Algorithmische Eigenschaften von Wahlsystemen I

Ausgewählte Folien zur Vorlesung

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Websites

- Vorlesungswebsite:

  http://ccc.cs.uni-duesseldorf.de/~rothe/voting1

- Anmeldung nicht nur im LSF, sondern auch unter

  http://ccc.cs.uni-duesseldorf.de/verwaltung

  (CCC-System für alle meine Veranstaltungen)

Literature

- **A Richer Understanding of the Complexity of Election Systems**, P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Chapter 14 in *Fundamental Problems in Computing: Essays in Honor of Professor Daniel J. Rosenkrantz*, pp. 375–406, S. Ravi and S. Shukla, Editors. Springer, Berlin, Heidelberg, New York, 2009.

- **Computational Aspects of Approval Voting**, D. Baumeister, G. Erdélyi, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Chapter 10 in *Handbook on Approval Voting*, pp. 199–251, R. Sanver and J. Laslier, Editors. Springer-Verlag, Berlin, Heidelberg, 2010.

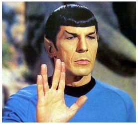
Literature

- **Voting Procedures**, S. Brams and P. Fishburn. Chapter 4 in Volume 1 of the *Handbook of Social Choice and Welfare*, pp. 173–236, K. Arrow, A. Sen, and K. Suzumura, Editors. North-Holland, 2002.
- **Chaotic Elections! A Mathematician Looks at Voting**, D. Saari. American Mathematical Society, 2001.
- **Original Papers** cited in this book and these book chapters.
- . . .

Elections

The Captain of Starship Enterprise is to be elected: 

Candidates:

Voters:

Elections

Definition

- An *election* (or *preference profile*) (C, V) is specified by a set

$$C = \{c_1, c_2, \ldots, c_m\}$$

of candidates and a list

$$V = (v_1, v_2, \ldots, v_n)$$

of votes over C.

- How the voters' preferences are represented depends on the voting system used, e.g., by
  - a *linear order (strict ranking)* or
  - an *approval vector*.

Elections

Definition

A *linear order* (or *strict ranking*) $>$ *on C* is a binary relation on C that is

- *total*: for any two distinct $c, d \in C$, either $c > d$ or $d > c$;
- *transitive*: for all $c, d, e \in C$, if $c > d$ and $d > e$ then $c > e$;
- *asymmetric*: for all $c, d \in C$, if $c > d$ then $d > c$ does not hold.

Remark:

1. Asymmetry of $>$ implies irreflexivity of $>$.
2. We often omit the symbol $>$ in the linear orders and write, e.g.,

$$b \quad c \quad a \quad e \quad d \qquad \text{instead of} \qquad b > c > a > e > d$$

to indicate that this voter (strictly) prefers b to c, c to a, a to e, and e to d. So the leftmost candidate is the most preferred one.

Elections

Remark:

3. Occasionally, by dropping asymmetry voters are allowed to be *indifferent* between candidates, as in:

$$b > c = a > e = d$$

If so, it will be mentioned explicitly.

4. One may distinguish between *weighted* and *unweighted* voters. Default case: unweighted voters (i.e., each voter has weight one).

5. Votes may be represented either *succinctly* or *nonsuccinctly*. Default case: nonsuccinct (i.e., one ballot per voter).

Elections

Example

Election (C, V) with $C = \{a, b, c, d, e\}$ and $V = \{v_1, \ldots, v_7\}$:

$$
\begin{array}{llllll}
v_1: & c & b & a & e & d \\
v_2: & a & e & d & c & b \\
v_3: & b & a & c & e & d \\
v_4: & b & d & e & a & c \\
v_5: & c & b & a & e & d \\
v_6: & c & d & b & e & a \\
v_7: & e & d & a & b & c
\end{array}
$$

Who should win this election?

Election Systems

Definition

An *election system* is a rule determining the winner(s) of a given election (C, V). Formally, letting

- $\mathcal{P}(C)^n$ denote the set of all n-vote preference profiles (e.g., n linear orders or n approval vectors) over the set C of candidates and
- $\mathfrak{P}(S)$ the set of all subsets of a set S,

an election system defines a *social choice correspondence*

$$f : \mathcal{P}(C)^n \to \mathfrak{P}(C).$$

Given a preference profile $P \in \mathcal{P}(C)^n$, $f(P) \subseteq C$ is the *set of winners* (which may be empty and may have more than one winner).

Election Systems

Remark:

- A *social choice function* is a mapping

$$f : \mathcal{P}(C)^n \rightarrow C$$

  that assigns a single winner to each given preference profile.
- Letting $\mathcal{R}(C)$ denote the set of all transitive, total preference relations over C, a *social welfare function* is a mapping

$$f : \mathcal{P}(C)^n \rightarrow \mathcal{R}(C)$$

  that assigns a complete (possibly nonstrict) ranking to each given preference profile.

Election Systems: An Incomplete Taxonomy

- Preference-based Systems:
  - Positional scoring protocols (plurality, veto, k-approval, Borda, ...)
  - Majority-based voting (simple majority, Bucklin voting, ...)
  - Pairwise-comparison-based voting procedures (Condorcet, Dodgson, Young, Kemeny, Copeland, Llull, ...)
  - Point distribution voting procedures (single transferable vote, ...)
- Nonranked Systems:
  - Approval voting
  - Negative voting
  - Plurality voting
  - Multistage voting procedures (plurality with run-off, ...)
- Hybrid Systems:
  - Sincere-strategy preference-based approval voting
  - Fallback voting

Election Systems: Plurality, Antiplurality, k-Approval

Definition

- *Plurality-rule elections*: The winners are precisely those candidates who are ranked first by the most voters.
- *Antiplurality-rule (a.k.a. veto) elections*: The winners are precisely those candidates who are ranked last by the fewest voters.
- *k-approval*: Each voter gives one point to each of the k most preferred candidates. Whoever scores the most points wins.

In our above example, c is the plurality winner, e is the antiplurality winner, and both a and b are 3-approval winners.

Election Systems: Borda Count

Definition

- *Borda Count*: With m candidates, each voter gives:
  - $m-1$ points to the candidate ranked at first position,
  - $m-2$ points to the candidate ranked at second position,

    $\vdots$
  - 0 points to the candidate ranked at last position.

  Whoever scores the most points wins.

In our above example, b is the Borda winner.

Election Systems: Borda Count

points :	4	3	2	1	0
v_1 :	c	b	a	e	d
v_2 :	a	e	d	c	b
v_3 :	b	a	c	e	d
v_4 :	b	d	e	a	c
v_5 :	c	b	a	e	d
v_6 :	c	d	b	e	a
v_7 :	e	d	a	b	c

Viewed as a social welfare function, the Borda system yields:

ranking	b	$>$	c	$>$	a	$>$	e	$>$	d
points	17	$>$	15	$>$	14	$>$	13	$>$	11

Election Systems: Scoring Protocols

Definition

A *scoring protocol* for m candidates is specified by a *scoring vector*, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, satisfying

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m.$$

Votes are linear orders. Each vote contributes

- α_1 points to that vote's most preferred candidate,
- α_2 points to that vote's second most preferred candidate,

  $\vdots$
- α_m points to that vote's second least preferred candidate.

Whoever scores the most points wins.

Election Systems: Scoring Protocols for m Candidates

Voting System	Scoring Vector
Plurality	$\alpha = (1, \overbrace{0, \dots, 0}^{m-1})$
Antiplurality (Veto)	$\alpha = (\overbrace{1, \dots, 1}^{m-1}, 0)$
k-Approval ($(m-k)$-Veto)	$\alpha = (\overbrace{1, \dots, 1}^{k}, \overbrace{0, \dots, 0}^{m-k})$
Borda Count	$\alpha = (m-1, m-2, \dots, 0)$
$\vdots$	$\vdots$

Simple Majority and Condorcet Voting

Definition

A candidate c wins by *simple majority* if c is ranked first by more than half of the voters.

In our above example, no candidate wins by simple majority. This obstacle is avoided by, e.g., Bucklin voting.

Definition

A candidate c is a *Condorcet winner* if c defeats every other candidate by a strict majority in pairwise comparisons.

In our above example, there is no Condorcet winner (as we have a top-3-cycle). This obstacle is avoided by, e.g., Dodgson, Young, Copeland, and Kemeny voting.

Election Systems: Dodgson, Young, and Copeland

Let (C, V) be a given election where votes are linear orders.

- *Dodgson*: The *Dodgson score of $c \in C$* (denoted by $DScore(c)$) is the smallest number of sequential switches needed to make c a Condorcet winner. Whoever has the smallest Dodgson score wins.
- *Young*: The *Young score of $c \in C$* (denoted by $YScore(c)$) is the size of a largest sublist of V for which c is a Condorcet winner. Whoever has the maximum Young score wins.
- *Copeland*: For each $c, d \in C$, $c \neq d$, let $N(c, d)$ be the number of voters who prefer c to d. Let $C(c, d) = 1$ if $N(c, d) > N(d, c)$ and $C(c, d) = 1/2$ if $N(c, d) = N(d, c)$. The *Copeland score of c* is $CScore(c) = \sum_{d \neq c} C(c, d)$. Whoever has the maximum Copeland score wins.

Election Systems: Bucklin Voting

Definition

- The *strict majority threshold for a list V of voters* is

$$maj(V) = \lfloor \|V\|/2 \rfloor + 1.$$

- Given an election (C, V) and a candidate $c \in C$, define the *level i score of c in (C, V)* (denoted by $score^i_{(C,V)}(c)$) as the number of votes in V that rank c among their top i positions.
- The *Bucklin score of c in (C, V)* is the smallest i such that

$$score^i_{(C,V)}(c) \geq maj(V).$$

- All candidates with a smallest Bucklin score, say k, and a largest level k score are the *Bucklin winners (BV winners) in (C, V)*.