$$T = \int_0^{x_1} \sqrt{\frac{1+y'^2}{2gy}} dx \tag{1}$$

$$L(y, y') = \sqrt{\frac{1+y'^2}{2gy}} \tag{2}$$

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \tag{3}$$

根据

$$\frac{dL}{dx} = \frac{\partial L}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial L}{\partial y'}\frac{\partial y'}{\partial x} \tag{4}$$

$$= \frac{\partial L}{\partial y} y' + \frac{\partial L}{\partial y'} y'' \tag{5}$$

$$\frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right) = y''\frac{\partial L}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \tag{6}$$

$(5)-(6)$可以得到

$$\frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} y' - y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \tag{7}$$

$$= y'\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right) \tag{8}$$

根据(3)和(8)，可以得到

$$\frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) = 0 \tag{9}$$

$$L - y'\frac{\partial L}{\partial y'} = C \tag{10}$$

$$\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} \tag{11}$$

$$L - y'\frac{\partial L}{\partial y'} = \sqrt{\frac{1+y'^2}{2gy}} - \frac{y'^2}{\sqrt{2gy}\sqrt{1+y'^2}} \tag{12}$$

$$= \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} \tag{13}$$

$$= C \tag{14}$$

我们可以得到

$$y(1+y'^2) = \frac{1}{2gC^2} \tag{15}$$

令$1/2gC^2 = 2r, x = x(\theta), y' = \cot\frac{\theta}{2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

可以得到

$$y(1+y'^2) = y\left(1 + \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right) \tag{16}$$

$$= y\frac{1}{\sin^2\frac{\theta}{2}} = 2r \tag{17}$$

$$y = 2r\sin^2\frac{\theta}{2} \qquad = r(1-\cos\theta) \tag{18}$$

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} \tag{19}$$

$$= \cot\frac{\theta}{2}\frac{dx}{d\theta} = r\sin\theta \tag{20}$$

$$\frac{dx}{d\theta} = \frac{r\sin\theta}{\cot\frac{\theta}{2}} \tag{21}$$

$$= 2r\sin^2\frac{\theta}{2} \tag{22}$$

$$= r(1-\cos\theta) \tag{23}$$

$$x = r(\theta - \sin\theta) + C_2 \tag{24}$$

x, y过零点,所以当$\theta = 0$时，$x = y = 0$

x,y过极值点,所以当$\theta = \pi$时,$x = x_1, y = y_1$

$$x(0) = C_2 = 0 \tag{25}$$

$$y(0) = 0 \tag{26}$$

$$x(\pi) = r\pi = x_1 \tag{27}$$

$$y(\pi) = 2r = y_1 \tag{28}$$

摆线**顶点**构成的集合是一条直线

$$y = \frac{2}{\pi}x \tag{29}$$

参考链接:

1. 寻找“最好”（2）——欧拉-拉格朗日方程
2. matlab拉格朗日曲线_欧拉神作之四——从“最速降曲线”问题到创立新学科
3. 从最速降线问题到欧拉-拉格朗日方程