (1):: $V \to W$; $\boldsymbol{Ordered} Basis\ of\ V = S = \{e_1 = v_1, e_2 = v_2, e_3 = v_3\}$

$$\boldsymbol{Ordered} Basis\ of\ W = T = \{w_1, w_2, w_3\}$$

Then $[L(X)]_T = A[X]_S$ $\quad for\ all\ X \in V$

$A = [[L(v_1)]_T [L(v_2)]_T [L(v_3)]_T]$with respect to S and T.

$$[w_1 w_2 w_3 |\ L(v_1) |\ L(v_2)\ \ |\ \ L(v_3)]$$

$$[I \ \ |\ \ A]$$

Theorem (2): $: W \to V$; $Ordered\ Basis\ of\ V = S = \{v_1, v_2, v_3\}$

$$Ordered\ Basis\ of\ W = T = \{w_1, w_2, w_3\}$$

Then $[L(X)]_S = A\,[X]_T$ $\quad for\ all\ X \in W$

$A = [[L(w_1)]_S [L(w_2)]_S [L(w_3)]_S]$with respect to T and S.

$$[v_1 v_2 v_3 |\ L(w_1) |\ L(w_2)\ \ |\ \ L(w_3)]$$

$$[I \ \ |\ \ A]$$

Theorem (3):: $V \rightarrow W$; $Ordered\ Basis\ of\ V = S = \{v_1, v_2, v_3\}$

Then $L[X] = A\,[X]_S \quad for\ all\ X \in V$

$A = \left[L[v_1] \quad L[v_2] \quad L[v_3]\right]$with respect to S.

$$[v_1 v_2 v_3 |\ L(v_1)|\ L(v_2)\ |\ \ L(v_3)]$$

$$[I \ \ |\ A]$$

Theorem (4): $: V \rightarrow W$; $Ordered\ Basis\ of\ V = T = \{w_1, w_2, w_3\}$

Then $[L(X)]_T = A\,X \quad for\ all\ X \in V$

$A = \left[L[w_1] \quad L[w_2] \quad L[w_3]\right]$with respect to T.

$$[w_1 w_2 w_3 |\ L(w_1)|\ L(w_2)\ |\ \ L(w_3)]$$

$$[I \ \ |\ A]$$

Solution to some important problem Exercise 6.3

1. Let $L : R^2 \rightarrow R^2$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix}.$$

Let S be the natural basis for R^2 and let

$$T = \left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\}.$$

Find the representation of L with respect to

(a) S; **(b)** S and T; **(c)** T and S; **(d)** T.

Solution:Given

$$S = \left\{v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\ and\ T = \left\{w_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\}$$

(a) Matrix of linear transformation with respect to S =?

$$[v_1 v_2 |\ L(v_1)|\ L(v_2)] \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$L(v_1) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$L(v_2) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Put these values in (1) we have

$$\begin{bmatrix} 1 & 0 & | & 1 & | & 2 \\ 0 & 1 & | & 2 & | & -1 \end{bmatrix}$$

Matrix of linear transformation with respect to S is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

(b) Matrix of linear transformation with respect to S and T=?

$$[w_1 w_2 |\ L(v_1)|\ L(v_2)] \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Put values of $L(v_1)$and $L(v_2)$from part (a) in (2) we have

$$\begin{bmatrix} -1 & 2 & | & 1 & | & 2 \\ 2 & 0 & | & 2 & | & -1 \end{bmatrix}$$

$$Do\ yourself \qquad RREF \sim \begin{bmatrix} 1 & 0 & | & 1 & | & -1/2 \\ 0 & 1 & | & 1 & | & 3/4 \end{bmatrix}$$

Matrix of linear transformation with respect to S and T is

$$A = \begin{bmatrix} 1 & -1/2 \\ 1 & 3/4 \end{bmatrix}$$

(c) Matrix of linear transformation with respect to T and S=?

$$L\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix}$$

$$[v_1 v_2 |\ L(w_1)|\ L(w_2)] \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$L(w_1) = L\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$L(w_2) = L\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Put values of $L(w_1)$and $L(w_2)$ in (3) we have

$$\begin{bmatrix} 1 & 0 & | & 3 & | & 2 \\ 0 & 1 & | & -4 & | & 4 \end{bmatrix}$$

Matrix of linear transformation with respect to T and S is

$$A = \begin{bmatrix} 3 & 2 \\ -4 & 4 \end{bmatrix}$$

(d) Matrix of linear transformation with respect to T=?

$$L\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix}$$

$$[w_1 w_2 |\ L(w_1)|\ L(w_2)] \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Putvalues of $L(w_1)$and $L(w_2)$from part (c) in (4) we have

$$\begin{bmatrix} -12| & 3 & |2 \\ 2\ 0| & -4 & |4 \end{bmatrix}$$

$$Do\ yourself \quad RREF \sim \begin{bmatrix} 10| & -2 & |2 \\ 01| & 1/2 & |2 \end{bmatrix}$$

Matrix of linear transformation with respect to T is

$$A = \begin{bmatrix} -2 & 2 \\ 1/2 & 2 \end{bmatrix}$$

Home Work: Question 13 is similar to Question 1

3. Let $L: R^4 \to R^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

Let S and T be the natural bases for R^4 and R^3, respectively, and consider the ordered bases

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and}$$

$$T' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

for R^4 and R^3, respectively. Find the representation of L with respect to (a) S and T; (b) S' and T'.

Solution:(a) Matrix of linear transformation with respect to S and T natural/standard basis is same as given in the definition of linear transformation, i.e.,

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \ and\ T = \left\{ w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Matrix of linear transformation with respect to $S$and T?

Recall Theorem1:- $A = [[L(v_1)]_T \quad [L(v_2)]_T \quad [L(v_3)]_T]$ with respect to S and T.

$$[w_1 \quad w_2 \quad w_3 |\, L(v_1) |\, L(v_2) \ | \ L(v_3) | \ L(v_4)] ----(1)$$

$$[I \ | \ A]$$

$$L(v_1) = L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$L(v_2) = L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\-2\end{bmatrix}$$

$$L(v_3) = L\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$L(v_4) = L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Put these values in (1) we have

$$\left[\begin{array}{ccc|c|c|c|c}1 & 0 & 0 & 1 & 0 & 1 & 1\\0 & 1 & 0 & 0 & 1 & 2 & 1\\0 & 0 & 1 & -1 & -2 & 1 & 0\end{array}\right]$$

Matrix of linear transformation with respect to S and T is

$$A = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}$$

(b) Matrix of linear transformation with respect to S' and T'=?

$$L(v_1) = L\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix}$$

$$L(v_2) = L\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\-2\end{bmatrix}$$

$$L(v_3) = L\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$L(v_4) = L\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 2 & 1\\-1 & -2 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$$

$$[w_1\ \ w_2\ \ w_3 |\ L(v_1)|\ L(v_2)\ |\ \ L(v_3)\ |\ \ L(v_4)] ----(1)$$

$$\left[\begin{array}{ccc|c|c|c|c}1 & 0 & 0 & 1 & 0 & 2 & 1\\0 & 1 & 0 & 1 & 1 & 3 & 3\\1 & 1 & 1 & -3 & -2 & 1 & -1\end{array}\right]$$

$$Do\ your\ self\ \ \ RREF \sim \left[\begin{array}{ccc|c|c|c|c}1 & 0 & 0 & 1 & 0 & 2 & 1\\0 & 1 & 0 & 1 & 1 & 3 & 3\\0 & 0 & 1 & -5 & -3 & -4 & -5\end{array}\right]$$

Hence Matrix representing L with respect to S' and T' is

$$A = \begin{bmatrix}1 & 0 & 2 & 1\\1 & 1 & 3 & 3\\-5 & -3 & -4 & -5\end{bmatrix}$$

8. Let $L: M_{22} \to M_{22}$ be defined by

$$L(A) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} A$$

for A in M_{22}. Consider the ordered bases

$$S = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

and

$$T = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$$

for M_{22}. Find the representation of L with respect to
(a) S; **(b)** T; **(c)** S and T; **(d)** T and S.

Solution: Given $S = \{v_1, v_2\ , v_3, v_4\}\ and\ T = \{w_1, w_2, w_3, w_4\}$

(a) Matrix of linear transformation with respect to S =?

$$[v_1\ \ v_2\ \ v_3\ \ v_4 |\ L(v_1)|\ L(v_2)|\ L(v_3)|\ L(v_4)] \ldots..(1)$$

$$L(v_1) = L\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

$$L(v_2) = L\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 3\end{bmatrix}$$

$$L(v_3) = L\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\4 & 0\end{bmatrix}$$

$$L(v_4) = L\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 2\\0 & 4\end{bmatrix}$$

Put these values in (1) we have

$$[v_1\ \ v_2\ \ \ v_3\ \ v_4|\ L(v_1)|\ L(v_2)|\ L(v_3)|\ L(v_4)]\ \ldots\ldots(1)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0|1 & |0 & |2 & |0 \\ 0 & 1 & 0 & 0|0 & |1 & |0 & |2 \\ 0 & 0 & 1 & 0|3 & |0 & |4 & |0 \\ 0 & 0 & 0 & 1|0 & |3 & |0 & |4 \end{bmatrix}$$

$$[I\ |\ A]$$

Matrix of linear transformation with respect to S is $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$

(c) Matrix of linear transformation with respect to $S\ and\ T$=?

$$[w_1 w_2 w_3 w_4|\ L(v_1)|\ L(v_2)|\ L(v_3)|\ L(v_4)]\ \ldots\ldots(2)$$

Put values of $L(v_1), L(v_2), L(v_3)$and $L(v_4)$from part (a) in (2) we have

$$\begin{bmatrix} 1110|1|0|2|0 \\ 0101|0|1|0|2 \\ 0010|3|0|4|0 \\ 1000|0|3|0|4 \end{bmatrix}$$

$$RREF \sim \begin{bmatrix} 1000| & 0 & | & 3 & | & 0 & | & 4 \\ 0100| & -2 & | & -3 & | & -2 & | & -4 \\ 0010| & 3 & | & 0 & | & 4 & | & 0 \\ 0001| & 2 & | & 4 & | & 2 & | & 6 \end{bmatrix}$$

Matrix of linear transformation with respect to $S\ and\ T$ is

$$A = \begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix}$$

Solution of part (b) and (d) given below

8. (a) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 3 & 0 & 3 \\ -6 & -5 & -4 & -3 \\ 3 & 3 & 7 & 0 \\ 8 & 6 & 4 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 7 & 0 \\ 4 & 3 & 0 & 3 \end{bmatrix}$.

Question 22: First time in our exercise, he gave "Matrix w.r.t. Ordered basis"

22. Let the representation of $L: R^3 \rightarrow R^2$ with respect to the ordered bases $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) Compute $[L(\mathbf{v}_1)]_T$, $[L(\mathbf{v}_2)]_T$, and $[L(\mathbf{v}_3)]_T$.

(b) Compute $L(\mathbf{v}_1)$, $L(\mathbf{v}_2)$, and $L(\mathbf{v}_3)$.

(c) Compute $L\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$.

Solution: (a) Given $A = [[L(v_1)]_T [L(v_2)]_T [L(v_3)]_T] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$$[L(v_1)]_T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; [L(v_2)]_T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; [L(v_3)]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(Recall:-$[v]_T = ?$; $[v]_T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ *such that* $v = c_1 w_1 + c_2 w_2$

$[L(v)]_T = ?$; $[L(v)]_T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ *such that* $L(v) = c_1 w_1 + c_2 w_2$)

(b) $L(v_1) = 1w_1 - 1w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$;

$$L(v_2) = 2w_1 + 1w_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$L(v_3) = 1w_1 + 0w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(c) $[L(X)]_T = A[X]_S$; $X = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$; $L(X) = L\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = ?$

$$[X]_S = ?\, c_1 v_1 + c_2 v_2 + c_3 v_3 = X \text{ --(1)} \textit{Non homogeneous system}$$

$$[X]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$[A|X] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$RREF\begin{bmatrix} 100 & 2 \\ 010 & -1 \\ 001 & 4 \end{bmatrix}$$

$$[X]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$[L(X)]_T = A[X]_S$$

$$RHS = A[X]_S = \begin{bmatrix} 1 & 21 \\ -1 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} = LHS = [L(X)]_T$$

$$[L(X)]_T = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$L(X) = 4w_1 - 3w_2 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$