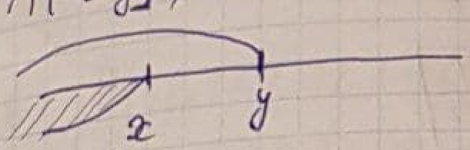
Домашня робота

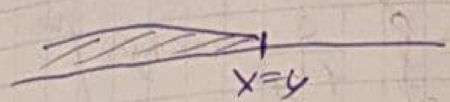
~~13~~ ~~14~~ ~~15~~ ~~16~~ ~~17~~ ~~18~~ ~~19~~ +~~20~~

$\boxed{11.15}$ ξ, $F_\xi(x)$

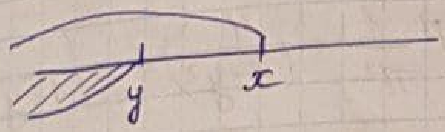
$$F_{(\xi,\xi)}(x,y) = P(\xi \le x, \xi \le y) = P(\xi \in (-\infty; x] \cap (-\infty; y]) =$$



$x < y$



$x = y$



$x > y$

$$= \begin{cases} F_\xi(x), & x \le y \\ F_\xi(y), & x \ge y \end{cases}$$

$\boxed{11.16}$ (ξ_1, ξ_2) $\quad p(x,y) = \dfrac{c}{1+x^2+x^2y^2+y^2}$

а) c – ?

Умова нормування

$$1 = \iint_{\mathbb{R}^2} p_{\xi_1,\xi_2}(x,y)\,dx\,dy$$

$$1 = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{c}{(1+x^2+x^2y^2+y^2)} dx\,dy = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{c}{(1+x^2)(1+y^2)} dx\,dy =$$

$$= \int\limits_{-\infty}^{\infty} \left(\frac{c}{(1+y^2)} \operatorname{arctg} x \Big|_{-\infty}^{+\infty}\right) dy = \int\limits_{-\infty}^{+\infty} c\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(\frac{1}{1+y^2}\right) dy$$

$$= \pi c \cdot \operatorname{arctg} y \Big|_{-\infty}^{\infty} = \pi^2 c = 1$$

$$\pi^2 c = 1 \Rightarrow \boxed{c = \frac{1}{\pi^2}}$$

б) $$p_{\xi_1}(x) = \int\limits_{-\infty}^{+\infty} \frac{c}{(x^2+1)(y^2+1)} dx = \int\limits_{-\infty}^{+\infty} \frac{c}{1+y^2} \operatorname{arctg} x \Big|_{-\infty}^{+\infty} =$$

$$= \frac{c}{1+y^2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi c}{1+y^2} = \pi \cdot \frac{1}{\pi^2}\left(\frac{1}{1+y^2}\right) = \frac{1}{\pi(1+y^2)}$$

$$p_{\xi_2}(x) = \int\limits_{-\infty}^{+\infty} \frac{c}{(x^2+1)(y^2+1)} dy = \frac{\pi c}{1+x^2} = \frac{1}{\pi(1+x^2)}$$

Чи є незалежними?

ξ_1, ξ_2 незал $\Leftrightarrow p_{(\xi_1,\xi_2)}(x,y) = p_{\xi_1}(x)\, p_{\xi_2}(y)$

$$\frac{1}{\pi(1+y^2)} \cdot \frac{1}{\pi(1+x^2)} = \frac{1}{\pi^2(1+x^2)(1+y^2)} = p_{\xi_1}(x)\, p_{\xi_2}(y)$$

$$p_{(\xi_1,\xi_2)}(x,y) = \frac{c}{(1+x^2)(1+y^2)} = \left\{c = \frac{1}{\pi^2}\right\} =$$

в)

$$= \frac{1}{\pi^2(x^2+y^2+x^2y^2+1)}$$

$$\Rightarrow \frac{1}{\pi^2(x^2+y^2+x^2y^2+1)} = \frac{1}{\pi^2(x^2+1)(y^2+1)}$$

$\Rightarrow \xi_1, \xi_2$ - независимы.

в) $P(|\xi_1| \le 1, |\xi_2| \le 1) = P(\xi_1 \in [-1,1], \xi_2 \in [-1,1]) =$

$= P((\xi_1, \xi_2) \in \Delta) = \left\{ \Delta = \{(x,y) \in \mathbb{R}^2 : \begin{array}{l} x \in [-1,1] \\ y \in [-1,1] \end{array} \} \right\}$

$$= \iint_\Delta p_{(\xi_1,\xi_2)}(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \frac{c\,dx\,dy}{(1+x^2)(1+y^2)} =$$

$$= \int_{-1}^{1} \left(\frac{c}{(1+y^2)} \operatorname{arctg} x \Big|_{-1}^{1} \right) dy = \int_{-1}^{1} \frac{c}{1+y^2} \left(\operatorname{arctg}(1) - \operatorname{arctg}(-1) \right) dy = \int_{-1}^{1} \frac{c}{1+y^2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) dy =$$

$$= \frac{c\pi}{2} \operatorname{arctg} y \Big|_{-1}^{1} = \frac{\pi c}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4} \cdot \frac{1}{\pi^2} = \frac{1}{4}$$

$\boxed{11.13}$ ξ, η – незал вел велич

зі щільністю $p(x) = \frac{e^{-|x|}}{2}$

Тоді: $p_{\xi+\eta}(x) = \int_{-\infty}^{\infty} p_\xi(x-y) p_\eta(y) dy =$

$= \int_{-\infty}^{\infty} \frac{e^{-|x-y|}}{2} \frac{e^{-|y|}}{2} dy = \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-y|-|y|} dy$

Тоді розглянемо 3 випадки: $x<0$, $x>0$, $x=0$

1) $x<0$

$\frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-y|-|y|} dy = \frac{1}{4}\Big(\int_{-\infty}^{x} e^{-x+y+y} dy + \int_{x}^{0} e^{x-y+y} dy +$

$+ \int_{0}^{+\infty} e^{x-y-y} dy\Big) = \frac{1}{4} \int_{-\infty}^{x} e^{-x+2y} dy + \frac{1}{4} \int_{x}^{0} e^{x} dy +$

$+ \frac{1}{4} \int_{0}^{+\infty} e^{x-2y} dy = \frac{1}{4} \frac{e^{-x}}{2} e^{2y} \Big|_{-\infty}^{x} + \frac{1}{4} e^{x} y \Big|_{x}^{0} +$

$+ \left(-\frac{1}{4} \frac{e^{x}}{2}\right) e^{-2y} \Big|_{0}^{+\infty} =$

$= \frac{e^{-x}}{8}\left(e^{2x} - e^{-\infty}\right) + \frac{e^{x}}{4}(0-x) - \frac{1}{8} e^{x} \big(e^{-\infty} - 1\big)$

$= \frac{e^{-x}}{8}\left(e^{2x} - 0\right) - \frac{x e^{x}}{4} - \frac{e^{x}}{8}(0-1) =$

$$= \frac{e^{x}}{8} - \frac{xe^{x}}{4} + \frac{e^{x}}{8} = \cancel{\frac{xe^x}{4}}\ \frac{1}{4}e^{x} - \frac{x}{4}e^{x} =$$

$$= \frac{1}{4}e^{x}(1-x)$$

2) $x>0$.

$$\frac{1}{4}\int_{-\infty}^{0} e^{-x+y+y}\,dy + \frac{1}{4}\int_{0}^{x} e^{-x}\,dy + \int_{x}^{+\infty} e^{-2y+x}\,dy =$$

$$= \frac{1}{4}\int_{-\infty}^{0} e^{-x+2y}\,dy + \frac{1}{4}\int_{0}^{x} e^{-x}\,dy + \frac{1}{4}\int_{x}^{+\infty} e^{-2y+x}\,dy =$$

$$= \frac{1}{4}\frac{e^{-x}}{2}e^{2y}\Big|_{-\infty}^{0} + \frac{1}{4}e^{-x}y\Big|_{0}^{x} + \frac{e^{x}}{4(-2)}e^{-2y}\Big|_{x}^{+\infty} =$$

$$= \frac{1}{8}e^{-x}(e^{0} - e^{-\infty}) + \frac{e^{-x}}{4}(x-0) - \frac{e^{x}}{8}(e^{-\infty} - e^{-2x})$$

$$= \cancel{\frac{1}{8}e}\ \frac{1}{8}e^{-x}(1-0) + \frac{e^{-x}x}{4} - \frac{e^{x}}{8}(0 - e^{-2x}) =$$

$$= \frac{1}{8}e^{-x} + \frac{e^{-x}x}{4} + \frac{e^{-x}}{8} = \frac{1}{4}e^{-x}(1+x)$$

3) $x=0$.

$$\frac{1}{4}\int_{-\infty}^{+\infty} e^{-|-y|-|y|}\,dy = \frac{1}{4}\int_{-\infty}^{0} e^{-(-y)+y}\,dy + \frac{1}{4}\int_{0}^{+\infty} e^{-y-y}\,dy =$$

$$= \frac{1}{4}\int_{-\infty}^{0} e^{2y}\,dy + \frac{1}{4}\int_{0}^{+\infty} e^{-2y}\,dy = \frac{1}{4\cdot 2}e^{2y}\Big|_{-\infty}^{0} + \frac{1}{-4\cdot 2}e^{-2y}\Big|_{0}^{+\infty} =$$

$$= \frac{1}{4\cdot 2}\left(e^{0} - e^{-\infty}\right) - \frac{1}{8}\left(e^{-\infty} - e^{0}\right) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$\Rightarrow p_{\xi+\eta}(x) = \frac{1}{4} e^{x}(1-x)\, e^{-x}(1+x)$$

$\boxed{11.20}$

ξ_1, ξ_2 – незал. вип. величини

$\xi_1 \sim U([0,1])$

$\xi_2 \sim U([0,1])$

$$p_{\xi_1}(x) = \frac{1}{1-0}\,\mathbb{1}(x \in [0,1])$$

$$p_{\xi_2}(y) = \frac{1}{1-0}\,\mathbb{1}(y \in [0,1])$$

$$P(\xi_1 < x,\ \xi_1 + 2\xi_2 < y) = P((\xi_1, \xi_2) \in \Delta) =$$

$$= \left|\ \Delta = \left\{ (u, v) \in \mathbb{R}^2 :\ \begin{matrix} u < x \\ u + 2v < y \end{matrix} \right\} \right|$$

$$= \iint_{\Delta} p_{(\xi_1,\xi_2)}(u, v)\, du\, dv = \left\{ \begin{matrix} \text{оскільки за} \\ \text{умов величини} \\ \text{незалежні, то} \\ \text{можемо отримати} \end{matrix} \right\}$$

$$= \iint_{\Delta} p_{\xi_1}(u)\, p_{\xi_2}(v)\, du\, dv =$$

$$= \iint \mathbb{1}(u < x)\,\mathbb{1}(u+2v)\,du\,dv \;\mathbb{1}(u,v \in [0;1]) =$$

$$= \int_0^{\min(x,y)} \int_0^{\frac{y-u}{2}} du\,dv = \int_0^{\min(x,y)} v \Big|_0^{\frac{y-u}{2}} du =$$

$$= \int_0^{\min(x,y)} \left(\frac{y-u}{2}\right) du = \int_0^{\min(x,y)} \frac{1}{2}(y-u)\,du =$$

$$\frac{y \cdot u}{2} - \frac{u^2}{4} \Big|_0^{\min(x,y)} = \frac{y}{2}(\min(x,y)) - \frac{1}{4}(\min(x,y))^2$$

$\boxed{11.19}$ ξ_1, ξ_2 – незав. випадк. величи

$\xi_i \sim N(0,1)$

$$p_{\xi_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

$$F_{\xi_1^2+\xi_2^2}(z) = P(\xi_1^2 + \xi_2^2 \le z) = \mathbb{1}(z \ge 0)\,P((\xi_1, \xi_2) \in \Delta) =$$

$$\Delta = \{(x,y) \in \mathbb{R}^2 : \quad x^2 + y^2 \le z\}$$

$$= \mathbb{1}(z \ge 0) \iint_\Delta p_{(\xi_1,\xi_2)}(x,y)\,dx\,dy = \left\{ \begin{matrix} \text{ост велич} \\ \text{незалежні} \end{matrix} \right\}$$

$$= \mathbb{1}(z \ge 0) \iint_\Delta p_{\xi_1}(x)\, p_{\xi_2}(y)\,dx\,dy =$$

$$= \mathbb{1}(z \ge 0) \iint_{x^2+y^2 \le z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dx\,dy =$$

$$= \mathbb{1}(z \geq 0) \iint\limits_{x^2+y^2 \leq z} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx\,dy =$$

Перейдемо до полярних коорд.

$$\begin{cases} x = r\sin\varphi & x^2+y^2 = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2 \\ y = r\cos\varphi & dx\,dy = r\,dr\,d\varphi, \quad r^2 \leq z \end{cases}$$

$r \leq \pm\sqrt{z}$, оскільки $\mathbb{1}(z \geq 0)$, то підходить $\sqrt{+z}$

$$= \mathbb{1}(z \geq 0) \int_0^{\sqrt{z}}\int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r\,dr\,d\varphi =$$

$$= \mathbb{1}(z \geq 0) \int_0^{\sqrt{z}}\int_0^{2\pi} -\frac{1}{2\pi} \cdot d(e^{-\frac{r^2}{2}})\,d\varphi =$$

$$= \mathbb{1}(z \geq 0) \int_0^{\sqrt{z}} \left(-\frac{1}{2\pi} d(e^{-\frac{r^2}{2}})\right) \cdot \varphi \Big|_0^{2\pi} = \int_0^{\sqrt{z}} -\left(\frac{1}{2\pi}\right) \cdot 2\pi\, d(e^{-\frac{r^2}{2}}) =$$

$$= -e^{-\frac{r^2}{2}} \Big|_0^{\sqrt{z}} = -(e^{-\frac{z}{2}} - 1) = \left(1 - e^{-\frac{z}{2}}\right) \mathbb{1}(z \geq 0)$$

$$p_{\xi_1^2+\xi_2^2}(z) = \left(1 - e^{-\frac{z}{2}}\right)' = \frac{1}{2} e^{-\frac{z}{2}}\ \mathbb{1}(z \geq 0)$$

$\boxed{11.14}$ $F_{\xi+\eta}(x) = P(\xi+\eta \le x) = \sum_{k=-\infty}^{+\infty} P(\xi+\eta < x \mid \eta = k) \times$
$\times P(\eta = k)$

Оскільки ξ, η – незал величини,

$$F_{\xi+\eta}(x) = \sum_{k=-\infty}^{+\infty} P(\eta = k) P(\xi + k < x)$$

$0 \le \xi \le -k + x$, $k \le |x|$, тому:

$$F_{\xi+\eta}(x) = \sum_{k<|x|} \min(1, x-k) p_k.$$

$\boxed{11.18}$

а) $F_{\xi_1}(x) = P\{\min(\xi_1, .. \xi_n) \le x\}$

$F_{\xi_1}(x) = 1 - P\{\min(\xi_1 .. \xi_n) \le x\}$ – протилежна подія

$F_{\xi_1}(x) = 1 - P(\xi_1 > x, \dots, \xi_n > x) =$ оскільки вели незалежн $=$

$= 1 - P(\xi_1 < x) \dots P(\xi_n > x)$

Перейдемо до протилежної події:

$$F_{\xi_1}(x) = 1 - (1 - P(\xi_1 \le x)) \dots (1 - P(\xi_n \le x))$$

$$F_{\xi_1}(x) = 1 - (1 - P(\xi_1 \le x))^n = 1 - (1 - F(x))^n$$

в)

$$F_{\xi_n}(x) = P(\xi_n \le x) = P(\max(\xi_1 \dots \xi_n) \le x)$$

$$F_{\xi_n}(x) = P(\xi_1 \le x, \dots, \xi_n \le x) =$$ оск величини незалежні

$$= P(\xi_1 \le x) \dots P(\xi_n \le x)$$

$$F_{\xi_n}(x) = (P(\xi_1 \le x))^n = (F(x))^n$$

c) $F_{\xi_m}(x) = P(\xi_n \le x)$

$$\xi_1 \le \xi_2 \le \dots \le \xi_n \le \dots \le \xi_n.$$

Якщо $\xi_m \le x$, то $\xi_1 \dots \xi_m \le x$

$$F_{\xi_n}(x) = \sum_{i=m}^{n} C_n^i (P(\xi \le x))^i (P(\xi \ge x))^{n-i} =$$

$$= \sum_{i=m}^{n} C_n^i (F(x))^i (1 - P(\xi \le x))^{n-i} =$$

$$= \sum_{i=m}^{n} C_n^i (F(x))^i (1 - F(x))^{n-i}$$

$\boxed{11.17}$ ξ, η – незал. вип. величини

$\xi \sim Exp(\lambda) \qquad \lambda > 0$

$\eta \sim Exp(\lambda) \qquad \lambda > 0$

Довести, що $\frac{\xi}{\xi+\eta}$ має рівн. розп. на $[0,1]$

$$F_{\frac{\xi}{\xi+\eta}}(x) = P\left\{\frac{\xi}{\xi+\eta} \le x\right) = M \mathbb{1}\left(\frac{\xi}{\xi+\eta} \le x\right) =$$

$$\left\{\begin{array}{l} \xi, \eta \text{ – незал.} \Rightarrow \; p_{(\xi;\eta)}(x,y) = p_\xi(x)\, p_\eta(y) = \\ = \mathbb{1}(x \ge 0)\, \mathbb{1}(y \ge 0)\, \lambda e^{-\lambda x}\, \lambda e^{-\lambda y} \end{array}\right\}$$

$$= F_{\frac{\xi}{\xi+\eta}}(x) = M \mathbb{1}\left(\frac{\xi}{\xi+\eta} \le x\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbb{1}\left(\frac{z}{z+y} \le x\right) p_{(\xi,\eta)}(z,y)\, dz\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbb{1}\left(\frac{z}{z+y} \le x\right) \mathbb{1}(z \ge 0\; y \ge 0)\, \lambda^2 e^{-\lambda z} e^{-\lambda y}\, dz\, dy =$$

$$= \int_{0}^{+\infty}\int_{0}^{+\infty} \mathbb{1}\left(\frac{z}{z+y} \le x\right) \lambda^2 e^{-\lambda z} e^{-\lambda y}\, dz\, dy =$$

$$= \int_{0}^{+\infty}\int_{0}^{+\infty} \mathbb{1}\left\{z(1-x) \le yx\right\} \lambda^2 e^{-\lambda z} e^{-\lambda y}\, dz\, dy =$$

$$= \int_{0}^{+\infty}\int_{0}^{+\infty} \mathbb{1}\left(z \le \frac{xy}{1-x}\right) \lambda^2 e^{-\lambda z} e^{-\lambda y}\, dz\, dy =$$

$$= \begin{cases} 0, & x<0 \\ \int_0^{+\infty}\int_0^{\frac{xy}{1-x}} \lambda^2 e^{-\lambda z} e^{-\lambda y}\,dz\,dy, & 0<x<1 \\ 1, & x>1 \end{cases}$$

$$\lambda^2 \int_0^{+\infty}\int_0^{\frac{xy}{1-x}} e^{-\lambda z} e^{-\lambda y}\,dz\,dy = \lambda^2 \int_0^{+\infty} \left(\frac{1}{-\lambda}\right) e^{-\lambda z}\Big|_0^{\frac{xy}{1-x}} e^{-\lambda y}\,dy$$

$$= -\lambda \int_0^{+\infty} \left(e^{-\frac{\lambda x}{1-x} y} - 1\right) e^{-\lambda y}\,dy =$$

$$= -\lambda \int_0^{+\infty} \left(e^{-\frac{\lambda x}{1-x} y - \lambda y} - e^{-\lambda y}\right) dy = -\lambda \int_0^{+\infty} e^{-\lambda\left(\frac{x+1-x}{1-x}\right)y}\,dy$$

$$= -\lambda \int_0^{+\infty} e^{-\lambda \frac{1}{1-x} y}\,dy = -\lambda\left(\frac{1}{-\lambda}\right)(1-x)\, e^{-\frac{\lambda}{1-x} y}\Big|_0^{+\infty} =$$

$$= (1-x)\left(e^{-\infty} - e^0\right) = -(1-x)$$

$$\lambda \int_0^{+\infty} e^{-\lambda y}\,dy = \lambda \frac{1}{(-\lambda)} e^{-\lambda y}\Big|_0^{+\infty} = -1\left(e^{-\infty} - e^0\right) = 1$$

$$= -1 + x + 1 = x$$

$$\Rightarrow$$

$$F_{\frac{\xi}{\xi+\eta}}(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1] \\ 1, & x > 1 \end{cases}$$

$$F_{\frac{\xi}{\xi+\eta}}(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x > 1 \end{cases}$$

$\Rightarrow \frac{\xi}{\xi+\eta}$ має рівном розподіл на $[0,1]$