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Game-theoretic Analysis of Strategyproofness in Cake-cutting Protocols

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Bachelorarbeit

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Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

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Alina Elterman

Abstract

1. What is the problem?

2. Why interesting?

3. How to solve?

4. What is the result?

Contents

1 Introduction

Cake cutting is an interdisciplinary field which is commonly researched and part of economics, political science, mathematics, operations research, and computer science. Game theory is fulfilling the same property. Except for this fact, they have hardly something in common. While cake cutting is about fair division of a heterogeneous divisible good, game theory studies the strategies people use when making decisions.

The role of strategies have not been widely researched yet in the context of cake cutting. But their importance is indisputable. Just imagine the following situation:

2   Preliminaries

2.1   Preliminaries of Game Theory

GT is about ... Game theory consist of two fields: cooperative and non-cooperative. The first one is about single selfish player, and in the second one the main focus is on forming coalitions. For compter scientist especially the algorithmic game theory is of major interest.

Definition. *(Game) A*

Definition. *(Pay-off) A Egalitarian and utiletarian*

Example. *(Cooperative Cake-Cutting)*
After the win of the election in Turingville the member of President Church coalition want to divide succesfully the power in the country.

2.2 Preliminaries of Cake-cutting

2.2.1 Basics

Before starting, it is necessary to define the components of cake-cutting. Example 2.2.1 sketches the problem statement.

Example. *Cook, Younger and Kasami got a cheese-chocolate-straberry-cake.*

Now, what exactly is cake-cutting about? First of all, it involves a discrete set of $n \in \mathbb{N}$ agents (or players) $P_n = \{p_1, \cdots, p_n\}$. It is assumed that each of them want to get as much as possible of the divided good. The goal is to allocate it in a manner that the players are satisfied. Hereby only the value of their own piece imply the degree of satisfiabily of the players. The allocation of a single, divisible and heterogeneous good is included in the consideration of this work.

It is common to use for the visualization a rectangular cake. The division is performed
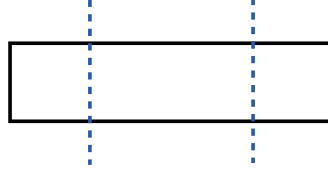

Figure 1: Cake
[Example for a visualisation of a cake and two cuts]

by parallel cuts.

The cake X is represented by the unit interval $I = [0, 1] \subseteq \mathbb{R}$. Each subinterval $I' \subseteq I$ or a union of subintervals $\bigcup_{m \in \mathbb{N}} I_m$ with $I_m \subseteq I$ is called a bundle (piece), which are disjoint.

The bundle of the cake, which the player p_i receives is denoted as X_i. When all bundles of the cake are owned by players, this state is called an <u>allocation</u>. Each piece has a public length, which can be computed as the sum of all border differences, and the private value of each player.

Every player $p_i \in P_n$ has a <u>valuation function (valuation)</u> $v_i : \{X' | X' \subseteq X\} \mapsto [0, 1]$ on the cake X with the following properties:

1. Non-negativity: $v_i(C) \geq 0$ for all $C \subseteq [0, 1]$.

2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$.

3. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq [0, 1]$.[1]

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[1]Monotonicity: If $C' \subseteq C$ then $v_i(C') \leq v_i(C)$ follows from additivity, because for the assumption $C' \subseteq C$ and $A := C \backslash C'$: $v_i(C) = v_i(A \cup C') = v_i(A) + v_i(C') = \underbrace{v_i(C \backslash C')}_{\geq 0} + v_i(C') \geq v_i(C')$

4. Divisibility: For all $C \subseteq [0,1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, exist a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.

5. v_i is continuous: If $0 < x < y \leq 1$ with $v_i([0,x]) = \alpha$ and $v_i([0,y]) = \beta$, than for every $\gamma \in [\alpha, \beta]$ there exist a $z \in [x,y]$ so that $v_i([0,z]) = \gamma$.

6. Emptyness of single points: $v_i([x,x]) = 0$ for all $x \in [0,1]$.

Different Types of Fairness

Especially the fairness plays an important role in fair division. But what is fair? It can be seen as an valuation criteria of an allocation, which can be normalized and gives a possibility to compare different allocations. Usually the fairness criteria are distinguished between the following:

Definition. *(Proportional or simple fair)*
An allocation is <u>proportional (simple fair)</u>, if $v_i(X_i) \geq 1/n$ for each player $p_i \in P_N$.

Definition. *(Envy-freeness)*
An allocation is <u>envy-free</u>, if $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_N$.

Definition. *(Equity)*
An allocation is <u>equitable</u>, if $v_i(X_i) = \alpha$ for each player $p_i \in P_N$ and $0 \leq \alpha \leq 1$ [2].

Correlation between the Fairness-Properties

Theorem. *For all allocations:*

1. *Every envy-free allocation is proportional.*

2. *For two players an allocation is envy-free iff it is proportional.*

Proof.   1. Every envy-free allocation is proportional.

2. For two players an allocation is envy-free iff it is proportional.

$\square$

A slighly different criteria to valuate the performance of an allocation is the efficiency. The correlation between the fairness criterion and efficiency can be found in [CLPP10] and [BJK07].

Definition. *(Efficiency)*
An allocation $A = \{A_1, \ldots, A_n\}$ is <u>efficient (Pareto optimal)</u> if there is no other allocation $A' = \{A'_1, \ldots, A'_n\}$ such that $v_i(A_i) \leq v_i(A'_i)$ for all players $p_i \in P_n$ and for at least one player the inequality is strict.

Theorem. *For all allocations:*

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[2] If $\alpha = 1/n$ the allocation is called <u>exact</u>.

1. *Envy-freeness does not imply efficiency.*

2. *Pareto optimality does not ensure fairness*

Proof.   1. Envy-freeness does not imply efficiency.

2. Allocating all the cake to one player is Pareto optimal, but definitely not proportional and so not envy-free.

$\square$

2.2.2 Strategyproofness

In the wide spectrum of literature diffent definitions and specifications about the honesty of players occur. Since the origin is game-theoretic and the main research is done in mechanism design, which is a part of game theory, the start will be there. In practise players are usually selfish, and try to increase the value of their bundle. In order to do so, they may report false valuations on parts of the cake. The goal is to prevent those happening.

Definition. *(Risk Aversion) A player is <u>risk averse</u> if .*

Definition. *(Strategyproofness)*

Definition. *(Truthfully) An allocation is <u>truthful</u> if the value of the cake obtained by a player by reporting false is not greater than by reporting the truth.*

Definition. *(Truthfully in expectation) An allocation is <u>truthful in expectation</u> if there .*

Only such divisions are interesting, in which each player acts truthfully by following his best strategy Σ_i. It is common that if a player deviates from Σ_i he risks to loose the guaranty for his fair share. At the end of an allocation it can be proven whether every player got his fair share. Each protocol

Different Types of Protocols

Due to the fact that the protocols are going to be analysed in this work, it is very important to understand the types, structure and design of them.

Classes of Protocols

Informal: (Algorithm)
In mathematics, computer science and related subjects, an <u>algorithm</u> is an effective method for solving a problem, which can be denoted as a finite set of sequences.

Definition. *(Protocol (Cake-Cutting-Protocol))*
A <u>protocol (cake-cutting-protocol)</u> is an adaptive algorithm with a fixed number of players and the following properties:

- *It consists of rules and strategies.*
  <u>Rules</u> are requirements, which has to be followed by the players without knowledge of their valuations.
  <u>Strategies</u> are recommendations, which can be followed for getting the guaranteed fair share.

- *If a player does not follow the strategy of a protocol, he looses his guarantee to get a fair piece of cake after a fixed number of steps. His actions does not harm the other players.*

- *Each player should be able to cut the cake at a required moment independent of other players.*

- *The protocol has no information about the valuation of the players, except of those, it got from the steps before. And so cannot prove whether a player follows the strategy of the protocol.*

Example. *Cut and Choose*

Cut & **Choose for** $n = 2$			
Rules	***Player*** p_1 ***strategy***	***Player*** p_2 ***strategy***	
1. Player p_1 partition the cake X into two pieces $\{X', X - X'\}$	*Partition X into two pieces of equal size*		
2. Player p_2 chooses one piece		*Choose the bigger piece*	
3. Player p_1 get the remaining piece			

Theorem. *Cut & choose is strategyproof.*

Proof. **Options for not following the recomended strategy:**

- Player p_2 take the smaller piece. This can not be in his intention, because then he is less satiesfied after the basic assumption.

- Player p_1 cut the cake into two not equal pieces. The chance to get less is equal to the chance to get more of the cake. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case: $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$ and in the dishonest case: $\frac{1}{2} \cdot X' + \frac{1}{2} \cdot (X - X') = \frac{1}{2} \cdot \underbrace{X}_{=1} = \frac{1}{2}$. It is defined, that in the case with equal outcomes the player would stay honest.

$\square$

Definition. *A cake-cutting protocol is called <u>proportional</u> or <u>envy-free</u>, if independent of the valuation of the players, each allocation is proportional or envy-free under the requirement, that all players follows the rules and strategies given by the protocol.*

The development of such protocols is one of the main goals of cake-cutting.

Definition. *(finite (discrete)/ continuous)*
A <u>finite (discrete)</u> protocol gives a solution after a finite number of queries (valuations, marks, . . .), in comparison in a <u>continuous</u> protocol has a player to make infinitely many queries.

Definition. *(finite bounded/ finite unbounded))*
A <u>finite bounded</u> protocol has an upper bound of steps for all possible valuations. The number of those steps is only correlated, in some cases, with the number of players. A <u>finite unbounded</u> protocol has no approximated number of steps.

Definition. *(Moving Knife))*
A <u>moving knife</u> protocol ...

The most desirable protocols are the finite bounded, because they are the easiest to realise in real life applications.

2.3   The Degree of Guaranteed Envy-Freeness

In the last sixty years the number of proportional finite bounded protocols have grown for arbitrary n.
But still no envy-free finite bounded protocol for an arbitrary n exist. The biggest group we can divide a cake in a fixed number of steps, so that it is envy-free, is sadly three.
As a compromise between envy-freeness and finite boundness and a possibility to value proportional cake-cutting protocols between each other is the degree of guaranteed envy-freeness from [LR09].

Definition. *For an allocation of the cake $X = \bigcup\limits_{i=1}^{n} X_i$ for the players $P = \{p_1, \ldots, p_n\}$:*

- *An <u>envy relation</u> $\Vdash$ is a binary relation on P : p_i envies p_j ($p_i \Vdash p_j$), $1 \leq i, j \leq n$, $i \neq j$, if $v_i(X_i) < v_i(X_j)$.*

- *An <u>envy-free relation</u> $\nVdash$ is a binary relation on P : p_i not envies p_j ($p_j \nVdash p_j$) ,$1 \leq i, j \leq n$,$i \neq j$, if $v_i(X_i) \geq v_i(X_j)$.*

<u>Properties of $\Vdash$ and $\nVdash$</u>:

- $\Vdash$ is irreflexive.

- $\nVdash$ is reflexive.

- $\Vdash$ and $\nVdash$ are not transitive

Proof.   - $v_i(X_i) < v_i(X_i)$ is never possible.

- $v_i(X_i) \geq v_i(X_i)$ is always satisfied, but the trivial relation $p_i \nVdash p_i$ does not count.

- Let $p_i \Vdash p_j$ and $p_j \Vdash p_k$, but about $v_i(X_k)$ nothing can be concluded: $p_i \nVdash p_k$ is still possible

$\square$

$\Rightarrow$ No proposition can be concluded about symmetry. So the following is possible:

1. Two-Way-Envy: $p_i \Vdash p_j$ and $p_j \Vdash p_i$
   (Exchange of the portions makes both happy.)

2. Two-Way-Envyfreeness: $p_i \nVdash p_j$ and $p_j \nVdash p_i$
   (No exchange necessary.)

3. One-Way-Envy: $p_i \Vdash p_j$ and $p_j \nVdash p_i$
   One-Way-Envyfreeness: $p_j \Vdash p_i$ and $p_i \nVdash p_j$

<u>DGEF</u> = Smallest number of envy-free-relations for all possible valuations.

Protocol. *Chomsky and Turing use cut & choose at a conference with n other scientists.*
$DGEF$: $2(n-1) + (n-2)(n-2) = 2n - 2 + n^2 - 4n + 4 = n^2 - 2n + 2$

Theorem. 1. *Each envy-free cake-cutting protocol with $n \geq 1$ players has a $\boldsymbol{DGEF} = n(n-1)$.*

2. *Let $d(n)$ be the notation for the $\boldsymbol{DGEF}$ of a proportional cake-cutting protocols with $n \geq 2$ players. Then: $n \leq d(n) \leq n(n-1)$.*

NEU MACHEN!!!

Proof. 1. $p_i \nVdash p_i$ for all i, $1 \leq i \leq n$, can be ignored, so has each of the n player to each other player, an envy-free-relation, summarized to $n(n-1)$.

2. $n = 2$ Obviosly: $d(2) = 2$, because the cake-cutting protocol is proportional:
$v_1(X_1) \geq \frac{1}{2}$ and $v_2(X_2) \geq \frac{1}{2} \Rightarrow v_1(X_1) \geq v_1(X_2)$ and $v_2(X_2) \geq v_2(X_1)$

$n \geq 3$ Because $p_i \nVdash p_i$ for all i are ignored, $d(n) \leq n(n-1)$.

In a proportional allocation:
$v_i(X_i) \geq \frac{1}{n}$ for $1 \leq i \leq n$.

$\Rightarrow$ None of the n players can simultaneously envy all other players, because:
In the contrary assumption: $p_1 \nVdash p_2$
$\Rightarrow v_1(X_2) > v_1(X_1) \geq \frac{1}{n}$
$\Rightarrow v_1((X - X_1) - X_2) < \frac{n-2}{n}$
$\Rightarrow (X - X_1) - X_2$ cannot be allocated in $n - 2$ portions, so that $v_i(X_j) \geq \frac{1}{n}$ for all $j, 3 \leq j \leq n$.
$\Rightarrow$ it exists a $j, 3 \leq j \leq n$, so that $v_i(X_j) < \frac{1}{n}$.
$\Rightarrow p_i \nVdash p_j$

So has each of the n players at least one guaranteed envy-free-relation to an other player: $n \leq d(n)$

$\square$

STOP!

Theorem. *If the rules/strategies of a proportional cake-cutting protocols with $n \geq 2$ player never require from a player to value the portion of an other player the $\boldsymbol{DGEF} = n$.*

Proof. $n = 2$ Proportionality $\Rightarrow$ Envy-freeness
$\boldsymbol{DGEF} = 2 \cdot 1 = n(n-1)$

$n \geq 3$ Imagine the following case: For a given allocation $X = \bigcup\limits_{i=1}^{n} X_i$, which is proportional, but require no other limitations, lets assume the following valuations:
For each $i, 1 \leq i \leq n$, value of p_i:

- the own piece X_i with $v_i(X_i) = \frac{1}{n} = \frac{n}{n^2} \Rightarrow$ proportional!
- the piece X_j of a player $p_j, j \neq i : v_i(X_j) = \frac{2}{n} < \frac{1}{n}$
- each of the $n - 2$ other pieces X_k of the players $p_k, |i, j, k| = 3, v_i = (X_k) = \frac{n+1}{n^2} > \frac{1}{n}$

In summary for each $i, 1 \le i \le n$:

1. $v_i(X) = v_i(\bigcup\limits_{j=1}^{n} X_j) \stackrel{\text{Additivity}}{=} \sum\limits_{j=1}^{n} v_i(X_j) = \frac{1}{n^2}(n + 2 + (n-2)(n+1)) = \frac{1}{n^2}(n + 2 + n^2 + n - 2n - 2) = 1$

2. p_i has $n - 2$ envy-relations and only one envy-free-relation
   $\Rightarrow$ There are n guaranteed envy-free-relations, each for every player.

$\square$

Overview: DGEF of proportional cake-cutting protocols

Protocol	DGEF	$n = 4$ (12 EF)
Last Diminisher	$2 + \frac{n \cdot (n-1)}{2}$	8
Lone Chooser	n	4
Lone Divider	$2n - 2$	6
Cut your Own Piece	n	4
Cut your Own Piece (left-right rule)	$2n - 2$	6
Divide & Conquer	$n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1}$	6
Minimal-Envy Divide & Conquer	$n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1}$	6
Recursive Divide & Choose	n	4
Parallel Last Diminisher	$\lceil \frac{n^2}{2} \rceil + 1$	9

Corollary. *The performance of the protocols have the following order for $n \ge 2$:*
$n \le 2n - 2 \le n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1} \le 2 + \frac{n(n-1)}{2} \le \lceil \frac{n^2}{2} \rceil + 1$

Proof. $n = 2$: $2 \le 2 \le 2 \le 3 \le 3$
$n \mapsto n + 1$
$n + 1 \le 2n \rightarrow 2n - n - 1 = n - 1 \ge 0$
$2n \le n \cdot \lfloor log(n+1) \rfloor + 2n + 2 - 2^{\lfloor log(n+1) \rfloor + 1} \rightarrow n \cdot \lfloor log(n+1) \rfloor + 2n + 2 - 2^{\lfloor log(n+1) \rfloor + 1} - 2n = n \cdot \lfloor log(n+1) \rfloor - 2^{\lfloor log(n+1) \rfloor} \ge 0$ $\square$

3 Proportional Protocols and their DGEF

Each proportional protocol consists of rules and strategies. Since the rules are compulsory, it is optional for the player to follow the strategies.

In this chapter the strategies of common used procedures are described. First of all, the protocols are rewritten into a game-theoretic manner. Then their strategyproofness is analysed. During complications the effect on the DGEF is shown. Complete protocols in the standart describtion can be found in [RW98].

3.1 The Steinhaus-Kuhn Lone Divider Procedure

3.2   The Banach-Knaster Last-Diminisher Procedure

The generalization of "I cut, you choose".

Theorem. *If the valuations of the players are not equal, there exist a non truthful strategy for the first player in each round of getting a piece with $v_i(S_i) > 1/N$.*

Proof. The player p_1 got the piece X_1. So $v_1(X_1) = 1/n$ and for all p_i with $2 \leq i \leq n$ $v_i(X_1) < 1/n$. W.l.o.g. in the assumption, let $\delta = 1/n - v_2(X_1)$. It is obvious that $\delta > 0$. Step 1: The piece which was cutted schould have the value $1/N + \epsilon$ with $\epsilon =$ residue of S_1.
Destinction of cases: Either player p_1 get this piece or player p_i cut it and so player p_1 can get a piece with same properties in the next roundor during Cut and Choose in the end. $\qquad\square$

Theorem. *The Last-Diminisher-protocol has a $\boldsymbol{DGEF}$ of $\frac{n(n-1)}{2} + 2$.*

3.3 The Fink Lone-Chooser Procedure

Theorem. *The Lone-Chooser procedure has a $\boldsymbol{DGEF} = n$.*

Proof. No player value a foreign piece $\overset{\text{Theorem}}{\Longrightarrow}$ $\boldsymbol{DGEF} = n$ ☐

For the description of this protocol between two groups of players will be separated. Players in the first group $P_I = p_1, \ldots, p_i$ in the $i - 1$ round have already their proportional piece. Assume that player p_1 owns the whole cake before the first round. The performance in each round is the following:

The Fink Lone-Chooser Procedure for arbitrary n		
Rules	**Players in P_I strategy**	**Player p_{i+1} strategy**
1. Players P_I partition their piece X_i into $i + 1$ pieces $\{X_{i,1}, \ldots, X_{i,i+1}\}$	Partition X_i into $i + 1$ pieces of equal size	
2. Player p_{i+1} chooses one piece of each player		Choose the biggest piece of each player

Theorem. *Fink Lone-Chooser Procedure is strategyproof.*

Proof. **Options for not following the recomended strategy:**

- Player p_{i+1} take the smaller piece. This can not be in his intention, because then he is less satiesfied after the basic assumption.

- Players in P_I cut the cake into $i + 1$ not equal pieces. The chance that player p_{i+1} take a certain piece is $\frac{1}{i+1}$. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case at least: $i \cdot \frac{1}{i+1} \cdot \frac{1}{i} = \frac{1}{i+1}$ (because he had at least $\frac{1}{i}$ before) and in the dishonest case:
Assume the value of the piece will be distributed on s pieces with $1 \leq s \leq i + 1$. Then $\frac{1}{i} \cdot \frac{i+1-s}{i+1} + \frac{s-1}{si} \cdot \frac{s}{i+1} = \frac{1}{i+1}$. Even if the value will be distributed inequally on the s pieces, it would not affect the outcome, because the probability of the piece the player p_{i+1} take is uniformly distributed.

It is defined, that in the case with equal outcomes the player would stay honest.

☐

3.4   The Cut-Your-Own-Piece Procedure

3.5 The Divide-and-Conquer Procedure

3.6   The Extended Procedure

3.7   The Extension of the Extended Procedure

4 Related Work

5   Conclusions and Open Questions/Problems

To take a closer look on the DGEF in the egalitarien point of view. Since a protocol is only fair if every player get his fair share, it seems to be intuitive more fair, if $n-1$ player envies one rather than one player envies all other. The definition could be:

MDGEF= minimal guaranteed degree of envy-freeness = $\min\limits_{i\in\mathbb{N}}\{\sum\limits_{j\in\mathbb{N},i\neq j} p_i \nVdash p_j\}$

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