Measurement in Bell Basis

The Bell Basis is an orthonormal basis represented by basis states:

$$|\beta_{00}\rangle = \cancel{\frac{|00\rangle + |11\rangle}{\sqrt{2}}} \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\beta_{01}\rangle = \cancel{|01\rangle} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

They are generated using a CNOT gate an a H gate as follows:

ckt. (i)

The initial state $|\psi\rangle$ is one of the 4 computational basis states – $|00\rangle$, $|01\rangle$, $|10\rangle$ or $|11\rangle$.

If $\cancel{|\psi\rangle}$ $|\psi\rangle = |00\rangle$, then

$$(H \otimes I)|\psi\rangle = H|0\rangle \otimes I|0\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$$

$$= \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

After the CNOT gate, we get $\frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\beta_{00}\rangle$.

Similarly, if $|\psi\rangle = |01\rangle$, we have,

$$(H \otimes I)|\psi\rangle = \cancel{to}\; H|0\rangle \otimes I|1\rangle$$

$$= \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |1\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

After the CNOT gate, $\frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\beta_{01}\rangle$.

We can similarly show that the circuit converts $|10\rangle$ and $|11\rangle$ to $|\beta_{10}\rangle$ and $|\beta_{11}\rangle$, resp.

<u>Note</u>: The Bell states are all <u>entangled</u> states.

Similarly, to convert back to one of the four computational basis states from a Bell state, we have the following circuit:

Ckt. (ii)

If the initial state, $|\phi\rangle$, is $|\beta_{10}\rangle$, i.e.,
$|\phi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$, then after the CNOT gate,
the state is $\frac{|00\rangle - |10\rangle}{\sqrt{2}} = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$.

The Hadamard gate causes the state of the 1st qubit to change to $|1\rangle$, i.e.,

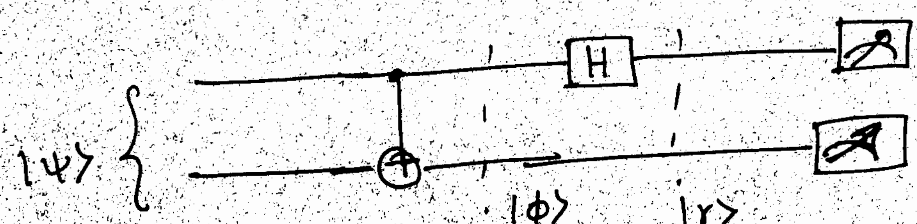
$$(H \otimes I)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |0\rangle\right) = H\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle = |1\rangle|0\rangle$$
$$= |10\rangle.$$

The 2nd ckt. is used to perform measurements in the Bell-Basis. For instance, suppose the state of a 2 qubit system in the Bell Basis is:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|\beta_{00}\rangle + \frac{1}{2}|\beta_{10}\rangle \quad \text{—— (1)}$$

$$= \frac{\sqrt{3}}{2}\left[\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right] + \frac{1}{2}\left[\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right].$$

Then to perform a measurement in the Bell Basis, we convert the state $|\psi\rangle$ from the Bell Basis to the computational basis by using ckt. (ii) and then measure in the computational $\{|0\rangle, |1\rangle\}$ basis as usual.



ckt (iii)

We have,
$$|\phi\rangle = \frac{\sqrt{3}}{2}\left[\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right] + \frac{1}{2}\left[\frac{|00\rangle - |10\rangle}{\sqrt{2}}\right].$$

$$= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)|00\rangle + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)|10\rangle$$

$$|r\rangle = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|0\rangle.$$

$$\Rightarrow |r\rangle = \left[\frac{\sqrt{3}+1+\sqrt{3}-1}{4}\right]|00\rangle + \left[\frac{\sqrt{3}+1-\sqrt{3}+1}{4}\right]|10\rangle$$

$$= \frac{2\sqrt{3}}{4}|00\rangle + \cancel{2-2}\ \frac{2}{4}|10\rangle$$

$$\therefore |r\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|10\rangle \quad \text{—— } (2)$$

Comparing (1) & (2), we see that $|r\rangle$ is a representation of $|\psi\rangle$ in the computational Basis.

Now if we measure ~~as~~ (in the computational basis) and get $|00\rangle$ we know that the corresponding state in the Bell Basis was $|\beta_{00}\rangle$. This measurement occurs with probability $\frac{3}{4}$.

Similarly, if we get $|10\rangle$, we know that it corresponds to the state $|\beta_{10}\rangle$ in the Bell Basis. This measurement outcome occurs with probability $\frac{1}{4}$.

Thus we are able to perform a measurement in the Bell Basis by using Circuit (iii).

Note: Had we written $|\psi\rangle$ in the computational basis directly, we would have:

$$|\psi\rangle = \frac{\sqrt{3}}{2}\left[\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right] + \frac{1}{2}\left[\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right] = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)|00\rangle + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)|11\rangle$$

On measuring in the computational basis, we get either $|00\rangle$ or $|11\rangle$ with probabilities $\left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2$ and $\left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right|^2$, respectively.

These donot correspond to the measurement results obtained from ckt. (iii) (In fact, the state $|11\rangle$ when transformed to the Bell Basis using ckt. (i) corresponds to the Bell basis state, $|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$).

Two important points to note:

① For $|\psi\rangle = \frac{\sqrt{3}}{2}|\beta_{00}\rangle + \frac{1}{2}|\beta_{10}\rangle$, measuring in the Bell basis corresponds to applying either $|\beta_{00}\rangle\langle\beta_{00}|$ or $|\beta_{10}\rangle\langle\beta_{10}|$ to $|\psi\rangle$. The probabilities with which we get states $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ are $\frac{3}{4}$ and $\frac{1}{4}$, respectively.

② Fig (iii) is a circuit where a corresponding measurement is performed in the computational basis, i.e., we apply $|00\rangle\langle 00|$ or $|10\rangle\langle 10|$ to $|r\rangle$ where $|r\rangle$ is the state obtained after transforming $|\psi\rangle$ from the Bell Basis to the computational basis. Note that measurement outcome $|00\rangle$ corresponds to the Bell state outcome $|\beta_{00}\rangle$. Similarly, measurement outcome $|10\rangle$ corresponds to the Bell state outcome $|\beta_{10}\rangle$. Therefore, the probabilities for obtaining $|00\rangle$ and $|10\rangle$ (in $|r\rangle$) must match the probabilities for obtaining $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$, resp. (in $|\psi\rangle$). This is indeed the case as can be observed from equations ① + ②, i.e.,

$$|\psi\rangle = \frac{\sqrt{3}}{2}|\beta_{00}\rangle + \frac{1}{2}|\beta_{10}\rangle$$

$$|r\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|10\rangle .$$

Some Applications:

- We will observe 2 communication protocols — Superdense coding and quantum teleportation. Both are inherently quantum — there are no classical protocols which behave in the same way. Both involve 2 parties ~~which~~ who wish to perform some communication task between them. Usually, one of the parties is named 'Alice' and the other 'Bob'.

A communication channel refers to a communication line (eg. a fiber optic cable) which can carry qubits between 2 remote locations.

A classical channel is one which can carry classical bits (but not qubits).

Both protocols require that Alice and Bob initially share an entangled pair of qubits in the Bell state $|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. This state is sometimes referred to as an EPR state. Such a state will have to be created ahead of time, when the qubits are in a lab together and can be made to interact in a way which will give rise to entanglement between them. After the state is created, Alice and Bob each take one of the two qubits away with them. If they are careful not to let them interact with the environment (i.e., keep them isolated), or any other quantum system, Alice and Bob's joint state will remain entangled. This entanglement becomes a resource which Alice and Bob can use to achieve protocols such as the following.

SUPERDENSE CODING:

Suppose Alice ~~and Bob~~ wishes to send Bob two classical bits of information. Superdense coding is a way of achieving this task over a quantum channel, requiring only that Alice send one qubit to Bob. Alice and Bob must initially share the Bell state: ~~$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[|00\rangle + \frac{1}{\sqrt{2}}|11\rangle]$~~.

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Alice keeps one of the qubits and Bob the other. Suppose Alice keeps qubit 1 and Bob qubit 2. Next, suppose Alice performs one of the operations on his qubit – I, X, Z, ZX – the combined state of the system will be one of the following states:

$$(I \otimes I)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\beta_{00}\rangle$$

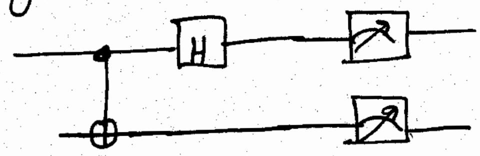
$$(X \otimes I)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|01\rangle = |\beta_{01}\rangle$$

$$(Z \otimes I)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle = |\beta_{10}\rangle$$

$$(Z \cdot X \otimes I)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle = |\beta_{11}\rangle.$$

Therefore, depending on what operation – I, X, Z or ZX – Alice performs on her qubit, the two qubits are together in one of the 4 Bell states. Alice uses this to transmit 2 classical bits of information. If she ~~wo~~ wishes to send bits 00 to Bob, she does nothing to her qubit. If she wishes to send 01, she applies the X gate to her qubit. If she wishes to send 10, she applies the Z gate. If she wishes to send 11, she applies the X gate followed by the Z gate (Z.X). She then sends her <u>single qubit</u>

to Bob. Bob next performs a measurement in the Bell Basis, i.e., he passes the 2 qubits through the following circuit:



~~He then performs a measure~~

The outcome of the measurement is $|00\rangle$, $|01\rangle$, $|10\rangle$ or $|11\rangle$ depending on whether the initial state was $|\beta_{00}\rangle$, $|\beta_{01}\rangle$, $|\beta_{10}\rangle$ or $|\beta_{11}\rangle$.

Note: Alice used only a single qubit to transmit 2 classical bits of information. However, this qubit had to be entangled with Bob's qubit $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$ before the communication ~~protocol~~ protocol was used.

QUANTUM TELEPORTATION:

This is a means of moving quantum states around, even in the absence of a quantum communications channel linking the sender of a quantum state to the recipient.

Suppose Alice and Bob met long ago but now live far apart. While together they generated an EPR pair $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$, each taking one qubit of the EPR pair when they separated.

Many years later, Bob is in hiding and Alice's mission is to deliver a qubit $|\psi\rangle$ to Bob. She does not know the state of the qubit and can only send <u>classical information</u> to Bob, i.e., bits (0 or 1) and not qubits ($\alpha|0\rangle + \beta|1\rangle$).

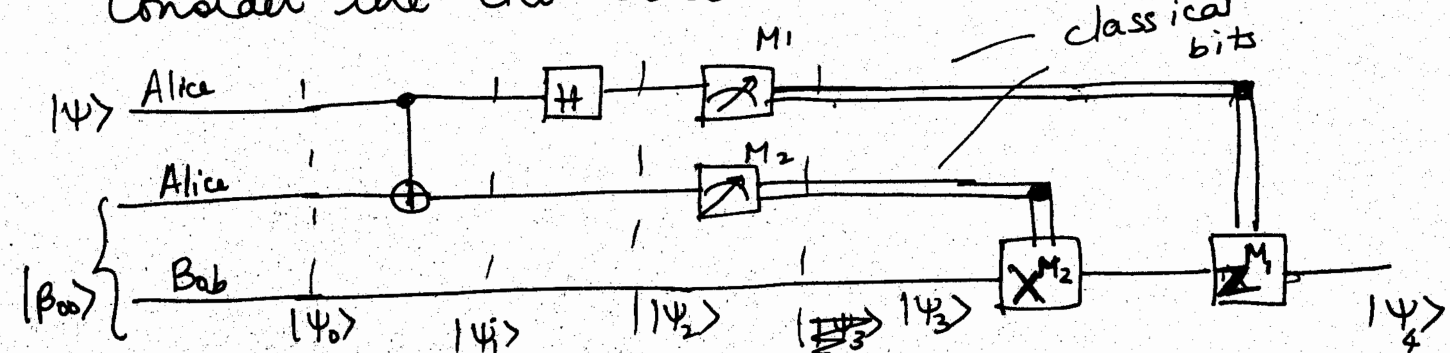
~~Alice~~ Intuitively, ~~that~~ things look pretty bad for Alice. She doesn't know the state $|\psi\rangle$ and ~~she can't create as~~ the laws of quantum mechanics prevent her from determining the state when she only has a single copy of ~~$|\psi\rangle$~~ $|\psi\rangle$ in her possession. What's worse is that even if she did know $|\psi\rangle$, describing it precisely takes an infinite amount of classical information since $|\psi\rangle$ takes values in a continuous space (~~e~~ If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, α and β can be <u>any</u> 2 complex nos. as long as $|\alpha|^2 + |\beta|^2 = 1$. Since Alice can only communicate 'bits' to Bob, she would ~~require an infinite~~ take forever in order to communicate the two complex amplitudes with infinite precision). Fortunately for Alice, quantum teleportation is a way of utilizing the entangled ~~EPR~~ EPR pair in order to send $|\psi\rangle$ to Bob exactly over a classical channel.

Teleportation is a protocol which allows Alice to communicate the state of a qubit <u>exactly</u>

to Bob, sending only 2 bits of classical information to him.

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be the state Alice needs to transmit. In addition to this qubit, Alice possesses one of the qubits of an EPR pair shared with Bob, $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Consider the ckt. below:



$\therefore$ The initial state is $|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle$

$$\Rightarrow |\psi_0\rangle = (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$\therefore |\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\Big]$$

Here qubits 1 & 2 belong to Alice and qubit 3 to Bob.

Alice next performs a CNOT gate operation between her 2 qubits (with qubit 1 as control & 2 as target):

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle[|10\rangle + |01\rangle]\Big]$$

Alice now performs a Hadamard gate on qubit 1:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[\alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(|00\rangle+|11\rangle\right) + \beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(|01\rangle+|10\rangle\right)\right]$$

$$= \frac{1}{2}\left[\alpha|000\rangle+\alpha|011\rangle+\alpha|100\rangle + \alpha|111\rangle + \beta|001\rangle+\beta|010\rangle-\beta|101\rangle-\beta|110\rangle\right]$$

$$= \frac{1}{2}\left[|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |01\rangle(\alpha|1\rangle+\beta|0\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle) + |11\rangle(\alpha|1\rangle-\beta|0\rangle)\right]$$

Now Alice performs a measurement on her two qubits. Observing the state $|\psi_2\rangle$, ~~if~~ following are the outcomes of the measurements:

$M_1\ M_2$

$00 \longrightarrow |\psi_3\rangle = \alpha|0\rangle+\beta|1\rangle$ (Here, $|\psi_3\rangle$ is the state of qubit 3 only)

$01 \longrightarrow |\psi_3\rangle = \alpha|1\rangle+\beta|0\rangle = X(\alpha|0\rangle+\beta|1\rangle)$

$10 \longrightarrow |\psi_3\rangle = \alpha|0\rangle-\beta|1\rangle = Z(\alpha|0\rangle+\beta|1\rangle)$

$11 \longrightarrow |\psi_3\rangle = \alpha|1\rangle-\beta|0\rangle = \cancel{ZX}\ XZ(\alpha|0\rangle+\beta|1\rangle)$

Therefore, depending on the measurement outcomes M_1 & M_2, the state of Bob's qubit $|\psi_3\rangle$ is either $|\psi\rangle$, $X|\psi\rangle$, $Z|\psi\rangle$ or $XZ|\psi\rangle$.

~~Alice now~~ Note that each of these outcomes occur with probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

~~Bob~~ Alice sends the 2 classical bits after the measurements to ~~Alice~~ Bob.

Bob then performs ~~the operation~~ one of the 4 gate operations — I, X, Z, ZX — on his qubit upon observing the 2 classical bits M_1 & M_2. This ~~res~~ can be generalized by ~~st~~ observing that Bob performs the gate operation $Z^{M_1}X^{M_2}\cancel{Z^{M_1}}$ on his qubit.

For instance, if $M_2 = 0$, $M_1 = 0$, ~~(i.e.~~ then

$\cancel{Z^{M_1}X^{M_2}Z^{M_1} = X^0 X^0 Z^0 = I}$. $Z^{M_1}X^{M_2} = Z^0X^0 = I$

If $M_1 = 1$, $M_2 = 0$, then $Z^{M_1}X^{M_2}\cancel{Z^{M_1}} = Z^1X^0\cancel{Z} = Z$.

This makes sense since if the results of the measurement is 10, the state $|\psi_3\rangle$ is

$\alpha|0\rangle - \beta|1\rangle$.

$\therefore \quad Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$.

Therefore $|\psi\rangle$ can be used recovered.

Similarly if $M_1 = 1$, $M_2 = 1$, then $|\psi_3\rangle = \alpha|1\rangle - \beta|0\rangle$.

$\cancel{|\psi_4\rangle = X^{M_2}Z^{M_1}|\psi_3\rangle}$. In this case, Bob performs ~~$|\psi_4\rangle =$~~ the operation $Z^1X^1Z^1 = ZX\cancel{Z}$ on his qubit, i.e., $ZX\cancel{Z}|\psi_3\rangle = ZX\cancel{Z}(\alpha|1\rangle - \beta|0\rangle)$

$$= Z(\alpha|0\rangle - \beta|1\rangle)$$

$$= \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$