INSTITUT FÜR
INFORMATIK
Lehrstuhl für Komplexitätstheorie und
Kryptologie

Universitätsstr. 1 D–40225 Düsseldorf

HEINRICH HEINE
UNIVERSITÄT DÜSSELDORF

# Game-theoretic Analysis of the Strategyproofness of Cake-cutting Protocols

**Alina Elterman**

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Gutachter: Prof. Dr. Jörg Rothe
Prof. Dr. Peter Kern

## Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

Alina Elterman

## Abstract

In cake-cutting a protocol instructs the participants how to divide a resource between them in a satisfactory manner. A part of those instructions, namely the strategies, are optional and their advantage can be examined. If this is not the case, the players have no intention to use that protocol, which can be overruled in this case. Otherwise the protocol is strategyproof.
Game Theory is designed to determine better strategies. By using a game-theoretic illustration of the cake-cutting problem it is possible to compare all strategies.
The strategy recommended by the protocol appears to be the best one in the well-known protocols with one exception.

# Contents

# 1 Introduction

It is Christmas party in the cakes4people agency. Everybody is waiting expectantly on the big promised cake at the end of the party. In the past years those cakes have been spectacular. A lot of different cake layers, different fruits on top and even chocolate sprinkles over parts of the cake have been so delicious. So it's no wonder that everyone wants to get as much as possible of this culinary treat, and especially of their individual favourite part. Nevertheless, the people do not want to penalise their colleagues.
A new employee is also celebrating with the group. Rumors have been told a lot about him, but no one has managed to assess him or his preferences properly. Nevertheless, he also takes part in the big cake division. He even wants to change the allocation procedure. He promises that everyone can keep their wishes private, each of them just needs to make a couple of simple decisions and will get their best possible share.
But the people become suspicious. Different questions occur in their minds: "What if he has a strategy he is not telling us about, which promises him a better piece? What if he is lying about his preferences? Why should we trust him?" The chief sees the mistrust and knows how to reassure the people. He is a game theory enthusiast and promises to show them that the proposed procedure is strategyproof. Hereby, only by taking actions truthful a participant can always get his best possible piece.
Strategyproofness of an allocation procedure ...

# 2 Preliminaries

## 2.1 Basics of Cake-cutting

It is necessary to define the components and challenges of cake-cutting. But first, what exactly is cake-cutting about? It involves a set of $n \in \mathbb{N}$ players $P_n = \{p_1, \ldots, p_n\}$. It is assumed that each of them wants to get as much as possible of the divided resource. The goal is to find an allocation of a single, divisible and heterogeneous good between the $n$ players. Such an allocation has to be of a special kind, so that the involved players are

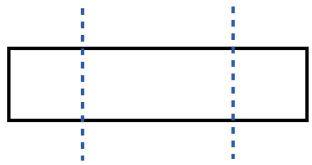

Figure 1: Cake
Example for a visualisation of a cake with two cuts

pleased with the outcome. For the visualisation it is common to use a rectangular cake. The division is performed by parallel cuts. The cake $X$ is represented by the unit interval $X = [0, 1] \subseteq \mathbb{R}$. Each subinterval $X' \subseteq X$ or a sequence of disjoint subintervals

$$\bigcup_{m \in \mathbb{N}} X'_m$$

with $X'_m \subseteq X$ is called a *portion (or piece)*. The portion of the cake which is received by player $p_i$ is denoted $X_i$. The state is called an *allocation*, when all portions of the cake are owned by players. Each piece has a public size, which can be computed as the sum of all border differences, and the private value of each player, which is constituted by the lower defined valuation function.

Every player $p_i \in P_n$ has a *valuation function (valuation)* $v_i : \{X' | X' \subseteq X\} \to [0, 1] \subseteq \mathbb{R}$ with the following properties:

1. Non-negativity: $v_i(C) \geq 0$ for all $C \subseteq [0, 1]$.
2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$.
3. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq [0, 1]$.[1]
4. Divisibility: For all $C \subseteq [0, 1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, there exists a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.
5. $v_i$ is continuous: If $0 < x < y \leq 1$ with $v_i([0, x]) = \alpha$ and $v_i([0, y]) = \beta$, then for every $\gamma \in [\alpha, \beta]$ there exists a $z \in [x, y]$ so that $v_i([0, z]) = \gamma$.
6. Non-atomic: $v_i([x, x]) = 0$ for all $x \in [0, 1]$.

[1] Monotonicity: If $C' \subseteq C$ then $v_i(C') \leq v_i(C)$. Monotonicity follows from additivity, because for the assumption $C' \subseteq C$ and $A := C \backslash C'$: $v_i(C) = v_i(A \cup C') = v_i(A) + v_i(C') = \underbrace{v_i(C \backslash C')}_{\geq 0} + v_i(C') \geq v_i(C')$.

## 2.2 Concepts in Game Theory

A brief introduction into the basic concepts of game theory is given and directly applied to the cake-cutting problem. For subsequent applications a probability model is introduced.In particular the possible representations of games are in the priority of this chapter. For further reading see [McCain, 2010], [Holler et al., 2008] and [Meir, 2009].

**Definition 1.** ***(Game)***
A *non-cooperative game* $\Gamma = (P_n, S, u)$ consists of the set of players $P_n$, the set of strategies $S$ and the set of utility functions of all players $u$.

- Each player in the set $P_n = \{p_1, \cdots, p_n\}$ behaves selfish and rational.
- Each player $p_i$ has his own set of strategies $S_i = \{S_{i,1}, S_{i,2}, \ldots\}$. Hereby a *pure strategy* $S_{i,j}$ with $j \in \mathbb{N}$ is a single action.
- Utility is measuring player's happiness for $p_i$ is $u_i \in \mathbb{R}$.

Each game has end-states, which are called outcomes. In cake-cutting an outcome is an allocation. The utility function in cake-cutting is the valuation function. The utility of an allocation for a player is the value of the piece this player obtain in it. From a bigger value follows more happiness for a player.

**Definition 2.** ***(Strategies)***
Assume two strategies $S_1$ and $S_2$ for a player $p_1$ and the value $v_1(X_{1,S_1,i})$ and $v_1(X_{1,S_2,i})$ for $i \in \mathbb{N}$ number of possible different allocations.
The strategy $S_1$ *dominates* the strategy $S_2$ if $v_1(X_{1,S_1,i}) \geq v_1(X_{1,S_2,i})$ for all $i$.

- The strategy $S_1$ *strictly dominates* the strategy $S_2$ if $v_1(X_{1,S_1,i}) > v_1(X_{1,S_2,i})$ for all $i$.
- The strategy $S_1$ *weakly dominates* the strategy $S_2$ if $v_1(X_{1,S_1,i}) > v_1(X_{1,S_2,i})$ for at least one $i$ and $v_1(X_{1,S_1,i}) \geq v_1(X_{1,S_2,i})$ for all other $i$.

The strategy $S_1$ is *dominated* by the strategy $S_2$ if $v_1(X_{1,S_1,i}) \leq v_1(X_{1,S_2,i})$ for all $i$.

- The strategy $S_1$ is *strictly dominated* by the strategy $S_2$ if $v_1(X_{1,S_1,i}) < v_1(X_{1,S_2,i})$ for all $i$.
- The strategy $S_1$ is *weakly dominated* by the strategy $S_2$ if $v_1(X_{1,S_1,i}) < v_1(X_{1,S_2,i})$ for at least one $i$ and $v_1(X_{1,S_1,i}) \leq v_1(X_{1,S_2,i})$ for all other $i$.

The strategy $S_1$ can neither dominates nor be dominated in regard to another strategy $S_2$, so $v_1(X_{1,S_1,i}) < v_1(X_{1,S_2,i})$ for at least one $i$ and $v_1(X_{1,S_1,i}) > v_1(X_{1,S_2,i})$ for at least an other $i$. Also $v_1(X_{1,S_1,i}) = v_1(X_{1,S_2,i})$ for all $i$ is possible, but in this case the strategies can be seen as equal and the player is indifferent between them.

Another important aspect to consider in studying of games is the amount of information available to the players about past moves of the game or the intentions of his fellow players.

**Definition 3.** ***(Perfect / Imperfect Information)***
In a game of *perfect information* every player always knows every move that other players have made before. In a game of *imperfect information* some players sometimes do not know the strategy choices other players have made.

**Definition 4.** ***(Incomplete / Complete Information)***
The games with *incomplete / complete information*, by contrast are about the information of the circumstances under which the game is played.

**Definition 5.** ***(Mutual / Common Knowledge)***
An event is *mutual knowledge* if all players know it. *Common knowledge* also requires that all players know the event, all players know that all players know it, and so on ad infinitum.

By applying game theory in cutting a cake it is especially important that the valuation functions are private and that no player knows each others type or his own position in the game, but the players know the moves of other players before them. So cake-cutting is a game with perfect but incomplete imformations. In game theory for finding solution in such concepts the unavailable informations are imitated through randomness and those games are played against nature.
For the further analysis some assumption have to be made and a probability to be defined. In cake-cutting the number of different valuation functions over the cake is infinite, so it makes no sense to define a probability for one special valuation. Hence, the possible events in the following chapters depend on special partitions of the cake. The preferences of each player $p_j$ over a partition $X^1, \ldots, X^n$ cutted by a player $p_i$ with $1 \leq i, j \leq n, i \neq j$ can be defined as follows:

**Definition 6.** ***(Event)***
An *event* is the selection of a single piece $X^j$ by a player $p_j$ from $X^1, \ldots, X^n$ with $1 \leq j \leq n$, $i \neq j$.

**Definition 7.** ***(Probability)***
For $n$ given disjoint pieces of a cake $X = X^1, \ldots, X^n$ the *probability* for a player $p_j$ to choose a piece $X^j$ is given by

$$P(v_j(X^k) \leq v_j(X^n)) = {}^1\!/_n$$

for $1 \leq j \leq n$, $j \neq i$ and $1 \leq k < n$

**Definition 8.** ***(Expected Value)***
The *expected value* for the piece of player $p_i$ is defined by

$$\mathbb{E}(v_i(X_i)) = {}^1\!/_n \cdot \sum_{k=1}^{n} v_i(X^k).$$

A game can be rather strategic, where all player move simoultaneous or extensive, where the players move in a fixed or variable order. A game can be represented in normal or in extended form. The normal form is advantageous for games where players move simultaneous, but is not suitable for cake-cutting since it does not respect the order of moves, which is an important part of the following allocation procedures. A better fitting model to represent a cake-cutting game is the extended form as a game tree.

**Definition 9.** *(Normal Form Game)*
A *normal form game* is a representation of a game in a tabular. In the case for two players the first player chases the row while the second choses the column. Each cell contains a tuple with the values of the obtained pieces, one per player.

> The player $p_1$ has two possible strategies $Strategy_a$ and $Strategy_b$.
>
> The player $p_2$ has two possible strategies $Strategy_I$ and $Strategy_II$.
>
> If player $p_1$ choose the strategy $Strategy_a$ and player $p_2$ the strategy $Strategy_I$: Player $p_1$ obtains a piece with the value $v_1(X_{1,I,a})$ and player $p_2$ obtains a piece with the value $v_2(X_{2,I,a})$

| | $Strategy_a$ | $Strategy_b$ |
|---|---|---|
| $Strategy_I$ | $(v_1(X_{1,I,a}), v_2(X_{2,I,a}))$ | $(v_1(X_{1,I,b}), v_2(X_{2,I,b}))$ |
| $Strategy_{II}$ | $(v_1(X_{1,II,a}), v_2(X_{2,II,a}))$ | $(v_1(X_{1,II,b}), v_2(X_{2,II,b}))$ |

Table 1: Game in normal form

**Definition 10.** *(Extended Form Game)*
An *extended form game* is a tree representation of a game. The tree starts on the left and has the leafs on the right side. The inner nodes are the decision points of the players. The number upon the node is the index of the player whose turn it is.

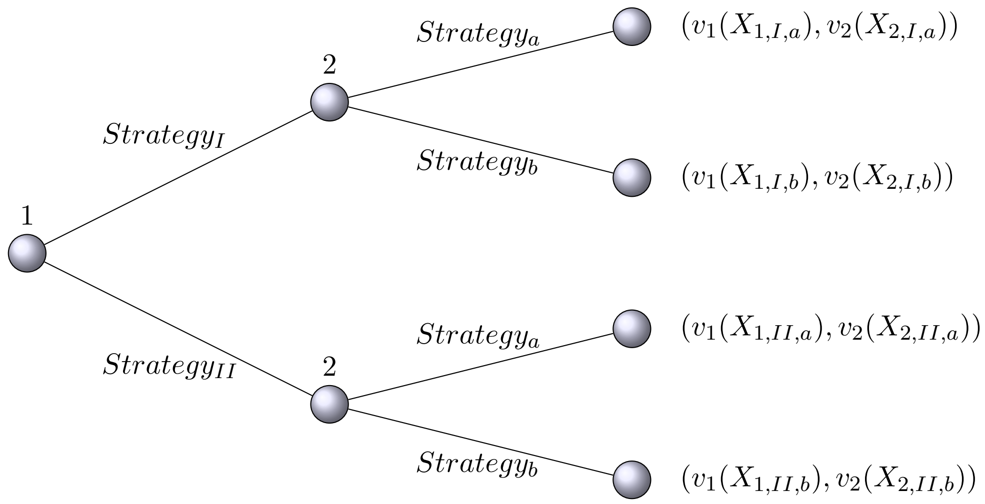


Figure 2: Game in extended form

So player $p_1$ choose his strategy first, and then it is player $p_2$'s move. Hereby, only player $p_2$ knows how player $p_1$ moved. The leafs are the end-states of the game with the values the players obtain on the right from them. Around the vertices is the strategy of the acting player. If a path is red it is not good for the first player. If after the obtained values is a lightning the path is not good for the second player.

After some basics it would be interesting to see how game-theory is applicable to cake-cutting. Example 1 illustrates the problem in a game-theoretic manner.

**Example 1.**

*John Cocke and Tadao Kasami want to divide a chocolate-strawberry-cake. The cake is half chocolate from the left and the right part is strawberry. John Cocke got the first move and is thinking about making three different cuts. After the cuts the two pieces would have the following values:*

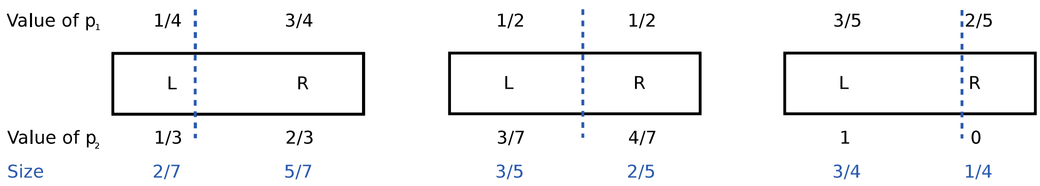


Figure 3: C&K cake game

*Before doing so, he analyses his situation via the normal form:*

| | Leftcut | Middlecut | Rightcut |
|---|---|---|---|
| L | $(3/4, ?)$ | $(1/2, ?)$ | $(2/5, ?)$ |
| R | $(1/4, ?)$ | $(1/2, ?)$ | $(3/5, ?)$ |

Table 2: C&K cake game in normal form

*Since the valuation is a private function, he does not know the preferences of his colleague and has to assume that Tadao is indifferent between the two pieces. Tadao is waiting for John's move and will choose his best strategy in the extended game form:*

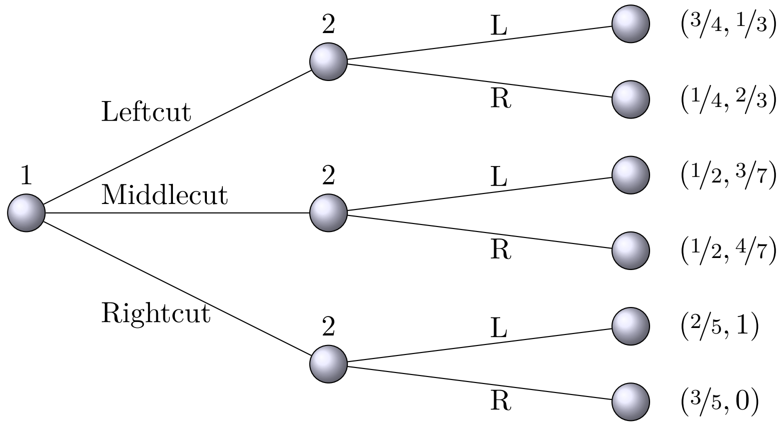


Figure 4: C&K cake game in extended form

*For John the graph shows that if he stays secure he would obtain* $v_1(X_1) = 1/2$*. Otherwise he would get* $1/4$ *or* $2/5$*. So to make the middlecut is his best possible move.*

## 2.3 Different Types of Fairness

As indicated by the name, fairness plays an important role in fair division. But how is fairness defined? It can be seen as a valuation criterion of an allocation, which can be normalized and gives a possibility to compare different allocations. Usually the fairness criteria are distinguished between the following:

**Definition 11.** ***(Proportional or Simple Fair)***
An allocation is *proportional (simple fair)* if $v_i(X_i) \geq 1/n$ for each player $p_i \in P_n$.

**Definition 12.** ***(Envy-Freeness)***
An allocation is *envy-free* if $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_n$.

Theorem 1 shows the correlation between the two types of fairness.

**Theorem 1.**

1. *Every envy-free allocation is proportional.*
2. *An allocation between two players is envy-free if and only if it is proportional.*

*Proof.*

1. Proof by contradiction:
   Assume $A$ is an envy-free allocation, but not proportional. From envy-freeness follows $v_i(X_i) \geq v_i(X_j)$ for each pair of players $p_i, p_j \in P_n$ and so each player has at least an as much valuable piece of cake as each other player in his own valuation. Hereby each player owns in his own valuation at least as much as each of the $(n-1)$ other players. The smallest value for which it is possible that $(n-1)$ players and the considered players own the same value in the valuation of the considered player is $1/((n-1)+1)$. Each player owns in his valuation at least $1/n$. ↯ The allocation $A$ is proportional.
   Therefore, all envy-free allocations are proportional.
2. "$\Leftarrow$" For two players $p_1$ and $p_2$ an allocation is proportional if each player has $p_i(X_i) \geq 1/2$ for $i \in \{1, 2\}$. The whole cake has the value $v_i(X) = 1$ for $i \in \{1, 2\}$. And so for each player $v_i(X - X_i) \leq 1 - 1/2 = 1/2$ for $i \in \{1, 2\}$. The value of the piece not obtained by a player is smaller or equal to his obtained piece. So the player $p_1$ does not envy the other player $p_2$ and vice versa.
   "$\Rightarrow$" Follows from part 1.

□

A different criterion to valuate the quality of an allocation is efficiency. Further correlations between the fairness criteria and efficiency can be found in [Caragiannis et al., 2009].

**Definition 13.** ***(Efficiency)***
An allocation

$$A = \{X_1, \dots, X_n\}$$

is *efficient (Pareto optimal)* if there is no other allocation

$$A' = \{X'_1, \dots, X'_n\}$$

such that

$$v_i(X_i) \leq v_i(X'_i)$$

for all players $p_i \in P_n$ and for at least one player the inequality is strict.

**Theorem 2.**

1. *Envy-freeness and proportionality do not imply efficiency.*
2. *Efficiency does not imply proportionality and so envy-freeness.*

*Proof.*

1. Imagine the following allocation with three players. Each player's portion can consist up to three pieces. The value of the whole portion is (because of the additivity of the valuation) the sum of the pieces :

| | $X_1 = X'_1 \cup X''_1$ | $X_2 = X'_2 \cup X''_2$ |
|---|---|---|
| $p_1$ | $1/2 = 1/3 + 1/6$ | $1/2 = 1/12 + 11/12$ |
| $p_2$ | $1/3 = 1/6 + 1/6$ | $2/3 = 2/3 + 0$ |

Table 3: Example for envy-freeness does not imply efficiency

This allocation is obviously envy-free, since $v_1(X_1) = v_1(X_2)$ and $v_2(X_2) > v_2(X_1)$. It is not efficient, because if the player $p_1$ would get $p_2$'s portion $X''_2$, $p_1$ would get a more valuable piece of the cake and $p_2$ would not get a less valuable piece. Since in Theorem 1 was shown that envy-freeness implies proportionality, this example also demonstates that proportionality does not imply efficiency.

2. Allocating the whole cake to one player is efficient, but definitely not proportional and therefore not envy-free.

□

In [Brams and Taylor, 1996] the authors show a general argument that no finite bounded protocol can exist for such an allocation that is both proportional and efficient at the same time.

### 2.4 Different Types of Protocols

It is very important to understand the types, structure and design of protocols, which are analysed in this work.

Informal: (Algorithm)
An *algorithm* is an effective method for solving a problem, which is composed of a finite sequence of instructions.

**Definition 14.** ***(Cake-Cutting-Protocol)***
A *cake-cutting-protocol* (protocol for short) is an algorithm with a fixed number of players and the following properties:

- A protocol consists of rules and strategies.
  *Rules* are requirements, which *have to* be followed by the players. Furthermore, it is possible to ensure that the players obey the rules.
  *Strategies* are recommendations, which *can* be followed for getting the guaranteed fair share. It is impossible to verify whether a player follows the strategy of the protocol.

- Each player should be able to cut the cake at a specific moment independent of other players.

- The protocol has no information about the valuation of the players, except of those it got from the steps before.

**Comment:** Only such protocols are interesting where the actions of one player does not harm the other players.

**Definition 15.** ***(Proportional/ Envy-Free Protocol)***
A cake-cutting *protocol* is called *proportional* or *envy-free* if independent of the players' valuations, each allocation is proportional or envy-free provided that all players follow the rules and strategies given by the protocol.

The development of such protocols is one of the main goals of cake-cutting [Robertson and Webb, 1998].

**Definition 16.** ***(Finite (Discrete) / Continuous (Moving-Knife) Protocol)***
A *finite (discrete) protocol* gives a solution after a finite number of queries (valuations, marks, ...). In a *continuous (moving-knife) protocol* a player has to make up to infinitely many queries.

**Definition 17.** ***(Finite Bounded / Finite Unbounded Protocol))***
A *finite bounded protocol* has an upper bound of steps for all possible valuations. The number of those steps is correlated with the number of players only in some cases. A *finite unbounded protocol* has no approximated number of steps.

The most desirable protocols are the finite bounded because of the ease of their implementation.

In the last sixty years the number of proportional finite bounded protocols has grown for an arbitrary number of players. But still no envy-free finite bounded protocol for an arbitrary $n$ is known [Chen et al., 2010]. Only for three or less players a cake can be divided in a fixed number of steps, so that it is envy-free. For this reason, only proportional protocols are considered in the further work.

# 3 Strategyproofness

In the scope of this work it is assumed that players are selfish and try to increase the value of their portion. In order to do so, they may misrepresent their valuation on the cake. The goal is to prevent this. It is assumed that a set of players and the divisible ressource are given. Each player knows his preferences, but has not valued the ressource yet. The protocol shows the cake or parts of it to a certain player and requires from him to follow the rules, so to make a mark or a cut, or to choose at least one piece of the cake. The protocol also makes a recommendation to the player by using a strategy. The player can deside whether to follow it or to made up an other strategy.

**Definition 18.** ***(Non-Truthful (Cheating) / Honest Player)***
Every strategy is called *non-truthful* except of the strategy recommended by the protocol. A player who follows a non-truthful strategy will be called a *non-truthful (cheating) player*. Otherwise the player is called *honest*.

**Definition 19.** ***(True Value Function)***
A *true value function* provides the value of the piece a player would receive by following the recommended strategy. This value is at least proportional in a proportional protocol.

A strategy $S_1$ is better for player $p_i$ than another strategy $S_2$ if the value of the obtained piece by following $S_1$ is bigger than by following $S_2$.

**Definition 20.** ***(Risk Aversion [Brams et al., 2006])***
A player is *risk averse* if he will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a piece of a guaranteed size.

**Definition 21.**
***(Strategyproofness of a Proportional Protocol [Lindner and Rothe, 2009])***
A proportional cake-cutting protocol is said to be *strategyproof for risk averse players* (SPP for short) if a cheating player is no longer guaranteed a proportional share, whereas all other players (provided they play truthful) are still guaranteed to receive their proportional share.

**Definition 22.** ***(Strategyproofness in the sense of [Brams et al., 2008])***
A protocol is *strategyproof* if no player has a strategy that is assuredly better than his true value function.

The strategyproofness in the sense of Definition 22 will be called weak strategyproofness (WSP for short) since it is always true for a proportional protocol, see also [Hill and Morrison, 2010].

**Example 2.**

*Assume the case when all valuations over the cake are equal, and all players, except of the cheating one follow the strategy provided by the protocol. Each of the honest players will get his proportional share, because the protocol is proved to be proportional. The cheating player also values those pieces as* $1/n$ *or more. Sharing a cake with* $(n-1)$ *other players means for the cheater that* $(n-1)/n$ *or a more valuable part of the cake is allocated to other players and so only the value of* $1/n$ *or less remains for him independent of his strategy. He would never obtain more than a proportional piece.*

As depicted in Example 2, there always exists a valuation such that in an allocation a cheating player will never obtain more than a proportional piece. Hence, weak strategyproofness is not significant.
A stronger condition comes from the social choice literature:

**Definition 23.** ***(Strategyproofness in the sense of [Thomson, 2006])***
A protocol is *strategyproof* if the true value function dominates every other strategy.

In order to prevent misunderstandings, in this work strategyproofness in the sense of Definition 23 will be called strong strategyproofness (SSP for short). It can be shown that none of the known cake-cutting protocols is able to fulfill the strong strategyproofness criteria, if the valuation of the players is not equal. All protocols shown in Chapter 3 work for two players in exactly the same way as Cut & Choose. Example 3 is similar to the one in [Chen et al., 2010].

**Example 3.**

*John Warner Backus and Peter Naur are celebrating and Donald E. Knuth has brought a huge marzipan cake with an enormous cherry on the left side. John loves cherries and hates marzipan, and Peter is just very hungry. The pioneers of computer science apply Cut & Choose (see Chapter 3.2). Peter is the cutter, and his best strategy would be to separate the cake from the cherry. If Peter had full knowledge (which would not violate the preconditions of strategyproofness in [Thomson, 2006]) about the valuations of John, he would benefit from cheating. From Table 4 he would know, that John would always take the left piece and so he could easily maximize the value of his portion. Hence, this algorithm is not strongly strategyproof.*

| | Only Cherry | Middlecut |
|---|---|---|
| L | $(9/10, 1)$ | $(1/2, 1)$ |
| R | $(1/10, 0)$ | $(1/2, 0)$ |

Table 4: B&N cake game in normal form

In strong strategyproofness a player would never get a more valuable piece by lying independent of the valuation of the other players.

After a counterexample in [Hill and Morrison, 2010] the definition of strategyproofness in [Brams et al., 2008] was restricted to the case with non-equal valuations and for the general case changed to:

**Definition 24.** ***(Strategyproofness in the sense of [Magid, 2008])***
A protocol is *strategyproof* (SP for short) if no player has a strategy that is at least as well and sometimes better than his true value function.

Imagine a non-truthful strategy with one allocation where the cheating player would obtain a bit less valuable piece than by the recommended strategy and one thousand other possible allocations this player would get the whole cake, while in the recommended strategy he would get just his proportional share. By the upper defnitions this player would stay honest, but which selfish and rational player would really do this?
A possibility to handle those situations gives a game-theoretical approach in the following definition:

**Definition 25.** ***(Game-Theoretic Strategyproofness)***
A protocol is *game-theoretic strategyproof* (GTSP for short) if no player has a strategy with a higher expected value than the expected value of his true value function.

## 3.1 Correlation between Strategyproofness Criteria

**Theorem 3.** *The relation between different strategyproofness criteria is the following:*

$$\begin{array}{ccccc} SPP & \overset{1}{\Longrightarrow} & SP & \overset{2}{\Longrightarrow} & WSP \\ & & {\scriptstyle 3}\Big\Uparrow & & \\ & & GTSP & & \end{array}$$

*Proof.*

1. If a protocol is $SPP$, then for every player in each strategy except of the recommended one exists at least one allocation $A_c$. In $A_c$ the cheating player $p_c$ receives a piece $X_c$ which is not proportional. Since in a proportional protocol all allocations guarantee a proportional share to every player, the value of his piece $X_c$ is less than he would obtain by following the recommended strategy. So the protocol is $SP$.

2. If a protocol is $SP$, then for every player in each strategy except of the recommended one exists at least one allocation $A_c$. In $A_c$ the cheating player $p_c$ receives a piece $X_c$ instead of $X_{\neg c}$, which he had received by following the recommended strategy. The value of $X_c$ is smaller or equal to the value of $X_{\neg c}$. So no non-truthful strategy exist where the cheating player gets in all allocations a more valuable piece than in the recommended one. The protocol is $WSP$.

3. **Proof by contradiction:**

   Assumption: If a protocol is *notSP* then it is not *notGTSP*.
   If a protocol is *notSP*, then there is one player with a strategy, which is not the recommended one. And for all allocations $A_{S_c}$ the cheating player $p_c$ receives a piece $X_{S_c}$ instead of $X_{S_{\neg c}}$, which he had received by following the recommended strategy. The value of $X_{S_c}$ is at least equal to the value of $X_{S_{\neg c}}$. In one allocation $A_{S'_c}$ the value of $X_{S'_c}$ is bigger than the value of $X_{S'_{\neg c}}$. Let $r$ be the number of all possible allocations. The expected value of the strategy $S_c$ at least equal with the expected value of the recommended strategy for the $r-1$ allocations since the expected value for the received pieces $X_{S_c}$ is at least equal to the expected value for the pieces $X_{S_{\neg c}}$. The expected value for the piece $X_{S'_c}$ is bigger than the expected value for $X_{S'_{\neg c}}$. The general expected value is $\mathbb{E}(S_c) > \mathbb{E}(S_{\neg c})$ and is particularly not $\mathbb{E}(S_c) \leq \mathbb{E}(S_{\neg c})$. ↯ So the protocol is *notGTSP*.
   So $notSP \Rightarrow notGTSP$ and therefore $GTSP \Rightarrow SP$.

□

**Theorem 4.** *The relation between different strategyproofness criteria is the following:*

$$\begin{array}{ccccc} SPP & \overset{1}{\not\Leftarrow} & SP & \overset{2}{\not\Leftarrow} & WSP \\ & & {}^{3}\not\Downarrow & & \\ & & GTSP & & \end{array}$$

*Proof.*

Assume three different protocols for two players $p_1$ and $p_2$. All of them have two possible allocations $A_1$ and $A_2$, which have the same probability. In the first protocol and in the third protocol by using the recommended strategy $S_\neg c$ and the non-truthful strategy $S_c$ the player $p_2$ gets the same values in $A_1$ and $A_2$. In the second protocol he obtains strictly more by using the non-truthful strategy $S_c$ than the recommended strategy $S_\neg c$ in $A_1$ and in $A_2$.The values of $p_1$ are shown in the four tables below:

1 $SPP \not\Leftarrow SP$
   By following the non-truthful strategy $S_c$ the player $p_1$ gets always $p_1(X_1) = 1/2$. In the allocation $A_2$ he obtains less by following the non-truthful strategy $S_c$ in particular $1/2 < 2/3$. With that outcome the player $p_1$ would stay honest and this protocol is *SP*. But $1/2$ is proportional so the protocol is *notSPP*.

2 $SP \not\Leftarrow WSP$
   By following the non-truthful strategy $S_c$ and the recommended strategy $S_{\neg c}$ the player obtains the same value in the allocation $A_2$: $3/4 = 3/4$. So the player $p_1$ has not got a more valuable piece and so he would stay honest and this protocol is *WSP*. But the value by following the non-truthful strategy $S_c$ is bigger in his own valuation

| | in $A_1$ | in $A_2$ |
|---|---|---|
| $p_1(X_1)$ by $S_{\neg}c$ | $1/2$ | $2/3$ |
| $p_1(X_1)$ by $S_c$ | $1/2$ | $1/2$ |

Protocol 1: $SPP \not\Leftarrow SP$

| | in $A_1$ | in $A_2$ |
|---|---|---|
| $p_1(X_1)$ by $S_{\neg}c$ | $7/8$ | $3/4$ |
| $p_1(X_1)$ by $S_c$ | 1 | $3/4$ |

Protocol 2: $WSP \not\Leftarrow SP$

| | in $A_1$ | in $A_2$ |
|---|---|---|
| $p_1(X_1)$ by $S_{\neg}c$ | $1/2$ | $4/7$ |
| $p_1(X_1)$ by $S_c$ | 1 | $3/7$ |

Protocol 3: $GTSP \not\Leftarrow SP$

Table 5: Counter-examples for the correlation between the strategyproofness criteria

because $1 > 7/8$ in $A_1$. In the other allocation $A_2$ the obtained values are equal. So the player $p_1$ has a more valuable piece in one allocation and the same value in the other so the protocol is *notSP*.

3 $GTSP \not\Leftarrow SP$
Since in the allocation $A_2$ by following the non-truthful strategy $S_c$ the player $p_1$ get $3/7$ smaller than $4/7$ by following the recommended strategy $S_{\neg c}$. He would stay honest and this protocol is $SP$. But the expected value for the recommended strategy is $(1/2+4/7)/2 = 15/28$, which is smaller than $(1+3/7)/2 = 20/28$. So the protocol is *notGTSP*.

$\square$

## 3.2 The Cut & Choose Protocol

Representation is inspired by [Barbanel, 1995]

| **Cut & Choose for $n = 2$** | | |
|---|---|---|
| **Rules** | **Player $p_1$ strategy** | **Player $p_2$ strategy** |
| 1. Player $p_1$ partitions the cake $X$ into two pieces $\{X', X - X'\}$ | Partition $X$ into two pieces of equal value | |
| 2. Player $p_2$ chooses one piece | | Choose the bigger value |
| 3. Player $p_1$ gets the remaining piece | | |

Table 6: Cut & Choose rules and strategies

**Theorem 5.** *Cut & Choose is game-theoretic strategyproof.*

*Proof.*

**Options for not following the recommended strategy:**

- Player $p_2$ takes the less valuable piece. This can not be his intention, because then he has a piece with less value.
- Player $p_1$ cuts the cake into two unequal pieces. The chance to get less in his valuation is equal to the chance to get more in his valuation of the cake. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case:

$$\begin{aligned}\mathbb{E}(v_1(X_1)) &= P(v_2(X') \leq v_2(X - X')) \cdot v_1(X') + P((X') \leq v_2(X')) \cdot v_1(X - X') \\ &= 1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/2\end{aligned}$$

and in the dishonest case:

$$\begin{aligned}\mathbb{E}(v_1(X_1)) &= P(v_2(X') \leq v_2(X - X')) \cdot v_1(X') + P(v_2(X - X') \leq v_2(X')) \cdot v_1(X - X') \\ &= 1/2 \cdot v_1(X') + 1/2 \cdot v_1(X - X') = 1/2 \cdot \underbrace{v_1(X)}_{=1} = 1/2\end{aligned}$$

According to the definition of game-theoretical strategyproofness in the case with equal expected values the player would stay honest. Cut & Choose is game-theoretical strategyproof.

$\square$

**Theorem 6.** *Cut & Choose is strategyproof for proportional protocols.*

*Proof.*

For the proof a general presentation of Cut & Choose game in extended form is used. W. l. o. g. the non-truthful cut is performed on the right part of the cake. The variables have the following restrictions:

$$0 \leq a \leq 1,\ 0 < \epsilon \leq 1/2,\ 0 \leq \delta \leq 1{-}a$$

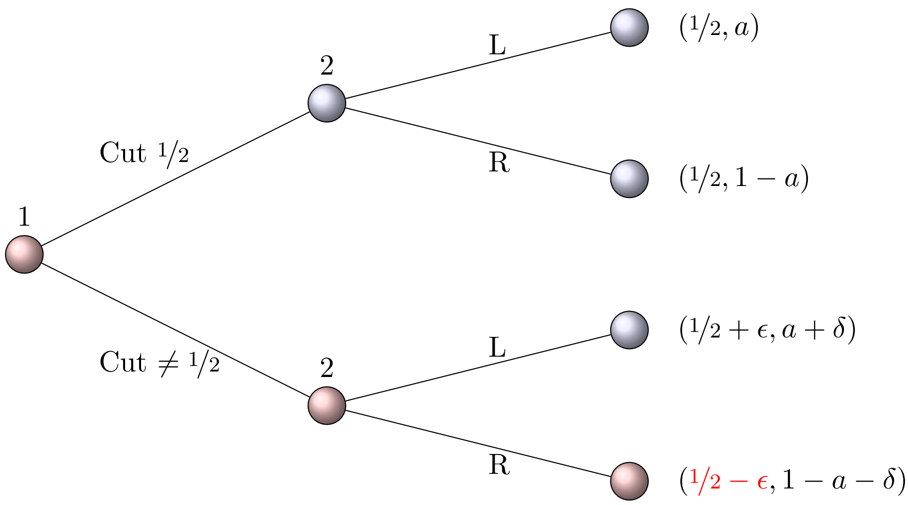


Figure 5: Cut & Choose game in extended form

The red path is the general case for an allocation, where by not following the recommended strategy player $p_1$ always becomes a less valuable piece than his proportional share. If Player $p_2$ would not choose the recommended strategy, he has to take a piece less valuable than his proportional share. Both players would stay honest. Thus Cut & Choose is strategyproof for proportional protocols. □

**Remark 1.** *According to Theorem 3, Theorem 6 and Theorem 5 Cut & Choose is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof and weak strategyproof.*

# 4 Strategyproofness of Proportional Protocols

The goal in this chapter is to analyse the strategyproofness of well-known protocols. First of all, they are rewritten into game-theoretic manner. Since each player has a truthful and a non-truthful strategy, a protocol with $n$ active players has at least $2n$ strategies. If the obtained value is not equal in different non-truthful strategies they have to be separated. So the amount of strategies would grow and would make the analysis very tedious.
Luckily a protocol consists of a lot of repeats and actually each of the well-known protocols can be simplified to an interaction between two kinds of players. So the analysis of the whole process is unnecessary.
The proceed is as follows, the interactions between two kinds of players are represented in tables. A separation between rules and strategies is given. Afterwards the different strategies of a protocol are represented as an extended form game and the different types of strategyproofness are analysed.
The complete protocols in the standard description as well as the proofs of their proportionality can be found in [Robertson and Webb, 1998].

## 4.1 The Kuhn à la Dawson Lone-Divider Protocol

The players are separated into two groups. In the first group is the player $p_1$ and the second group consists of $P_{n-1} = \{p_2, \ldots, p_n\}$, so all other players $p_i$ with $2 \leq i \leq n$.
This protocol is more complicated than the rules described here, but the details are not important for the illustration of strategies.
In the third step exist two possible cases. If an allocation is possible, first the players with only one acceptable piece choose, then the other players from player $p_n$ in descending order. The cutter is the last chooser.
If no allocation is possible, the conflicting players form a new piece of cake. For further details see the second part of the proof or [Brams and Taylor, 1996].

| **Kuhn à la Dawson Lone-Divider protocol for arbitrary $n$** | | |
|---|---|---|
| **Rules** | **Player $p_1$ strategy** | **Players in $P_{n-1}$ strategy** |
| 1. Player $p_1$ cuts the cake $X$ into $n$ pieces $\{X_1, \ldots, X_n\}$ | Cut $X$ into $n$ pieces of equal value | |
| 2. Players in $P_{n-1}$ mark $s$ pieces with $1 \leq s \leq n$ | | Mark $X_j$ if $v_i(X_j) \geq 1/n$ for $1 \leq j \leq n$ and $2 \leq i \leq n$ |
| 3. If an allocation is impossible: Detect the critical pieces and form them to a new cake and exchange the cutter (cutter leaves with a non-desirable piece) | | |

Table 7: Lone-Divider rules and strategies

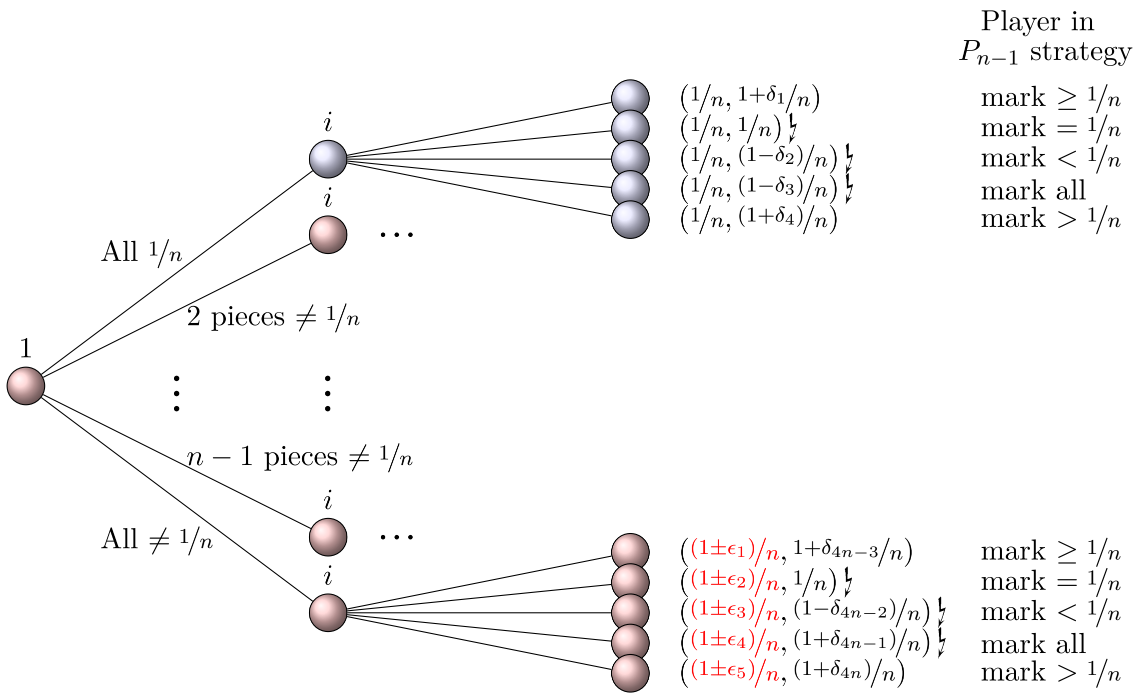


Figure 6: Lone-Divider cake game in extended form

**Explanation of the obtained values in the game tree:**

Every possible strategy is in the game tree. The variables have the restrictions:

$$0 \leq (1 \pm \epsilon_i) \leq n,\ 0 \leq \delta_{4m+1}, \delta_{4m+1} \leq (n-1),\ -1 \leq \delta_{4m+2}, \delta_{4m+3} \leq 0$$

$$for\ \ 0 \leq m \leq (n-1)$$

· Player $p_1$ obtains certainly $1/n$ if he cuts all pieces equal in his own valuation.
· It is possible that player $p_1$ obtains $(1{+}\epsilon_i)/n$ if he cuts at least one piece bigger than $1/n$ in his own valuation and get this in the allocation.
· It is possible that player $p_1$ obtains $(1{-}\epsilon_i)/n$ if he cuts at least one piece smaller than $1/n$ in his own valuation and get this in the allocation.

For every player $p_i$ in the group $P_{n-1}$:

· Player $p_i$ obtains certainly $1/n$ if he marks all pieces equal to $1/n$ in his own valuation.
· It is possible that player $p_i$ obtains $(1{+}\delta_i)/n$ if he marks pieces bigger or equal than $1/n$ in his own valuation.
· It is possible that player $p_i$ obtains $(1{+}\delta_i)/n$ if he marks pieces bigger than $1/n$ in his own valuation.
· It is possible that player $p_i$ obtains $(1{-}\delta_i)/n$ if he marks all pieces and especially pieces smaller than $1/n$ in his own valuation and get this in the allocation.

**Theorem 7.** *Lone-Divider protocol is not strategyproof.*

*Proof.*

For showing that Lone-Divider is not $SP$ it is necessary to have at least one cheating player $p_c$. This player has a non-truthful strategy with $v_c(X_c) > v_c(X_t)$ in one allocation and $v_c(X_c) \geq v_c(X_t)$ for all other allocations.
The proof is divided into two parts. In the first part the case $v_c(X_c) > v_c(X_t)$ is illustrated with an example. The second part uses a case distinction to show that this player will never get a less valuable piece. The non-truthful strategy is to mark $X_j$ if $v_c(X_j) > 1/n$ for $1 \leq j \leq n$, or in the only case when all pieces have the same value to mark all of them.

**Part I: Example for a successful non-truthful strategy**

Imagine the following allocation with three players. The player $p_1$ is going to cheat.

| | $X_L$ | $X_M$ | $X_R$ |
|---|---|---|---|
| $p_3$ (divider) | $1/3$ | $1/3$ | $1/3$ |
| $p_2$ (rank 2) | $3/5$ | $2/5$ | 0 |
| $p_1$ (rank 1) | $1/2$ | $1/3$ | $1/6$ |

Table 8: Example for a successful non-truthful strategy

The acceptable pieces

By following the recommended strategy:

| | $X_L$ | $X_M$ | $X_R$ |
|---|---|---|---|
| $p_3$ (divider) | ✓ | ✓ | ✓ |
| $p_2$ (rank 2) | ✓ | ✓ | ✗ |
| $p_1$ (rank 1) | ✓ | ✓ | ✗ |

By following the non-truthful strategy:

| | $X_L$ | $X_M$ | $X_R$ |
|---|---|---|---|
| $p_3$ (divider) | ✓ | ✓ | ✓ |
| $p_2$ (rank 2) | ✓ | ✓ | ✗ |
| $p_1$ (rank 1) | ✓ | ✗ | ✗ |

Table 9: Acceptable pieces in a successful non-truthful strategy

An allocation is possible since each pieces is acceptable for at least one different player.

If player $p_1$ chooses the recommended strategy $S_{\neg}c$:
There are no players with just one acceptable piece, so the player with the highest rank can choose first. Player $p_2$ chooses $X_L$ and player $p_1$ gets $X_M$ with $p_1(X_M) = 1/3$. The divider gets the last piece.

If player $p_1$ chooses the non-truthful strategy $S_c$:
There is one player with just one acceptable piece, so this player chooses first. Player $p_1$ chooses $X_L$ with $p_1(X_L) = 1/2$ and player $p_2$ gets $X_M$. The divider gets the last piece.
The value of player $p_1$'s piece in the non-truthful strategy $S_c$ is $p_1(X_L) = 1/2 > 1/2 = p_1(X_M)$ than in the recommended strategy $S_{\neg}c$. So the player $p_1$ is at least in one allocation better off with the non-truthful strategy than in the recommended one.

**Part II: Case distinction for a successful non-truthful strategy**

Case I: Allocation possible with $S_c$ and possible with $S_{\neg c}$

Assume the player $p_c$ has rank $k$ with the strategy $S_{\neg c}$ and rank $l$ by using the strategy $S_{\neg c}$ with $l \leq k$:

· $l = k$ : The player $p_c$ takes by using $S_c$ and $S_{\neg c}$ the same piece $\Rightarrow v_c(X_c) = v_c(X_t)$

· $l < k$ : The player $p_c$ has a smaller amount of acceptable pieces during the allocation by using $S_c$ than by using $S_{\neg c} \Rightarrow v_c(X_c) \geq v_c(X_t)$.

Case II: Allocation not possible with $S_c$ and possible with $S_{\neg c}$

Because an allocation is possible with $S_{\neg c}$ but not with $S_c$ the player $p_c$ gets a piece with $v_c(X_t) = 1/n$, otherwise those piece has been acceptable by using strategy $S_c$ aswell and an allocation has been also possible.

By using $S_c$ the player $p_c$ has to form a new cake $X_{new}$ with $r$ other players ($1 \leq r \leq (n-2)$, $-2$ because $p_c$ and the cutter cannot be included). So $X_{new}$ consists of $r+1$ pieces, since $r+1$ players are involved. This means that $n-r-1$ pieces were succesfully allocated (even if they had separate to be formed to a new cake). Each of this $n-r-1$ pieces $X_b$ has the value $v_c(X_b) \leq 1/n$, or otherwise they would be included into $X_{new}$. The proportional part of $X_{new}$ for $p_c$ will be:

$$\frac{v_c(X_{new})}{(r+1)} = \frac{(v_c(X) - (n-r-1) \cdot v_c(X_b))}{(r+1)} \geq \frac{(v_c(X) - (n-r-1) \cdot \frac{1}{n})}{(r+1)} =$$

$$\frac{(1 - (n-r-1) \cdot \frac{1}{n})}{(r+1)} = \frac{(\frac{n}{n} - (n-r-1) \cdot \frac{1}{n})}{(r+1)} = \frac{(\frac{(n-n+r+1)}{n})}{(r+1)} = \frac{(\frac{(r+1)}{n})}{(r+1)} = \frac{1}{n}$$

And so $p_c$ obtains $v_c(X_c) \geq 1/n \Rightarrow v_c(X_c) \geq v_c(X_t)$.

Case III: Allocation possible with $S_c$ and not possible with $S_{\neg c}$ $\lightning$

In $S_{\neg c}$ is the amount of acceptable pieces bigger or equal to $S_c$, so there is at least one non-conflicting piece which $p_c$ receives by using $S_c$ and those piece $p_c$ would also receive by using $S_{\neg c}$.

Case IV: Allocation not possible with $S_c$ and not possible with $S_{\neg c}$

· The player $p_c$ is the cutter $\Rightarrow v_c(X_c) = v_c(X_t)$

· The player $p_c$ is not the cutter $\Rightarrow$ Case I-IV possible.

□

**Remark 2.** *According to Theorem 3 and Theorem 7 Kuhn à la Dawson Lone-Divider is not strategyproof for proportional protocols, not strategyproof and not game-theoretical strategyproof. According to Example 2 it is weak strategyproof.*

## 4.2 The Banach-Knaster Last-Diminisher Protocol

The Last-Diminisher protocol consists of $(n-2)$ rounds.
The players are separated into two groups. In the first group is the player $p_i$ (with $i$ number of the round) and in the second group $P_{n-i+1}$ are all other players $p_j$ with $i < j \leq n$.
At the end of each round one player gets a piece and leaves the game. If it is player $p_i$ then player $p_{i+1}$ as $p_{i+1}$ takes his place, otherwise player $p_i$ becomes player $p_{i+1}$ in the next round $i+1$ and players in group $P_{n-i+1}$ will be consecutively numbered. In the last round the two remaining players apply Cut & Choose.

| The Banach-Knaster Last-Diminisher protocol for arbitrary $n$ | | |
|---|---|---|
| **Rules** | **Player $p_i$ strategy** | **Players in $P_{n-i+1}$ strategy** |
| 1. Player $p_i$ cut a piece $I_i$ | Cut a piece with value $1/n$ | |
| 2. Players $p_j$ trim or pass | | If $v_j(I_i) > 1/n$ trim so that $v_j(I_i) = 1/n$, else pass |
| 3. Last trimmer take it | | |

Table 10: Last-Diminisher rules and strategies

**Theorem 8.** *Last-Diminisher protocol is strategyproof for proportional protocols.*

*Proof.*

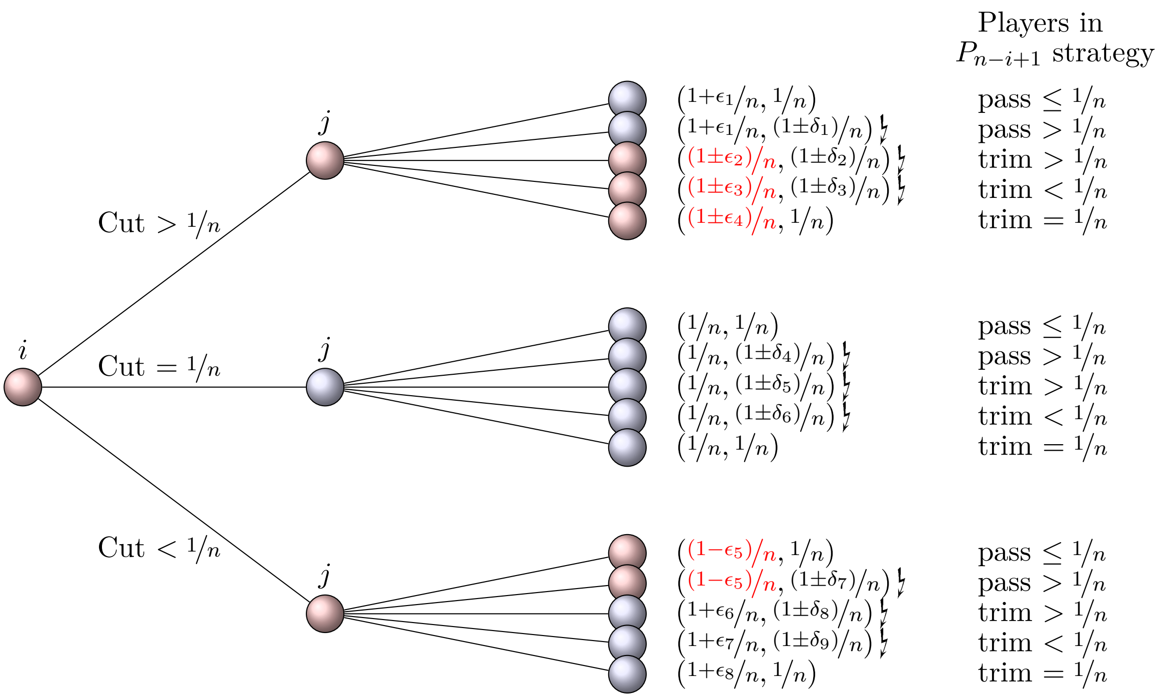


Figure 7: Last-Diminisher cake game in extended form

For the proof a presentation of Last-Diminisher game in extended form is used. The variables have the following restrictions:

$$0 < (1 \pm \epsilon_2), (1 \pm \epsilon_3), (1 \pm \epsilon_4) < n,\ 0 < \epsilon_5 \leq 1,\ 0 < \epsilon_1, \epsilon_6, \epsilon_7, \epsilon_8 \leq (n-1)$$

and :

$$0 \leq (1\pm\delta_1), (1\pm\delta_4), (1\pm\delta_7) \leq n, 0 < (1\pm\delta_2), (1\pm\delta_3), (1\pm\delta_5), (1\pm\delta_6), (1\pm\delta_8), (1\pm\delta_9) < n$$

**Explanation of the obtained values in the game tree:**

· It is possible that player $p_i$ or player $p_j$ obtains $v_{i,j}(X_{i,j}) = {}^{(1-\delta_i)}/_n$ if he pass, trim or cut a piece with $v_{i,j}(X_k) > {}^1/_n$ in his own valuation.
Because a subsequent player $p_l$ could trim it s.t. $v_{i,j}(X_k) > v_{i,j}(X_l) > {}^1/_n$ and get $X_l$ in the allocation. For the players $p_j$ and $p_j$ a proportional share is not longer guaranteed.
· It is possible that player $p_i$ or player $p_j$ obtains $v_{i,j}(X_{i,j}) = {}^{(1-\delta_i)}/_n$ if he trim or cut a piece with $v_{i,j}(X_k) < {}^1/_n$ in his own valuation.
Because if all subsequent player $p_l$ pass it the player $p_j$ or depending on $p_j$ get $v_{i,j}(X_k) = {}^{(1-\delta_i)}/_n < {}^1/_n$.
· Otherwise player $p_j$ and player $p_i$ follow the recommended strategy and would obtain $v_{i,j}(X_{i,j}) \geq {}^1/_n$, since Last-Diminisher is proportional.

The consequences for each player $p_j$ in $P_{n-i+1}$ by choosing the strategy to pass a piece $X_i$ with $v_j(X_i) > {}^1/_n$ or to trim a piece $X_i$ to $v_j(X_i) > {}^1/_n$ are the same as for the player $p_i$ by cutting a piece with $v_i(X_i) > {}^1/_n$. The upper red paths in Figure 7 show that by choosing this non-truthful strategy a player can obtain a less valuable piece. For that to happen, a following player $p_s$ needs to trim the piece s.t. $v_i(X_s) > {}^1/_n$ and $v_j(X_s) > {}^1/_n$ and for the reason that he is the last trimmer to get this piece.

The consequences for each player $p_j$ in $P_{n-i+1}$ by choosing the strategy trim a piece $X_i$ to $v_j(X_i) < {}^1/_n$ are the same as for the player $p_i$ by cutting a piece with $v_i(X_i) < {}^1/_n$. The lower red paths in Figure 7 show that by choosing this non-truthful strategy a player can obtain a less valuable piece. For that to happen, no following player $p_s$ is allowed to trim the considered piece. So the cheating player will get this undesirable piece.

The only strategy which does not conceal this risk, is for all players the recommended one. □

**Remark 3.** *According to Theorem 3 and Theorem 8Banach-Knaster Last-Diminisher is strategyproof for proportional protocols, strategyproofand weak strategyproof.*

### 4.3 The Fink Lone-Chooser Protocol

For the description of this protocol the players will be separated in two groups. In the $(i-1)$ round players in the first group $P_i = \{p_1, \dots, p_i\}$ have already their proportional piece. Assume that player $p_1$ owns the whole cake before the first round. The performance in each round is the following:

| The Fink Lone-Chooser protocol for arbitrary $n$ | | |
|---|---|---|
| **Rules** | **Players in $P_i$ strategy** | **Player $p_{i+1}$ strategy** |
| 1. Players in $P_i$ partition their piece $X_i$ into $i+1$ pieces $\{X_{i,1}, \dots, X_{i,i+1}\}$ | Partition $X_i$ into $i+1$ pieces of equal value | |
| 2. Player $p_{i+1}$ chooses one piece of each player's cake | | Choose the most valuable piece of each player's cake |

Table 11: Lone-Chooser rules and strategies

**Theorem 9.** *Fink Lone-Chooser protocol is strategyproof for proportional protocols.*

*Proof.*

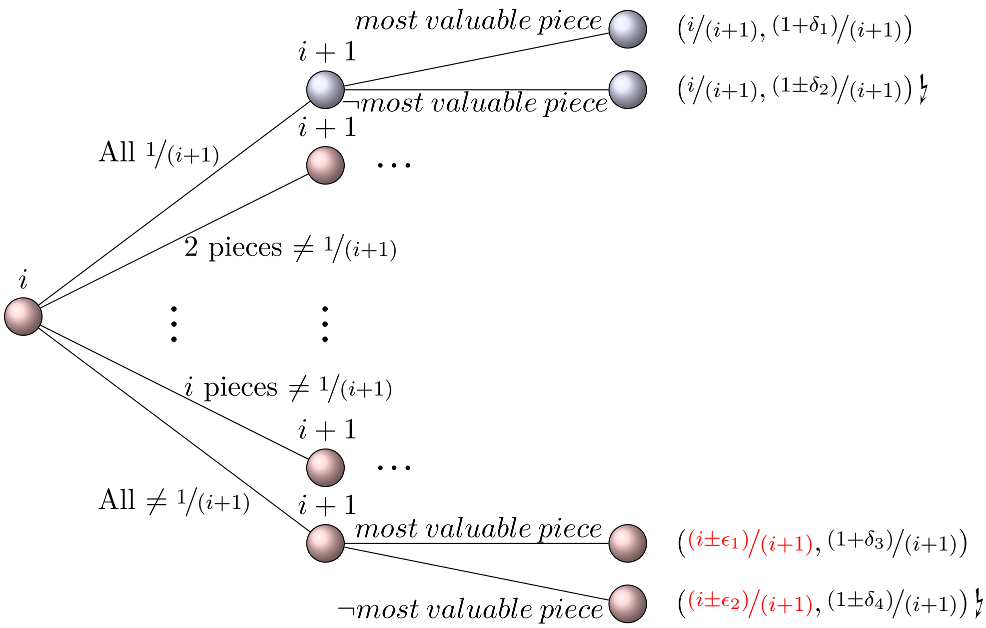


Figure 8: Lone-Chooser cake game in extended form

For the proof a presentation of Lone-Chooser game in extended form is used.
The variables have the following restrictions:

$$0 \leq (i \pm \epsilon_1), (i \pm \epsilon_2) \leq (i+1),\ 0 \leq \delta_1, \delta_3 \leq (i^2 + i - 1)$$

$$0 \leq (i \pm \delta_2), (i \pm \delta_4) \leq (i+1) \cdot i,\ \delta_2 \leq \delta_1,\ \delta_4 \leq \delta_3$$

**Explanation of the obtained values in the game tree:**

· Player $p_{i+1}$ chooses not the most valuable piece:
Then the value of $(1+\delta_2)$ can be bigger or smaller than 1, for example when one of the $i+1$ pieces has the whole value.
· It is possible that player $p_i$ obtains ${}^{(1-\epsilon_i)}/_{(i+1)}$ if he cuts at least one piece bigger than ${}^{1}/_{(i+1)}$ in his own valuation and this one would the player $p_{i+1}$ take.
· It is possible that player $p_i$ obtains ${}^{(1+\epsilon_i)}/_{(i+1)}$ if he cuts at least one piece smaller than ${}^{1}/_{(i+1)}$ in his own valuation and this one would the player $p_{i+1}$ take.

The red path is the not succesful allocation by following the non-truthful strategy for a player in $P_i$. Here every player gets less than by following his recommended strategy. Especially it is the case, where the player $p_{i+1}$ takes a piece which a player in $P_i$ values more than ${}^{1}/_{(i+1)}$.
If Player $p_2$ would not choose the recommended strategy, he has to take a piece less valuable than in his best possibility. Both kind of players would stay honest. Thus Lone-Chooser is strategyproof for proportional protocols. □

**Theorem 10.** *Fink Lone-Chooser protocol is game-theoretic strategyproof.*

*Proof.* Proof by induction:

- Case $i = 2$: Cut & Choose is game-theoretic strategyproof by Theorem 5.

- Case $(i-1) \to i$:

  **Options for not following the recommended strategy:**

  - Player $p_{i+1}$ takes not the biggest piece. Then he has a less valuable piece than by following the recommended strategy.
  - Players in $P_i$ cut the cake into $i+1$ non-equal pieces. The chance that player $p_{i+1}$ takes a certain piece $X^j$ is ${}^{1}/_{(i+1)}$. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case (because he had at least ${}^{1}/_{i}$ before) at least:

$$\mathbb{E}(v_i(X^1 + X^2 + \ldots + X^{j-1} + X^{j+1} + \ldots + X^{i+1})) =$$

$$P(v_n(X^k) \leq v_n(X^j)) \cdot v_i(X_1) + P(v_n(X^k) \leq v_n(X^j)) \cdot v_i(X_2) + \ldots$$

$$+ P(v_n(X^k) \leq v_n(X^j)) \cdot v_i(X_{i+1}) =$$

$${}^{1}/_{(i+1)} \cdot {}^{1}/_{i} + {}^{1}/_{(i+1)} \cdot {}^{1}/_{i} + \ldots + {}^{1}/_{(i+1)} \cdot {}^{1}/_{i} =$$

$$i \cdot {}^{1}/_{(i+1)} \cdot {}^{1}/_{i} =$$

$${}^{1}/_{(i+1)}$$

Assume the value of the piece will be distributed on $s$ pieces with $1 \leq s \leq i+1$. There are $s$ pieces with the value

$$v_i(X_1), \ldots, v_i(X_s) = {}^{1}/_{i \cdot s}$$

and

$$v_i(X_{s+1}), \ldots, v_i(X_{i+1}) = 0$$

So the probability that $X^j \in \{X_{s+1}, \ldots, X_{i+1}\}$ is

$$P(X^j \in \{X_{s+1}, \ldots, X_{i+1}\}) = {}^{(i+1-s)}/_{(i+1)}$$

and the probability for $X^j \in \{X_1, \ldots, X_s\}$ is

$$P(X^j \in \{X_1, \ldots, X_s\}) = {}^{(s)}/_{(i+1)}$$

So the expected value will be in the dishonest case:

$$\mathbb{E}(v_1(X^1 + X^2 + \ldots + X^{j-1} + X^{j+1} + \ldots + X^{i+1})) =$$

$${}^{s}/_{(s \cdot i)} \cdot P(X^j \in \{X_{s+1}, \ldots, X_{i+1}\}) + {}^{(s-1)}/_{(s \cdot i)} \cdot P(X^j \in \{X_1, \ldots, X_s\}) =$$

$${}^{1}/_{i} \cdot {}^{(i+1-s)}/_{(i+1)} + {}^{(s-1)}/_{(s \cdot i)} \cdot {}^{s}/_{(i+1)} =$$

$${}^{1}/_{(i+1)}$$

Even if the value is distributed unequally on the $s$ pieces, it would not affect the value of the piece in the allocation, because the probability of the piece the player $p_{i+1}$ take is uniformly distributed.
According to the definition of game-theoretic strategyproofness in the case with equal expected values the player would stay honest.

□

**Remark 4.** *According to Theorem 3, Theorem 10 and Theorem 9 Fink Lone-Chooser is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof and weak strategyproof.*

## 4.4 The Even & Paz Divide-&-Conquer Protocol

The players are separated in two groups. The player $p_i$ in the group $P_{n-1}$ make the mark on the cake and then get from left to right numbered in ascending order. So the player with the leftmost cut is numbered as player $p_1$. The piece $X'$ that this first player $p_1$ marks is labelled as $X'_1$ and so on. For each new allocation the players get new numbers.

| **Divide-&-Conquer protocol for arbitrary $n$** | | |
|---|---|---|
| **Rules** | **Players in $P_{n-1}$ strategy** | **Player $p_n$ strategy** |
| 1. Each player in $P_{n-1}$ marks the cake $X$ into two pieces $\{X', X - X'\}$ | Mark $X$ to two pieces with the value-ratio of $\lfloor n/2 \rfloor : \lceil n/2 \rceil$ | |
| 2. Cut in the ratio of the player with the $\lfloor n/2 \rfloor$-th cut from the left border | | |
| 3. Player $p_n$ chooses one piece | | If $v_n(X'_{\lfloor n/2 \rfloor}) \geq \lfloor n/2 \rfloor / n$ choose $X'$, otherwise $X - X'$ |
| 4. Repeat the protocol with the new groups separately until each player has his own piece | | |

Table 12: Divide-&-Conquer rules and strategies

**Theorem 11.** *The Divide-&-Conquer protocol is strategyproof for proportional protocols.*

*Proof.*

The proof via induction is similar to [Brams et al., 2007]:

$n = 2$ Assume two players $p_1$ and $p_2$ cutting cake. Player $p_1$ may be the cheating player. The truthful $1/2$ points are shown below, then by making a cut at $|$ would give $p_1$ and $p_2$ more than $1/2$:

$$0 - - - - - - - - - - - - a - | - b - - - - - - - - - - - - - 1$$

But if player $p_1$ should move his $1/2$ point to $a$, he gets into the risk of getting less than $1/2$ if the other player choose player $p_1$ favourite part.

$n \mapsto (n+1)$ Assume in the same sketch the $|$ is the mark of the $\lfloor n/2 \rfloor$-player. If the player $p_1$ should report false that his $\lfloor n/2 \rfloor / n$ point, which is $a$ is more to the left. He risks to cross the $|$ mark and to be assigned to the group of players with whom the value he would obtain is smaller than his proportional piece.

□

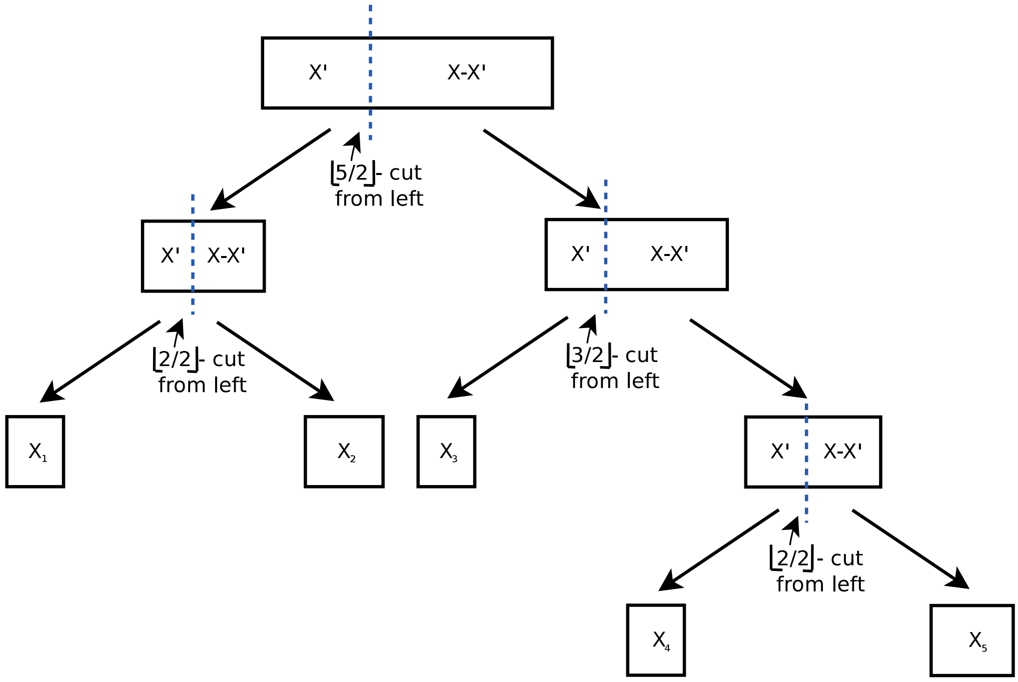


Figure 9: D&C execution for $n = 5$

**Theorem 12.** *The Divide-&-Conquer protocol is game-theoretic strategyproof.*

*Proof.*

- The player $p_n$ has no intention to take the smaller piece.

- Every player $p_i$ in the set $P_{n-1}$ has to assume that he is the player with the $\lfloor n/2 \rfloor$-th cut, since he does not know the valuation functions of the other players. There is a possibility to get or to have to divide the right or the left piece with other players. For the same argument as in Cut & Choose the expected values are equal and so the player would stay honest.

□

**Remark 5.** *According to Theorem 3 and Theorem 11 and Theorem 12 Even & Paz Divide-&-Conquer is strategyproof for proportional protocols, strategyproof game-theoretical strategyproof and weak strategyproof.*

# 5 Conclusion

The strategyproofness in the context of cake-cutting has not been widely researched yet. In this work an overview over the occured definitions of the last five years is given. The applicability of them was proven. Hereby a proportional cake-cutting protocol is always weak and never strong strategyproof. For cake-cutting applicable definitions an overview over the correlation is given.

Then the well-known proportional cake-cutting protocols have been rewritten into a game-theoretic manner and analysed on whether a non-truthful strategy could yield a more advantageous situation for a non-truthful player. It was possible to approve game-theoretically that the only strategy which promises the best outcome is the strategy recommended by the protocol.

| Protocol | WSP | GTSP | SP | SPP | SSP |
|---|---|---|---|---|---|
| Cut & Choose | ✓ | ✓ | ✓ | ✓ | ✗ |
| Last-Diminisher | ✓ | (✓) | ✓ | ✓ | ✗ |
| Lone-Chooser | ✓ | ✓ | ✓ | ✓ | ✗ |
| Lone-Divider | ✓ | ✗ | ✗ | ✗ | ✗ |
| Divide-&-Conquer | ✓ | ✓ | ✓ | ✓ | ✗ |

Table 13: Overview: Strategyproofness of proportional cake-cutting protocols

## 5.1 Related Work

Recently, two papers with the focus on strategyproofness have been published. In [Chen et al., 2010] they weakened the basic concepts of cake cutting by including the free disposal assumption, which can lead to a not complete allocation of the cake and allow only piecewise uniform valuations. The second restriction is indeed very hard. Their goal was to give a proportional, envy-free, polynomial and strong strategyproof protocol. In [Mossel and Tamuz, 2010] the authors invented new procedures including a referee, who has full knowledge. This extension is a restriction of cake-cutting as well. They partly considered the case with indivisible items, where for example in [Lipton et al., 2004] research in strategyproofness been done earlier, but is not practical applicable to cake-cutting. [Chen et al., 2010] and [Mossel and Tamuz, 2010] also researched truthfulness in expectation for randomized protocols.

In pie-cutting [Thomson, 2006] showed that a strategyproof and efficient mechanism must be dictatorial. The definition of strategyproofness in this paper is the strongest condition. Also pie-cutting slightly differs from cake-cutting, since the pie ist represented as a circular object and the cuts are wedges. The results for pie- and cake-cutting do not carry over to each other, but the definition of strategyproofness does. [Brams et al., 2008] gave more details in the context of pie-cutting and strategyproofness.

The start of reaserching strategyproofness was [Brams et al., 2006], where the authors introduced a fitting definition of strategyproofness (weak strategyproof in this work) and proved that two procedures called SP and EP are weak strategyproof.
A response on their work was a counterexample by [Hill and Morrison, 2010]. The authors argued that each proportional protocol is weak strategyproof. After admitting their mistake, in [Magid, 2008] they restricted their first definition to cases with non-equal valuation functions and introduced a new general definition for strategyproof cake-cutting.
In [Brams et al., 2007] and in the revisited version of this work [Brams et al., 2010] the authors focused on the Divide-and-Conquer protocol and showed that it is strategyproof for risk averse players. In the later work they call this property truth-inducing.
[Lindner and Rothe, 2009] parallelyzed the Last-Diminisher and proved that this new protocol is also strategyproof for risk-averse players.

# 6 Open Questions and Future Research

An interesting aspect would be to take a closer look on groupstrategyproofness. Hereby, groups have public valuations for group members.

An other approach about strategyproofness could be a consecutively allocation of several cakes. Then the cake-cutting-game could be interpreted as a repeated game.
Actually only a few game-theoretic methods are applied in this work. A lot more intense reasearch could yield towards different results and different applications in the field of cake-cutting. Some possible application field could be the Last Diminisher protocol where the players know the stage in the game or the Divide-and-Conquer protocol with the focus on whether the participants can gain some advantages from knowing with how many players the are dividing a part of the cake and how many marks are on the left or right side of their mark.
To search for strategies which fulfill the strategyproofness criterion but not the game-theoretic strategyproofness criterion.
The probability model that is used in this work is very primitive, since even if the player $p_1$ cut a cake in two pieces $\{X', X - X'\}$ with the value $v_1(X') = 1 - \epsilon$ and $v_1(X - X') = \epsilon$ the probabilty that player $p_2$ takes $X'$ or $X - X'$ is equal. Certainly there are valuation where it is possible, but the sense in those is the main question. An further application could be to correlate the valuation of the cutter and the probability of the other player in taking this piece.

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