Weighted Spaces of Besov. Embedding and Interpolation Theorems

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Intro 1. Sobolev spaces

In this work, scales of weighted spaces $X_p^s(\Omega; \rho, v_s)$ for functions in n-dimensional domains with normalization of the type l_p $(s \geq 0, 1 < p < \infty)$ are introduced. In particular, scales of Sobolev spaces $W_p^m(\Omega; \rho, v_m)$ and Besov spaces $B_p^s(\Omega; \rho, v_s)$.

Let $D = (D_1, D_2, ..., D_n)$, $D_i = \frac{\partial}{\partial x_i}$, $D_i^m = \frac{\partial^m}{\partial x_i^m}$, $i = 1, 2, ..., n$. For a weight ω on Ω

$$\|u\|_{L_p(\Omega;\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx \right)^{1/p},$$

$$\sum_{i=1}^n \|D_i^m u\|_{L_p(\Omega;\omega)} = \sum_{i=1}^n \left(\int_\Omega |D_i^m u(x)|^p \omega(x)\, dx \right)^{1/p}.$$

Intro 1. Sobolev spaces

$L_p(\Omega; \mathrm{loc})$ is the space of functions f in Ω such that $f \in L_p(F)$ for every compact $F \subset \Omega$. Assume $L_p(\Omega) = L_p(\Omega; \omega)$ when $\omega = 1$; $L_p(\mathbb{R}^n) = L_p$.

The classes $C^m(\Omega)$, $C^\infty(\Omega)$, $C_0^\infty(\Omega)$ are respectively the class of functions having continuous derivatives up to order m inclusive in Ω, the class of infinitely differentiable functions in Ω, the class of functions from $C^\infty(\Omega)$ with compact support *suppf* in Ω. By definition $C^\infty(\bar{\Omega}) = \{f : f = F|_{\bar{\Omega}}, F \in C^\infty(\mathbb{R}^n)\}$.

Intro 1. Sobolev spaces

Denote for an integer $m > 0$, $mp > n$, the space $W_p^m(\Omega; \rho, v)$ of functions f in Ω, having in Ω (generalized) derivatives $D_i^m = \frac{\partial^m}{\partial x_i^m}$ of order m ($i = 1, 2, ..., n$), and such that the norm (on equivalence classes)

$$\|u\|_{W_p^m(\Omega;\rho,v)} = \sum_{i=1}^{n} \|D_i^m u\|_{L_p(\Omega;\rho)} + \|u\|_{L_p(\Omega;v)} < \infty, \tag{1}$$

$W_p^m(\Omega; \rho, v)$ is a two-weighted Sobolev space defined as the completion of the class $C^\infty W_p^m(\Omega; \rho, v) = C^\infty(\Omega) \cap W_p^m(\Omega; \rho, v)$ by the norm (1). If $\rho = v = 1$ the space $W_p^m(\Omega; \rho, v)$ will be the known Sobolev space $W_p^m(\Omega)$ with norm

$$\|u; W_p^m(\Omega)\| = \sum_{i=1}^{n} \|D_i^m u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}.$$

Intro 2. Localization. Weights

Let I^n be the collection of all cubes of the form

$$Q = Q_h(x) = \{y \in \mathbb{R}^n : |y_i - x_i| < h/2, i = 1, 2, ..., n\}.$$

For $a > 0$, the cube $aQ_h(x) = Q_{ah}(x)$.
Let ρ and $h(\cdot)$ be functions in Ω, satisfying the conditions:
1) $\rho(x) > 0$, $1 \geq h(x) > 0$ in Ω,
2) $Q(x) = Q_{h(x)}(x) \subset \Omega$, $x \in \Omega$;
3) there exist such $\varkappa > 1$ and $\tau_0 \in (0, 1)$ that

$$\varkappa^{-1} \leq \frac{\rho(y)}{\rho(x)}, \frac{h(y)}{h(x)} \leq \varkappa, \text{if } y \in \tau_0 Q(x).$$

Intro 3. Besicovitch Coverings and Partitions of Unity

From the family $\{\varepsilon Q(x), x \in \Omega\}$, $0 < \varepsilon \leq \tau_0$, one can extract a countable Besicovitch covering $\{\varepsilon Q(x^j), j \in \mathcal{J}\}$ ($\mathcal{J} \subset \mathbb{N}$) of the domain Ω (see [1]).

From a covering $\{\varepsilon Q(x), x \in \Omega\}$ with $0 < \varepsilon \leq \tau_0$, we derive a countable Besicovitch covering $\{\varepsilon Q(x^j)\}$ characterized by $\varkappa_1$-multiplicity and decomposa into $\varkappa_2$ disjoint subfamilies.

$\forall \tau$, $0 < \varepsilon < \tau < \tau_0$, $\{\tau Q(x^j), j \in \mathcal{J}\}$ is also a B-cover of Ω (cond. 3)).

Associated partitions of unity $\{\psi_j\}$ adhere to:
- $0 \leq \psi_j \leq 1$ on $\varepsilon Q(x^j)$, with $\operatorname{supp}\psi_j \subset \tau Q(x^j)$ and $\sum \psi_j = 1$ over Ω,
- Derivatives up to order m of ψ_j satisfy $\sup_{x\in\Omega} |\nabla_m \psi_j| \leq c h^{-m}(x^j)$ ([2], 2.3.1]). Here $|\nabla_m f(x)| = \left(\sum_{|\alpha|=m} |D^\alpha f(x)|^2\right)^{1/2}$.

Intro 4. Functional Spaces and Norms

Below, as a rule, a double B-covering $\{\widehat{Q}^j, \widetilde{Q}^j\}$ of the domain Ω, where $\widetilde{Q}^j = \tau Q(x^j)$, $\tau = 3\tau_0/4$, $\widehat{Q}^j = \frac{3}{4}\widetilde{Q}^j$, and $\hat{\varkappa}_i = \hat{\varkappa}_i(n)$, $\tilde{\varkappa}_i = \tilde{\varkappa}_i(n, \varkappa)$ $(i = 1, 2)$ – corresponding multiplicities and divisibility coefficients.

Let $\{X_{(s)}\}, s \geq 0$ be a family of Banach spaces of functions $f : \mathbb{R}^n \to \mathbb{R}$, satisfying the conditions:
i_1) $\mathcal{D} = C_0^\infty(R^n)$ is dense in $X_{(s)}$;
i_2) $X_{(s)} \hookrightarrow X_{(t)}$, if $s > t \geq 0$;
i_3) There exists such p, $1 < p < \infty$, that $X_{(0)} = L_p$ and

$$W_p^m \hookrightarrow X_{(s)}, \text{if } m > [s],$$

where W_p^m is the Sobolev space, equipped with the norm

$$\|f; W_p^m\| = \|\nabla_m f\|_p + \|f\|_p. \tag{2}$$

Intro 4. Functional Spaces and Norms

Let $1 \le q \le p$, and let

$$v_s(x) = \rho(x) h^{-s}(x) \ (x \in \Omega).$$

Due to conditions 3) and i_3), the functional

$$\|f; X^s_p(\Omega; \rho, v_s)\| \stackrel{def}{=} \left[\sum_{j \ge 1} \rho^p(x^j) \|\psi_j f; X_{(s)}\|^p + v^p_s(x^j) \|\psi_j f\|^p_p \right]^{1/p} \ll$$

$$\ll \|f; W^m_q(\Omega; \rho, v_s)\| \sim \|\rho \nabla_m f; L_q(\Omega)\|_q + \|v_m f; L_q(\Omega)\|_q < \infty \qquad (3)$$

in the class $C^\infty_0(\Omega)$ and, thereby, is a norm.

Intro 4. Functional Spaces and Norms

Let $0 < \tau_i \le 3\tau_0/4$, $\varepsilon_i = 3\tau_i/4$, $i = 1, 2$. Consider double B-coverings $\mathcal{M}_1 = \{\varepsilon_1 Q(x^j), \tau_1 Q(x^j)\}$, $\mathcal{M}_2 = \{\varepsilon_2 Q(t^k), \tau_2 Q(t^k)\}$ and their norms

$$\left[\|f; X_{(s)}(\Omega; \rho, v_s)\|^{(1)}\right]^p = \sum_{j \ge 1} \left(\rho^p(x^j)\|\psi_j f; X_{(s)}\|^p + v_s^p(x^j)\|\psi_j f\|_p^p\right),$$

$$\left[\|f; X_{(s)}(\Omega; \rho, v_s)\|^{(2)}\right]^p = \sum_{k \ge 1} \left(\rho^p(t^k)\|\varphi_k f; X_{(s)}\|^p + v_s^p(t^k)\|\varphi_k f\|_p^p\right);$$

here $\{\psi_j\}$, $\{\varphi_k\}$ are partitions of unity associated with $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively

If the norms $\|f; X_{(s)}(\Omega; \rho, v_s)\|^{(i)}$, $i = 1, 2$, are equivalent, then we can correctly define the space $X_p^s(\Omega; \rho, v_s)$ as the completion of the class $C_0^\infty(\Omega)$ by the norm $\|f; X_p^s(\Omega; \rho, v_s)\|$ (3).

Intro 5. Overview of Weighted Besov Spaces

This work considers spaces constructed on the scales of Bessel potentials $\{H_p^s\}$ and Besov spaces $\{B_p^s = B_{p,p}^s\}$, $s \geq 0,\ 1 < p < \infty$.
The space H_p^s can be defined as the completion of the class of fundamental functions $\mathcal{D}$ $(= C_0^\infty(\mathbb{R}^n))$ by the norm

$$\|f; H_p^s\| = \|\mathcal{J}_s f\|_p, \tag{4}$$

where J_s is an operator with the symbol $k(\xi) = (1 + |\xi|^2)^{s/2}$. For natural m, we have $H_p^m = W_p^m$ (see, for example, [10, 13]).

Intro 5. Overview of Weighted Besov Spaces

Let $\mathcal{G}$ be a domain in $\mathbb{R}^n$, $s > 0$, $1 < p < \infty$, $1 \leq q < \infty$. $B^s_{p,q}(\mathcal{G})$ is the space of all $f \in L_{loc}(\mathcal{G})$ with finite norm

$$\|f; B^s_{p,q}(\mathcal{G})\| = \|f; b^s_{p,q}(\mathcal{G})\| + \|f; L_p(\mathcal{G})\|, \tag{5}$$

$$\|f; b^s_{p,q}(\mathcal{G})\| = \sum_{|\alpha|=k} \left[\int \left(|t|^{-(s-k)} \|\Delta^m_t D^\alpha f; L_p(\mathcal{G}_{t,m})\| \right)^q |t|^{-n} dt \right]^{1/q}, \tag{6}$$

m, k are integers: (xx) $0 \leq k < s$, $m > s + k$, $\mathcal{G}_{t,m} = \cap^m_{j=0} \{x : x + jt \in \mathcal{G}\}$,

$\Delta^m_j f(x) = \sum^m_{j=0} (-1)^{m+j} \begin{pmatrix} m \\ j \end{pmatrix} f(x + jt)$ is the m-th difference of f (at x) with step t, $\Delta^0_t f(x) = f(x)$, $\Delta_t f(x) = f(x + t) - f(x)$.

Intro 5. Overview of Weighted Besov Spaces

The semi-norms $\|f; b^s_{p,q}(\mathcal{G})\|$ are equivalent for a pair of integers m, k, satisfying condition (xx).

Therefore, for $s > 0$ – non-integers, take $k = [s]$, $m = 1 > s - [s] = \{s\}$ where $[s]$ – integer, $\{s\}$ – fractional parts of s; for $s \in \mathbb{N}$ $k = s - 1$, $m = 2$.

Set $B^s_p(\mathcal{G}) = B^s_{p,p}(\mathcal{G})$, $B^0_p(\mathcal{G}) = L_p(\mathcal{G})$. In the space $B^s_p = B^s_{p,p}(\mathbb{R}^n)$ the semi-norm is given as follows:

$$\|f; b^s_p\| = \sum_{|\alpha|=m} \left[\int \left(|t|^{s-k} \|\Delta^m_t f\|_p \right)^p |t|^{-n} dt \right]^{1/p}.$$

([7, 13]).

Intro 5. Overview of Weighted Besov Spaces

Let $m \in \mathbb{N}$, $m > t > s \geq 0$, $1 < p < \infty$. From the known embedding inequalities

$$\|f; H_p^s\| \leq c_1 \|f; H_p^t\|,$$
$$\|f; B_p^s\| \leq c_2 \|f; H_p^s\|, \; 2 \leq p < \infty,$$
$$\|f; B_p^s\| \leq c_3 \|f; B_{p,2}^t\| \leq c_4 \|f; H_p^t\|, \; 1 < p \leq 2$$

it follows that the families $\{H_p^s\}$, $\{B_p^s\}$ $(s \geq 0, 1 < p < \infty)$ satisfy condition i_3), and the inequalities

Intro 5. Overview of Weighted Besov Spaces

$$\|f;H_p^s(\Omega;\rho,v_s)\| \leq c_1\|f;W_p^m(\Omega;\rho,v_m)\| < \infty,\ \ f\in\mathcal{D},$$

$$\|f;B_p^s(\Omega;\rho,v_s)\| \leq c_2\|f;W_p^m(\Omega;\rho,v_m)\| < \infty,\ \ f\in\mathcal{D},$$

are valid (see [13], 2.3.3, 2.8.1).

The spaces $H_p^s(\Omega;\rho,v_s)$ were introduced in the work [13]. There, the interpolati spaces $(H_p^{s_0}(\Omega;\rho,v_{s_0}),H_p^{s_1}(\Omega;\rho,v_{s_1}))_\theta$, obtained by the complex method, are described.

Assertion *. In the class $C_0^\infty(\Omega)$ the norms $\|B_{p,q}^s(\Omega;\rho,v_s)\|^{(i)}$ $(i=1,2)$ are equivalent.

Main Results

Throughout, $\{\psi_j\}$ denotes a partition of unity associated with the family $\{\widehat{Q}^j, \widetilde{Q}^j\}$, where $\widehat{Q}^j = \frac{3}{4}\widetilde{Q}^j$, $\widetilde{Q}^j = \tau Q^j$, and $\tau = 3\tau_0/4$, with $Q^j = Q(x^j)$.
Let $\overline{A} = \{A_0, A_1\}$ be an interpolation pair of Banach spaces A_i $(i = 0, 1)$, and let $(A_0, A_1)_{\theta,q}$ be the interpolation space obtained by the real interpolation method (Petree space) ([13], 1.2).

Theorem 1. Let $0 \le m_1 < m_0$ be integers, $s = (1-\theta)m_0 + \theta m_1$, $0 < \theta < 1$, $1 < p < \infty$. Then

$$\left(W_p^{m_0}(\Omega; \rho, v_{m_0}), W_p^{m_1}(\Omega; \rho, v_{m_1})\right)_{\theta,p} = B_p^s(\Omega; \rho, v_s). \tag{7}$$

Theorem 2. Let $0 \le \lambda_0 < \lambda_1$, $s = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$, $1 < p < \infty$. Then

$$\left(B_p^{\lambda_0}(\Omega; \rho, v_{\lambda_0}), B_p^{\lambda_1}(\Omega; \rho, v_{\lambda_1})\right)_{\theta,p} = B_p^s(\Omega; \rho, v_s). \tag{8}$$

On the Proof of Results

We will use the notation: $U : A \to B$ represents a bounded linear operator from $A = D(A)$ into B (both A and B are Banach spaces), and $l_p(A_j)$ is the space of sequences $\{a_j\} \in \sqcap_{j\geq 1} A_j$ with a finite norm

$$\|\{a_j\};\ l_p(A_j)\| = \Big(\sum_{j\geq 1} \|a_j\|_{A_j}^p\Big)^{1/p},$$

where A_j $(j \geq 1)$ are Banach spaces.
The proofs will be guided by reduction to an interpolation equality of the form

$$\big(l_{p_0}(A_j), l_{p_1}(B_j)\big)_{\theta,p} = l_p\big((A_j, B_j)_{\theta,p}\big), \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{9}$$

About the Proof of Results

Definition. An operator $R : A \to B$ is called a retraction if there exists an operator $S : B \to A$ such that the composition $RS = I_s$, where I_s is the identity operator in B. The operator S is called a coretraction corresponding to the retraction R.

Theorem T1 (fundamental, [13]) Let $R : \sum \overline{A} \to \sum \overline{B}$, $S : \sum \overline{B} \to \overline{A}$. Such operators, where the restrictions $R/A_i : A_i \to B_i$ are retractions with respect to $S/B_i : B_i \to A_i$, and let F be an interpolation functor. Then the restriction $S/_{F(\overline{B})}$ isomorphically maps the space $F(\overline{B})$ onto the closed subspace

$$W = \{b : b = SRa,\ a \in F(\overline{A})\} \subset F(\overline{A}). \tag{10}$$

In this work, the interpolation functor F is defined as

$$F(\overline{A}) = (A_0, A_1)_{\theta,q} \quad (0 < \theta < 1,\ 1 < q < \infty).$$

About the Proof of Results

Proposition 1. Let λA $(\lambda > 0)$ be the Banach space A with the norm $\|a; \lambda A\| = \lambda \|a\|_A$. The following equality holds:

$$\left(\lambda_0 A_0, \lambda_1 A_1\right)_{\theta,p} = \lambda_0^{1-\theta} \lambda_1^{\theta} (A_0, A_1)_{\theta,p}. \tag{11}$$

Lemma L1. Let $0 \leq m_1 < m_0$ be integers, $s = (1-\theta)m_0 + \theta m_1$, $1 < p < \infty$. The equality

$$\left(W^{m_0}_{p,(h)}, W^{m_1}_{p,(h)}\right)_{\theta,p} = B^s_{p,(h)} \tag{12}$$

is valid.

About the Proof of Results

Lemma L2. Let $0 \le \lambda_1 < \lambda_0$, $s = (1-\theta)\lambda_0 + \lambda_1\theta$, $1 < p < \infty$. The following equality is established:

$$\left(B^{\lambda_0}_{p,(h)}, B^{\lambda_1}_{p,(h)}\right)_{\theta,q} = B^{s}_{p,(h)}. \tag{13}$$

Proposition P1*. [LK,dis.p1] For $1 \le p < \infty$, the following statements hold within $W^m_p(Q_h)$:

a)

$$\sum_{k=0}^{m} h^{k-m} \|\nabla_k u; L_p(Q_h)\| \le c\left(\|\nabla_m u; L_p(Q_h)\| + h^{-m}\|u; L_p(Q_h)\|\right).$$

About the Proof of Results

b)

$$\|D^\alpha(\psi u); L_p(Q_h)\| \leq c_1\big(\|\nabla_m u; L_p(Q_h)\| + h^{-m}\|u; L_p(Q_h)\|\big),$$

where $\psi \in C_0^m(Q_h)$ and satisfies

$$\max_{Q_h} |D^\alpha \psi| \leq h^{-|\alpha|}, |\alpha| \leq m.$$

Proposition P2. In $W_p^m(\Omega; \rho, v_m)$, the norm

$$\|f; W_p^m(\Omega; \rho, v_m)\| = \|H_p^m(\Omega; \rho, v_m)\| \sim \|\rho\nabla_m u; L_p(\Omega)\| + \|v_m u; L_p(\Omega)\|.$$

is equivalent.

Th1

Alongside the double B-covering $\{\widehat{Q}^j, \widetilde{Q}^j\}$, with $Q^j = Q(x^j)$, we consider the B-covering $\{\widetilde{Q}^j, \tau_0 Q^j\}$ and its associated partition of unity $\{\varphi_j\}$ with functions $\varphi_j \in C_0^\infty(\tau_0 Q^j)$, and $\varphi_j = 1$ on $\widetilde{Q}^j$.
Let $h_j = h(x^j)$ for $j \geq 1$, and define:

$$Sf = \{f_j\}, \quad f_j = \varphi_j f \quad (j \geq 1), \; f \in W_p^m,$$

$$R(\{f_k\}) = \sum_{k \geq 1} \psi_k f_k, \quad f_k \in W^m_{p,(h_k)}.$$

The operator R is shown to be bounded from $l_p(\rho(x^k)A_k)$ (with $A_k = W^m_{p,h_k}$) to the weighted space $W_p^m(\Omega; \rho, v_m)$.

Th1

For $f \in W_p^m(\Omega;\rho,v_m)$ $\big(RS\big)f(x) = f(x)$.

$$\|R\{f_k\}; W_p^m(\Omega;\rho,v_m)\| \ll c_5\|\{f_k\}; l_p(\rho(x^k)A_k\|,$$

$c_5 = c_5(m,n,p,\varkappa)$.

$$\|Sf; l_p(\rho(x^j)W^m_{p,(h_j)}\| \leq c_6\|f; W_p^m(\Omega;\rho,v_m)\|,$$

$c_6 = c_6(n,m,p,\varkappa)$.
Therefore, R and S are retraction and coretraction satisfying Theorem T1.

Th1

Lemma L1 $\Rightarrow \left(W_p^{m_0}(\Omega;\rho,v_{m_0}),\, W_p^{m_1}(\Omega;\rho,v_{m_1})\right)_{\theta,p} =$
$= \left\{ f = R\{f_k\},\ \{f_k\} \in l_p(\rho(x^k)B^s_{p,(h_k)}) \right\}$
and $B_p^s(\Omega;\rho,v_s) \subset W_\theta$ for $f \in B_p^s(\Omega;\rho,v_s)$.

Otherwise, from the fact that there are finite families of coverings of Ω, and from the known interpolation equalities

$$\left(W_p^{m_0}, W_p^{m_1}\right)_{\theta,p} = B_p^s,\quad \left(W^{m_0}_{p,(h_j)}, W^{m_1}_{p,(h_j)}\right)_{\theta,p} = B^s_{p,(h_j)},$$

we obtain $(c_7 = c_7(n,p,m_0,m_1,\theta,\varkappa))$

$$\|R\{f_k\}; B_p^s(\Omega;\rho,v_s)\|^p \le c_7^p \sum_{k\ge 1} \rho^p \|f_k; B^s_{p,(h_k)}\|^p = c_7^p\ \|\{f_k\}\|^p_{l_p\left(\rho(x^k)B^s_{p,(h_k)}\right)}.$$

Th2. Assertion *

From Theorem T1 and the lemmas, it follows:

$$\Big(B_p^{\lambda_0}(\Omega;\rho,v_{\lambda_0}),\ B_p^{\lambda_1}(\Omega;\rho,v_{\lambda_1})\Big)_{\theta,p} =$$

$$= \{f = R\{f_k\},\ f_k \in l_p(\rho(x^k)B^s_{p,(h_k)})\} = B_p^s(\Omega;\rho,v_s).$$

Proof of Assertion *. We have: $s = (1-\theta)m_0 + \theta m_1$, $0 < \theta < 1$, $0 \leq m_1 < m_0$ are integers, $1 < p < \infty$. Due to Theorem 1, the following implication is valid:

$$\|f; B_p^s(\Omega;\rho,v_s)\|^{(1)} \sim \|(W_p^{m_0}(\Omega;\rho,v_{m_0}),\ W_p^{m_1}(\Omega;\rho,v_{m_1}))_{\theta,p}\| \sim$$

$$\sim \|f; B_p^s(\Omega;\rho,v_s)\|^{(2)}.$$

Remarks

Remark 1. Let $m \in \mathbb{N}$, $1 < p < \infty$, $mp > n$. Let also the weight function $\upsilon \in L^{+}_{loc}$ satisfy the condition

$$(i) \ \left(|Q|^{-1}\right) \upsilon(Q) \geq 1, Q \in I^{n}).$$

Then the function (Otelaev's running average)

$$\upsilon^{*}(x) = (h^{*}_{m,p}(x;\upsilon) = \sup\{h > 0 : h^{mp-n}\upsilon((Q_h(x)) \leq 1\},$$

defined by L. Kusainova ([2], is positive in $\mathbb{R}^n$, $\sup h^{*}(x) \leq 1$, and for any $\tau, 0 < \tau < 1$,

$$1 - \tau \leq \frac{h^{*}(y)}{h^{*}(x)} \leq (1-\tau)^{-1} \ \ if \ \ y \in \tau Q^{*}(x),$$

where $Q^{*}(x) = Q_h(x)$ at $h = h^{*}(x)$.

Remarks

If moreover, there exist such $0 < \delta, \gamma < 1$, that

(ii) $\upsilon(e) \le \gamma \upsilon(Q)$, as soon as $e \subset Q$ $|e| \le \delta |Q|$ $\forall Q \subset Q^*(x)$, then $\upsilon_m(x) = h^*(x)^{-m}$ and $W_p^m(1,\upsilon) = W_p^m(1,\upsilon_m)$.

Let $l, m \in \mathbb{N}$, $l < m$, $mp > n$. From Theorem 1, it follows that

$$\left(W_p^m(1,\upsilon), W_p^l(1,\upsilon^{*-l})\right)_{\theta,p} = \mathfrak{B}_p^s(1,\upsilon^{*-s}).$$

Remarks

Remark 2. Theorem 1 was formulated in [1]. This work provides an expanded proof of this theorem. Earlier in ([1], ch. 2) for weight functions υ, satisfying conditions $(i)-(ii)$, an interpolation equality was obtained:

$$\left(W_p^m\left(1,\upsilon\right), W_p^l\left(1,\upsilon\upsilon^{*p(m-l)}\right)\right)_{\theta,p} = \mathfrak{B}_p^s\left(1,\upsilon^{*-s}\right)$$

for the case $\frac{n}{p} < l < m$, $1 < p < \infty$.

Remarks

Obviously, the embedding

$$W_p^l\left(1, \upsilon^{*-l}\right) \hookrightarrow W_p^l\left(1, \upsilon\upsilon^{*p(m-l)}\right).$$

The equality

$$W_p^l\left(1, \upsilon^{*-l}\right) = W_p^l\left(1, \upsilon\upsilon^{*p(m-l)}\right)$$

is valid for weights $\upsilon \geq 1$ with bounded oscillation of the form $c_0^{-1} \leq \upsilon(y)/\upsilon(x) \leq c_1$, if $y \in Q_{h_x}(x)$, where $h_x = (c\upsilon(x))^{-1/mp}$ $(c_0,\ c_1 > 1)$.

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Finally

Thank you