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Series Solution

Homogeneous Second Order Linear Differential Equation with variable Coefficients

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{d}{dx}y + P_2(x)y = 0$$

~~P_0, P_1, P_2~~ $P_0(x), P_1(x)$ & $P_2(x)$ → Polynomials in powers of x

Homogeneous D.E → R.H.S zero

Second Order → Second Derivative

Linear D.E → Max Power of $y, \frac{d}{dx}y, \frac{d^2y}{dx^2}$ → 1

variable Coefficients → ∵ P_0, P_1 & P_2 are polynomials in x

As $$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \rightarrow (1)$$

So $$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

$$\frac{d^2y}{dx^2} + P(x)\frac{d}{dx}y + Q(x)y = 0 \rightarrow (2)$$

Equation (2) is called Normal form/Canonical form or standard form of Homogeneous second order Linear Differential Equation with constant coefficients.

Here $P(x) = \frac{P_1(x)}{P_0(x)}$ & $Q(x) = \frac{P_2(x)}{P_0(x)}$

A series solution is a method used to solve differential equations by representing the solution as an infinite series of terms.

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Example:

Solve the D.E by using Series Solution

$$\frac{d^2}{dx^2}y + xy = 0 \rightarrow (A)$$

OR

$$y'' + xy = 0$$

The given D.E (A) is Homogeneous Second order Linear Differential Equation with variable coefficients.

Now by comparing given differential Equation (A) with its canonical form of Homogeneous second order Linear D.E with constant coefficients.

So, $$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{d}{dx}y + P_2(x)\,y = 0$$

$$\frac{d^2y}{dx^2} + 0 \qquad + xy = 0$$

Here $P_0(x) = 1$, $P_1(x) = 0$, $P_2(x) = x$

Now at point $x = 0$

$P_0(0) = 1$, so $P_0(0) \neq 0$

$\therefore x = 0$ is an ordinary point / Hint

$\downarrow$

(power series methods)

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Differential Equations

Series Solution of D.E

Series solution of D.E near ordinary point

Series solution of D.E near regular singular point

① Legendre's Equation:

$$(1-x^2)y'' - 2xy' + \underbrace{n(n+1)}_{\text{consecutive numbers}}y = 0$$

Points

Singular • Singular points of Legendre's Equation occur at $x = \pm 1$

Ordinary • The points where Legendre's Equation is regular (not singular) are all points except $x = \pm 1$, these points are called ordinary points.

Method • Power series method $[P_0(x) \neq 0]$

② Bessel's Equation:

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Singular • Singular Point of Bessel's Equation occurs at $x = 0$.

Ordinary • The points where Bessel's equation is regular (not singular) are all points except $x = 0$

Method • Frobenius Series Method $[P_0(x) = 0]$

③ Chebyshev's Equation:

$$(1-x^2)y'' - xy' + \underbrace{n^2}_{\text{square value of number}}y = 0$$

Points

Singular • Singular Points of Chebyshev's equation occur at $x = \pm 1$

ordinary • The points where chebyshev's equation is regular (not singular) are all points except $x = \pm 1$

Method • Power Series Method $[P_0(x) \neq 0]$

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Solve by power Series

$$y'' + xy = 0$$

$P_0(x) = 1$

$P_1(x) = 0$

$P_2(x) = x$

standard form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$$

we get at point $x = 0$

$$P_0(x) = 1$$

$$P_0(0) = 1 \quad \neq 0$$

we will use Power Series so solution will be

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Put the values

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

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$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 = 0$$

$$a_2 = 0$$

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$(n+2)(n+1)a_{n+2} = -a_{n-1}$$

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} a_{n-1}$$

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Legendre's Equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\underbrace{12}_{(3\times4)}y=0$$

$(3\times4) \longrightarrow$ multiplication of consecutive Num

$P_0(x)=1-x^2$, $P_1(x)=-2$, $P_2(x)=12$

$1-x^2=0 \Rightarrow x^2=1 \Rightarrow x=\pm1$

singular points $=\pm1$

$\ldots\ldots -3,-2,0,2,3,\ldots\ldots$ Ordinary points

Let [Power Series]

$$y=\sum_{n=0}^{\infty}a_nx^n=a_0x^0+a_1x^1+a_2x^2+\ldots\ldots$$

$$dy/dx=y'=\sum_{n=1}^{\infty}a_n(n)x^{n-1}$$

$$d^2y/dx^2=y''=\sum_{n=2}^{\infty}a_n(n)(n-1)x^{n-2}$$

As Equation is

$$(1-x^2)y''-2xy'+12y=0$$

Put the values

$$(1-x^2)\left(\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}\right)-2x\left(\sum_{n=1}^{\infty}na_nx^{n-1}\right)+12\left(\sum_{n=0}^{\infty}a_nx^n\right)=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^n-2\sum_{n=1}^{\infty}na_nx^n+12\sum_{n=0}^{\infty}a_nx^n=0$$

only $\boxed{\text{Put } n=n+2}$ [Power "x^n" is same]

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=2}^{\infty}n(n-1)a_nx^n-2\sum_{n=1}^{\infty}na_nx^n+12\sum_{n=0}^{\infty}a_nx^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=2}^{\infty}n(n-1)a_nx^n-2a_1x-2\sum_{n=2}^{\infty}na_nx^n+12a_0+12a_1x+12\sum_{n=2}^{\infty}a_nx^n=0$$

Comparing Coefficients

$$(n+2)(n+1)a_{n+2}-n(n-1)a_n-2na_n+12a_n=0$$

$$(n+2)(n+1)a_{n+2}=n(n-1)a_n+2na_n-12a_n$$

$$a_{n+2}=\frac{(n(n-1)+2n-12)}{(n+2)(n+1)}a_n$$

$2a_2+12a_0=0$

$a_2=-6a_0$ $\quad$ $6a_3+12a_1-2a_1=0$

$a_3=-5/3\,a_1$

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Bassel Equation

$$2x(1-x)y'' + (1-x)y' + 3y = 0$$

$P_0(x) = 2x(1-x)$, $P_1(x) = (1-x)$, $P_2(x) = 3$

$P_0(x) = 0 \Rightarrow 2x(1-x) = 0 \Rightarrow x = 0, 1$ [singular Points]

$P_0(x) = 2x(1-x) \Rightarrow P_0(0) = 2(0)(1-0) \Rightarrow P_0(0) = 0$ Finite/Analytic

Let [Frobenius Series Method]

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$\sum_{n=0}$ → n will not be affected b/c of "m"

$$\frac{dy}{dx} = y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$\frac{d^2y}{dx^2} = y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

Put values in Equation

$$2x(1-x)y'' + (1-x)y' + 3y = 0$$

$$(2x-2x^2)y'' + (1-x)y' + 3y = 0$$

$$(2x-2x^2)\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2} + (1-x)\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + 3\sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-1} - 2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} + \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} - \sum_{n=0}^{\infty}(m+n)a_n x^{m+n} + 3\sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2\sum_{n=0}^{\infty}(m+n+1)(m+n)a_{n+1} x^{m+n} - 2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} + \sum_{n=0}^{\infty}(m+n+1)a_{n+1} x^{m+n} - \sum_{n=0}^{\infty}(m+n)a_n x^{m+n} + 3\sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

Comparing Coefficients

$$2(m+n+1)(m+n)a_{n+1} - 2(m+n)(m+n-1)a_n + (m+n+1)a_{n+1} - (m+n)a_n + 3a_n = 0$$

$$\left[2(m+n+1)(m+n) + (m+n+1)\right]a_{n+1} = 2(m+n)(m+n-1)a_n + (m+n)a_n - 3a_n$$

$$a_{n+1} = \frac{\left[2(m+n)(m+n-1) + (m+n) - 3\right]a_n}{\left[2(m+n+1)(m+n) + (m+n+1)\right]}$$

Power Series

Chebyshev's Equation

$$(1-x^2)y'' - xy' + 4y = 0$$

$n^2 = (2)^2 \rightarrow$ chebyshev's D.E

$P_0(x) = (1-x^2)$, $P_1(x) = -x$, $P_2(x) = 4$

$P_0(x) = 0 \Rightarrow 1-x^2 = 0 \Rightarrow x = \pm 1$ [singular Points]

$\ldots\ldots, -3, -2, 0, 2, 3, \ldots\ldots$ [ordinary points]

Let [Power Series Method]

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow a_0 x^0 + a_1 x^1 + a_2 x^2 \ldots\ldots$$

$$dy/dx = y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$d^2y/dx^2 = y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Put values in equation

$$(1-x^2)y'' - xy' + 4y = 0$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$ [$x^n \rightarrow$ same]

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n + 4a_0 + 4a_1 x + 4\sum_{n=2}^{\infty} a_n x^n = 0$$ [$\sum_{n=2}^{\infty} \downarrow$ same]

Comparing Coefficients

$(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 4a_n = 0$	$6a_3 - a_1 + 4a_1 = 0$	$2a_2 + 4a_0 = 0$
$(n+2)(n+1)a_{n+2} = n(n-1)a_n + na_n - 4a_n$	$6a_3 - 3a_1 = 0$	$2a_2 = -4a_0$
$a_{n+2} = \dfrac{[n(n-1)+n-4]a_n}{(n+2)(n+1)}$	$a_3 = \frac{1}{2}a_1$	$a_2 = -2a_0$