Simulation and Performance Evaluation
Homework 1 - Report

Exercise 1.

Let X be the random variable representing the final result of the draw and Y the random variable representing the chosen Gaussian distribution, with $Y \in \{1, 2, 3, 4\}$.

Random selection of a Gaussian distribution

To randomly select a Gaussian distribution, we generate a uniform random number u in [0, 1] and use the CDF inversion law for discrete random variables:

- If $u \le 0.15$, $Y = 1, \mu = -2, \sigma^2 = 2$
- If $0.15 < u \le 0.40$, $Y = 2, \mu = 4, \sigma^2 = 1$
- If $0.40 < u \le 0.75$, $Y = 3, \mu = 10, \sigma^2 = 3$
- If $u > 0.75$, $Y = 4, \mu = 15, \sigma^2 = 2$

Conditional expectation formula

To compute the global expectation of X, we will directly use the expectations of the given Gaussian distributions. Since Y is a discrete random variable, we can use this formula:

$$E[X] = E_Y[E_X[X \mid Y]] = \sum_{y=1}^{4} E[X \mid Y = y]\, P\{Y = y\}$$

Applied to our problem, we have:

$$E[X] = 0.15 \cdot (-2) + 0.25 \cdot 4 + 0.35 \cdot 10 + 0.25 \cdot 15 = 7.95$$

Conditional variance formula

To compute the variance of X, we will use the conditional variance formula:

$$Var(X) = E[Var(X \mid Y)] + Var(E[X \mid Y])$$

On the one hand: $$E[Var(X \mid Y)] = \sum_{y=1}^{4} P\{Y = y\}\, Var(X \mid Y = y)$$

$$E[Var(X \mid Y)] = 0.15 \cdot 2 + 0.25 \cdot 1 + 0.35 \cdot 3 + 0.25 \cdot 2 = 2.1$$

On the other hand: $$Var(E[X \mid Y]) = \sum_{y=1}^{4} (E[X \mid Y = y] - E[X])^2$$

$$\begin{aligned} Var(E[X \mid Y]) = {} & 0.15\,(-2 - 7.95)^2 + 0.25\,(4 - 7.95)^2 \\ & + 0.35\,(10 - 7.95)^2 + 0.15\,(15 - 7.95)^2 = 32.6475 \end{aligned}$$

Finally: $$Var(X) = 2.1 + 32.6475 = 34.7475$$

By drawing a set of at least 1,000,000 numbers at random using the provided Python program, we can verify that the obtained empirical mean and empirical variance are very similar to the theoretical ones.

Exercise 2.

(*Facultative*) Can you justify the result with theoretical arguments?

In Exercise 2, we aim to calculate the probability $P(X > Y)$ where:

X - an exponential random variate of mean μ = 1;

Y - a random variate uniformly distributed in the interval [0, 5].

Exponential Distribution (X)

The exponential random variable X with mean μ=1 follows the probability density function (PDF). This is the standard PDF for an exponential distribution with rate parameter $\lambda = 1$.

$$f_X(x) = e^{-x},\ x \geq 0$$

Uniform Distribution (Y)

The uniform random variable Y is distributed over the interval [0,5], with a PDF given by

$$f_Y(y) = \frac{1}{5},\ 0 \leq y \leq 5$$

Y has an equal probability of taking any value in the range from 0 to 5.

Since X and Y are independent random variables, their joint PDF is the product of their individual PDFs.

$$f(x, y) = f_X(x) * f_Y(y) = \frac{1}{5}e^{-x},\ x \geq 0 \ and\ 0 \leq y \leq 5$$

We need to find the probability that $X > Y$ (the probability that the exponential random variable is greater than the uniform random variable).

$$P(X > Y) = \int_0^5 \left(\int_y^\infty \frac{1}{5}e^{-x}dx\right)dy = \int_0^5 \frac{1}{5}e^{-y}dy = \frac{1}{5}\left[-e^{-y}\right]_0^5 = \frac{1-e^{-5}}{5}$$

$$P(X > Y) \approx 0.19865$$

0.19865 is the theoretical probability that X is greater than Y.

To validate this result, we conducted a **Monte Carlo simulation** by generating one million random samples of X and Y. The simulation estimated *P(X > Y)* to be approximately **0.1985**, which closely matches the theoretical value.