Worksheet 3

Each question is worth 5 marks.

```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.pyplot import plot, xlabel, ylabel, contourf, title,
show, colorbar
from matplotlib import ticker, cm
import math as math
import scipy as scp
from scipy.linalg import expm
import seaborn as sns
from numpy.linalg import multi_dot
from scipy.linalg import expm, sinm, cosm
from IPython.display import display, Latex, Markdown
from numpy import sqrt, cos, arange, linspace, meshgrid, ma, exp, pi,
zeros, dot, conj, matrix, kron, array, eye, linalg
```

1

Consider a simple harmonic oscillator with Hamiltonian

$$\hat{H} = \hbar \omega_0 \hat{a}^\dagger \hat{a}.$$

(a) Write down the Heisenberg equation for the annihilation operator $\hat{a}$ and show that its solution is

$$\hat{a}(t) = {\rm e}^{-{\rm i}\omega_0 t} \hat{a}(0).$$

a) Simple Harmonic Oscillator with Hamiltonian

$$\hat{H} = \hbar\omega_0 \hat{a}^+ \hat{a}$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = \frac{i}{\hbar}[\hbar\omega_0 \hat{a}^+\hat{a}, \hat{a}] = i\omega_0[\hat{a}^+\hat{a}, \hat{a}] = i\omega_0[\hat{a}^+[\hat{a},\hat{a}] + [\hat{a}^+, \hat{a}]\hat{a}]$$

$$= i\omega_0(0 - \hat{a}) = -i\omega_0\hat{a}$$

$$\frac{d\hat{a}(t)}{dt} = -i\omega_0\hat{a}(t)$$

Here we want $\hat{a}(t) = e^{-i\omega_0 t}\hat{a}(0)$, $\frac{d\hat{a}(t)}{dt} = -i\omega_0 e^{-i\omega_0 t}\hat{a}(0) = -i\omega_0\hat{a}(t)$

(b) Find the general analytical solution for the Heisenberg evolution of the coordinate $\hat{q}(t)$ and momentum $\hat{p}(t)$ in terms of the initial conditions $\hat{q}(0)$ and $\hat{p}(0)$.

b) General analytic solution for the Heisenberg evolution of $\hat{q}(t)$ and $\hat{p}(t)$ in terms of $\hat{q}(0)$ and $\hat{p}(0)$.

$$\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{q} + \frac{i\hat{p}}{m\omega_0}\right) \qquad \hat{a}^{\dagger} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{q} - \frac{i\hat{p}}{m\omega_0}\right)$$

$$\hat{q}(t) + \frac{i\hat{p}}{m\omega_0} = e^{-i\omega_0 t}\left(\hat{q}(0) + \frac{i\hat{p}(0)}{m\omega_0}\right) \qquad \hat{a}(t) = e^{-i\omega_0 t} a(0)$$

$$\hat{q}^{\dagger}(t) - \frac{i\hat{p}}{m\omega_0} = e^{i\omega_0 t}\left(q(0) - \frac{i\hat{p}(0)}{m\omega_0}\right) \qquad \hat{a}^{\dagger}(t) = e^{i\omega_0 t} a^{\dagger}(0)$$

We need to add these two.

$$e^{-i\omega_0 t} + e^{i\omega_0 t} = \cos(\omega_0 t) - i\sin(\omega_0 t) + \cos(\omega_0 t) + i\sin(\omega_0 t) = \underline{2\cos(\omega_0 t)}$$

$$e^{-i\omega_0 t} - e^{i\omega_0 t} = \cos(\omega_0 t) - i\sin(\omega_0 t) - \cos(\omega_0 t) - i\sin(\omega_0 t) = \underline{-2i\sin(\omega_0 t)}$$

$$2\hat{q}(t) = \left(e^{-i\omega_0 t} + e^{i\omega_0 t}\right)\hat{q}(0) + i\left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right)\frac{\hat{p}(0)}{m\omega_0}$$

$$\hookrightarrow \hat{q}(t) = \cos(\omega_0 t)\hat{q}(0) + \sin(\omega_0 t)\frac{\hat{p}(0)}{m\omega_0}$$

$$\frac{2ip(t)}{m\omega_0} = \left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right)\hat{q}(0) + i\left(e^{-i\omega_0 t} + e^{i\omega_0 t}\right)\frac{\hat{p}(0)}{m\omega_0}$$

$$\frac{2ip(t)}{m\omega_0} = -2i\sin(\omega_0 t)\,\hat{q}(0) + 2i\cos(\omega_0 t)\frac{\hat{p}(0)}{m\omega_0}$$

$$p(t) = \cos(\omega_0 t)\hat{p}(0) - \sin(\omega_0 t)\, m\omega_0\, \hat{q}(0)$$

(c) Compute the expectation values $\langle \hat{q}(t) \rangle$, $\langle \hat{q}(t)^2 \rangle$, $\langle \hat{n}(t) \rangle$ and $\langle \hat{n}(t)^2 \rangle$ given the initial state

$$¿\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right),$$

where $¿n\rangle$ are Fock states.

c) Compute the expectation values $\langle \hat{q}(t)\rangle$, $\langle \hat{q}(t)^2\rangle$, $\langle \hat{n}(t)\rangle$, $\langle \hat{n}(t)^2\rangle$ given:

$|\Psi(0)\rangle = 1/\sqrt{2}(|1\rangle - |2\rangle)$ $\quad$ $|n\rangle$: Fock states.

$\hat{q}(t)$

$$\langle \hat{q}(t)\rangle = \cos(\omega_0 t)\langle q(0)\rangle + \sin(\omega_0 t)\frac{\langle \hat{p}(0)\rangle}{m\omega_0}$$

$$\langle \hat{q}(0)\rangle = \langle \Psi(0)|\hat{q}(0)|\Psi(0)\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle \Psi(0)|(\hat{a}+\hat{a}^+)|\Psi(0)\rangle =$$

$$= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\frac{1}{2}\right)\left[(\langle 1| - \langle 2|)\hat{a}(|1\rangle - |2\rangle) + (\langle 1| - \langle 2|)\hat{a}^+(|1\rangle - |2\rangle)\right] =$$

$$= \frac{1}{2}\sqrt{\hbar/2m\omega_0}\left[(\langle 1| - \langle 2|)(|0\rangle - \sqrt{2}|1\rangle) + (\langle 1| - \langle 2|)(\sqrt{2}|2\rangle - \sqrt{3}|3\rangle)\right] =$$

$$= \frac{1}{2}\sqrt{\hbar/2m\omega_0}\left[-\sqrt{2}\langle 1|1\rangle - \sqrt{2}\langle 2|2\rangle\right] = 1/2\sqrt{\hbar/2m\omega_0}\,(2\sqrt{2}) = \sqrt{2}\, q_0$$

$$\langle \hat{p}(0)\rangle = \langle \Psi(0)|\hat{p}(0)|\Psi(0)\rangle = \frac{i\hbar}{2}\sqrt{\frac{2m\omega_0}{\hbar}}\langle \Psi(0)|\hat{a}^+ - \hat{a}|\Psi(0)\rangle =$$

$$= i\sqrt{\frac{\hbar m\omega_0}{2}}\cdot\frac{1}{2}\left[(\langle 1| - \langle 2|)\hat{a}^+(|1\rangle - |2\rangle) - (\langle 1| - \langle 2|)\hat{a}(|1\rangle - |2\rangle)\right] =$$

$$= i/2\sqrt{\frac{\hbar m\omega_0}{2}}\left[-\sqrt{2} + \sqrt{2}\right] = 0$$

$$\langle \hat{q}(t)\rangle = \sqrt{2}\, q_0 \cos(\omega_0 t)$$

$$\langle \hat{q}(t)^2\rangle = \left\langle \left(\cos(\omega_0 t)\, q(0) + \sin(\omega_0 t)\frac{p(0)}{m\omega_0}\right)^2\right\rangle =$$

$$= \left\langle \cos^2(\omega_0 t)\, q(0)^2 + \sin^2(\omega_0 t)\frac{p(0)^2}{m^2\omega_0^2} + \cos(\omega_0 t)\sin(\omega_0 t)\{q(0), p(0)\}\right\rangle =$$

$$= \cos^2(\omega_0 t)\langle q(0)^2\rangle + \sin^2(\omega_0 t)\frac{\langle p(0)^2\rangle}{m^2\omega_0^2} + 2\cos(\omega_0 t)\sin(\omega_0 t)\langle\{q(0), p(0)\}\rangle$$

$$\langle q(0)^2\rangle = \langle \Psi(0)|q(0)^2|\Psi(0)\rangle$$

$$= \hbar/2m\omega_0\langle \Psi(0)|(\hat{a}+\hat{a}^+)(\hat{a}+\hat{a}^+)|\Psi(0)\rangle = \hbar/2m\omega_0\langle \Psi(0)|\hat{a}\hat{a} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+|\Psi(0)\rangle$$

$$= \hbar/4m\omega_0\,(\langle 1| - \langle 2|)(\underbrace{\cancel{\hat{a}\hat{a}}}_{0} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \underbrace{\cancel{\hat{a}^+\hat{a}^+}}_{0})(|1\rangle - |2\rangle) =$$

$$= \hbar/4m\omega_0\left[(\langle 1| - \langle 2|)\hat{a}\hat{a}^+(|1\rangle - |2\rangle) + (\langle 1| - \langle 2|)\hat{a}^+\hat{a}(|1\rangle - |2\rangle)\right] =$$

$$= \hbar/4m\omega_0\left[(\langle 1| - \langle 2|)\hat{a}(\sqrt{2}|2\rangle - \sqrt{3}|3\rangle) + (\langle 1| - \langle 2|\,\hat{a}^+(|0\rangle - \sqrt{2}|1\rangle)\right] =$$

$$= \hbar/4m\omega_0\left[(\langle 1| - \langle 2|)(\sqrt{2}\sqrt{2}|1\rangle - \sqrt{3}\sqrt{3}|2\rangle) + (\langle 1| - \langle 2|)(|1\rangle - \sqrt{2}\sqrt{2}|2\rangle\right) =$$

$$= \hbar/4m\omega_0\left[(2\langle 1|1\rangle + 3\langle 2|2\rangle + \langle 1|1\rangle + 2\langle 2|2\rangle\right] = \hbar/4m\omega_0\left[2 + 3 + 1 + 2\right] = \frac{\hbar}{4m\omega_0}(8) =$$

$$= 2\hbar/m\omega_0 = 4\,\hbar/2m\omega_0 = 4\sqrt{\hbar/2m\omega_0}^{\,2} = 4q_0^2$$

$$\langle p(0)^2\rangle = \langle \psi(0) \,|\, p(0)^2 \,|\, \psi(0)\rangle$$

$$= -\frac{\hbar^2}{4}\cdot\frac{2m\omega_0}{\hbar}\langle\psi(0)|(\hat{a}^+ - \hat{a})(\hat{a}^+ - \hat{a})|\psi(0)\rangle =$$

$$= -\frac{\hbar m\omega_0}{2}\cdot\frac{1}{2}\left(\langle 1| - \langle 2|\right)\left(\cancelto{0}{\hat{a}^+\hat{a}^+} - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ + \cancelto{0}{\hat{a}\hat{a}}\right)\left(|1\rangle - |2\rangle\right) =$$

$$= \frac{\hbar m\omega_0}{4}\left[\left(\langle 1| - \langle 2|\right)\hat{a}^+\hat{a}\left(|1\rangle - |2\rangle\right) + \left(\langle 1| - \langle 2|\right)\hat{a}\hat{a}^+\left(|1\rangle - |2\rangle\right)\right] = \frac{\hbar m\omega_0}{4}\,(8) =$$

$$= 2\hbar m\omega_0 = 4\frac{\hbar}{2m\omega_0}\,m^2\omega_0^2 = 4m^2\omega_0^2 q_0^2$$

$$\langle q(t)^2\rangle = \cos^2(\omega_0 t)\langle q(0)^2\rangle + \sin^2(\omega_0 t)\frac{\langle p(0)^2\rangle}{m^2\omega_0^2} + 2\cos(\omega_0 t)\sin(\omega_0 t)\langle\{q(0), p(0)\}\rangle$$

$$= 4q_0^2\cos^2(\omega_0 t) + 4q_0^2\frac{m^2\omega_0^2}{m^2\omega_0^2}\sin^2(\omega_0 t) = 4q_0^2\left(\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right) = 4q_0^2$$

$$\langle \hat{n}(t)\rangle = \langle\psi(0)|\hat{n}(t)|\psi(0)\rangle = \langle\psi(0)|\hat{a}^+\hat{a}|\psi(0)\rangle = \frac{1}{2}\left(\langle 1| - \langle 2|\right)(\hat{a}^+\hat{a})\left(|1\rangle - |2\rangle\right) =$$

$$= \frac{1}{2}\left(\langle 1| - \langle 2|\right)\left(\sqrt{1}\sqrt{1}\,|1\rangle - \sqrt{2}\sqrt{2}\,|2\rangle\right) = \frac{1}{2}\left(\langle 1|1\rangle + 2\langle 2|2\rangle\right) = \frac{3}{2}$$

$$\langle \hat{n}^2(t)\rangle = \langle\psi(0)|\hat{n}^2(t)|\psi(0)\rangle = \langle\psi(0)|\hat{a}^+\hat{a}\,\hat{a}^+\hat{a}|\psi(0)\rangle = \frac{1}{2}\left(\langle 1| - \langle 2|\right)(\hat{a}^+\hat{a}\,\hat{a}^+\hat{a})\left(|1\rangle - |2\rangle\right)$$

$$= \frac{1}{2}\left(\langle 1| - \langle 2|\right)\left(\sqrt{1}\sqrt{1}\sqrt{1}\sqrt{1}\,|1\rangle - \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\,|2\rangle\right) = \frac{1}{2}(1+4) = 5/2$$

(d) Plot the number fluctuations $\Delta\hat{n} = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$ and the dimensionless position fluctuations $\Delta\hat{q}/q_0$ as a function of dimensionless time $\omega_0 t$ up to $\omega_0 t = 10$.

```
import numpy as np
import matplotlib.pyplot as plt
import math as math
import scipy as scp
import seaborn as sns
from numpy.linalg import multi_dot
from scipy.linalg import expm, sinm, cosm
from IPython.display import display, Latex, Markdown

# Functions describing average coordinate and coordinate squared
def q(t):
    return -2**0.5*np.cos(t)
def q2(t):
    return 4

# Generate the time axis and fluctuations, and plot
time = np.linspace(0,10,1000)
d_q = [(q2(t)-q(t)**2)**0.5 for t in time]
```

```
d_n = [(2.5-1.5**2)**0.5 for t in time]
plt.plot(time, d_q, label = '$\Delta\hat{q}/q_0$')
plt.plot(time, d_n, label = '$\Delta\hat{n}$')
plt.xlabel('$\omega_0t$')
plt.legend()
```

```
<matplotlib.legend.Legend at 0x7fad4b8c67c0>
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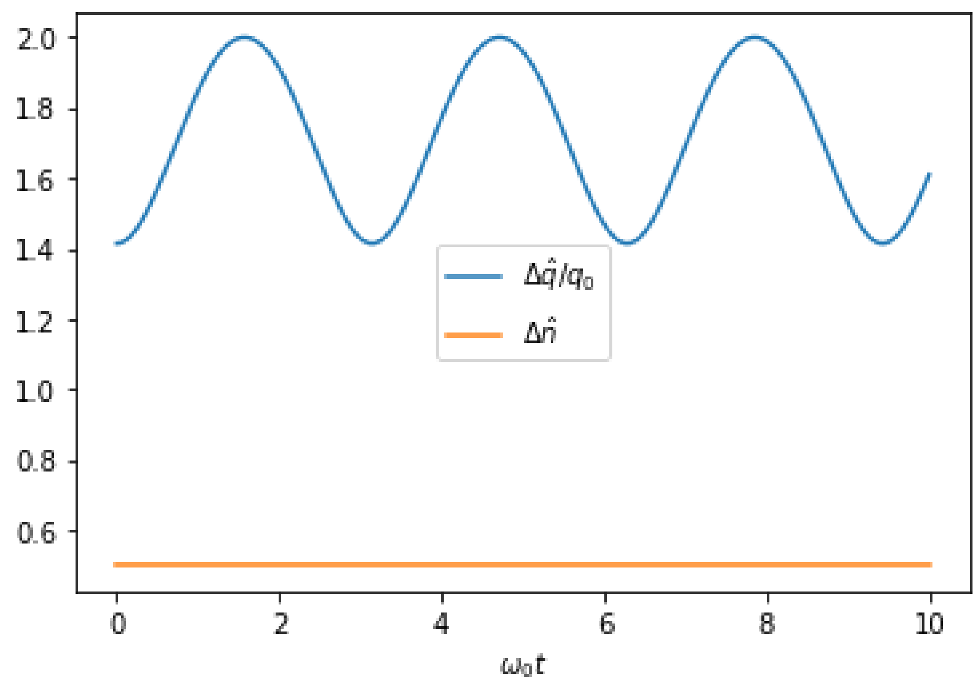


2

A driven harmonic oscillator is described by the Hamiltonian

$$ \hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar F\cos(\omega_{\rm d} t+\varphi)\left(\hat{a}+\hat{a}^\dagger\right),$$

where F, $\omega_{\rm d}$, and φ denote the strength, the frequency, and the phase of the drive, respectively.

(a) Move to an interaction picture with respect to the free Hamiltonian $\hat{H}_0 = \hbar\omega_{\rm d}\hat{a}^\dagger\hat{a}$ and show that the interaction-picture Hamiltonian under the rotating-wave approximation takes the form

$$\hat{H}_I = \hbar\Delta \hat{a}^\dagger\hat{a} + \frac{\hbar F}{2}\left({\rm e}^{{\rm i}\varphi} \hat{a}+{\rm e}^{-{\rm i}\varphi} \hat{a}^\dagger \right),$$

where $\Delta =\omega_0-\omega_{\rm d}$. State the conditions for this approximation to be valid.

a) Free Hamiltonian: $\hat{H}_0 = \hbar\omega_d \hat{a}^\dagger \hat{a}$. Show that the interaction-picture Hamiltonian under the rotating wave approximation takes the form:

$$\hat{H}_I = \hbar\Delta \hat{a}^\dagger \hat{a} + \frac{\hbar F}{2}\left(e^{i\varphi \hat{a}} + e^{-i\varphi \hat{a}^\dagger}\right) \quad \text{where: } \Delta = \omega_0 - \omega_d$$

$\hat{H}_0 = \hbar\omega_d \hat{a}^\dagger \hat{a}$ $\qquad \hat{H}_I = U_0^\dagger(t)\,\hat{H}_1\,U_0(t)$ $\qquad U_0(t) = e^{-iH_0 t/\hbar} = e^{-i\omega_d \hat{a}^\dagger \hat{a} t}$

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \hbar\omega_d \hat{a}^\dagger \hat{a} + \hbar(\omega_0 - \omega_d)\hat{a}^\dagger \hat{a} + \hbar F\cos(\omega_0 t + \varphi)(\hat{a} + \hat{a}^\dagger)$$

$$\hat{H}_1 = \hbar\Delta \hat{a}^\dagger \hat{a} + \hbar F\cos(\omega_d t + \varphi)(\hat{a} + \hat{a}^\dagger)$$

$$\hat{H}_I = \hbar\Delta\, e^{-i\omega_d \hat{a}^\dagger \hat{a} t}\hat{a}^\dagger \hat{a}\, e^{i\omega_d \hat{a}^\dagger \hat{a} t} + \hbar F\cos(\omega_d t + \varphi)\left(e^{-i\omega_d \hat{a}^\dagger \hat{a}}\hat{a}\, e^{i\omega_d \hat{a}^\dagger \hat{a} t} + \dots\right.$$

$$\left.\dots + e^{-i\omega_d \hat{a}^\dagger \hat{a} t}\, e^{i\omega_d \hat{a}^\dagger \hat{a}^\dagger}\right] =$$

$$= \hbar\Delta \hat{a}^\dagger \hat{a} + \frac{\hbar F}{2}\left[e^{i(\omega_d t + \varphi)} + e^{-i(\omega_d t + \varphi)}\right]\left[e^{-i\omega_d t}\hat{a} + e^{i\omega_d t}\hat{a}^\dagger\right] =$$

$$= \hbar\Delta \hat{a}^\dagger \hat{a} + \frac{\hbar F}{2}\left[e^{i\varphi}\hat{a} + e^{-i\varphi}\hat{a}^\dagger + e^{-i(2\omega_d t + \varphi)}\hat{a} + e^{i(2\omega_d t + \varphi)}\hat{a}^\dagger\right] =$$

$$= \hbar\Delta \hat{a}^\dagger \hat{a} + \frac{\hbar F}{2}\left(e^{i\varphi}\hat{a} + e^{-i\varphi}\hat{a}^\dagger\right)$$

(b) Consider the case of a resonant drive, $\omega_{\rm d} = \omega_0$. Write an explicit expression for the time evolution operator in the interaction picture. Thus find the interaction-picture state at time t, ¿$\psi_I(t)\rangle$, given that the system is initially in the vacuum state, ¿.

b) Resonant drive $\omega_d = \omega_0$. Write an analytic expression for time evolution in the interaction picture. Find the interaction-picture at time t: $|\Psi_I(t)\rangle$ given a vacuum state $|\Psi(0)\rangle = |0\rangle$

$$\omega_d = \omega_0 \rightarrow \Delta = 0$$

$$\hat{H}_I = \frac{\hbar F}{2}\left(e^{i\varphi}\hat{a} + e^{-i\varphi}\hat{a}^+\right)$$

$$|\Psi_I(0)\rangle = |0\rangle$$

$$|\Psi_I(t)\rangle = \mathcal{U}_I(t)|0\rangle = \vec{T}\exp\left[\frac{1}{i\hbar}\int_0^t dt'\,\hat{H}_I(t')\right]|0\rangle =$$

$$= \vec{T}\exp\left[\frac{F}{2i}\left(e^{i\varphi}\int_0^t \hat{a}(t')dt' + e^{-i\varphi}\int_0^t \hat{a}^+(t')dt'\right)|0\rangle =$$

$$= \vec{T}\exp\left[\frac{F}{2i}\left[e^{i\varphi}\int_0^t e^{-i\omega_0 t'}\hat{a}(0)dt' + e^{-i\varphi}\int_0^t e^{i\omega_0 t}\hat{a}^+(0)\,dt'\right]\right]|0\rangle$$

$$= \vec{T}\exp\left[\frac{F}{2i}e^{i\varphi}\left(\frac{1}{-i\omega_0}\right)\hat{a}(t) + e^{-i\varphi}\left(\frac{1}{i\omega_0}\right)\hat{a}^+(t)\right]|0\rangle =$$

$$= \exp\left[\frac{F}{2\omega_0}e^{i\varphi}\hat{a}(t) - F/2\omega_0\, e^{-i\varphi}\hat{a}^+(t)\right]|0\rangle =$$

$$= \exp\left[\frac{F}{2\omega_0}e^{i\varphi}e^{-i\omega_0 t}\hat{a}(0) - \frac{F}{2\omega_0}e^{-i\varphi}e^{+i\omega_0 t}\hat{a}^+(0)\right]|0\rangle =$$

$$= \exp\left[-\frac{F}{2\omega_0}e^{-i(\omega_0 t-\varphi)}\hat{a}^+(0) + \frac{F}{2\omega_0}e^{+i(\omega_0 t-\varphi)}\hat{a}(0)\right]|0\rangle$$

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp\left(-\frac{[\hat{A}][\hat{B}]}{2}\right)$$

$$\hat{A} = \frac{F}{2\omega_0}e^{i(\omega_0 t-\varphi)}\hat{a}^+ \qquad \hat{B} = \frac{F}{2\omega_0}e^{-i(\omega_0 t-\varphi)}\hat{a}$$

$$[\hat{A},\hat{B}] = \frac{F^2}{4\omega_0^2}[\hat{a}^+,\hat{a}] = -\frac{F^2}{4\omega_0^2}$$

$$= \exp\left[-F^2/8\omega_0^2\right]\exp\left[F/2\omega_0\, e^{+i(\omega_0 t-\varphi)}\hat{a}^+\right]\exp\left[\frac{F}{2\omega_0}e^{-i(\omega_0 t-\varphi)}\hat{a}\right]|0\rangle =$$

$$= \exp\left[-F^2/8\omega_0^2\right]\exp\left[F/2\omega_0\, e^{+i(\omega_0 t-\varphi)}\hat{a}^+\right]\exp(0)|0\rangle =$$

$$= \exp\left(-\frac{F^2}{8\omega_0^2}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{F}{2\omega_0}e^{+i(\omega_0 t-\varphi)}\right)^n(\hat{a}^+)^n|0\rangle = \exp\left[-\frac{F^2}{8\omega_0^2}\right]\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{F}{2\omega_0}e^{i(\omega_0 t-\varphi)}\right)^n|n\rangle$$

$$\alpha = \frac{F}{2\omega_0}e^{i(\omega_0 t-\varphi)} \qquad |\alpha|^2 = F^2/4\omega_0^2$$

$$|\Psi_I(t)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

(c) Using your answer from part (b), write down the state in the Schrödinger picture at time t. Draw a sketch of the corresponding Wigner function and how it evolves in time.

c) Write the Schrödinger picture at time t. Sketch the Wigner function and how it evolves in time.

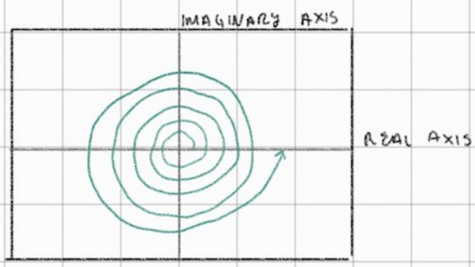
$$|\Psi_S(t)\rangle = \exp[-iH_0 t/\hbar]\,|\Psi_I(t)\rangle = \exp[-i\omega_d \hat{a}^\dagger \hat{a}\, t]\exp[-|\alpha|^2/2]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle =$$

$$= \exp[-|\alpha|^2/2]\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega_d n t}}{\sqrt{n!}}|n\rangle = \exp[-|\alpha|^2/2]\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega_d t})^n}{\sqrt{n!}}|n\rangle$$

$$|\alpha e^{-i\omega_d t}|^2 = |\alpha|^2 = e^{-|\alpha e^{-i\omega_d t}|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega_d t})^n}{\sqrt{n!}}|n\rangle = |\alpha e^{-i\omega_d t}\rangle$$

Weigner function: $W(\alpha) = \frac{2}{\pi}\, e^{-2|\alpha - \alpha e^{-i\omega_d t}|^2}$

$$\hat{\rho} = |\alpha e^{-i\omega_d t}\rangle\langle \alpha e^{-i\omega_d t}|$$



$t=0$, $W(\alpha) = 2/\pi$

$t \to \infty$, $W(\alpha) \to 2/\pi \exp[-2|\alpha|^2]$

(d) Now consider the case where $\omega_{\rm d} \neq \omega_0$. Show that the interaction-picture Hamiltonian is diagonalised by the unitary change of basis $\hat{S}={\rm e}^{\hat{G}}$ with

$$\hat{G} = \frac{F}{2\Delta}\left({\rm e}^{{\rm i}\varphi} \hat{a}-{\rm e}^{-{\rm i}\varphi} \hat{a}^\dagger \right).$$

Specifically, find the transformed Hamiltonian $\hat{S}^\dagger \hat{H}_I \hat{S}$ and, therefore, give general expressions for eigenvalues and eigenstates of $\hat{H}_I$.

d) Consider $\omega_d \neq \omega_0$. Show that the interaction-picture Hamiltonian is diagonalised by the unitary change of basis $\hat{S} = e^{\hat{G}}$ with

$$\hat{G} = \frac{F}{2\Delta}\left(e^{i\varphi}\hat{a} - e^{-i\varphi}\hat{a}^{+}\right)$$

Find $\hat{S}^{+}\hat{H}_I\hat{S}$ and give the eigenvalues $\hat{H}_I$

$\alpha = \frac{F}{2\Delta}e^{i\varphi}$ $\qquad \alpha^* = \frac{F}{2\Delta}e^{i\varphi}$

$$\hat{H}_I = \hbar\Delta\hat{a}^{+}\hat{a} + \hbar\Delta(\alpha^*\hat{a} + \alpha\hat{a}^{+}) \qquad \hat{S} = e^{\hat{G}}$$

$$\hat{G} = \alpha^*\hat{a} - \alpha\hat{a}^{+} \qquad \hat{S}^{+} = (e^{\hat{G}})^{+} = e^{-\hat{G}}$$

$$\hat{g} = -\hat{G} = \alpha\hat{a}^{+} - \alpha^*\hat{a} \qquad \hat{S}^{+}H_I\hat{S} = e^{-\hat{G}}\hat{H}_I e^{\hat{G}} = e^{+g}\hat{H}e^{-g}$$

$$[g, H_I] = [g, \hbar\Delta\hat{a}^{+}\hat{a}] + [g, \hbar\Delta(\alpha^*\hat{a} + \alpha\hat{a}^{+})] =$$

$$= \hbar\Delta\alpha[\hat{a}^{+}, \hat{a}^{+}\hat{a}] - \hbar\Delta\alpha^*[\hat{a}, \hat{a}^{+}\hat{a}] + \hbar\Delta[g, \alpha^*\hat{a}] + \hbar\Delta[g, \alpha\hat{a}^{+}] =$$

$$= -\hbar\Delta\alpha\hat{a}^{+} - \hbar\Delta\alpha^*\hat{a} - \hbar\Delta\alpha^*\alpha - \hbar\Delta\alpha^*\alpha = -\hbar\Delta(\alpha\hat{a}^{+} + \alpha^*\hat{a} + 2|\alpha|^2)$$

$$[g, [g, H_I]] = [g, -\hbar\Delta\alpha\hat{a}^{+} - \hbar\Delta\alpha^*\hat{a} - 2\hbar\Delta|\alpha|^2] = -\hbar\Delta\alpha[g, \hat{a}^{+}] - \hbar\Delta\alpha^*[g, \hat{a}] - 0$$

$$= \hbar\Delta\alpha^*\alpha + \hbar\Delta\alpha^*\alpha = 2\hbar\Delta|\alpha|^2$$

$$e^{-G}\hat{H}_I e^{G} = \hbar\Delta\hat{a}^{+}\hat{a} + \hbar\Delta(\alpha^*\hat{a} + \alpha\hat{a}^{+}) - \hbar\Delta(\alpha^*\hat{a} + \alpha\hat{a}^{+}) - 2\hbar\Delta|\alpha|^2 + \tfrac{1}{2}(2\hbar\Delta|\alpha|^2)$$

$$\hat{A} = e^{-G}\hat{H}_I e^{G} = \hbar\Delta(\hat{a}^{+}\hat{a} - |\alpha|^2) = \hbar\Delta\left(\hat{a}^{+}\hat{a} - \frac{F^2}{4\Delta^2}\right)$$

$$H_0|\psi_n\rangle = E_n|\psi_n\rangle$$

$$\hat{A}|E\rangle = E|E\rangle$$

$$[\hat{A}, \hat{a}] = \hbar\Delta[\hat{a}^{+}\hat{a}, \hat{a}] - 0 = -\hbar\Delta\hat{a}$$

$$[\hat{A}, \hat{a}^{+}] = \hbar\Delta[\hat{a}^{+}\hat{a}, \hat{a}^{+}] - 0 = \hbar\Delta\hat{a}^{+}$$

$$\hat{A}\hat{o}|E\rangle = [\hat{A}, \hat{a}]|E\rangle + \hat{a}\hat{A}|E\rangle$$

$$\hat{A}\hat{a}|E\rangle = -\hbar\Delta\hat{a}|E\rangle + E\hat{a}|E\rangle = (E - \hbar\Delta)\hat{a}|E\rangle$$

$$\hat{A}\hat{a}^{+}|E\rangle = [\hat{A}, \hat{a}^{+}]|E\rangle + \hat{a}^{+}\hat{A}|E\rangle = \hbar\Delta\hat{a}^{+}|E\rangle + E\hat{a}^{+}|E\rangle = (E + \hbar\Delta)\hat{a}^{+}|E\rangle$$

$$\hat{A}|0\rangle = -\frac{\hbar F^2}{4\Delta}|0\rangle$$

$$\hat{A}(\hat{a}^{+})^n|0\rangle = \left(-\frac{\hbar F}{4\Delta} + \hbar\Delta n\right)(\hat{a}^{+})^n|0\rangle$$

$$E_n = \hbar\Delta\left(n - F^2/4\Delta^2\right)$$

3

Consider two harmonic oscillators with frequencies ω_1 and ω_2 that are coupled together with strength g. The situation is described by the Hamiltonian

$$\hat{H} = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{\hbar g}{2}\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger\right),$$

where $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ are creation operators that create one quantum of energy in mode 1 and mode 2 , respectively. We have the canonical commutation relations $\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}$ and $\left[\hat{a}_j, \hat{a}_k\right] = 0$ for $j,k \in \{1,2\}$.

(a) Write down the Heisenberg equations for the annihilation operators $\hat{a}_1$ and $\hat{a}_2$.

QUESTION 3

Simple Harmonic Oscillator, frequencies: ω_1, ω_2 coupled with strength g.
The situation is described by the Hamiltonian:

$$\hat{H} = \hbar\omega_1 \hat{a}_1^+ \hat{a} + \hbar\omega_2 \hat{a}_2^+ \hat{a}_2 + \frac{\hbar g}{2}\underbrace{(\hat{a}_1^+ \hat{a}_2 + \hat{a}_1 \hat{a}_2^+)}_{\text{creation operators}}$$

$[\hat{a}_j, \hat{a}_k^+] = \delta_{jk}$ $\quad [\hat{a}_j, \hat{a}_k] = 0$

$j,k \in \{1,2\}$

a) Write Heisenberg equations for the annihilation operators $\hat{a}_1$ and $\hat{a}_2$.

$$\frac{d\hat{a}_1}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}_1] = \frac{i}{\hbar}[\hbar\omega_1 \hat{a}_1^+ \hat{a}_1 + \hbar\omega_2 \hat{a}_2^+ \hat{a}_2 + \frac{\hbar g}{2}(\hat{a}_1^+ \hat{a}_2 + \hat{a}_1 \hat{a}_2^+), \hat{a}_1] =$$

$$= i\omega_1 [\hat{a}_1^+ \hat{a}_1, \hat{a}_1] + i\omega_2 [\hat{a}_2^+ \hat{a}_2, \hat{a}_1] + \frac{ig}{2}[\hat{a}_1^+ \hat{a}_2, a_1] + \frac{ig}{2}[a_1 \hat{a}_2^+, \hat{a}_1] =$$

$$= i\omega_1 (0 - \hat{a}_1) + i\omega_2 (0+0) + ig/2 (0 - \hat{a}_2) + \frac{ig}{2}(0+0) = -i\omega_1 \hat{a}_1 - \frac{ig}{2}\hat{a}_2$$

$$\frac{d\hat{a}_1}{dt} = -i\omega_1 \hat{a}_1 - \frac{ig}{2}\hat{a}_2$$

$$\frac{d\hat{a}_2}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}_2] = \frac{i}{\hbar}[\hbar\omega_1 \hat{a}_1^+ \hat{a}_1 + \hbar\omega_2 \hat{a}_2^+ \hat{a}_2 + \frac{\hbar g}{2}(\hat{a}_1^+ \hat{a}_2 + \hat{a}_1 \hat{a}_2^+), \hat{a}_2] =$$

$$= i\omega_1 (0+0) + i\omega_2 (0 - \hat{a}_2) + \frac{ig}{2}(0+0) + \frac{ig}{2}(\hat{a}_1 (-1) + 0) = -i\omega_2 a_2 - \frac{ig}{2}\hat{a}_1$$

(b) Consider the new mode operators

$$\hat{b}_1 = \cos(\theta)\hat{a}_1 + \sin(\theta)\hat{a}_2, \hat{b}_2 = \cos(\theta)\hat{a}_2 - \sin(\theta)\hat{a}_1,$$

where $\tan(2\theta) = g/(\omega_1 - \omega_2)$. Demonstrate that these new operators obey canonical commutation relations $\left[\hat{b}_j, \hat{b}_k^\dagger\right] = \delta_{jk}$.

b) Considering $\hat{b}_1 = \cos(\theta)\hat{a}_1 + \sin(\theta)\hat{a}_2$ where $\tan(2\theta) = g/\omega_1 - \omega_2$

$\hat{b}_2 = \cos(\theta)\hat{a}_2 - \sin(\theta)\hat{a}_1$

Demonstrate they obey cannonical commutation relations: $[\hat{b}_j, \hat{b}_k^\dagger] = \delta_{jk}$

$j=1,\ k=1$

$[\hat{b}_1, \hat{b}_1^\dagger] = [\cos\theta\hat{a}_1 + \sin\theta\,\hat{a}_2,\ \cos\theta\,\hat{a}_1^\dagger + \sin\theta\,\hat{a}_2^\dagger] =$

$= \cos\theta[\hat{a}_1,\ \cos\theta\hat{a}_1 + \sin\theta\,\hat{a}_2^\dagger] + \sin\theta[\hat{a}_2,\ \cos\theta\hat{a}_1^\dagger + \sin\theta\,\hat{a}_2^\dagger] =$

$= \cos\theta[\hat{a}_1,\ \hat{a}_1^\dagger]\cos\theta + \cos\theta[\hat{a}_1,\ \hat{a}_2^\dagger]\sin\theta + \sin\theta[\hat{a}_2,\ \hat{a}_1^\dagger]\cos\theta + \sin\theta[\hat{a}_2, \hat{a}_2^\dagger]\sin\theta =$

$= \cos^2\theta + \sin^2\theta = 1$

$j=1,\ k=2$

$[\hat{b}_1, \hat{b}_2^\dagger] = [\cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2,\ \cos\theta\,\hat{a}_2^\dagger - \sin\theta\,\hat{a}_1^\dagger] =$

$= \cos\theta[\hat{a}_1,\ \cos\theta\,\hat{a}_2^\dagger - \sin\theta\,\hat{a}_1^\dagger] + \sin\theta[\hat{a}_2,\ \cos\theta\hat{a}_2^\dagger - \sin a_1^\dagger] =$

$= \cos\theta\,[\hat{a}_1,\ \hat{a}_2^\dagger]\cos\theta - \cos\theta[\hat{a}_1,\ \hat{a}_1^\dagger]\sin\theta + \sin\theta[\hat{a}_2, \hat{a}_2^\dagger]\cos\theta - \sin\theta[\hat{a}_2,\ \hat{a}_1^\dagger]\sin\theta =$

$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

$j=2,\ k=1$

$[\hat{b}_2, \hat{b}_1^\dagger] = [\cos\theta\,\hat{a}_2 - \sin\theta\,\hat{a}_1,\ \cos\theta\,\hat{a}_1^\dagger + \sin\theta\,\hat{a}_2^\dagger] =$

$= \cos\theta[\hat{a}_2,\ \cos\theta\,\hat{a}_1^\dagger + \sin\theta\hat{a}_2^\dagger] - \sin\theta\,[\hat{a}_1,\ \cos\theta\,\hat{a}_1^\dagger + \sin\theta\,\hat{a}_2^\dagger] =$

$= \cos\theta[\hat{a}_2,\ \hat{a}_1^\dagger]\cos\theta + \cos\theta[\hat{a}_2,\ \hat{a}_2^\dagger]\sin\theta - \sin\theta[\hat{a}_1,\ \hat{a}_1^\dagger]\cos\theta - \sin\theta[\hat{a}_1,\ \hat{a}_2^\dagger]\sin\theta =$

$= \cos\theta\sin\theta - \sin\theta\cos\theta = 0$

$j=2,\ k=2$

$[\hat{b}_1, \hat{b}_2^\dagger] = [\cos\theta\,\hat{a}_2 - \sin\theta\,\hat{a}_1,\ \cos\theta\,\hat{a}_2^\dagger - \sin\theta\,\hat{a}_1^\dagger]$

$= \cos\theta\,[\hat{a}_2,\ \cos\theta\,\hat{a}_2^\dagger - \sin\theta\,\hat{a}_1^\dagger] - \sin[\hat{a}_1,\ \cos\theta\,\hat{a}_2^\dagger - \sin\theta\,\hat{a}_1^\dagger] =$

$= \cos\theta[\hat{a}_2,\ \hat{a}_2^\dagger]\cos\theta - \cos\theta[\hat{a}_2,\ \hat{a}_1^\dagger]\sin\theta - \sin\theta[\hat{a}_1,\ \hat{a}_2^\dagger]\cos\theta + \sin\theta[\hat{a}_1,\ \hat{a}_1^\dagger]\sin\theta =$

$\cos^2\theta + \sin^2\theta = 1$

(c) The mode transformation can also be written as

$$\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}.$$

Show that the above matrix is unitary. Invert the transformation to find expressions for $\hat{a}_1$ and $\hat{a}_2$ in terms of $\hat{b}_1$ and $\hat{b}_2$. Thus, show that the Hamiltonian can be written in the form

$$\widehat{H} = \sum_{j=1}^{2} \hbar \nu_j \hat{b}_j^{\dagger} \hat{b}_j,$$

and state the corresponding eigenfrequencies ν_1 and ν_2.

We need to show that the matrix is unitary.

Let $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Personal Remainder, for a matrix to be unitary:

$UU^{\dagger} = U^{\dagger}U = UU^{\dagger} = \mathbb{1}$

$$U^{\dagger} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$UU^{\dagger} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$U^{\dagger}U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

U is unitary ü

Now, to find an expression for $\hat{a}_1$ and $\hat{a}_2$ in terms of $\hat{b}_1$ and $\hat{b}_2$

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad U^{\dagger} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$$

$$\hat{a}_1^{\dagger}\hat{a}_1 = (\cos\theta\, \hat{b}_1^{\dagger} - \sin\theta\, \hat{b}_2^{\dagger})(\cos\theta\, \hat{b}_1 - \sin\theta\, \hat{b}_2)$$
$$= \cos^2\theta\, \hat{b}_1^{\dagger}\hat{b}_1 - \cos\theta\sin\theta\, \hat{b}_1^{\dagger}\hat{b}_2 - \cos\theta\sin\theta\, \hat{b}_2^{\dagger}\hat{b}_1 + \sin^2\theta\, \hat{b}_2^{\dagger} b_2$$

$$\hat{a}_2^{\dagger}\hat{a}_2 = (\cos\theta\, \hat{b}_1^{\dagger} + \sin\theta\, \hat{b}_1^{\dagger})(\cos\theta\, \hat{b}_2 + \sin\theta\, \hat{b}_1) =$$
$$= \cos^2\theta\, \hat{b}_2^{\dagger}\hat{b}_2 + \cos\theta\sin\theta\, \hat{b}_2^{\dagger}\hat{b}_1 + \sin\theta\cos\theta\, \hat{b}_1^{\dagger}\hat{b}_2 + \sin^2\theta\, \hat{b}_1^{\dagger}\hat{b}_1$$

$$\hat{a}_1^{\dagger}\hat{a}_2 = (\cos\theta\, \hat{b}_1^{\dagger} - \sin\theta\, \hat{b}_2^{\dagger})(\cos\theta\, \hat{b}_2 + \sin\theta\, \hat{b}_1) =$$
$$= \cos^2\theta\, \hat{b}_1^{\dagger}\hat{b}_2 - \cos\theta\sin\theta\, \hat{b}_1^{\dagger}\hat{b}_1 - \cos\theta\sin\theta\, \hat{b}_2^{\dagger}\hat{b}_2 - \sin^2\theta\, \hat{b}_2^{\dagger}\hat{b}_1$$

$$\hat{a}_1\hat{a}_2^{\dagger} = \cos^2\theta\, \hat{b}_1\hat{b}_2^{\dagger} - \cos\theta\sin\theta\, \hat{b}_1\hat{b}_1^{\dagger} - \cos\theta\sin\theta\, \hat{b}_2\hat{b}_2^{\dagger} - \sin^2\theta\, \hat{b}_2\hat{b}_1^{\dagger}$$

Hamiltonian:

$$\hat{H} = \hbar\omega_1 \hat{a}_1^{\dagger}\hat{a}_1 + \hbar\omega_2 \hat{a}_2^{\dagger}\hat{a}_2 + \frac{\hbar g}{2}(\hat{a}_1^{\dagger}\hat{a}_2 + \hat{a}_1\hat{a}_2^{\dagger})$$

can be written as:

$$\hat{H} = (\hat{a}_1^{\dagger}, \hat{a}_2^{\dagger})\begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix}\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

With this transformation:

$$\hat{H} = \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hbar\omega_1 & \frac{\hbar g}{2} \\ \frac{\hbar g}{2} & \hbar\omega_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_1 \end{pmatrix} =$$

$$= \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hbar\omega_1\cos\theta + \hbar g/2\sin\theta & -\hbar\omega_1\sin\theta + \hbar g/2\cos\theta \\ \frac{\hbar g}{2}\cos\theta + \hbar\omega_2\sin\theta & -\hbar g/2\sin\theta + \hbar\omega_2\cos\theta \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} =$$

This should be there but no space

$$= \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} \cos\theta[\hbar\omega\cos\theta + \frac{\hbar g}{2}\sin\theta] + \sin\theta[\frac{\hbar g}{2}\cos\theta + \hbar\omega_2\sin\theta] & \cos\theta[-\hbar\omega_1\sin\theta + \frac{\hbar g}{2}\cos\theta] + \sin\theta[-\frac{\hbar g}{2}\sin\theta + \hbar\omega_2\cos\theta] \\ -\sin\theta[\hbar\omega_2\cos\theta + \frac{\hbar g}{2}\sin\theta] + \cos\theta[\frac{\hbar g}{2}\cos\theta + \hbar\omega_2\sin\theta] & -\sin\theta[-\hbar\omega_2\sin\theta + \frac{\hbar g}{2}\cos\theta] - \cos\theta[-\frac{\hbar g}{2}\sin\theta + \hbar\omega_2\cos\theta] \end{pmatrix}$$

$$= \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} (\cos^2\theta\hbar\omega_1 + \cos\theta\sin\theta\frac{\hbar g}{2} + \cos\theta\sin\theta\frac{\hbar g}{2} + \sin^2\theta\hbar\omega_2) & (-\cos\theta\sin\theta\hbar\omega_1 + \cos^2\theta\frac{\hbar g}{2} - \sin^2\theta\frac{\hbar g}{2} + \cos\theta\sin\theta\hbar\omega_2) \\ (-\cos\theta\sin\theta\hbar\omega_1 - \sin^2\theta\frac{\hbar g}{2} + \cos^2\theta\frac{\hbar g}{2} + \sin\theta\cos\theta\hbar\omega_2) & (\sin^2\theta\hbar\omega_1 - \cos\theta\sin\theta\frac{\hbar g}{2} - \cos\theta\sin\theta\frac{\hbar g}{2} + \cos^2\theta\hbar\omega_2) \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} \cos^2\theta\hbar\omega_1 + \sin^2\theta\hbar\omega_2 + \cos\theta\sin\theta\hbar g & \cos\theta\sin\theta\hbar(\omega_2-\omega_1) + \frac{\hbar g}{2}(\cos^2\theta + \sin^2\theta) \\ \hbar\cos\theta\sin\theta(\omega_2-\omega_1) + \frac{\hbar g}{2}(\cos^2\theta - \sin^2\theta) & \cos^2\theta\hbar\omega_2 + \sin^2\theta\hbar\omega_2 - \cos\theta\sin\theta\hbar g \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} \cos^2\theta\hbar\omega_1 + \sin^2\theta\hbar\omega_2 + \cos\theta\sin\theta\hbar g & \cos\theta\sin\theta\hbar(\omega_2-\omega_1) + \frac{\hbar g}{2}\cos(2\theta) \\ \cos\theta\sin\theta\hbar(\omega_2-\omega_1) + \frac{\hbar g}{2}\cos(2\theta) & \cos^2\hbar\omega_2 + \sin^2\theta\hbar\omega_1 - \cos\theta\sin\theta\hbar g \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{b}_1^+ & \hat{b}_2^+ \end{pmatrix} \begin{pmatrix} \cos^2\theta\hbar(\omega_1-\omega_2) + \hbar\omega_2 + \sin(2\theta)\frac{\hbar g}{2} & 0 \\ 0 & \cos^2\theta\hbar(\omega_2-\omega_1) + \hbar\omega_1 - \sin(2\theta)\frac{\hbar g}{2} \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$$

$$\hat{H} = \sum_{j=1}^{2} \hbar V_j \hat{b}_j^+ \hat{b}_j^+$$

$$\hat{H} = \hbar\underbrace{(\omega_1\cos^2\theta + \omega_2\sin^2\theta)}_{V_1}\hat{b}_1^+\hat{b}_1 + \hbar\underbrace{(\omega_1\sin^2\theta + \omega_2\cos^2\theta)}_{V_2}\hat{b}_2^+\hat{b}_1 =$$

d) Heisenberg equation for $\hat{b}_1$, $\hat{b}_2$

$$\frac{d\hat{b}_1}{dt} = \frac{i}{\hbar}[\hat{H}, b_1] = \frac{i}{\hbar}[\hbar V_1\hat{b}_1^+\hat{b}_1 + \hbar V_2\hat{b}_2^+\hat{b}_2, \hat{b}_1] = \frac{i}{\hbar}[(\hbar V_1\hat{b}_1^+\hat{b}_1, b_1) + (\hbar V_2\hat{b}_2^+\hat{b}_2, \hat{b}_1)] =$$

$$= iV_1(\hat{b}_1^+\hat{b}_1, \hat{b}_1) + iV_2(\hat{b}_2^+\hat{b}_2, \hat{b}_1) = iV_1(-\hat{b}_1) = -iV_1\hat{b}_1$$

$$\frac{d\hat{b}_2}{dt} = \frac{i}{\hbar}[\hat{H}, b_2] = \frac{i}{\hbar}[\hbar V_1\hat{b}_1^+\hat{b}_1 + \hbar V_2\hat{b}_2^+\hat{b}_2, b_2] = \frac{i}{\hbar}[(\hbar V_1\hat{b}_1^+\hat{b}_1, b_2) + (\hbar V_2\hat{b}_2^+\hat{b}_2, b_2)] =$$

$$= iV_1(\hat{b}_1^+\hat{b}_1, b_2) + iV_2(\hat{b}_2^+\hat{b}_2, b_2) = iV_2(-\hat{b}_2) = -iV_2\hat{b}_2$$

Also use those solutions for the Heisenberg equations for $\hat{a}_1$ and $\hat{a}_2$.

$$\frac{d\hat{b}_1}{dt} = -iV_1\hat{b}_1, \quad \hat{b}_1(t) = e^{-iV_1 t}\, b_1(0), \quad \frac{d\hat{b}_1(t)}{dt} = -iV_1 e^{-iV_1 t}\, b_1(0) = -iV_1\hat{b}_1(t)$$

$$\frac{d\hat{b}_2}{dt} = -iV_2\hat{b}_2, \quad \hat{b}_2(t) = e^{-iV_2}\, b_2(0), \quad \frac{d\hat{b}_2(t)}{dt} = -iV_2 e^{-iV_2}\, b_2(0) = -iV_2\hat{b}_2(t)$$

$$\left.\begin{array}{l} \hat{b}_1(t) = e^{-iV_1 t}\cos\theta\,\hat{a}_1(0) + e^{-iV_1 t}\sin\theta\,\hat{a}_2(0) \\ \hat{b}_2(t) = e^{-iV_2 t}\cos\theta\,\hat{a}_2(0) - e^{-iV_2 t}\sin\theta\,\hat{a}_1(0) \end{array}\right\} \quad \begin{array}{l} \hat{a}_1 = \cos\theta\,\hat{b}_1 - \sin\theta\,\hat{b}_2 \\ \hat{a}_2 = \cos\theta\,\hat{b}_2 + \sin\theta\,\hat{b}_1 \end{array}$$

$$\hat{a}_1(t) = \left(e^{-iV_1 t}\cos^2\theta + e^{-iV_2 t}\sin^2\theta\right)\hat{a}_1(0) + \left(e^{-iV_1 t} - e^{-iV_2 t}\right)\sin\theta\cos\theta\,\hat{a}_2(0)$$

$$\hat{a}_2(t) = \left(e^{-iV_1 t} - e^{-iV_2 t}\right)\sin\theta\cos\theta\,\hat{a}_1(0) + \left(e^{-iV_1 t}\sin^2\theta + e^{-iV_2 t}\cos^2\theta\right)\hat{a}_2(0)$$

$$\frac{d\hat{a}_1}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}_1] = i\omega_1[\hat{a}_1^\dagger\hat{a}_1, \hat{a}_1] + i\omega_2[\hat{a}_2^\dagger\hat{a}_2, \hat{a}_1] + \frac{\hbar g}{2}[\hat{a}_1^\dagger\hat{a}_2, \hat{a}_1] + \frac{\hbar g}{2}[\hat{a}_1\hat{a}_2^\dagger, \hat{a}_1] +$$
$$+ \hbar U[\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1, \hat{a}_1] =$$
$$= -i\omega_1\hat{a}_1 - \frac{ig}{2}\hat{a}_2 + \hbar U\underbrace{[\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1, \hat{a}_1]}$$

$$\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger[\hat{a}_1, \hat{a}_1] + \hat{a}_1^\dagger\hat{a}_1[\hat{a}_1^\dagger, \hat{a}_1]\hat{a}_1 + \hat{a}_1^\dagger[\hat{a}_1, \hat{a}_1]\hat{a}_1^\dagger\hat{a}_1 + [\hat{a}_1^\dagger, \hat{a}_1]\hat{a}_1\hat{a}_1^\dagger\hat{a}_1$$
$$= -\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 - \hat{a}_1\hat{a}_1^\dagger\hat{a}_1 = -\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 - \hat{a}_1\hat{a}_1^\dagger\hat{a}_1$$

$$\frac{d\hat{a}_1}{dt} = -i\omega_1\hat{a}_1 - \frac{ig}{2}\hat{a}_2 - \hbar U\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 - \hbar U\hat{a}_1\hat{a}_1^\dagger\hat{a}_1$$

(d) Write down the Heisenberg equations for the operators $\hat{b}_1$ and $\hat{b}_2$. Use these to find the solution for the Heisenberg equations for $\hat{a}_1$ and $\hat{a}_2$.

Both answers are in the answer sheets above

4

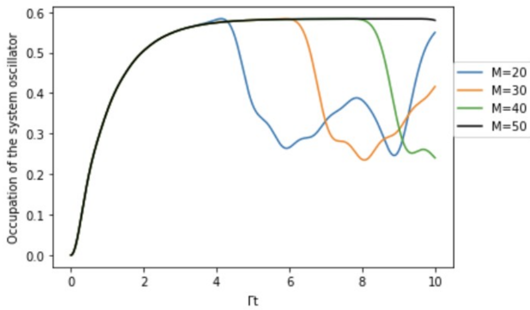
Consider the *cat state*

$$¿\psi_{+¿\rangle = \frac{1}{N_{+¿}¿¿}¿}$$

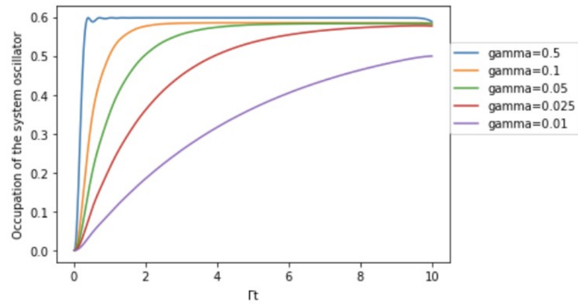
where $|\bar{\alpha}\rangle $ and $¿-\acute{\alpha}\rangle$ are coherent states.

(a) Assume that $¿\psi_{+¿\rangle¿}$ is normalised. What is the value of the normalisation constant $N_{+¿¿}$?

(d)



(e)

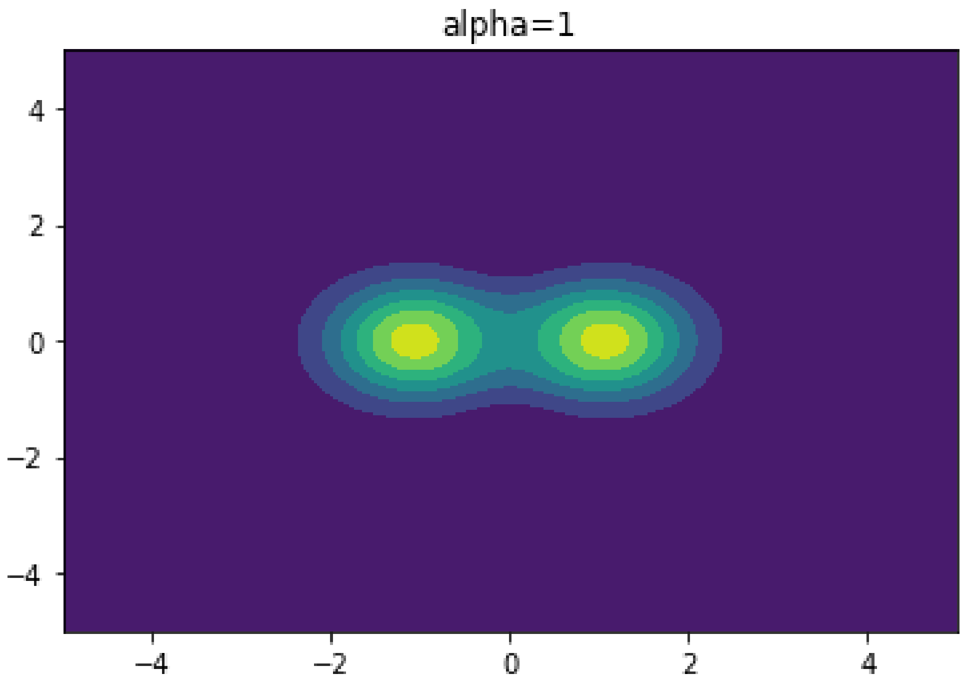


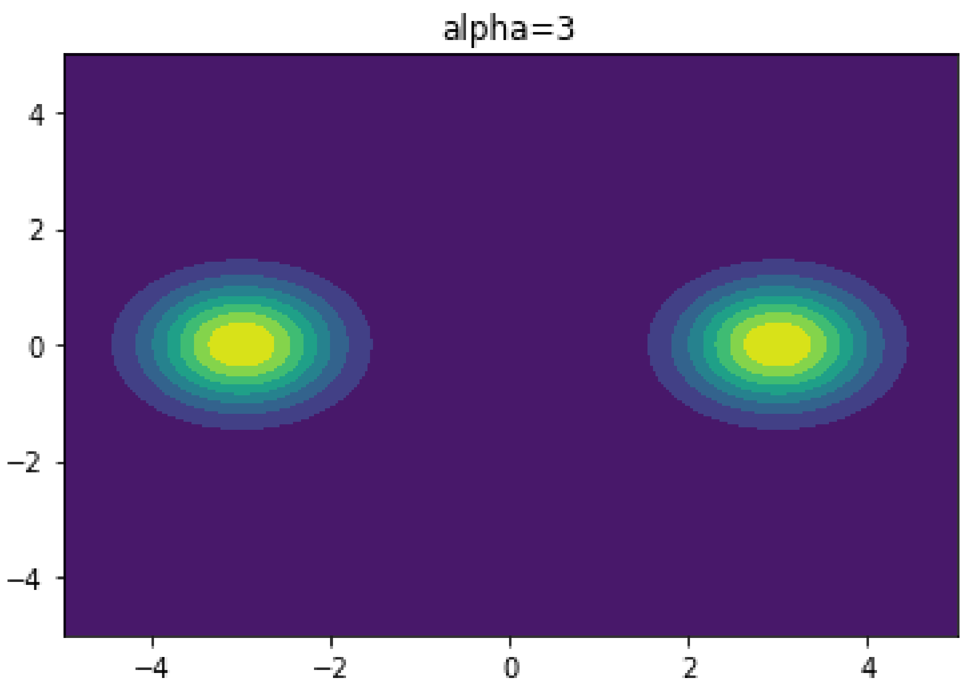
(b) Derive an expression for the Husimi Q-function for the state $¿\psi_{+¿)¿}$. Make a contour plot of the Q-function in the complex α-plane for $\acute{\alpha}=1$ and $\acute{\alpha}=3$.

```
def husimi_Q_function(a_bar,a): #definition of the Q-function
    #Equation
    e1 = np.exp(-abs(a_bar-a)**2)
    e2 = np.exp(-abs(-a_bar-a)**2)
    e3 = np.exp(-abs(a_bar-a)**2-2*a_bar*a)
    N = 2*(1+np.exp(-2*abs(a_bar)**2))
    return 1/(np.pi*N)*(e1+e2+e3)
#grid for contour plot
n=200  #number of plots
lim=5 #limits
alpha = np.linspace(-lim,lim,n) #alpha function
x,y = np.meshgrid(alpha,alpha)  #np.meshgrid returns the coordinate
matrices from the  vectors
val = x+1j*y
#PLOTS
#plot for α=1
plt.figure()
plt.title('alpha=1')
plt.contourf(x,y,husimi_Q_function (val,1))
#plot for α=3
plt.figure()
plt.title('alpha=3')
plt.contourf(x,y,husimi_Q_function (val,3))

/Users/alinagallardo/opt/anaconda3/lib/python3.9/site-packages/numpy/
ma/core.py:2825: ComplexWarning: Casting complex values to real
discards the imaginary part
  _data = np.array(data, dtype=dtype, copy=copy,

<matplotlib.contour.QuadContourSet at 0x7fad4bafbc40>
```





(c) Consider the mixed state

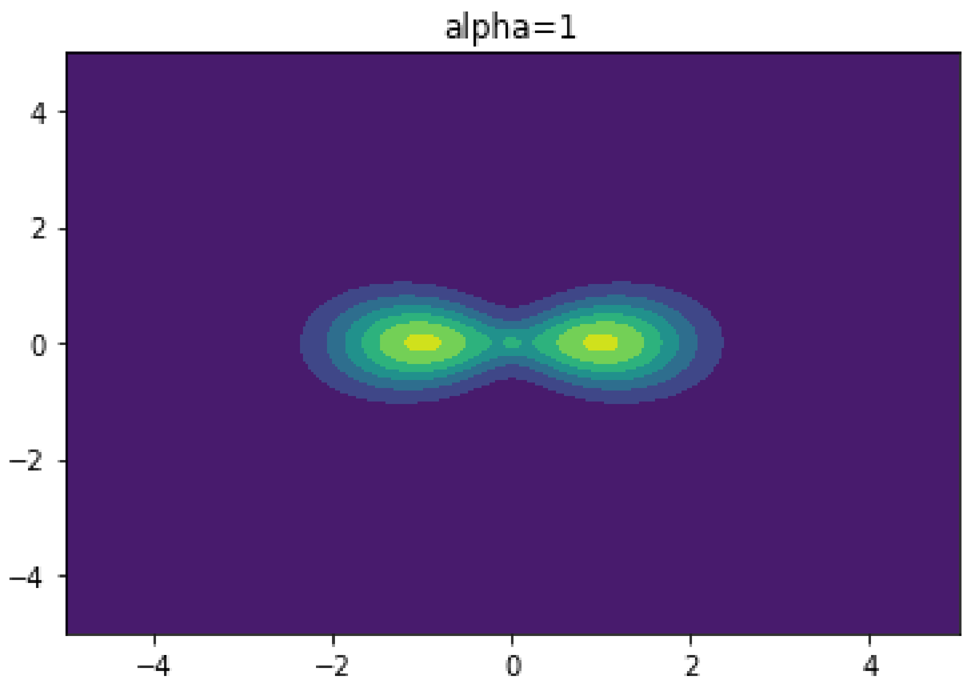
$$\hat{\rho} = \frac{1}{N_0}\left(|\acute{\alpha}\rangle\langle\acute{\alpha}| + |-\acute{\alpha}\rangle\langle-\acute{\alpha}|\right).$$

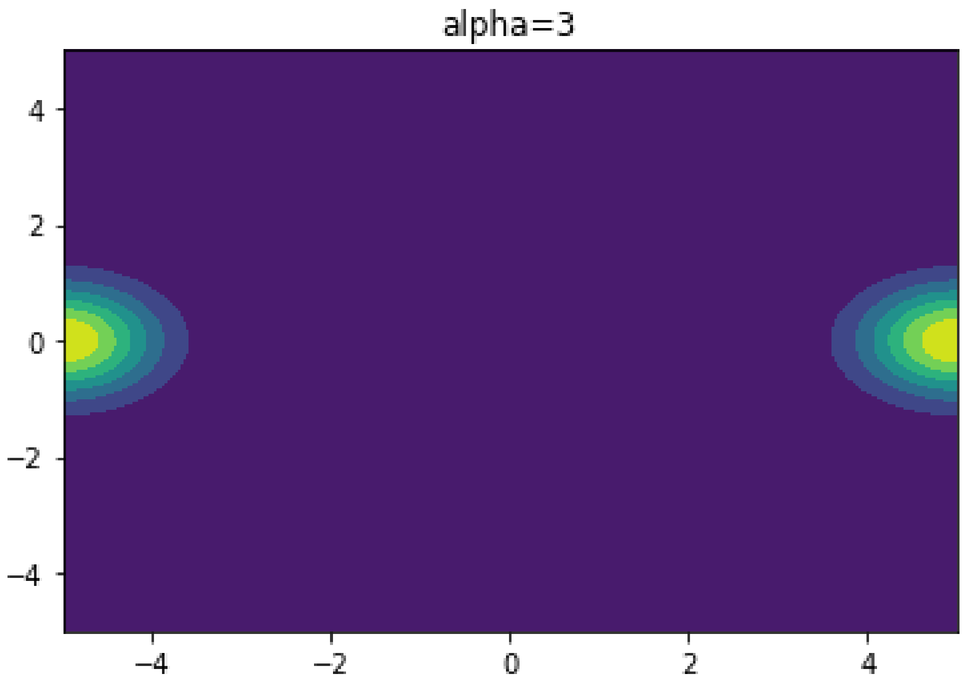
What is the value of N_0 that ensures that $\hat{\rho}$ is normalised? Derive the Q-function and make a contour plot for $\hat{\rho}$ for $\acute{\alpha}=1$ and $\acute{\alpha}=3$.

```
def husimi_Q_function(a_bar,a):
    e1 = np.exp(-2*abs(a_bar-a)**2)
    e2 = np.exp(-2*abs(a_bar+a)**2)
    N = 2*(2*np.pi*np.exp(-abs(a_bar)**2+2*abs(a_bar)))
    return 1/(np.pi*N)*(e1+e2)
n=200
lim=5
alpha = np.linspace(-lim,lim,n)
x,y = np.meshgrid(alpha,alpha)
val = x+1j*y

plt.figure()
plt.title('alpha=1')
plt.contourf(x,y,husimi_Q_function (val,1))
plt.figure()
plt.title('alpha=3')
plt.contourf(x,y,husimi_Q_function (val,3))
```

```
<matplotlib.contour.QuadContourSet at 0x7fad30c9a5e0>
```





(d) Now consider a (normalised) cat state with the opposite phase

$$¿\psi_{-}\rangle = \frac{1}{N_{-}}\left(\left|\acute{\alpha}\right\rangle - \right) - \acute{\alpha}\rangle\Big).$$

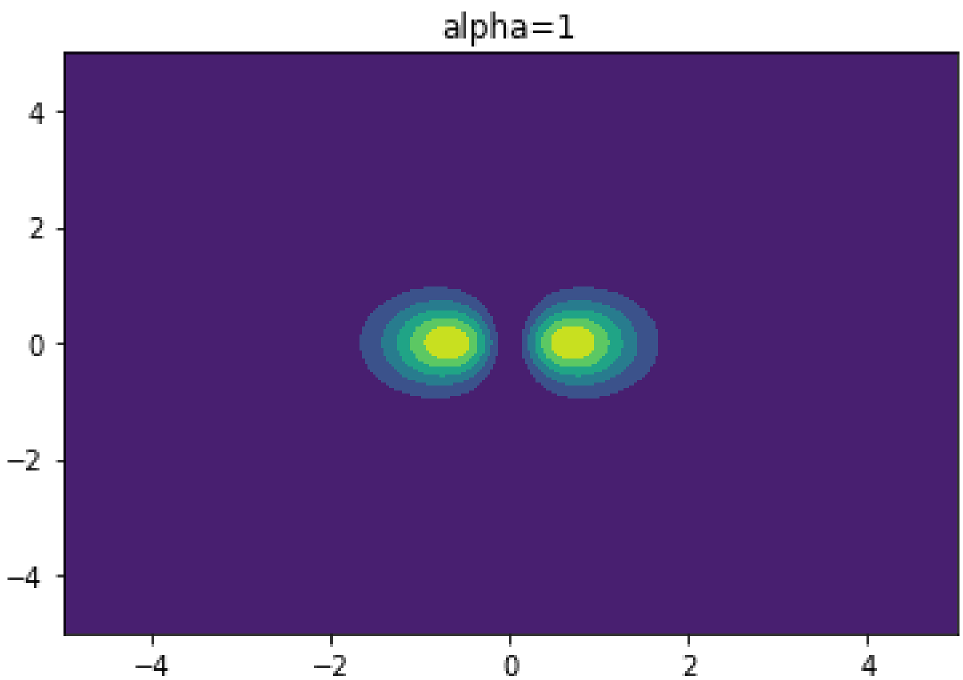
Derive the corresponding Q-function and make contour plots for $\acute{\alpha}=1$ and $\acute{\alpha}=3$.

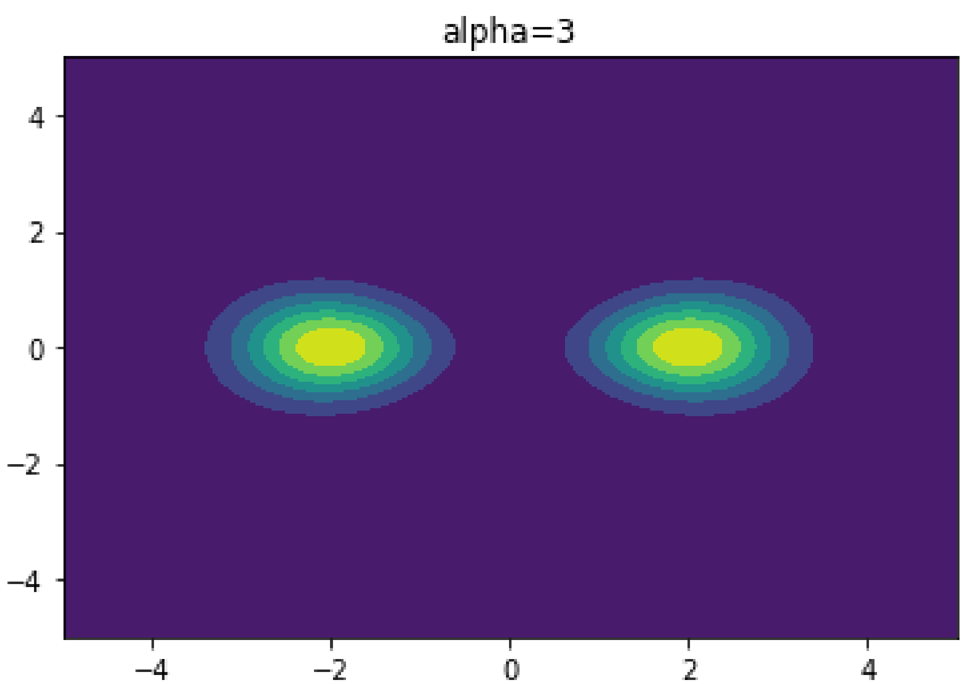
```
def husimi_Q_function(a_bar,a):
    e1 = np.exp(-abs(a_bar-a)**2)
    e2 = np.exp(-abs(a_bar+a)**2)
    e3 = -2*np.exp(-abs(a_bar)**2-abs(a)**2)
    N = 2*(2*np.pi*np.exp(-abs(a)**2+2*abs(a_bar)))
    return 1/(np.pi*N)*(e1+e2+e3)
n=200
lim=5
alpha = np.linspace(-lim,lim,n)
x, y = np.meshgrid(alpha,alpha)
val = x+1j*y
print(husimi_Q_function(1+1j,1))

plt.figure()
plt.title('alpha=1')
plt.contourf(x,y,husimi_Q_function (val,1))
plt.figure()
plt.title('alpha=3')
plt.contourf(x,y,husimi_Q_function(val,3))

0.0011193488201920482
```

```
<matplotlib.contour.QuadContourSet at 0x7fad286cd460>
```



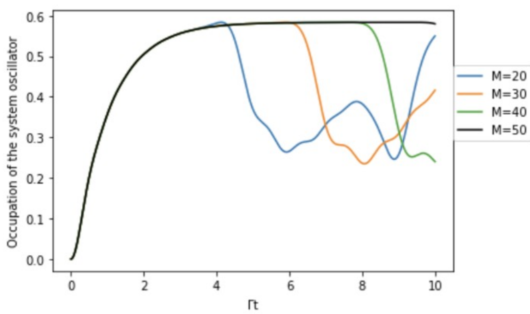


(e) Write a short paragraph that compares and contrasts the results for (b), (c) and (d).

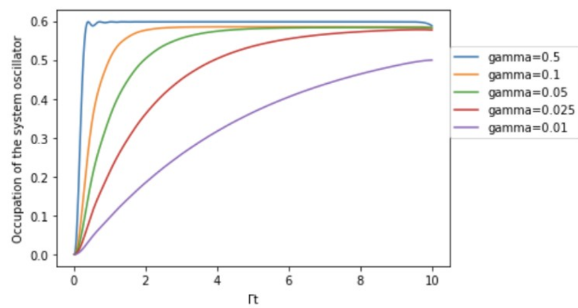
Each one of the plots show circular distrubutions in which the center of each one of the circules represents the maximum probability of both states. In the case of alpha bar equal to 1 when ¿ $\psi_{+¿)¿}$ there is constructive interference which can be observed by the largest concentration of points in the center. When ¿ $\psi_{-}\rangle$ there is a hole in the center implying destructive interference. In the case of alpha bar = 3 there is not a lot of visible difference between 4c and 4d but since ¿ $\psi_{+¿)¿}$ and ¿ $\psi_{-}\rangle$ are orthogonal we can say that even when not noticeable, there must larger distance between them.

Higher values of α get the states more separated.

(d)



(e)



5

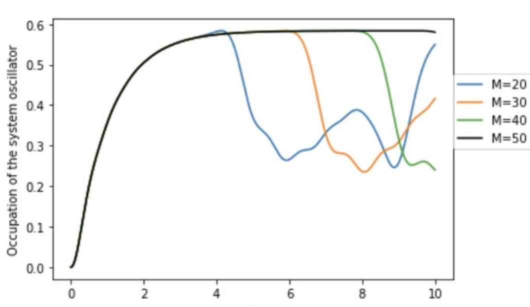
The harmonic oscillator is infinite-dimensional and therefore its states and operators cannot be exactly represented by any numerical array. However, one can find an approximate representation by *truncation* of the Hilbert space to a finite size.

(a) Let

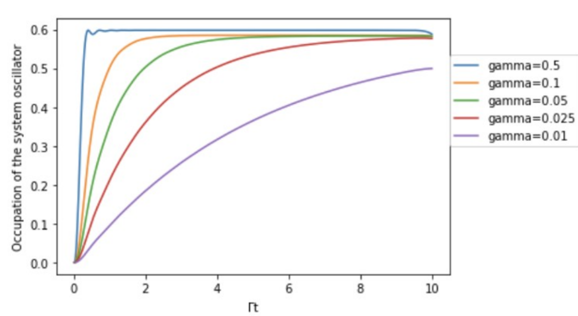
$$\hat{a} = \sum_{n=1}^{d-1} \sqrt{n} |n-1\rangle\langle n|$$

be the approximate annihilation operator in the d-dimensional truncated Hilbert space. (This becomes the exact annihilation operator as $d \to \infty$.) For finite d, find an analytical expression for the operator $\hat{a}^{\dagger}\hat{a}$ and compute the commutator $[\hat{a}, \hat{a}^{\dagger}]$.

(d)



(e)



(b) Define a Python function that generates the d-dimensional matrix representation of $\hat{a}$ for any integer d. For $d=5$, numerically test your function: check the commutation relation $[\hat{a}, \hat{a}^{\dagger}]$ and confirm that $\hat{a} \vee 4 \rangle$ yields the correct result, where $¿4 \rangle$ is a Fock state.

```
def d_dimensional_matrix(d): #create a function that generates a d-dimensional matrix
     #Receive the d-dimensional matrix as input and output the  matrix representation
    annihilation_operator=np.zeros((d,d))
    for n in range(d):
        annihilation_operator[n-1][n]=np.sqrt(n)
    return annihilation_operator
d=5 #relevant opertors for d=5
annihilation_operator=d_dimensional_matrix(d)
creation_operator=annihilation_operator.T.conj()
commutation=((annihilation_operator@creation_operator) -
(creation_operator@annihilation_operator)) #check the commutation relation
print('d-dimensional matrix representation for â', '\n')
print(commutation)
fock_state_confirmation=np.array([0,0,0,0,1]) #confirme that â|4⟩ yields the correct result, where |4⟩is a Fock state.
result=annihilation_operator@fock_state_confirmation
print('\n','fock state', '\n')
print(result)
```

```
d-dimensional matrix representation for â

[[ 1.  0.  0.  0.  0.]
 [ 0.  1.  0.  0.  0.]
 [ 0.  0.  1.  0.  0.]
 [ 0.  0.  0.  1.  0.]
 [ 0.  0.  0.  0. -4.]]

 fock state

[0. 0. 0. 2. 0.]
```

(c) Write another function that outputs the truncated displacement operator

$$\widehat{D}(\alpha)=\exp[\alpha \hat{a}^{\dagger} - \alpha^{¿} \hat{a}],$$

with arbitrary integer d and complex α as inputs. Print its output for $\alpha=0.5$ with $d=3$, $d=5$, and $d=10$. Comment on the difference between the results for different d. Is the truncated displacement operator unitary?

```
#displacement operator with different values
def truncated_displacement_operator(a,d): #another function that outputs the truncated displacement operator with arbitrary integer  d and complex  α as inputs
```

```
    annihilation_operator=d_dimensional_matrix(d)
#d_dimensional_matrix from b)
    creation_operator=annihilation_operator.T.conj()
    a_conjugate=np.conj(a)
    displacement=expm((a*creation_operator)-
(a_conjugate*annihilation_operator))   #expm compute the matrix
exponential of an array.
    return displacement
a=0.5
#output for  α=0.5 and d=3
d=3
print('\n','  α=0.5, d=3', '\n')
print(truncated_displacement_operator(a,d))
#output for  α=0.5 and d=5
d=5
print('\n','  α=0.5, d=5', '\n')
print(truncated_displacement_operator(a,d))
#output for  α=0.5 and d=10
d=10
print('\n','  α=0.5, d=10', '\n')
print(truncated_displacement_operator(a,d))
#is D unitary?
print('\n','Is the truncated displacement operator unitary?', '\n')
d_u=truncated_displacement_operator(a,d)
d_u_dag=d_u.T.conjugate()
print(np.round(d_u@d_u_dag,5))
```

```
   α=0.5, d=3

[[ 0.88261978 -0.43980233  0.1660007 ]
 [ 0.43980233  0.64785934 -0.62197442]
 [ 0.1660007   0.62197442  0.76523956]]

   α=0.5, d=5

[[ 0.88249693 -0.44124785  0.15601245 -0.04496584  0.01173405]
 [ 0.44124785  0.66186201 -0.54613558  0.24675339 -0.08993168]
 [ 0.15601245  0.54613558  0.46996952 -0.5734898   0.35725922]
 [ 0.04496584  0.24675339  0.5734898   0.25891541 -0.73563789]
 [ 0.01173405  0.08993168  0.35725922  0.73563789  0.56831097]]

   α=0.5, d=10

[[ 8.82496903e-01 -4.41248451e-01  1.56004886e-01 -4.50347315e-02
   1.12586829e-02 -2.51751803e-03  5.13886129e-04 -9.71162663e-05
   1.71595159e-05 -2.92662710e-06]
 [ 4.41248451e-01  6.61872677e-01 -5.46017101e-01  2.47691023e-01
  -8.44401215e-02  2.39164214e-02 -5.90968910e-03  1.31108046e-03
  -2.65906400e-04  5.14785476e-05]
```

```
 [ 1.56004886e-01  5.46017101e-01  4.68826479e-01 -5.81159643e-01
   3.20433913e-01 -1.25055827e-01  3.93350225e-02 -1.05928747e-02
   2.52509204e-03 -5.66142095e-04]
 [ 4.50347315e-02  2.47691023e-01  5.81159643e-01  3.01060142e-01
  -5.78564045e-01  3.77577956e-01 -1.65500004e-01  5.70322740e-02
  -1.65346168e-02  4.41561976e-03]
 [ 1.12586829e-02  8.44401215e-02  3.20433913e-01  5.78564045e-01
   1.56419062e-01 -5.52460508e-01  4.21055188e-01 -2.04726146e-01
   7.62191796e-02 -2.52922186e-02]
 [ 2.51751803e-03  2.39164214e-02  1.25055827e-01  3.77577956e-01
   5.52460508e-01  3.28922041e-02 -5.10982626e-01  4.52799950e-01
  -2.39816306e-01  1.06208210e-01]
 [ 5.13886129e-04  5.90968910e-03  3.93350225e-02  1.65500004e-01
   4.21055188e-01  5.10982626e-01 -7.18185992e-02 -4.61896846e-01
   4.62348658e-01 -3.16802322e-01]
 [ 9.71162663e-05  1.31108046e-03  1.05928747e-02  5.70322740e-02
   2.04726146e-01  4.52799950e-01  4.61896846e-01 -1.46878811e-01
  -3.56595188e-01  6.22583796e-01]
 [ 1.71595159e-05  2.65906400e-04  2.52509204e-03  1.65346168e-02
   7.62191796e-02  2.39816306e-01  4.62348658e-01  3.56595188e-01
  -3.74741079e-01 -6.74567750e-01]
 [ 2.92662710e-06  5.14785476e-05  5.66142095e-04  4.41561976e-03
   2.52922186e-02  1.06208210e-01  3.16802322e-01  6.22583796e-01
   6.74567750e-01  2.12236553e-01]]

 Is the truncated displacement operator unitary?

[[ 1.  0. -0.  0. -0. -0. -0. -0. -0. -0.]
 [ 0.  1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-0.  0.  1.  0. -0. -0. -0.  0.  0.  0.]
 [ 0.  0.  0.  1. -0. -0.  0. -0.  0.  0.]
 [-0.  0. -0. -0.  1. -0.  0.  0.  0. -0.]
 [-0.  0. -0. -0. -0.  1.  0. -0. -0.  0.]
 [-0.  0. -0.  0.  0.  0.  1.  0.  0.  0.]
 [-0.  0.  0. -0.  0. -0.  0.  1.  0.  0.]
 [-0.  0.  0.  0.  0. -0.  0.  0.  1.  0.]
 [-0.  0.  0.  0. -0.  0.  0.  0.  0.  1.]]
```

for d=3 and d=5 the terms are very similar but for d=10 there are more differences, this can be interpreted as when we increase the dimension of the matrix (d), we will have a greater displacement operator which is unitary

```
a=0.5
d=3
unitaryd3
=truncated_displacement_operator(a,d).T.conj()@truncated_displacement_
operator(a,d)
print('\n','d=3')
print(np.round(unitaryd3))
```

```
a=0.5
d=5
unitaryd5
=truncated_displacement_operator(a,d).T.conj()@truncated_displacement_
operator(a,d)
print('\n','d=5')
print(np.round(unitaryd5))
a=0.5
d=10
unitaryd10
=truncated_displacement_operator(a,d).T.conj()@truncated_displacement_
operator(a,d)
print('\n','d=10')
print(np.round(unitaryd10))
```

```
 d=3
[[ 1.  0.  0.]
 [ 0.  1. -0.]
 [ 0. -0.  1.]]

 d=5
[[ 1. -0.  0. -0.  0.]
 [-0.  1. -0. -0.  0.]
 [ 0. -0.  1. -0. -0.]
 [-0. -0. -0.  1. -0.]
 [ 0.  0. -0. -0.  1.]]

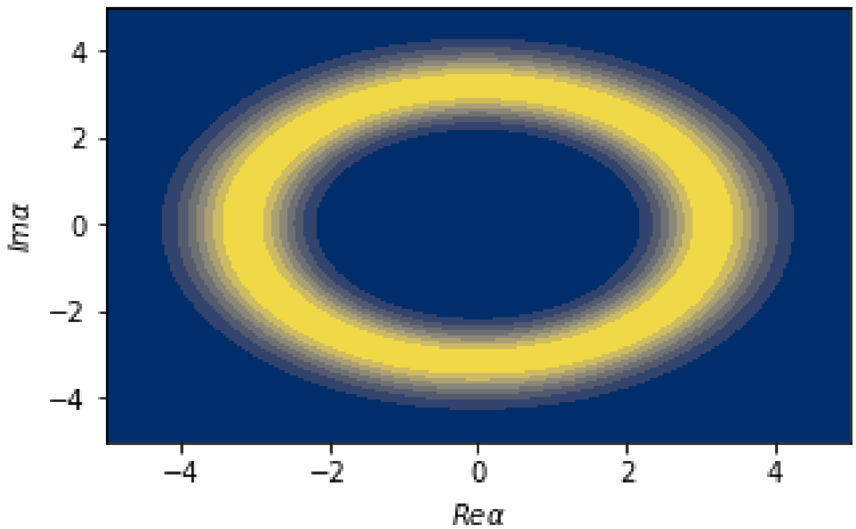
 d=10
[[ 1.  0. -0.  0. -0.  0. -0.  0.  0. -0.]
 [ 0.  1. -0. -0.  0. -0.  0. -0.  0. -0.]
 [-0. -0.  1. -0.  0. -0.  0. -0. -0. -0.]
 [ 0. -0. -0.  1.  0. -0.  0. -0.  0. -0.]
 [-0.  0.  0.  0.  1.  0. -0.  0. -0. -0.]
 [ 0. -0. -0. -0.  0.  1.  0.  0. -0.  0.]
 [-0.  0.  0.  0. -0.  0.  1. -0.  0.  0.]
 [ 0. -0. -0. -0.  0.  0. -0.  1.  0.  0.]
 [ 0.  0. -0.  0. -0. -0.  0.  0.  1. -0.]
 [-0. -0. -0. -0. -0.  0.  0.  0. -0.  1.]]
```

The displacement operator for each one of the values of 'd' are unitary

(d) Write a Python function that computes the Q-function $Q(\alpha)$ in the truncated Hilbert space, for any given complex value of α and an arbitrary d-dimensional pure state as input. Test your function by making a contour plot of the Q-function for a Fock state with $n=5$. Try to make your plot accurate (i.e., closely corresponding to the exact $d \to \infty$ result) without your code running for more than a couple of minutes! Comment your code to explain your parameter choices.

```python
#Truncated space for 0th dimension
def Q_function(a,dim):
    matrix=len(dim)
    displacement=truncated_displacement_operator(a,matrix)
    d_dimensional=displacement.T.conj()@dim
    Q = (np.abs(d_dimensional[0])**2)/np.pi
    return Q
def state(n,d):
    dim=np.zeros(d)
    dim[n]=1
    return dim
#Parameters for the state and the grid for Q
n=10
d=50
dim=state(n,d)
lim=5
s=100 #if I increase this value further, the program takes too long to compile and even the kernel dies
x=np.linspace(-lim,lim,s)
y=np.linspace(-lim,lim,s)
truncated_hilbert_space=np.zeros([s,s])

for i in range(len(x)):
    for j in range(len(y)):
        truncated_hilbert_space[j,i]=Q_function(x[i]+1j*y[j],dim)

_,ax = plt.subplots(figsize=(5,3))
plt.contourf(x,y,truncated_hilbert_space,cmap="cividis")
ax.set_xlabel('$Re \\alpha$')
ax.set_ylabel('$Im \\alpha$')
```

```
Text(0, 0.5, '$Im \\alpha$')
```



6

Let us now apply the truncated Hilbert-space representation to study the dynamics of the driven quantum harmonic oscillator, described by the Hamiltonian from question 2:

$$ \hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar F\cos(\omega_{\rm d} t)\left(\hat{a} +\hat{a}^\dagger\right).$$

(a) Write a function that generates this Hamiltonian (for any given t) in a truncated Hilbert space with $d=16$.

```
def Hamiltonian_truncated_Hilber_space(d,F,w0,wd,t): #function for the
Hilbert-space
    #creation and annihilation operators
    annihilation_operator=d_dimensional_matrix(d)
    creation_operator=annihilation_operator.T.conj()
    #return the hamiltonian
    return w0*(creation_operator@annihilation_operator)
+F*np.cos(wd*t)*(annihilation_operator+creation_operator)
```

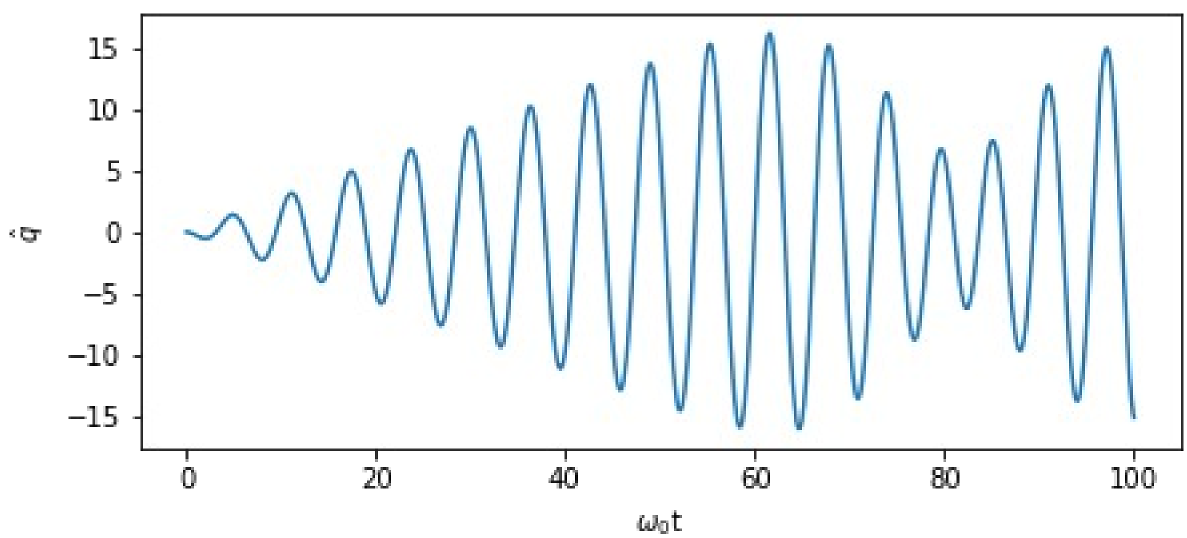
(b) Numerically calculate the dynamics of the quantum state in the Schrödinger picture starting from the vacuum state ¿, and plot the mean value of the coordinate, $\langle \hat{q}(t) \rangle$, as a function of time up to $\omega_0 t=100$. Choose parameters $F=0.1\omega_0$ and $\omega_{\rm d} = 0.99\omega_0$. (Recall that Worksheet 1 explored techniques to numerically calculate time evolution.)

```
def schrodinger_dynamics(F, w0,wd,t_tot,d, h): #function for the
dynamics of the quantum state in the Schrödinger picture starting
    t = np.linspace(0, t_tot, t_tot*10) #time interval
    values = []
    qt = ((1/2*d*w0)**(1/2))*(d_dimensional_matrix(d) +
d_dimensional_matrix(d).T.conj())
    vacuum_state = state(0,d)
```

```
    for i in range(len(t)): #Loop over time steps
        create_hamiltonian=h(d,F,w0,wd,t[i])
        time_evolution_operator = expm(-1j*create_hamiltonian*0.1)
        vacuum_state = time_evolution_operator@vacuum_state #time
evolution through time
        values.append(vacuum_state.T.conj()@qt@vacuum_state)
    return t,values
#variables and values
w0=1
F=0.1*w0
wd=0.99*w0
t_tot=100
d=16
a,b=schrodinger_dynamics(F,w0,wd,t_tot,d,Hamiltonian_truncated_Hilber_
space)#"schrodinger_dynamics" function with the established parameters

plt.subplots(figsize=(7,3))
plt.plot(a,b)
plt.xlabel('$\omega_0$t')
plt.ylabel('$\\hat{q}$')
```

```
/Users/alinagallardo/opt/anaconda3/lib/python3.9/site-packages/numpy/
core/_asarray.py:102: ComplexWarning: Casting complex values to real
discards the imaginary part
  return array(a, dtype, copy=False, order=order)
```

```
Text(0, 0.5, '$\\hat{q}$')
```



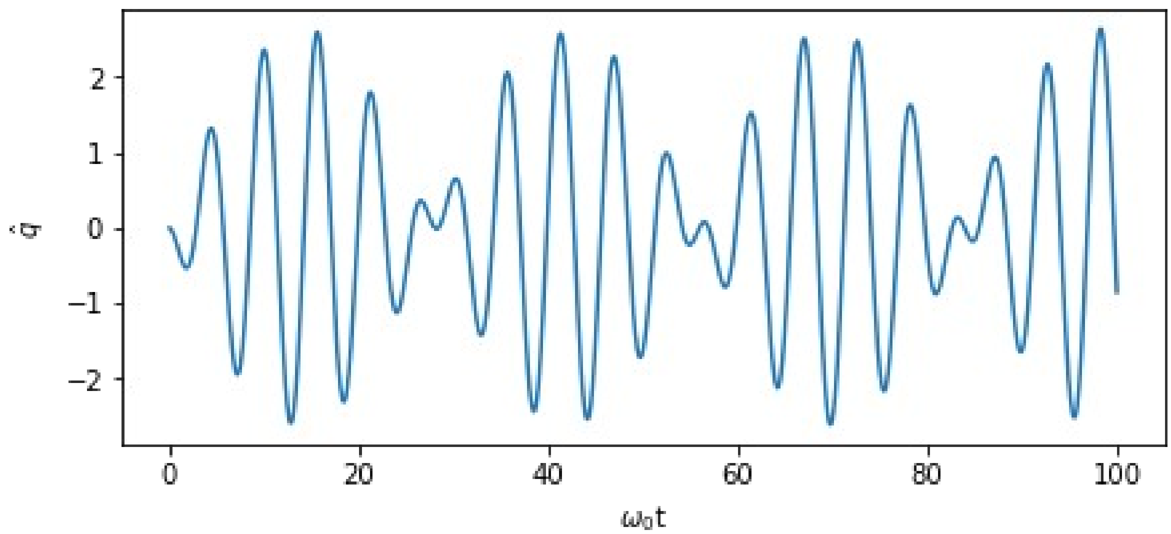
(c) Repeat the same calculation but with the Hamiltonian

$$ \hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar F\cos(\omega_{\rm d} t)\left(\hat{a}
+\hat{a}^\dagger\right) + \hbar U (\hat{a}^\dagger \hat{a})^2,$$

with $U = 0.2\,\omega_0$ and other parameters the same as in (b).

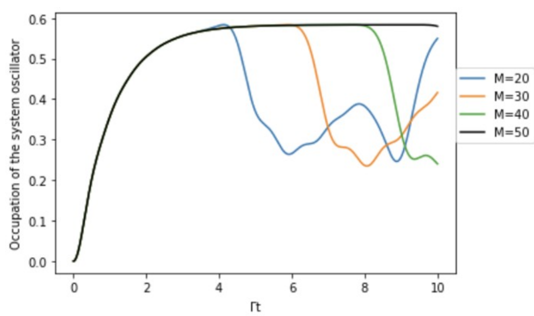
```
def Hamiltonian_truncated_Hilber_space(d,F,w0,wd,t): #function for the
Hilbert-space
    #creation and annihilation operators
    annihilation_operator=d_dimensional_matrix(d)
    creation_operator=annihilation_operator.T.conj()
    #return the hamiltonian
    return w0*(creation_operator@annihilation_operator)
+F*np.cos(wd*t)*(annihilation_operator+creation_operator+0.2*w0*(creat
ion_operator@annihilation_operator)**2)

def Hamiltonian_truncated_Hilber_space(d,F,w0,wd,t):#function for the
Hilbert-space
    #creation and annihilation operators
    annihilation_operator=d_dimensional_matrix(d)
    creation_operator=annihilation_operator.T.conj()
    #return the hamiltonian
    return w0*(creation_operator@annihilation_operator)
+F*np.cos(wd*t)*(annihilation_operator+creation_operator)
+0.2*w0*(creation_operator@annihilation_operator)**2
x,y=schrodinger_dynamics(F,w0,wd,t_tot,d,Hamiltonian_truncated_Hilber_
space)
plt.subplots(figsize=(7,3))
plt.plot(x, y)
plt.xlabel('$\omega_0$t')
plt.ylabel('$\\hat{q}$')
```

```
/Users/alinagallardo/opt/anaconda3/lib/python3.9/site-packages/numpy/
core/_asarray.py:102: ComplexWarning: Casting complex values to real
discards the imaginary part
  return array(a, dtype, copy=False, order=order)

Text(0, 0.5, '$\\hat{q}$')
```

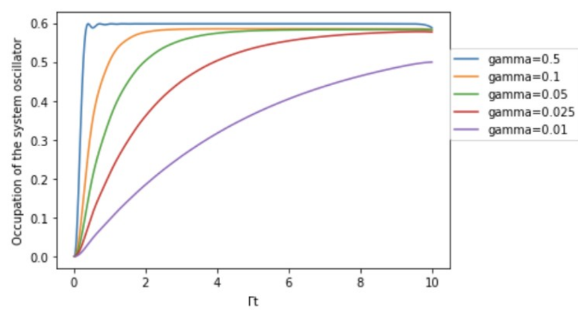


(d) Comment on the numerical accuracy and physical meaning of your results for (b) and (c).

(d)



(e)



b) the Hamiltonian causes a linear constant growth until w0t=60 where the amplitude decreases and increases again around 85 and 90. This behaviour can be due to the realtion between the annihilation and creation operators with the force F. The increase in the amplitude with time could be due to resonance.

c) This graph also has oscillatory behaviour but in shorter periods of time which could be due to the new quadratic term on the Hamiltonian, the energy of the quantum oscillator is higher and it experiences a major number of wave-packets in shorter periods of time, which means, a higher level of energy.

7

Another useful way to simulate *quadratic* Hamiltonians is to use the linearity of the Heisenberg equations. As in question 4, consider a two-mode system with annihilation operators $\hat{a}_1$ and $\hat{a}_2$. Convince yourself that the Hamiltonian can be written as a vector-matrix-vector product (a quadratic form)

$$\widehat{H} = \hat{a}^\dagger \cdot H \cdot \hat{a} = \sum_{j,k} H_{jk} \hat{a}_j^\dagger \hat{a}_k = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2^\dagger \end{pmatrix} \cdot \begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix} \cdot \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix},$$

where all three expressions above are just different ways of writing the same thing, with

$$\hat{a} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \hat{a}^\dagger = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2^\dagger \end{pmatrix}, H = \begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix},$$

while H_{jk} are the elements of the 2×2 Hermitian matrix H.

(a) Consider the displacement vector

$$d = \begin{pmatrix} \langle \hat{a}_1 \rangle \\ \langle \hat{a}_2 \rangle \end{pmatrix},$$

and the correlation matrix

$$C = \begin{pmatrix} \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle \\ \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \end{pmatrix},$$

or, equivalently, $d_j = \langle a_j \rangle$ and $C_{jk} = \langle\langle \hat{a}_k^\dagger \hat{a}_j \rangle\rangle = \langle \hat{a}_k^\dagger \hat{a}_j \rangle - \langle \hat{a}_k^\dagger \rangle \langle \hat{a}_j \rangle$ (note the order of indices j, k !). Use the Heisenberg equations to show that

$$ \frac{\rm d}{\rm dt}\mathbf{d} = \frac{1}{{\rm i}\hbar}\mathbf{H}\cdot \mathbf{d},\qquad \frac{\rm d}{\rm dt} \mathbf{C} = \frac{1}{{\rm i}\hbar}[\mathbf{H},\mathbf{C}].$$

QUESTION 7

Two-mode system with annihilation operators $\hat{a}_1$ and $\hat{a}_2$. Convince yourself that the Hamiltonian can be written as a vector-matrix-vecto product.

$$\hat{H} = \hat{a}^\dagger \cdot H \cdot \hat{a} = \sum_{j,k} H_{jk}\, \hat{a}_j^\dagger \hat{a}_k = (\hat{a}_1^\dagger\ \hat{a}_2^\dagger)\begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix}\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

You can write:

$$\hat{a} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \qquad \hat{a}^\dagger = (\hat{a}_1^\dagger\ \ \hat{a}_2^\dagger) \qquad H = \begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix}$$

a) Consider the displacement vector:

$$d = \begin{pmatrix} \langle \hat{a}_1 \rangle \\ \langle \hat{a}_2 \rangle \end{pmatrix}$$

and the matrix

$$C = \begin{pmatrix} \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle \\ \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \end{pmatrix}$$

Use Heisenberg equation to show that:

$$\frac{d}{dt} d = \frac{1}{i\hbar} H \cdot d \qquad \frac{d}{dt} C = \frac{1}{i\hbar}[H, C]$$

$$\frac{d}{dt}\vec{d} = \begin{pmatrix} \frac{d}{dt}\langle \hat{a}_1 \rangle \\ \frac{d}{dt}\langle \hat{a}_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle \frac{d}{dt}\hat{a}_1 \rangle \\ \langle \frac{d}{dt}\hat{a}_2 \rangle \end{pmatrix} = \begin{pmatrix} \frac{i}{\hbar}[H, \hat{a}_1] \\ \frac{i}{\hbar}[H, \hat{a}_2] \end{pmatrix} = \begin{pmatrix} \frac{1}{i\hbar}\left(\hbar\omega_1 \langle \hat{a}_1 \rangle + \frac{\hbar g}{2}\langle \hat{a}_2 \rangle\right) \\ 1/i\hbar\left(\frac{\hbar g}{2}\langle \hat{a}_1 \rangle + \hbar\omega_2 \langle \hat{a}_2 \rangle\right) \end{pmatrix}$$

$$= 1/i\hbar \begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix}\begin{pmatrix} \langle \hat{a}_1 \rangle \\ \langle \hat{a}_2 \rangle \end{pmatrix}$$

$$\frac{d}{dt}\vec{d} = \frac{1}{i\hbar}\vec{H}\cdot\vec{d}$$

We need to prove that: $\frac{d}{dt}\vec{C} = \frac{1}{i\hbar}[\vec{H}, \vec{C}]$

$$\frac{d}{dt}\vec{C} = \begin{pmatrix} \frac{d}{dt}\langle\langle a_1^\dagger a_1 \rangle\rangle & \frac{d}{dt}\langle\langle a_2^\dagger a_2 \rangle\rangle \\ \frac{d}{dt}\langle\langle a_1^\dagger a_2 \rangle\rangle & \frac{d}{dt}\langle\langle a_2^\dagger a_2 \rangle\rangle \end{pmatrix}$$

$$\frac{d}{dt}\langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle = \frac{d}{dt}\left(\langle \hat{a}_1^\dagger \hat{a}_1 \rangle - \langle \hat{a}_1^\dagger \rangle\langle \hat{a}_1 \rangle\right) =$$

$$\left\langle \frac{d\hat{a}_1}{dt}\hat{a}_1^\dagger \right\rangle + \left\langle \frac{da_1^\dagger}{dt}\hat{a}_1 \right\rangle - \langle \hat{a}_1^\dagger \rangle\left\langle \frac{d\hat{a}_1}{dt} \right\rangle - \left\langle \frac{d\hat{a}_1^\dagger}{dt} \right\rangle\langle \hat{a}_1 \rangle$$

$$\frac{d\hat{a}_1^+}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}_1^+] = \frac{i}{\hbar}\left[\hbar\omega_1 \hat{a}_1^+\hat{a}_1 + \hbar\omega_2 \hat{a}_2^+\hat{a}_2 + \frac{\hbar g}{2}(\hat{a}_1^+\hat{a}_2 + \hat{a}_1\hat{a}_2^+), \hat{a}_1^+\right] =$$

$$= i\omega_1[\hat{a}_1^+\hat{a}_1, \hat{a}_1^+] + i\omega_2[\hat{a}_2^+\hat{a}_2, \hat{a}_1^+] + \frac{ig}{2}[\hat{a}_1^+\hat{a}_2, \hat{a}_1^+] + \frac{ig}{2}[\hat{a}_1\hat{a}_2^+, \hat{a}_1^+]$$

$$= i\omega_1 \hat{a}_1^+ + 0 + 0 + \frac{ig}{2}\hat{a}_2^+ = -1/i\hbar\left(\hbar\omega_1\hat{a}_1^+ + \frac{\hbar g}{2}\hat{a}_2\right)$$

$$\frac{d}{dt}\langle\langle \hat{a}_1^+\hat{a}_1\rangle\rangle = \left\langle \frac{1}{i\hbar}\left(\cancel{\hbar\omega_1\hat{a}_1^+\hat{a}_1} + \frac{\hbar g}{2}\hat{a}_1^+\hat{a}_2\right)\right\rangle + \left\langle -1/i\hbar\left(\cancel{\hbar\omega_1\hat{a}_1^+\hat{a}_1} + \frac{\hbar g}{2}\hat{a}_2^+\hat{a}_1\right)\right\rangle - \ldots$$

$$\ldots - \langle\hat{a}_1^+\rangle\langle 1/i\hbar(\cancel{\hbar\omega_1\hat{a}_1} + \hbar g/2\,\hat{a}_2)\rangle - \langle -1/i\hbar(\cancel{\hbar\omega_1\hat{a}_1^+} + \frac{\hbar g}{2}\hat{a}_2^+)\rangle\langle\hat{a}_1\rangle$$

$$= 1/i\hbar\left(\frac{\hbar g}{2}\right)\left[\left(\langle\hat{a}_1^+\hat{a}_2\rangle - \langle\hat{a}_1^+\rangle\langle\hat{a}_2\rangle\right) - \left(\langle\hat{a}_2^+\hat{a}_1\rangle - \langle\hat{a}_2^+\rangle\langle\hat{a}_1\rangle\right)\right] =$$

$$= 1/i\hbar\,(\hbar g/2)\left[\langle\langle\hat{a}_1^+\hat{a}_2\rangle\rangle - \langle\langle\hat{a}_2^+\hat{a}_1\rangle\rangle\right]$$

$$\frac{d}{dt}\langle\langle\hat{a}_2^+\hat{a}_1\rangle\rangle = \frac{d}{dt}\left(\langle\hat{a}_2^+\hat{a}_1\rangle - \langle\hat{a}_2^+\rangle\langle\hat{a}_1\rangle\right) =$$

$$= \left\langle\hat{a}_2^+\frac{d\hat{a}_1}{dt}\right\rangle + \left\langle\frac{d\hat{a}_2^+}{dt}\hat{a}_1\right\rangle - \langle\hat{a}_2^+\rangle\left\langle\frac{d\hat{a}_1}{dt}\right\rangle - \left\langle\frac{d\hat{a}_2^+}{dt}\right\rangle\langle\hat{a}_1\rangle$$

$$\frac{d\hat{a}_2^+}{dt} = i/\hbar[\hat{H}, \hat{a}_2^+] = i/\hbar\left[\hbar\omega_1\hat{a}_1^+\hat{a}_1 + \hbar\omega_2\hat{a}_2^+\hat{a}_2 + \frac{\hbar g}{2}(\hat{a}_1^+\hat{a}_2 - \hat{a}_1\hat{a}_2^+), \hat{a}_2^+\right]$$

$$= i\omega_1[\hat{a}_1^+\hat{a}_1, \hat{a}_2^+] + i\omega_2[\hat{a}_2^+\hat{a}_2, \hat{a}_2^+] + \frac{ig}{2}[\hat{a}_1^+\hat{a}_2, \hat{a}_2^+] + \frac{ig}{2}[\hat{a}_1\hat{a}_2^+, \hat{a}_2^+]$$

$$= i\omega_2\hat{a}_2^+ + ig/2\,\hat{a}_1^+ = -1/i\hbar[\hbar\omega_2\hat{a}_2^+ + \hbar g/2\,\hat{a}_1^+]$$

$$\frac{d}{dt}\langle\langle\hat{a}_2^+\hat{a}_1\rangle\rangle = \left\langle\frac{1}{i\hbar}\left(\hbar\omega_1\hat{a}_2^+\hat{a}_1 + \frac{\hbar g}{2}\hat{a}_2^+\hat{a}_2\right)\right\rangle + \left\langle -1/i\hbar\left(\hbar\omega_2\hat{a}_2^+\hat{a}_1 + \frac{\hbar g}{2}\hat{a}_1^+\hat{a}_1\right)\right\rangle - \ldots$$

$$\ldots - \langle\hat{a}_2^+\rangle\langle 1/i\hbar(\hbar\omega_1\hat{a}_1 + \hbar g/2\,\hat{a}_2)\rangle - \langle -1/i\hbar(\hbar\omega_2\hat{a}_2^+ + \hbar g/2\,\hat{a}_1^+)\rangle\langle\hat{a}_1\rangle =$$

$$= 1/i\hbar\left[(\hbar\omega_1 - \hbar\omega_2)\left(\langle\hat{a}_2^+\hat{a}_1\rangle - \langle\hat{a}_2^+\rangle\langle\hat{a}_1\rangle\right)\right] + (\hbar g/2)\left(\langle\hat{a}_2^+\hat{a}_2\rangle - \langle\hat{a}_2^+\rangle\langle\hat{a}_2\rangle - \ldots\right.$$

$$\left.\ldots - \left(\langle\hat{a}_1^+\hat{a}_1\rangle - \langle\hat{a}_1^+\rangle\langle\hat{a}_1\rangle\right)\right] =$$

$$= 1/i\hbar\left[(\hbar\omega_1 - \hbar\omega_2)\langle\langle\hat{a}_2^+\hat{a}_1\rangle\rangle + \hbar g/2\left(\langle\langle\hat{a}_2^+\hat{a}_2\rangle\rangle - \langle\langle\hat{a}_1^+\hat{a}_1\rangle\rangle\right)\right]$$

$$\frac{d}{dt}\langle\langle\hat{a}_1^+\hat{a}_2\rangle\rangle = \frac{d}{dt}\left(\langle\hat{a}_1^+\hat{a}_2\rangle - \langle\hat{a}_1^+\rangle\langle\hat{a}_2\rangle\right) =$$

$$= \left\langle\hat{a}_1^+\frac{d\hat{a}_2}{dt}\right\rangle + \left\langle\frac{d\hat{a}_1^+}{dt}\hat{a}_2\right\rangle - \langle\hat{a}_1^+\rangle\left\langle\frac{d\hat{a}_2}{dt}\right\rangle - \left\langle\frac{d\hat{a}_1^+}{dt}\right\rangle\langle\hat{a}_2\rangle =$$

$$= \left\langle 1/i\hbar\left(\hbar\omega_2\hat{a}_1^+\hat{a}_2 + \frac{\hbar g}{2}\hat{a}_1^+\hat{a}_1\right)\right\rangle + \left\langle -1/i\hbar\left(\hbar\omega_1\hat{a}_1^+\hat{a}_2 + \frac{\hbar g}{2}a_2^+\hat{a}_2\right)\right\rangle - \ldots$$

$$\ldots - \langle\hat{a}_1^+\rangle\langle 1/i\hbar(\hbar\omega_2\hat{a}_2 + \hbar g/2\,\hat{a}_1)\rangle - \langle -1/i\hbar(\hbar\omega_1\hat{a}_1^+ + \frac{\hbar g}{2}\hat{a}_2^+)\rangle\langle\hat{a}_2\rangle =$$

$$= 1/i\hbar\left[(\hbar\omega_2 - \hbar\omega_1)\left(\langle\hat{a}_1^+\hat{a}_2\rangle - \langle\hat{a}_1^+\rangle\langle\hat{c}_2\rangle\right) + \left(\frac{\hbar g}{2}\right)\left[\langle a_1^+\hat{a}_1\rangle - \langle\hat{a}_1^+\rangle\langle\hat{a}_1\rangle - \ldots\right.\right.$$

$$\left.\left.\ldots - \left(\langle\hat{a}_2^+\hat{a}_2\rangle - \langle\hat{a}_2^+\hat{a}_2\rangle\right)\right]\right] =$$

$$= \frac{1}{i\hbar}\left[(\hbar\omega_2 - \hbar\omega_1) \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle + \frac{\hbar g}{2} \left(\langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle - \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \right) \right]$$

$$\frac{d}{dt}\langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle = \frac{d}{dt}\left(\langle \hat{a}_2^\dagger \hat{a}_2 \rangle - \langle \hat{a}_2^\dagger \rangle \langle \hat{a}_2 \rangle \right) = \left\langle \frac{d\hat{a}_2}{dt} \hat{a}_2^\dagger \right\rangle + \left\langle \frac{d\hat{a}_2^\dagger}{dt} \hat{a}_2 \right\rangle - \langle \hat{a}_2^\dagger \rangle \left\langle \frac{d\hat{a}_2}{dt} \right\rangle - \ldots$$

$$\ldots - \left\langle \frac{d\hat{a}_2^\dagger}{dt} \right\rangle \langle \hat{a}_2 \rangle =$$

$$= \left\langle \frac{1}{i\hbar} \left(\hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{\hbar g}{2} \hat{a}_2^\dagger \hat{a}_1 \right) \right\rangle + \left\langle -\frac{1}{i\hbar} \left(\hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{\hbar g}{2} \hat{a}_1^\dagger \hat{a}_2 \right) \right\rangle - \ldots$$

$$\ldots - \langle \hat{a}_2^\dagger \rangle \left\langle \frac{1}{i\hbar} \left(\hbar\omega_2 \hat{a}_2 + \frac{\hbar g}{2} \hat{a}_1 \right) \right\rangle - \left\langle -\frac{1}{i\hbar} \left(\hbar\omega_2 \hat{a}_2^\dagger + \frac{\hbar g}{2} \hat{a}_1^\dagger \right) \right\rangle \langle \hat{a}_2 \rangle =$$

$$\ldots = \frac{1}{i\hbar} \left(\frac{\hbar g}{2} \right) \left[\langle \hat{a}_2^\dagger \hat{a}_1 \rangle - \langle \hat{a}_2^\dagger \rangle \langle \hat{a}_1 \rangle - \left(\langle \hat{a}_1^\dagger \hat{a}_2 \rangle - \langle \hat{a}_1^\dagger \rangle \langle \hat{a}_2 \rangle \right) \right] =$$

$$\frac{d}{dt}\langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle = \frac{1}{i\hbar} \left(\frac{\hbar g}{2} \right) \left[\langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle - \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle \right]$$

$$\frac{1}{i\hbar}[H, C] = \frac{1}{i\hbar}\left[\vec{H} \cdot \vec{C} - \vec{C} \cdot \vec{H} \right] =$$

$$= \frac{1}{i\hbar}\left[\begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix} \begin{pmatrix} \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle \\ \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \end{pmatrix} - \begin{pmatrix} \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle \\ \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle & \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \end{pmatrix} \begin{pmatrix} \hbar\omega_1 & \hbar g/2 \\ \hbar g/2 & \hbar\omega_2 \end{pmatrix} \right]$$

$$= \frac{1}{i\hbar}\left[\begin{pmatrix} \hbar\omega_1 \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle + \frac{\hbar g}{2} \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle & \hbar\omega_1 \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle + \hbar g/2 \, \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \\ \frac{\hbar g}{2} \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle + \hbar\omega_2 \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle & \frac{\hbar g}{2} \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle + \hbar\omega_2 \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \end{pmatrix} - \ldots \right.$$

$$\left. \ldots - \begin{pmatrix} \hbar\omega_1 \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle + \frac{\hbar g}{2} \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle & \hbar g/2 \, \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle + \hbar\omega_2 \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle \\ \hbar\omega_1 \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle + \frac{\hbar g}{2} \langle\langle \hat{a}_2^\dagger a_2 \rangle\rangle & \hbar g/2 \, \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle + \hbar\omega_2 \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \end{pmatrix} \right] =$$

$$= \frac{1}{i\hbar} \begin{pmatrix} \hbar g/2 \left(\langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle - \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle \right) & (\hbar\omega_1 - \hbar\omega_2) \langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle + \frac{\hbar g}{2} \left(\langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle - \langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle \right) \\ (\hbar\omega_2 - \hbar\omega_1) \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle + \frac{\hbar g}{2} \left(\langle\langle \hat{a}_1^\dagger \hat{a}_1 \rangle\rangle - \langle\langle \hat{a}_2^\dagger \hat{a}_2 \rangle\rangle \right) & \hbar g/2 \left(\langle\langle \hat{a}_2^\dagger \hat{a}_1 \rangle\rangle - \langle\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\rangle \right) \end{pmatrix}$$

$$\frac{1}{i\hbar}[\vec{H}, \vec{C}] = \begin{pmatrix} d/dt \, \langle\langle a_1^\dagger a_1 \rangle\rangle & d/dt \, \langle\langle a_2^\dagger a_1 \rangle\rangle \\ d/dt \, \langle\langle a_1^\dagger a_2 \rangle\rangle & d/dt \, \langle\langle a_2^\dagger a_2 \rangle\rangle \end{pmatrix} = \frac{d}{dt} \vec{C}$$

(b) Convince yourself that the solutions to the above equations are

$$d(t) = V(t) \cdot d(0), C(t) = V(t) \cdot C(0) \cdot V^\dagger(t),$$

where $\mathbf{V}(t) = {\rm e}^{-{\rm i}\mathbf{H} t/\hbar}$ is a unitary matrix. Therefore, the dynamics can be simulated using the same iterative method as we have previously used for qubits in Worksheet 1, since

$$d(t+\Delta t) = V(\Delta t) \cdot d(t), C(t+\Delta t) = V(\Delta t) \cdot C(t) \cdot V^\dagger(\Delta t).$$

Use this method to calculate the dynamics of the system numerically, starting from the product state

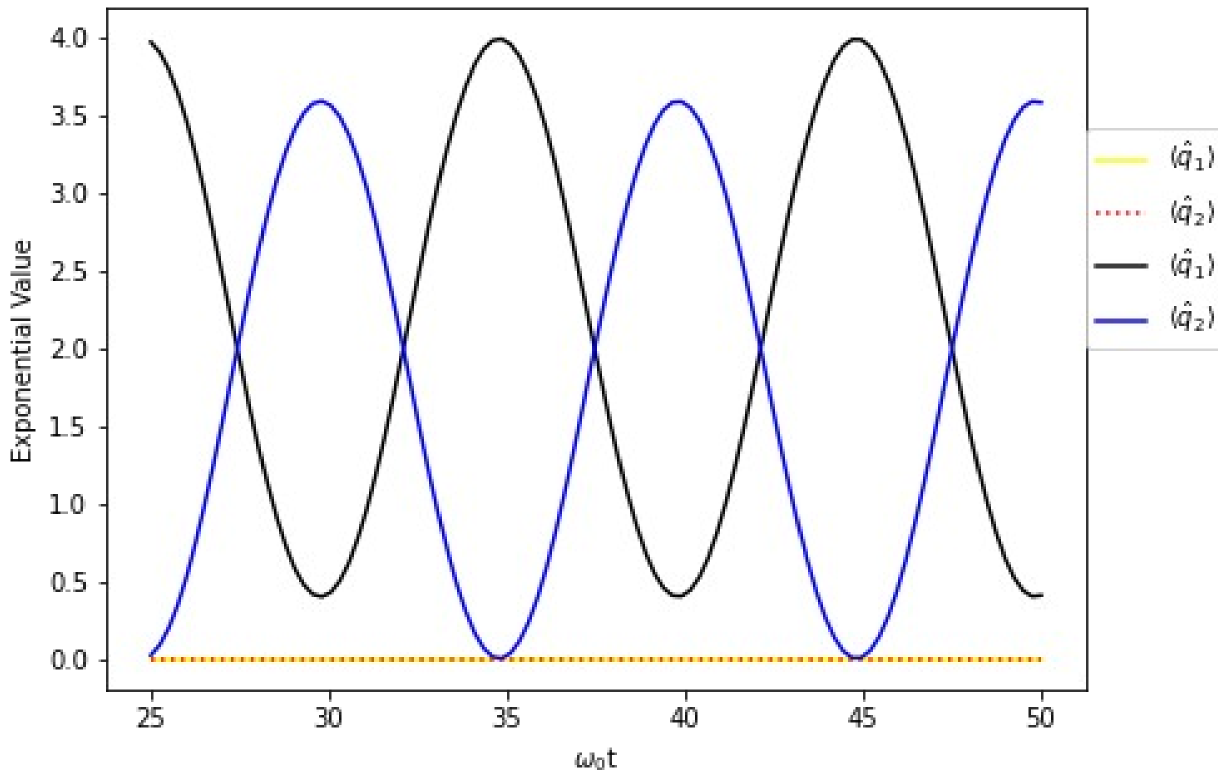
¿

corresponding to a state with 4 energy quanta in mode 1 and 0 quanta in mode 2. Generate plots of the mean coordinate $\langle \hat{q}_j(t) \rangle$ and the mean occupation number $\langle \hat{n}_j(t) \rangle$ for both modes, $j=1,2$. Choose parameters $\omega_2=0.9\,\omega_1$, $g=0.3\,\omega_1$, and evolve up to a time $\omega_1 t=50$.

```
time = np.linspace(25,50,100) #linspace return spaced numbers over a specified interval
expectation_values_q = np.zeros((2,len(time)))
expectation_values_n = np.zeros((2,len(time)))
#initial values
d=np.array([0,0])
c=np.array([[4,0],[0,0]])
w1=1
w2=0.9*w1
g=0.3*w1
hamiltonian=np.array([[w1,g/2],[g/2,w2]])
#time evolution
v=expm(-1j*hamiltonian*0.5)
#time step
dt=d
ct=c
 #Loop over time steps
for i in range(len(time)):
    dt=v@dt
    ct=v@ct@(v.T.conj())
    expectation_values_q[0][i]=dt[0].real
    expectation_values_q[1][i]=dt[1].real
    expectation_values_n[0][i]=(ct[0,0]+np.abs(dt[0]**2).real)
    expectation_values_n[1][i]=(ct[1,1]+np.abs(dt[1]**2).real)

plt.subplots(figsize=(7,5))
plt.plot(time,expectation_values_q[0,:],label='$\\langle \\hat{q}_{1}\\rangle$',color='yellow')
plt.plot(time,expectation_values_q[1,:],label='$\\langle \\hat{q}_{2}\\rangle$',color='red',linestyle='dotted')
plt.plot(time,expectation_values_n[0,:],label='$\\langle \\hat{q}_{1}\\rangle$',color='black')
plt.plot(time,expectation_values_n[1,:],label='$\\langle \\hat{q}_{2}\\rangle$',color='blue')
plt.xlabel('$\omega_0$t')
plt.ylabel('Exponential Value')
plt.legend(loc=(1.0,0.5))
```

```
/var/folders/07/02v51ss15857xv4ddkzq17_c0000gn/T/ipykernel_11309/4269321635.py:22: ComplexWarning: Casting complex
```

```
values to real discards the imaginary part
  expectation_values_n[0][i]=(ct[0,0]+np.abs(dt[0]**2).real)
/var/folders/07/02v51ss15857xv4ddkzq17_c0000gn/T/ipykernel_11309/42693
21635.py:23: ComplexWarning: Casting complex values to real discards
the imaginary part
  expectation_values_n[1][i]=(ct[1,1]+np.abs(dt[1]**2).real)

<matplotlib.legend.Legend at 0x7fad30e71430>
```



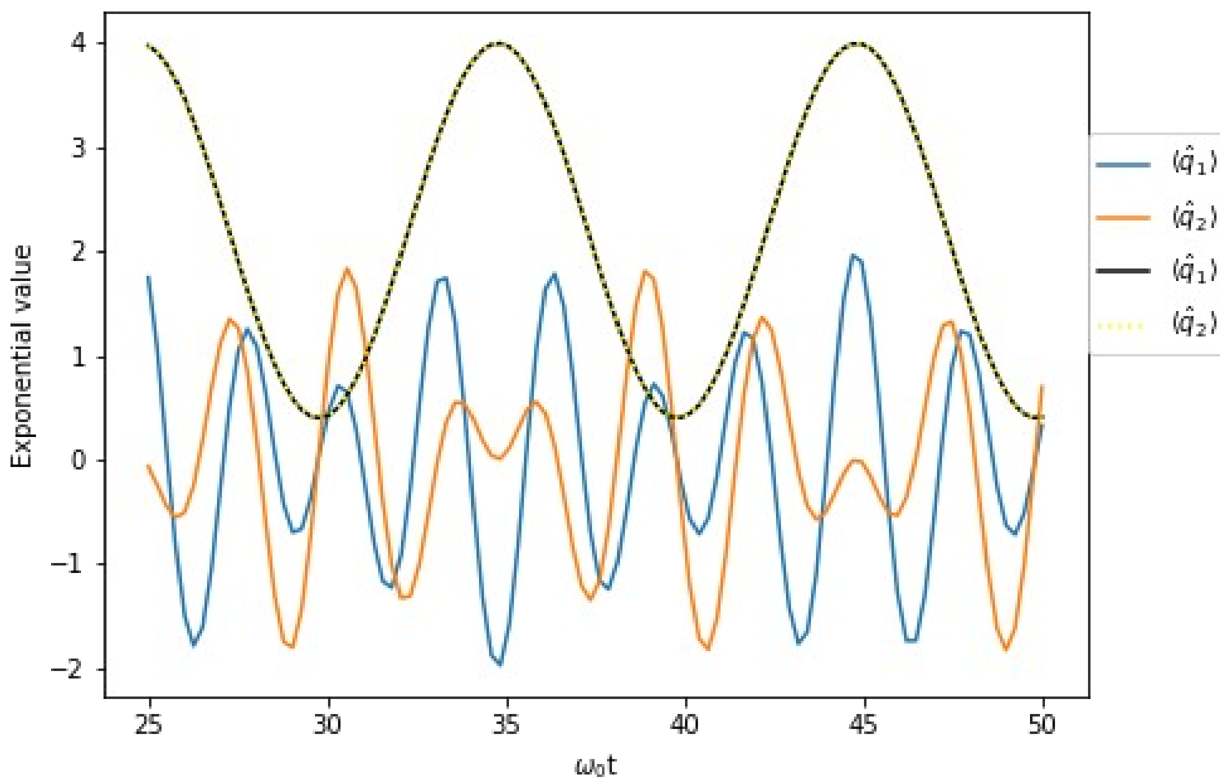
(c) Repeat the calculation for an initial product of coherent states,

¿

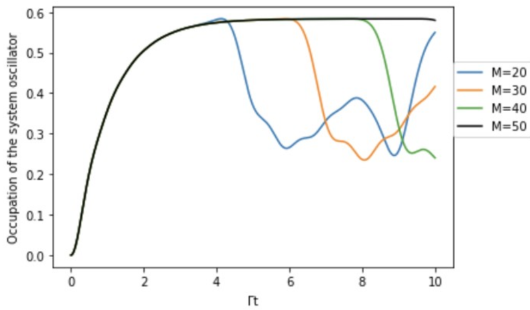
with $\alpha_1=2$ and $\alpha_2=0$. Comment on the similarities and differences between (b) and (c).

```
expectation_value_q2=np.zeros((2,len(time)))
expectation_value_n2=np.zeros((2,len(time)))
dt2=np.array([2,0])
ct2=np.array([[0,0],[0,0]])
for i in range(len(time)):
    dt2=v@dt2
    ct2=v@ct2@(v.T.conj())
    expectation_value_q2[0][i]=dt2[0].real
    expectation_value_q2[1][i]=dt2[1].real
    expectation_value_n2[0][i]=(ct2[0,0]+np.abs(dt2[0])**2).real
```

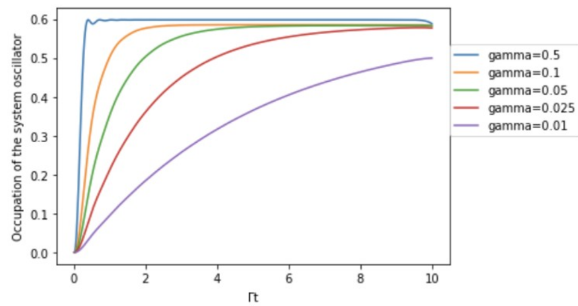
```
    expectation_value_n2[1][i]=(ct2[1,1]+np.abs(dt2[1])**2).real
plt.subplots(figsize=(7,5))
plt.plot(time,expectation_value_q2[0,:],label='$\\langle \\
hat{q}_{1}\\rangle$')
plt.plot(time,expectation_value_q2[1,:],label='$\\langle \\
hat{q}_{2}\\rangle$')
plt.plot(time,expectation_value_n2[0,:],label='$\\langle \\
hat{q}_{1}\\rangle$',color='black',linestyle='solid')
plt.plot(time,expectation_value_n2[0,:],label='$\\langle \\
hat{q}_{2}\\rangle$',color='yellow',linestyle='dotted')
plt.xlabel('$\omega_0$t')
plt.ylabel('Exponential value')
plt.legend(loc=(1.0,0.5))

<matplotlib.legend.Legend at 0x7fad30f1fd00>
```



(d)



(e)



Coordinates 'q' have a very different behaviour in both graphs.

b the mean coordinate of (b) is zero all the time of the evolution, also in this there is no generation of coherent states, because there is no coherence in the initial state of the density matrix.

c The initial states have coherences so the coherent states have an oscillating wave behavior.

Both graphs have the same mode occupations but the expectation value of the operator vary destructively for both modes, which means that the rabi oscillations are occuring.

8

Now consider a general M-mode quadratic Hamiltonian

$$\widehat{H} = \sum_{j,k=0}^{M-1} H_{jk} \hat{a}_j^\dagger \hat{a}_k$$

and let us assume for simplicity that $d(0)=0$. The matrix H is Hermitian and thus can be diagonalised by a unitary transformation:

$$U^\dagger \cdot H \cdot U = \hbar \Omega = \hbar \begin{pmatrix} \omega_0 & 0 & 0 \cdots \\ 0 & \omega_1 & 0 \cdots \\ 0 & 0 & \omega_2 \cdots \\ \vdots & \vdots & \vdots \ddots \end{pmatrix},$$

where $\hbar\omega_j$ are the eigenvalues of H. Therefore, $H = \hbar U \cdot \Omega \cdot U^\dagger$ or, in components,

$$H_{jk} = \sum_{l,m} U_{jl} \hbar \omega_l \delta_{lm} U_{km}^{¿}.$$

By plugging this into the Hamiltonian, one finds

$$\widehat{H} = \sum_{j=0}^{M-1} \hbar \omega_j \hat{b}_j^\dagger \hat{b}_j,$$

where $\hat{b}_j = \sum_k U^{¿}_{kj} \hat{a}_k$. Since $\left[\hat{b}_j, \hat{b}_k^\dagger\right] = \delta_{jk}$, this represents the Hamiltonian as a sum of M independent oscillators.

(a) Suppose that the system is in a thermal state, $\hat{\rho} = \frac{{\rm e}^{-\beta \hat{H}}}{Z}$, with $\beta = 1/k_B T$. Convince yourself that for this state,

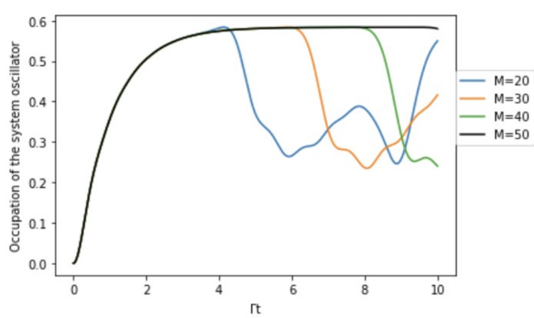
$$\langle \hat{b}_k \rangle = 0, \langle \hat{b}_j^\dagger \hat{b}_k \rangle = \acute{n}(\omega_k) \delta_{jk},$$

where $\acute{n}(\omega)$ is the Bose-Einstein distribution. Hence, show that the correlation matrix for a thermal state is

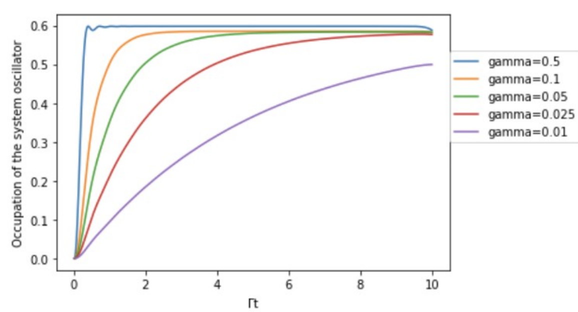
$$\mathbf{C} = \left({\rm e}^{\beta \mathbf{H}} - 1\right)^{-1}.$$

(The correlation matrix is defined as in question 7, i.e. $C_{jk} = \langle\langle \hat{a}_k^\dagger \hat{a}_j \rangle\rangle = \langle \hat{a}_k^\dagger \hat{a}_j \rangle - \langle \hat{a}_k^\dagger \rangle \langle \hat{a}_j \rangle$.)

(d)



(e)



(b) Let us now apply these results to simulating the dynamics of a single oscillator system in contact with a large many-mode bath. Consider the Hamiltonian $\widehat{H} = \widehat{H}_S + \widehat{H}_B + \widehat{H}_{SB}$, where \begin{align} &\hat{H}_S = \hbar\nu_0 \hat{a}_0^\dagger \hat{a}_0 \,\qquad &\hat{H}*B = |sum*{j=1}^{M-1} \hbar\nu_j \hat{a}*j^|dagger |hat{a}j + |sum{j=1}^{M-2} |frac{|hbar W}{2} |left(|hat{a}j^|dagger|hat{a}{j+1} + |hat{a}j|hat{a}{j+1}^|dagger |right),| & |hat{H}*{SB} = \frac{\sqrt{\Gamma W}}{2}\left(\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger\right). \end{align} This models a bath comprising a one-dimensional chain of oscillators coupled to the system at one of its ends ($j = 1$). Write a function that constructs the matrix H for the *full* system-bath Hamiltonian $\widehat{H} = \sum_{j,k} H_{jk} \hat{a}_j^\dagger \hat{a}_k$, taking the parameters $\{\nu_j\}$, W, Γ, and M as input. Write another function that constructs the corresponding thermal correlation matrix C. Test your function by printing the thermal correlation matrix for the parameters $M = 4$, $\nu_j = \nu_0 = 2$, $W = 0.5$, $\Gamma = 0.05$ and $\beta = 0.5$.

```
def full_system_hamiltonian(vj,w,g,M):#function for the construction of the fyll system-bath Hamiltonian, vj=diagonal values
    hamiltonian=np.zeros((M,M))
    for n in range(M):
        for i in range(M):
            if n==i:
```

```
                hamiltonian[n][i]=vj[i]
    off_diagonal_values=[]#empty list to store future values
    off_diagonal_values.append(np.sqrt(g*w)/2) #upper diagonal values
Ĥ _SB and allocate them with append
    for n in range(1,(M-2)+1):
        off_diagonal_values.append(w/2)
    hamiltonian=hamiltonian+np.diag(off_diagonal_values,1)
    hamiltonian=hamiltonian+np.diag(off_diagonal_values,-1)
    return hamiltonian
def thermal_cirrelation_matrix(vj,w,g,M,b):  # Function to define the
thermal correlation matrix
    hamiltonian=full_system_hamiltonian(vj,w,g,M) #another hamiltonian

    correlation_matrix=expm(b*hamiltonian)-np.eye(M)
    correlation_matrix=np.linalg.matrix_power(correlation_matrix,-1)
    return correlation_matrix
M=4
vj=np.array([2,2,2,2])
g=0.05
w=0.5
b=0.5
print('\n','Matrix  H for the full system-bath Hamiltonian','\n')
print(full_system_hamiltonian(vj,w,g,M))
print('\n','Thermal correlation matrix  C','\n')
print(thermal_cirrelation_matrix(vj,w,g,M,b))
```

```
 Matrix  H for the full system-bath Hamiltonian

[[2.         0.07905694 0.         0.        ]
 [0.07905694 2.         0.25       0.        ]
 [0.         0.25       2.         0.25      ]
 [0.         0.         0.25       2.        ]]

 Thermal correlation matrix  C

[[ 0.58356092 -0.03709499  0.00508955 -0.00063924]
 [-0.03709499  0.59965547 -0.1193261   0.01609456]
 [ 0.00508955 -0.1193261   0.61414057 -0.11912395]
 [-0.00063924  0.01609456 -0.11912395  0.59804602]]
```

(c) Now let us perform a dynamical simulation of a thermalisation process. Start from the product state

$$\hat{\rho}(0) = |0\rangle\langle 0|\otimes \frac{{\rm e}^{-\beta\hat{H}_B}}{Z_B},$$

where $¿0\rangle$ is the vacuum for mode $\hat{a}_0$. Construct the correlation matrix for this state and then evolve it as a function of time using the method of question 7, with parameters $v_j = v_0 = 2$, $W = 0.5$, $\Gamma = 0.05$, $\beta = 0.5$, $M = 51$, and up to a time $\Gamma t = 10$. Specifically, plot the occupation of the system oscillator, $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle$, as a function of time. Explore how this open-system evolution changes as the number of oscillators, M, and the system-reservoir coupling, Γ, is increased or decreased. Comment briefly on your results.

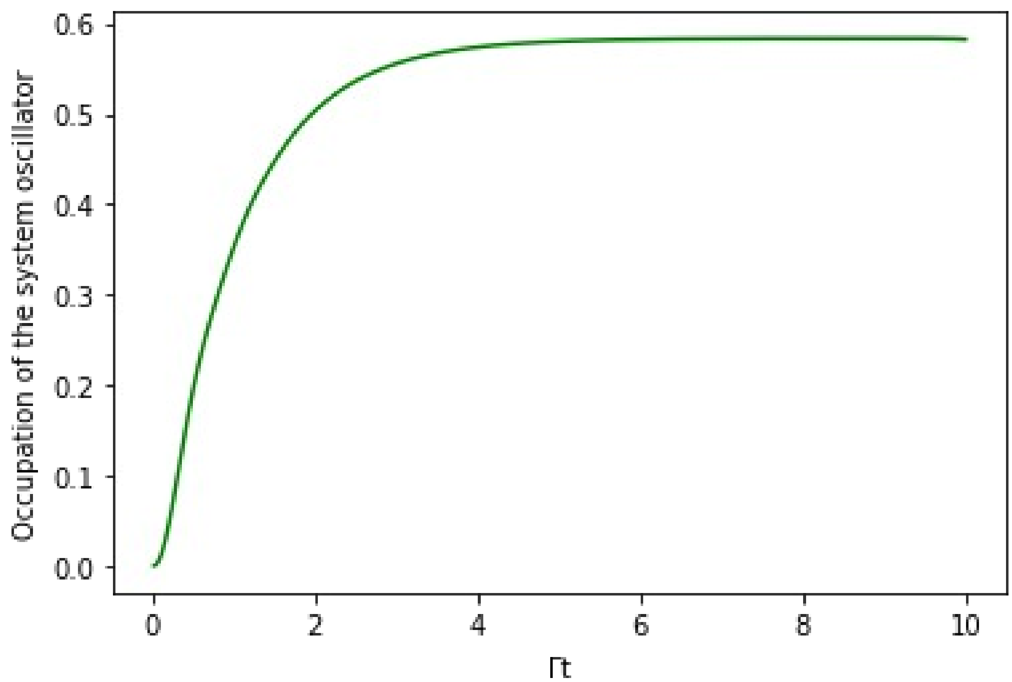
c) $\hat{\rho}(0) = |0\rangle\langle 0| \otimes \dfrac{e^{-\beta \hat{H}_B}}{Z_B}$

$$C_{jj} = \left(e^{\beta \hbar V_j} - 1\right)^{-1}, \quad j = \{0, 1, 2, \dots, M-1\}$$

$$C_{(j+1)(j)} = C_{(j)(j+1)} = \left(e^{\beta \hbar w/2} - 1\right)^{-1}, \quad j = \{1, 2, \dots, M-2\}$$

```
def occupation_system_oscillator(vj,M,w,g,b,gt): #function for the plot of the occupation of the system oscillator,
    time=np.arange(0.5,200,0.05)
    occupation_system=[]
    hamiltonian=full_system_hamiltonian(vj,w,g,M)
    c=thermal_cirrelation_matrix(vj,w,g,M,b)
    c[0,0]=0
    v=expm(-1j*hamiltonian*0.05)
    for i in range(len(time)): #time evolution
        c=v@c@(v.T.conj())
        occupation_system.append(c[0,0].real)
    return time,occupation_system
M=51
w=0.5
g=0.05
b=0.5
gt=10
vj=np.full(M,2)
time,occupation_system=occupation_system_oscillator(vj,M,w,g,b,gt)
print('\n','Occupation of the system oscillator as a function of Gamma*Time')
plt.plot(time*g,occupation_system,color='g',linestyle='solid')
plt.xlabel('$\Gamma$t')
plt.ylabel('Occupation of the system oscillator')
print('\n')
```

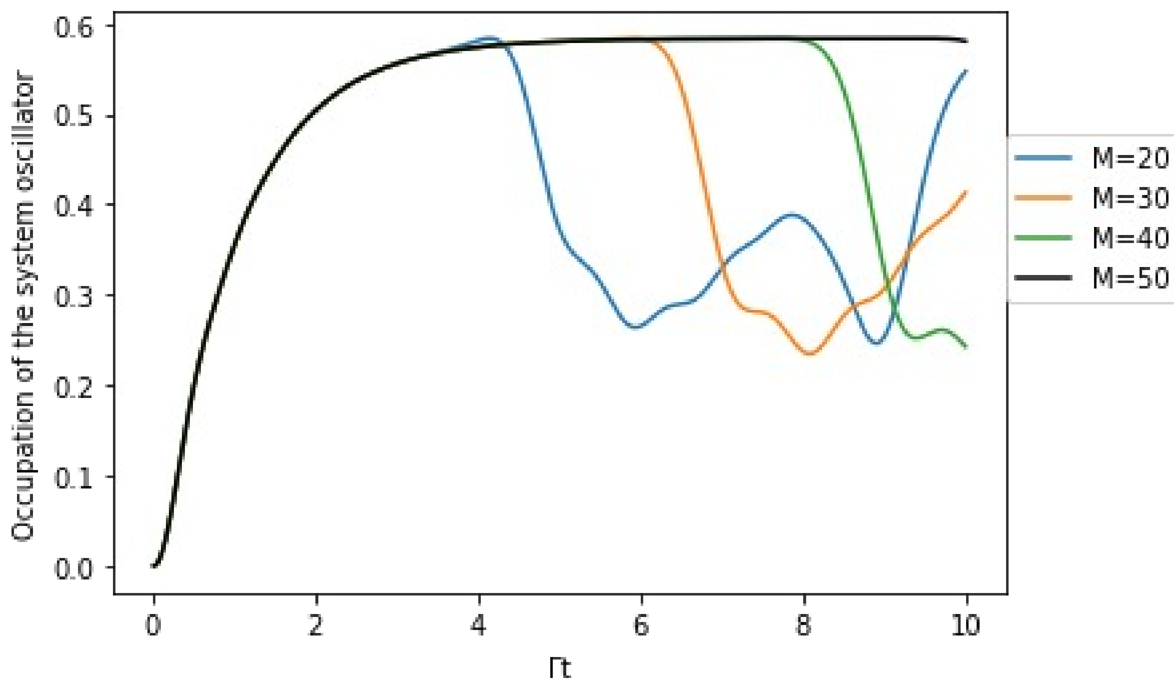
```
 Occupation of the system oscillator as a function of Gamma*Time
```




```
_,occupation_20=occupation_system_oscillator(vj,20,w,g,b,gt);
_,occupation_30=occupation_system_oscillator(vj,30,w,g,b,gt);
_,occupation_40=occupation_system_oscillator(vj,40,w,g,b,gt);
_,occupation_50=occupation_system_oscillator(vj,50,w,g,b,gt);
print('Occupation of the system oscillator as a function of
Gamma*Time')
plt.plot(time*g,occupation_20,label='M=20')
plt.plot(time*g,occupation_30,label='M=30')
plt.plot(time*g,occupation_40,label='M=40')
plt.plot(time*g,occupation_50,label='M=50',color='black',linestyle='so
lid')
plt.xlabel('$\Gamma$t')
plt.ylabel('Occupation of the system oscillator')
plt.legend(loc=(1.0,0.5))
```

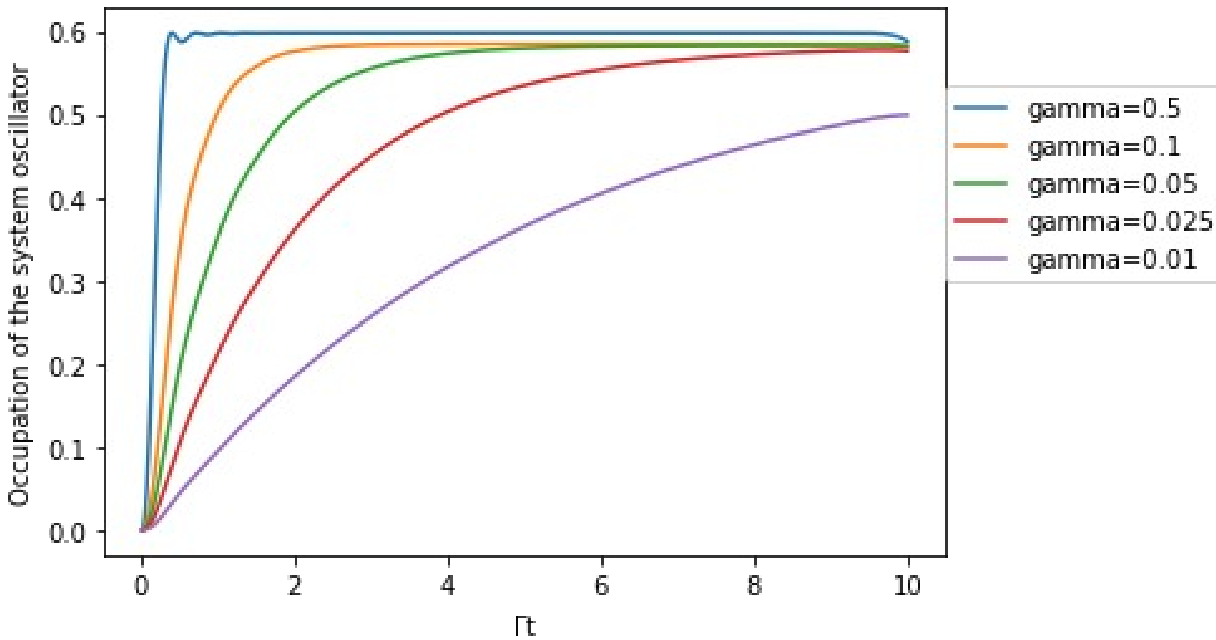
```
Occupation of the system oscillator as a function of Gamma*Time
```

```
<matplotlib.legend.Legend at 0x7fad385f6460>
```

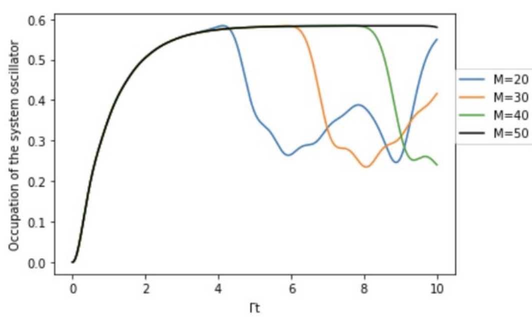



```
_,occupation_g1=occupation_system_oscillator(vj,M,w,0.5,b,10/g);
_,occupation_g2=occupation_system_oscillator(vj,M,w,0.1,b,10/g);
_,occupation_g3=occupation_system_oscillator(vj,M,w,0.05,b,10/g);
_,occupation_g4=occupation_system_oscillator(vj,M,w,0.025,b,10/g);
_,occupation_g5=occupation_system_oscillator(vj,M,w,0.01,b,10/g);
plt.plot(time*g,occupation_g1,label='gamma=0.5')
plt.plot(time*g,occupation_g2,label='gamma=0.1')
plt.plot(time*g,occupation_g3,label='gamma=0.05')
plt.plot(time*g,occupation_g4,label='gamma=0.025')
plt.plot(time*g,occupation_g5,label='gamma=0.01')
plt.xlabel('$\Gamma$t')
plt.ylabel('Occupation of the system oscillator')
plt.legend(loc=(1.0,0.5))
```

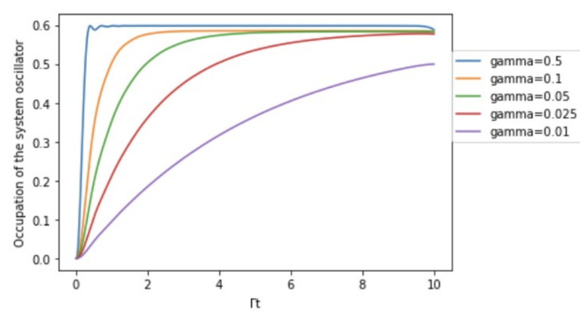
```
<matplotlib.legend.Legend at 0x7fad286c0400>
```



(d)



(e)



(d) At values of M=80 and 50, the system behaves very similarly. For the other values, as M increases, the system takes longer to decrease the occupancy. For M=20, the occupancy grows regularly until Γt=4, for M=30, the same happens until Γt=6 and for M=40, until Γt=8. The larger the value of M, the more stable the system, the smaller the value of M, the less oscillations and the more unstable the system.

(e) As the value of gamma increases, the rate at which the system is occupied increases, whereas if gamma decreases, the occupancy is slower. Similarly, at most values of gamma, the system occupancy becomes constant at approximately 'Γt'=6.