The Butterfly Effect through Differential Equations

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1.1 Abstract

The Butterfly Effect is the idea that that one small occurrence can make a drastic change within a large complex system, like how a single individual buying stock shares can have long-run ripple effects on international economies (The Decision Lab). The Lorenz equations mathematically illustrate this phenomenon and are explored in Section 2. This laboratory, which is the last in the four part series, also examines and compares four methods for numerically solving differential equations.

1.2 Introduction

Edward Norton Lorenz was an MIT meteorologist who used one of the first commercially available computers to evaluate weather models (Sauer). The scientist simplified the behaviour of rolling fluid convection into three equations that will be introduced and explored throughout this laboratory.

Over a wide range of parameters, the solutions for this system are known for oscillating unpredictably, serving as an introduction to Chaos Theory. This theory describes the point at which order moves to disorder (Straussfogel), or in this case, when a seemingly simple system of equations produces unpredictable results.

The solutions of the Lorenz equations can be observed in a phase space, a representation where each possible state of a system corresponds to a unique point (Espinosa-Paredes). The Lorenz Attractor is the name given to the state that a system evolves to over time (Harris). When this state settles into a region in the phase space, the system is said to maintain equilibrium (Espinosa-Paredes). Sometimes, the trajectories of the solutions are attracted to a region, but once in the region, the trajectories repel each other; these systems never reach final equilibrium and are referred to as the Strange Attractors (Harris).

The Lorenz equations are implemented numerically by solving the system similar to how one would other ordinary differential equations (ODEs). In this laboratory, we will investigate Euler's Method, the Explicit Trapezoid Method, Taylor's Method, and the Runge-Kutta family for solving ODEs.

Taken together, all of this research will allow us to acquire a deeper understanding of Chaos Theory and provide insight into how a butterfly flapping its wings could cause a typhoon.

2 Numerical Solutions of ODEs

Solving ODEs of different orders can be done in a variety of ways. In this section, we will mainly examine four methods. At the end of this section, using each of the methods we will compare solutions to:

$$(1) \qquad \frac{dy}{dt} = t + 2y, \qquad initial\ \ value : y(1) = 3$$

2.1 Euler's Method

Given an ODE, we can calculate the slope of a function for any given point in the domain. If we start at an initial condition (t_0, y_0) we can evaluate the ODE at this point and plot this tangent line. Then, we can move forward, making a 'step' in the positive direction, and evaluate the slope at this new point (t_1, y_1). We can continue taking 'steps' and plotting the tangent line until we arrive at a solution. In this way, Euler's method can help us solve ODEs for a grid of $n + 1$ points where $t_0 < t_1 < t_2 < ... < t_n$ along the t-axis, choosing equal steps of size h—this is a first order method (Sauer).

This process inevitably results in error, especially the larger the step size h. If the solution to an ODE is concave up, Euler's method will produce an overestimate (an approximation that lies above the function); if the solution is concave down, the opposite occurs and the approximation will instead be an underestimate (Schmoop Editorial Team).

2.2 Explicit Trapezoid Method

This method is similar to Euler's Method is generally a second order method (Sauer). It tends to be more accurate, because instead of just taking the slope for a point, the Trapezoid Method takes the average slope between pairs of points where:

$$w_0 = y_0, \qquad w_{i+1} = w_i + \frac{h}{2}(f(t_i, w_i) + f(t_i + h, w_i + hf(t_i, w_i)))$$

This method gets its name because when taking the average slope between points, the area underneath this tangent forms the shape of a trapezoid (GfG). The Trapezoid Method calculates the slope across intervals of size h, and decreasing the size of these intervals will increase the accuracy of this method (Engineering at Alberta).

2.3 Taylor Methods

The solution to an ODE can also be approximated using the Taylor expansion, where there exists a Taylor Method of order k for each positive integer k (Sauer). For a function f and step size h, the Taylor Method is as follows:

$$w_0 = y_0, \qquad w_{i+1} = w_i + hf(t_i, w_i) + ... + \frac{h^k}{k!} f^{(k-1)}(t_i, w_i)$$

The derivatives of f used in the equation above can be calculated using the initial condition, but this means the method does require knowledge of higher-order derivatives. Also, this method requires a function to be infinitely differentiable, which is not feasible for all ODEs (Biscani).

2.4 Runge-Kutta Methods

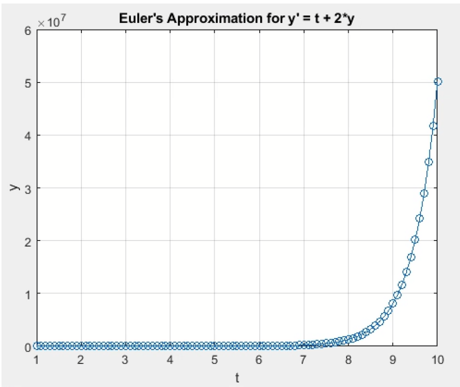
The Runge-Kutta family are methods of varying orders and includes the afformentioned Trapezoid Method (Sauer). The Midpoint Method, also of second-order, belongs to this family and uses the following:

$$w_0 = y_0, \qquad w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i))$$

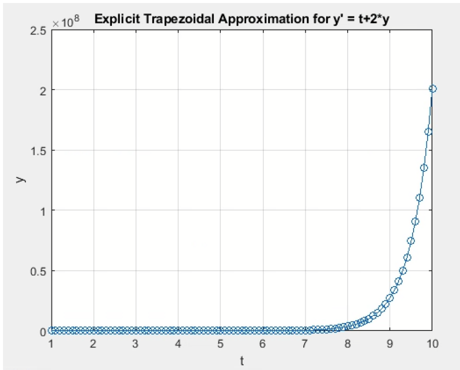
This family of numerical methods allow us to construct high-order, accurate solutions, without needing the higher-order derivatives of functions like we do in Taylor's method (Zheng).

2.5 Comparing the Methods

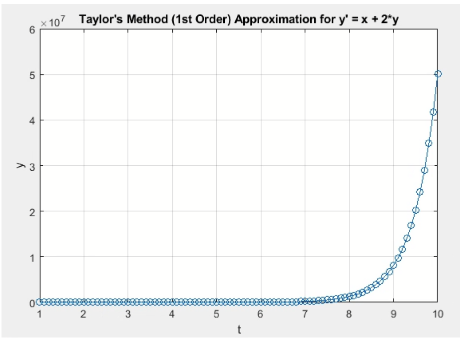
Solving equation (1) in Matlab with all four of the methods described above with step size $h = 0.1$ and $0 \leq t \leq 10$ yields the following solutions:



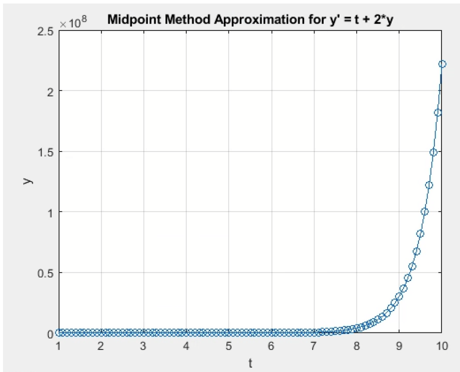
(a) Using Euler's Approximation



(b) Using the Explicit Trapezoid Rule



(c) Using Taylor's Method



(d) Using the Midpoint Method

We can see the Trapezoidal Method yields a similar, but more accurate solution to the ODE compared to Euler's method. We observe that the y-axis is similar for the solutions produced by the Midpoint and Explicit Trapezoid numerical methods; comparatively, the y-axis for Euler's and Taylor's solutions appear identical. This makes sense because the Trapezoid and Midpoint methods belong to the Runge-Kutta family so we would expect comparable results from these two methods.

Using our findings from these ODE solvers and approximators, we can now explore solutions to the Lorenz equations and their subsequent implications for Chaos Theory among other applications.

3 Applications of Numerical Solution of ODE to Q6

3.1 Introduction to the Lorenz Equations

Lorenz used the following three equations to model Rayleigh-Bénard convection, the movement of heat in a fluid, in a two-dimensional atmosphere:

$$(2) \qquad \begin{cases} x' = s(y - x) \\ y' = x(r - z) - y \\ z' = xy - bz \end{cases}$$

where x is the clockwise circulation velocity, y is the temperature difference between ascending and descending columns of air, and z is the deviation from a linear temperature profile in the vertical direction (Sauer). The parameters, s, r, and b are > 0, where s is the Prandtl number and r is the Rayleigh number (Sauer). These parameters are particularly important since the idea of Chaos Theory tells us that minute changes in these parameters may inexplicably and unpredictably lead to big changes in the solution. To see these equations represented in Matlab, see Figure 3.1 in the Code Appendix.

3.2 Finding the Fixed Equilibrium Points of the System

To find the equilibrium point of the Lorenz System we need to solve system (2), provided above. Solving these equations, we find the following equilibrium points:

1. Origin: $(0, 0, 0)$
2. Q_a: $\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$ (for $r > 1$)

The last two equilibria only exist when $r > 1$, indicating a bifurcation occurs when $r = 1$.
Linearizing around the origin, we find the system:

$$\mathbf{Y}' = \begin{bmatrix} -s & s & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

At the origin, the eigenvalues of this matrix are given by:

$$\lambda_1 = -b \quad \text{and} \quad \lambda_2, \lambda_3 = \pm\frac{1}{2}(-(s+1)\pm\sqrt{(s+1)^2 - 4s(1-r)})$$

These eigenvalues provide insight into the stability of the equilibrium point at the origin, and the presence of bifurcation at $r = 1$ suggests a qualitative change in the behavior of the system.

These equilibrium points provide insight into the behavior of the system in these regions, especially when we linearize around these points. Generally, the behavior of the linearized system around an equilibrium point can indicate whether the corresponding equilibrium point in the original nonlinear system is stable or unstable. In the case of the Lorenz system, due to its chaotic behavior, the linearized system might not fully capture the dynamics of the original system, especially far from the equilibrium points; however, near an equilibrium point, the linearized system can provide valuable information about local stability properties, especially near each equilibrium point.

3.3 Specifying (r,s,b) such that (x,y,z) = (0,0,0)

To specify the vector (r, s, b) such that $(x, y, z) = (0, 0, 0)$ is a fixed point (equilibrium point), we need to find the parameter values r, s, and b that satisfy the equilibrium condition $x' = y' = z' = 0$ for the Lorenz system, refer to system (2).
(3) For $x' = 0$:

$$-sx + sy = 0 \Rightarrow s(y - x) = 0$$

For $y' = 0$:

$$-xz + rx - y = 0 \Rightarrow -xz + rx = y$$

For $z' = 0$:

$$xy - bz = 0 \Rightarrow xy = bz$$

To find the values of r, s, and b that satisfy the equilibrium condition $x = y = z = 0$, we examine the equations derived from the Lorenz system.

From (3) $s(y - x) = 0$, we consider two cases:

- 1. If $s = 0$, then the trivial solution $(0, 0, 0)$ is obtained.(See plot 3.31 in the plot appendix)
- 2. If $y = x$, then $x^2 = bz$ (See plot 3.32 in the plot appendix.)
  Substituting $y = x$ and $x^2 = bz$ into $xz - rx + y = 0$, we solve for r:

  $$rx - 2x = 0 \Rightarrow r = 2$$

  For $y = x$ and $r = 2$, we find b from $x^2 = bz$:

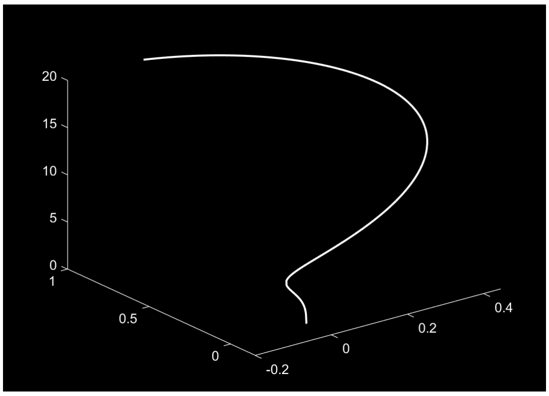
  $$x^2 = bx \Rightarrow x = b$$

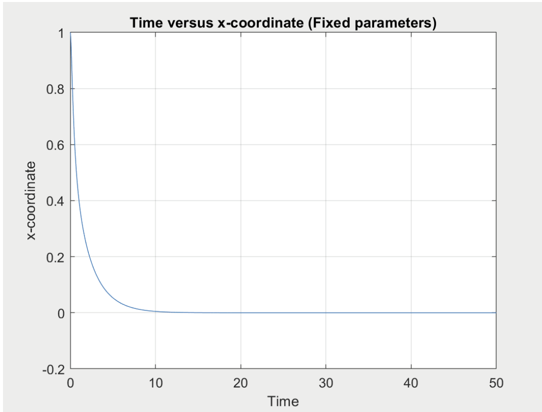
  Thus, the specific vector (r, s, b) such that $(x, y, z) = (0, 0, 0)$ is a fixed point is $(2, 0, 0)$. (See Plot 3.33 in the plot appendix)

3.4 How Trajectories Change Based on r

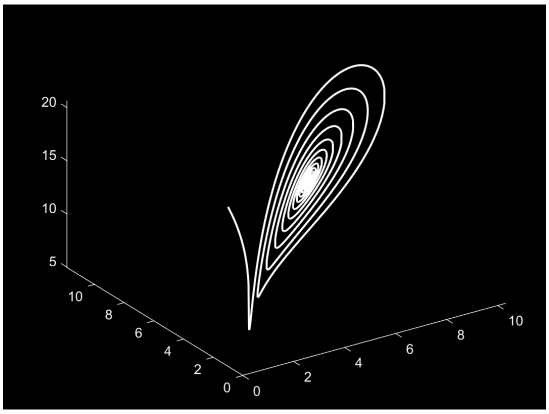
Parameter r is called the Rayleigh Number and it determines whether the heat transfer is primarily in the form of conduction or convection. To explore the importance of parameter r in the chaotic behavior of the Lorenz Equations we should fix the s and parameters. In this lab we used the most commonly used values for parameters s and b, where $s = 10$ and $b = \frac{8}{3}$. The code for the simulation is Figure 3.2 in Code Appendix. The graph of x-coordinates with respect to time is Figure 3.3 in Code Appendix.

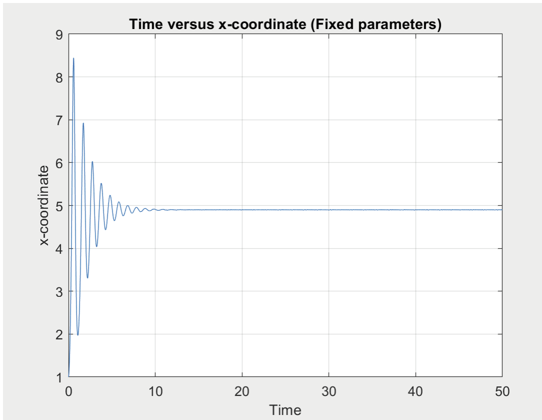
Case 1) When $0 < r < 1$, the origin is overall stable. Example for when $r = 0.5$:



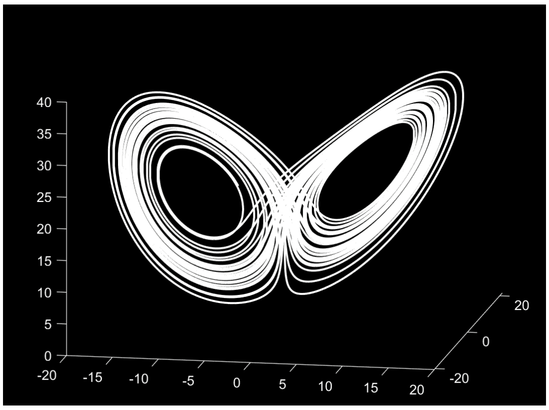


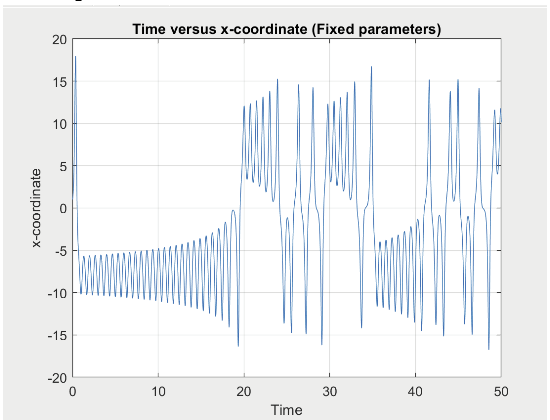
Case 2a) When $r > 1$, the origin is non stable. Example for when $r = 10$:



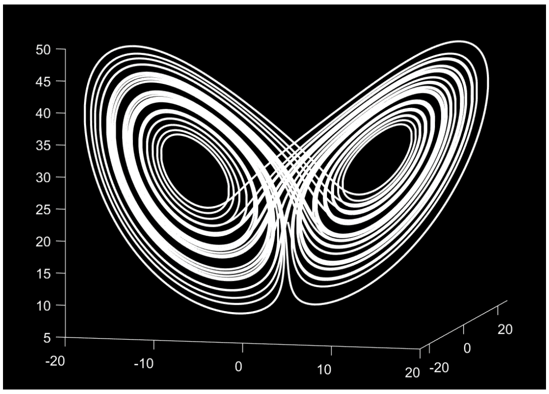


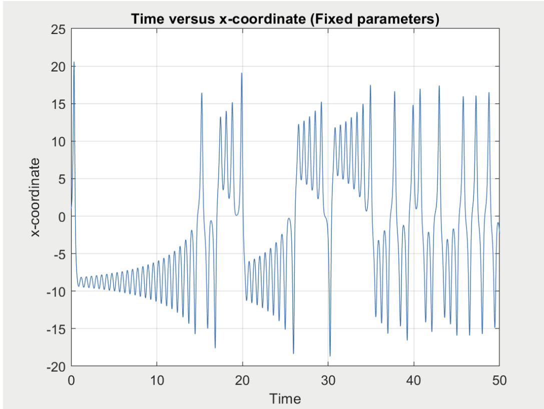
Case 2b) When $r > 1$, the origin is non stable. Example for when $r = 24$:





Case 2c) When $r > 1$, the origin is non stable. Example for when $r = 30$:





The analysis of the r parameter gives us the insight on the point of convergence: At the two other equilibrium points, the eigenvalues are the roots of a polynomial of degree 3. At $r = r^*$, a Hopf bifurcation occurs. The two stable points $C_\pm$ collide each with an unstable cycle and become unstable. For $s = 10$ and $b = \frac{8}{3}$, we have $r^* = \frac{470}{19} \approx 24.7$. For $r \in (1, 24.74)$ the fixed points will be stable $\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$
For other r the trajectory is unstable, for example well known Lorenz attractor.

3.5 Overall Observations

The solutions to the Lorenz equations never repeat exactly but remain in the bounded region of the phase space (Espinosa-Paredes). The lines never intersect themselves and never retrace their own paths (Harris). This is why this mathematical system is celebrated for its ability to illustrate the butterfly effect. Almost poetically, the plots of these systems are often said to resemble a butterfly's wings.

4 Applications of the Lorenz Model

- Lasers: the equations have been used to describe the dynamics of ejected particles from lasers in a fractal paradigm (Irimiciuc).
- Electric Circuits: the system and its integration can be simulated and modeled by electric circuits (Kruno).
- Chemical Reactions: the system can be used to describe chemical chaos, such as in complex chemical reactions like cooperative catalysis (Poland).

5 Conclusion

Our results show us that even in discrete systems, there are limitations to predictions (The Decision Lab). The plots demonstrate chaos because of the system's sensitivity to initial conditions, which becomes increasingly apparent as more time elapses and the system responds to interrelated phenomena (The Decision Lab).

This system created to model atmospheric convection has allowed for greater conclusions about Chaos Theory and regarding the sensitivity of a system initial conditions. Lorenz arrived at similar conclusions in his research, where he argued that accurately predicting weather is essentially impossible due to the knowledge required of the initial conditions. Not only is it virtually impossible to precisely determine the initial conditions of a system, for example with weather systems in meteorology, but also any minor changes will drastically change the outcome (The Decision Lab).

Our findings from the Lorenz equations can paint a broader conclusion about life. It tells us that small actions can make a big difference in our lives, often through random and unexpected ways. As Dan Kootz speaks to the butterfly effect in his novel From the Corner of His Eye, "kindness is passed on and grows each time it's passed until a simple courtesy becomes an act of selfless courage, years later, and far away. Likewise, each small meanness, each expression of hatred, each act of evil."

6 Team Work Statement

The writing for this lab was delegated evenly between both partners. The coding questions were done collaboratively by the team members, with Erina focusing more on the ODE solvers and Alina contributing more to the investigation on the Lorenz Equations.

7 Code Appendix

```
function dx = lorenz(t, x, Beta)
dx = [
Beta(1)*(x(2) - x(1));
x(1)*(Beta(2)-x(3))-x(2);
x(1)*x(2)-Beta(3)*x(3);
];
```

Figure 3.1: Lorenz system.m

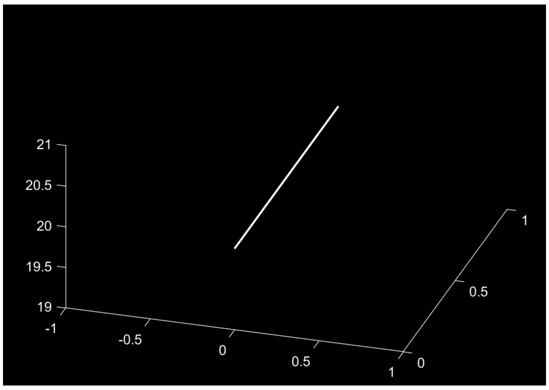
```
% Define fixed parameters
sigma = 10;  % Prandtl number
r = 30;      % Rayleigh number
b = 8/3;     % Geometric factor

% Define initial condition
x0 = [1; 1; 1];  % Initial condition

% Define time span
dt = 0.01;
tspan = 0:dt:50;

% Solve the system for fixed parameters
Beta = [sigma; r; b];
[~, x] = ode45(@(t, x) lorenz(t, x, Beta), tspan, x0);

% Plot time versus x-coordinate
figure;
plot(tspan, x(:,1));
xlabel('Time');
ylabel('x-coordinate');
title('Time versus x-coordinate (Fixed parameters)');
grid on;
```

Figure 3.2: Time versus x-coordinate.m

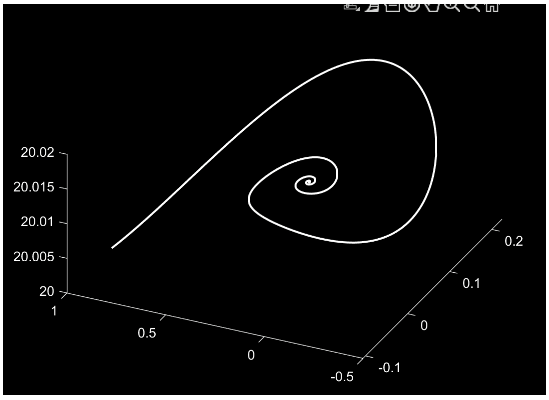
```
clear all, close all, clc
Beta = [10;30;8/3];
x0 = [0;1;20];
dt = 0.001;
tspan = dt:dt:50;

options = odeset('RelTol',1e-12,'AbsTol',1e-12*ones(1,3));
[t,x] = ode45(@(t,x)lorenz(t,x,Beta),tspan,x0, options);

plot3(x(:,1),x(:,2),x(:,3),'w','LineWidth',1.5);
set(gca,'color','k','xcolor','w','ycolor','w','zcolor','w');
set(gcf,'color','k')
```

Figure 3.3: Lorenz equation parameters.m

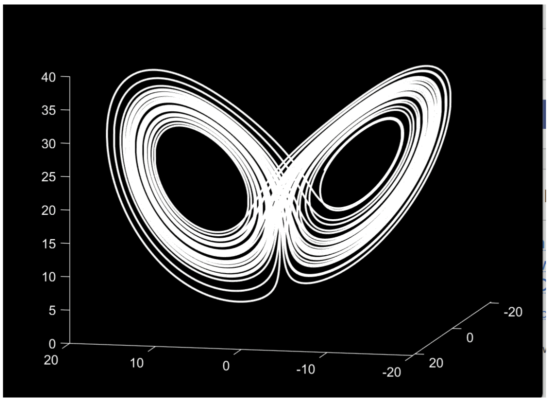
8 Plot Appendix



Plot 3.31: Beta = 0, 0, 0



Plot 3.32: Beta = 2, 0, 0



Plot 3.33: Beta = 10, 28, 8/3

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