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ASSIGNMENT-4

QUESTION 1

Given:

Eigenvalues = λ = 1, -2, -2

And Eigen vectors => $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Therefore

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

To find P^{-1}

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right] R_{23}$$

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right] R_1 + R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right] \frac{R_3}{-1}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - 2R_3 \\ \end{array}$$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$A = \cancel{\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & -2 \\ 2 & 2 & 0 \end{bmatrix}} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & -2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

QUESTION 2:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

We know $\quad A\underline{x} = \lambda \underline{x} \quad$ where $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\cancel{\Rightarrow \lambda\ \lambda A x}$

$\Rightarrow (A - \lambda I)\underline{x} = \underline{0} \longrightarrow (1)$

(1) is homogeneous system in three unknowns x_1, x_2, x_3
Now (1) has non-trivial solution only if

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

Expanding from R_1

$$1-\lambda \begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} -3 \begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} +3 \begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix} = 0$$

$$(1-\lambda)[(-5-\lambda)(1-\lambda)+9) - 3(-3+3\lambda+9) + 3(-9+15+3\lambda) = 0$$
$$(1-\lambda)(-5+5\lambda-\lambda+\lambda^2+9) - \cancel{9\lambda} - \cancel{18} + \cancel{18} + \cancel{9\lambda} = 0$$
$$(1-\lambda)(\lambda^2+4\lambda+4) = 0$$

$1-\lambda = 0 \quad ; \quad \lambda^2+4\lambda+4 = 0$

$\boxed{\lambda = 1} \quad ; \quad \lambda^2+2\lambda+2\lambda+4 = 0$

$(\lambda+2)(\lambda+2) = 0$

$\boxed{\lambda = -2} \; ; \; \boxed{\lambda = -2}$

$\lambda = 1, -2, -2$ are eigen values of A

When $\lambda = 1$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Then (i) becomes

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|O] = \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right]$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \; R_2 + R_3$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \; R_{31}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \; \frac{R_1}{3}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \; \frac{R_2}{-1}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & +0 & 0 \end{array}\right] \; \begin{array}{l} R_1 - R_2 \\ R_3 - 3R_3 \end{array}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \; \frac{R_2}{+3}$$

$x_1 + x_2 = 0$

$x_2 + x_3 = 0$

x_2 is a free variable

$x_2 = r$ where $r \in \mathbb{R}$

$x_1 = -r$

$x_3 = -r$

So for $\lambda = 1$ Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r \\ r \\ -r \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

For $\lambda = -2$ put in ①

$$A = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{X} \qquad \underline{0}$$

$$[A|\underline{0}] = \left[\begin{array}{ccc|c} +3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \Rightarrow \underset{\sim}{R} \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \frac{R_1}{3}$$

Here we have 2 free variables

$x_2 = r$ and $x_3 = s$ where $r, s \in \mathbb{R}$

$x_1 = -r - s$

Eigen vectors $= \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = -2$ there are two eigen vector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The matrix A is diagonalizable because for every eigen value there is different vector.