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Game-theoretic Analysis of Strategyproofness in Cake-cutting Protocols

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Bachelorarbeit

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Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

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Abstract

Contents

1 Introduction

Strategic play, cheating, incentive compatible, risk aversion, truthfulness, strategyproof and a lot more. All of them are keywords of whether an algorithm can resist the actions of selfish players and their greedyness.

1.1 Related Work

2 Preliminaries

2.1 Preliminaries of Game Theory

In this chapter some basic concepts from game theory are described and directly applied to the cake-cutting problem. The best fitting game-theoretic model is called bayesian game. The development to this result and use of the associated solution concepts is described. Game theory can be seen as a tool for describing interaction in lifes processes. The occuring problems can be simulated by games.

Definition. ***(Game)***
A non-cooperative (strategic) game $\Gamma = (P_n, S, u)$ *consists of the set of players* P_n *, the set of strategies* S *and the set of utility functions (pay-off) of all players* u_i*.*

- *Each player in the set* $P_n = \{p_1, \cdots, p_n\}$ *behave selfish and rational.*

Definition. ***(Strategies)*** *dominant, dominated, best response*

Definition. ***(Pay-off function)*** *A Egalitarian and utiletarian*

Classes of Special Games

Definition. ***(Zero-Sum Game)*** *A*

Cutting a cake is a non-zero-sum game, since it allows players to get better off than 1/n-th. An exception is a homogenius cake, then the valuations over the cake are equal and the the game zero-sum.

Definition. ***(Bayesian Game)*** *A*

A game can be represented in normal or in extended form.

2.2 Preliminaries of Cake-cutting

2.2.1 Basics

First, it is necessary to define the components of cake-cutting. Example **??** describes the problem.

Example. *Cocke, Younger and Kasami have a cheese-chocolate-straberry-cake.*

Now, what exactly is cake-cutting about? First of all, it involves a discrete set of $n \in \mathbb{N}$ agents (or players) $P_n = \{p_1, \cdots, p_n\}$. It is assumed that each of them wants to get as much as possible of the divided good. The goal is to allocate it in a manner that each player is satisfied. The allocation of a single, divisible and heterogeneous good is included

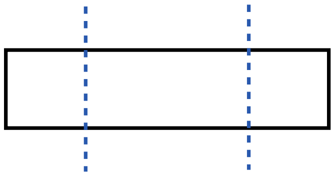

Figure 1: Cake
Example for a visualisation of a cake and two cuts

in the consideration of this work.
For the visualization it is common to use a rectangular cake. The division is performed by parallel cuts. The cake X is represented by the unit interval $I = [0,1] \subseteq \mathbb{R}$. Each disjoint subinterval $I' \subseteq I$ or a union of such subintervals

$$\bigcup_{m \in \mathbb{N}} I_m$$

with $I_m \subseteq I$ is called a bundle (or piece). The bundle of the cake, which the player p_i receives is denoted as X_i. The state is called an allocation, when all bundles of the cake are owned by players. Each piece has a public length, which can be computed as the sum of all border differences, and the private value of each player.

Every player $p_i \in P_n$ has a valuation function (valuation) $v_i : \{X' | X' \subseteq X\} \to [0,1]$ on the cake X with the following properties:

1. Non-negativity: $v_i(C) \geq 0$ for all $C \subseteq [0,1]$.
2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0,1]) = 1$.
3. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq [0,1]$.[1]
4. Divisibility: For all $C \subseteq [0,1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, exists a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.
5. v_i is continuous: If $0 < x < y \leq 1$ with $v_i([0,x]) = \alpha$ and $v_i([0,y]) = \beta$, then for every $\gamma \in [\alpha, \beta]$ there exists a $z \in [x,y]$ so that $v_i([0,z]) = \gamma$.
6. Non-atomic: $v_i([x,x]) = 0$ for all $x \in [0,1]$.

Different Types of Fairness

Especially the fairness plays an important role in fair division. But how is fairness defined? It can be seen as a valuation criteria of an allocation, which can be normalized and gives a possibility to compare different allocations. Usually the fairness criteria are distinguished between the following:

[1]Monotonicity: If $C' \subseteq C$ then $v_i(C') \leq v_i(C)$ follows from additivity, because for the assumption $C' \subseteq C$ and $A := C \backslash C'$: $v_i(C) = v_i(A \cup C') = v_i(A) + v_i(C') = \underbrace{v_i(C \backslash C')}_{\geq 0} + v_i(C') \geq v_i(C')$.

Definition. ***(Proportional or Simple Fair)***
An allocation is proportional (simple fair), if $v_i(X_i) \geq 1/n$ for each player $p_i \in P_N$.

Definition. ***(Envy-Freeness)***
An allocation is envy-free, if $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_N$.

Correlation between the Fairness-Properties

Theorem. *For all allocations:*

1. *Every envy-free allocation is proportional.*
2. *For two players an allocation is envy-free iff it is proportional.*

Proof.

1. Proof by contradiction:
   Assume A is an envy-free allocation, but not proportional. From envy-freeness follows $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_N$ and so each player has at least as much cake as each other player. Hereby each player owns at least as much as $(n-1)$ other players and so at least $1/n$. The allocation A is proportional. Therefore all envy-free allocations are proportional.
2. "$\Rightarrow$" For two players an allocation is proportional if each player has at least the half ot the cake. So the first player thinks the second player got at most half of the cake and vice versa. They would not envy each other.
   "$\Leftarrow$" Follows from part 1.

□

A slighly different criteria to valuate the performance of an allocation is the efficiency. The correlation between the fairness criterion and efficiency can be found in [?].

Definition. ***(Efficiency)***
An allocation $A = \{X_1, \ldots, X_n\}$ is efficient (Pareto optimal) if there is no other allocation $A' = \{X_1', \ldots, X_n'\}$ such that $v_i(X_i) \leq v_i(X_i')$ for all players $p_i \in P_n$ and for at least one player the inequality is strict.

Theorem. *For all allocations:*

1. *Proportionality does not imply efficiency.*
2. *Efficiency does not ensure fairness.*

Proof.

1. Imagine the following allocation with three players:

	X_1	X_2	X_3
p_1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
p_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
p_3	0	$\frac{7}{18}$	$\frac{11}{18}$

This allocation is obviously proportional, since $v_i(X_i) \geq 1/3$ for all $i \in \{1, 2, 3\}$. It is not efficient, because if the players p_1 and p_2 exchange their pieces, p_2 would get more of the cake and p_1 would not get less.

2. Allocating the whole cake to one player is efficient, but definitely not proportional and therefore not envy-free.

$\square$

In [?] the authors showed a general argument that no finite bounded procedure can exist for such an allocation that is both, proportional and efficient at the same time.

2.2.2 Strategyproofness

In the wide spectrum of literature different definitions and specifications about the honesty of players occur. Since the origin is game-theoretic and the main research is done in mechanism design, which is a part of game theory, the most results are from there. Nevertheless some definitions adapted to cake-cutting exist.
In practice players are selfish and try to increase the value of their bundle. In order to do so, they may report false valuations on parts of the cake. The goal is to prevent this.

Definition. ***(Risk Aversion)****([?]) A player is risk averse if he or she will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a piece of a guaranteed size.*

Definition. ***(Strategyproofness of a Proportinal Protocol)****([?])*
A proportional cake-cutting protocol is said to be strategyproof for risk averse players if a cheating player is no longer guaranteed a proportional share, whereas all other players (provided they play truthfully) are still guaranteed to receive their proportional share.

Definition. ***(Truthfully)*** *An allocation is truthful if the value of the cake obtained by a player by reporting false is not greater than by reporting the truth.*

Definition. ***(Strategyproofness)****([?])*
A procedure is strategyproof if no player has a strategy that dominates his true value function.

A stronger condition from social choice literature:

Definition. ***(Strategyproofness)****([?])*
A procedure is strategyproof if the true value function dominates every other strategy.

In order to prevent misunderstandings in this paper it will be called strong strategyproofness. It can be shown, that cake-cutting is not able to fulfill the strong strategyproofness criteria. The following example is similar to the one in [?].

Example. *John Warner Backus and Peter Naur are celebrating and Donald E. Knuth has brough a huge marzipan cake with an enormous cherry on top. John loves cherries and hate marzipan, and Peter is just very hungry. The pioneers of computer science apply cut & choose* [2]*. Peter is the cutter, and his best strategy is to separate the cake from the*

[2] see next page

cherry. And if Peter would have the fully knowledge (which should be no problem for the strategyproofness in [?]) about the valuations of John, he would always benefit by lying and so this algorithm is not strong strategyproof.

Different Types of Protocols

It is very important to understand the types, structure and design of protocols, which will be analysed in this paper.

Informal: (Algorithm)
In mathematics, computer science and related subjects, an algorithm is an effective method for solving a problem, which can be denoted as a finite set of sequences.

Definition. ***(Protocol (Cake-Cutting-Protocol))***
A protocol (cake-cutting-protocol) is an adaptive algorithm with a fixed number of players and the following properties:

- *It consists of rules and strategies.*
  Rules are requirements, which have to be followed by the players without knowledge of their valuations.
  Strategies are recommendations, which can be followed for getting the guaranteed fair share.
- *If a player does not follow the strategy of a protocol, he looses his guarantee to get a fair piece of cake after a fixed number of steps. His actions do not harm the other players.*
- *Each player should be able to cut the cake at a specific moment independent of other players.*
- *The protocol has no information about the valuation of the players, except of those it got from the steps before. It can not prove whether a player follows the strategy of the protocol.*

Example. *(Representation idea taken from [?])*

Cut & choose for $n = 2$		
Rules	***Player*** p_1 ***strategy***	***Player*** p_2 ***strategy***
1. Player p_1 *partition the cake* X *into two pieces* $\{X', X - X'\}$	*Partition* X *into two pieces of equal size*	
2. Player p_2 *chooses one piece*		*Choose the bigger piece*
3. Player p_1 *get the remaining piece*		

Theorem. *Cut & choose is strategyproof.*

Proof. **Options for not following the recomended strategy:**

- Player p_2 takes the smaller piece. This can not be his intention, because then he is less satiesfied after the basic assumption.
- Player p_1 cuts the cake into two not equal pieces. The chance to get less is equal to the chance to get more of the cake. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case:

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

and in the dishonest case:

$$\frac{1}{2} \cdot X' + \frac{1}{2} \cdot (X - X') = \frac{1}{2} \cdot \underbrace{X}_{=1} = \frac{1}{2}$$

According to the definition of truthfulness in the case with equal outcomes the player would stay honest.

□

Definition. ***(Proportional/ Envy-Free Protocol)***
A cake-cutting protocol is called proportional or envy-free, if independent of the players' valuation, each allocation is proportional or envy-free under the requirement, that all players follow the rules and strategies given by the protocol.

The development of such protocols is one of the main goals of cake-cutting [**?**].

Definition. ***(Finite (Discrete)/ Continuous (Moving-Knife))***
A finite (discrete) protocol gives a solution after a finite number of queries (valuations, marks, ...). In a continuous (a.k.a. moving-knife) protocol a player has to make up to infinitely many queries.

Definition. ***(Finite Bounded/ Finite Unbounded))***
A finite bounded protocol has an upper bound of steps for all possible valuations. The number of those steps is only correlated, in some cases, with the number of players. A finite unbounded protocol has no approximated number of steps.

The most desirable protocols are the finite bounded, because of the ease of their implementation.

2.3 The Degree of Guaranteed Envy-Freeness

In the last sixty years the number of proportional finite bounded protocols have grown for arbitrary n. But still no envy-free finite bounded protocol for an arbitrary n is known (compare [**?**]). The biggest number of players a cake can be divided for in a fixed number of steps, so that it is envy-free, is three.
As a compromise between envy-freeness and finite boundness and a possibility to value proportional cake-cutting protocols between each other is the degree of guaranteed envy-freeness from [**?**].

Definition. ***(Envy-/ Envy-Free-Relation)***
For an allocation of the cake $X = \bigcup_{i=1}^{n} X_i$ *for the players* $P = \{p_1, \ldots, p_n\}$*:*

- *An envy-relation* $\Vdash$ *is a binary relation on* P : p_i *envies* p_j $(p_i \Vdash p_j)$, $1 \leq i, j \leq n$, $i \neq j$, *if* $v_i(X_i) < v_i(X_j)$.
- *An envy-free-relation* $\nVdash$ *is a binary relation on* P : p_i *not envies* p_j $(p_j \nVdash p_j)$, $1 \leq i, j \leq n$,$i \neq j$, *if* $v_i(X_i) \geq v_i(X_j)$.

Properties of $\Vdash$ and $\nVdash$:

- $\Vdash$ is irreflexive.
- $\nVdash$ is reflexive.
- $\Vdash$ and $\nVdash$ are not transitive.

Proof.

- $v_i(X_i) < v_i(X_i)$ is not possible.
- $v_i(X_i) \geq v_i(X_i)$ is fulfilled, but the trivial relation $p_i \nVdash p_i$ will not count.
- Let $p_i \Vdash p_j$ and $p_j \Vdash p_k$, but nothing can be concluded about $v_i(X_k)$: $p_i \nVdash p_k$ is still possible.

□

No proposition can be concluded about symmetry. So the following is possible:

1. Two-Way-Envy: $p_i \Vdash p_j$ and $p_j \Vdash p_i$
(Exchange of the portions makes both happy.)
2. Two-Way-Envyfreeness: $p_i \nVdash p_j$ and $p_j \nVdash p_i$
(No exchange necessary.)
3. One-Way-Envy: $p_i \Vdash p_j$ and $p_j \nVdash p_i$
One-Way-Envyfreeness: $p_j \Vdash p_i$ and $p_i \nVdash p_j$

DGEF = Smallest number of envy-free-relations for all possible valuations.

Definition. ***(Degree of Guaranteed Envy-Freeness (DGEF))****([?])*
For $n \geq 1$ *players, the degree of guaranteed envy-freeness (DGEF) of a given cake-cutting protocol is defined to be the maximum number of envy-free-relations that exist in every division obtained by this protocol (provided that all players follow the rules and strategies of the protocol), i.e., the DGEF (which expressed as a function of n) is the number of envy-free-relations that can be guaranteed.*

Protocol. *Chomsky and Turing use cut & choose at a conference with* n *other scientists.*
DGEF :

$$\underbrace{2}_{Chomsky\,and\,Turing} \cdot \underbrace{(n-1)}_{do\,not\,envy\,other\,scientists} + \underbrace{(n-2)}_{other\,scientists} \cdot \underbrace{(n-2)}_{do\,not\,envy\,other\,scientists} =$$

$$2n - 2 + n^2 - 4n + 4 = n^2 - 2n + 2$$

Theorem. *1. Each envy-free cake-cutting protocol with $n \geq 1$ players has a* $\boldsymbol{DGEF} = n(n-1)$.

2. Let $d(n)$ be the notation for the $\boldsymbol{DGEF}$ *of a proportional cake-cutting protocols with $n \geq 2$ players. Then: $n \leq d(n) \leq n(n-1)$.*

Proof. 1. Since $p_i \nVdash p_i$ for all i, $1 \leq i \leq n$ will not count, the number of envy-free-relations adds up to

$$\underbrace{\sum_{1\geq i\geq n} \sum_{\substack{1\geq j\geq n\\ i\neq j}}}_{(n-1)} |(p_i \nVdash p_j)| = \underbrace{(n-1)+\cdots+(n-1)}_{n\ times} = n(n-1).$$

2. The upper boundary follows from part 1. For the lower bound consider the following situation:

$n = 2$ The cake-cutting protocol is proportional and envy-free: $d(2) = 2$

$n \geq 3$ Take a look at the biggest number of envy-relations in a proportional allocation with $v_i(X_i) \geq \frac{1}{n}$ for $1 \leq i \leq n$.

Contrary assumption: $p_i \Vdash p_j$ for $1 \geq i, j \geq n$ and $i \neq j$

In that case $p_1 \Vdash p_j$ for $2 \geq j \geq n$.

$\Rightarrow$ $v_1(X_1) = \frac{1}{n}$, because the allocation is proportional.

$\Rightarrow$ $\frac{1}{n} = v_1(X_1) < v_1(X_j) = \frac{a_j}{n}$ for $2 \geq j \geq n$.

Take $a = \min\limits_{2\geq j\geq n}\{a_j\}$ with $a_j > 1$ for all $2 \geq j \geq n$.

$\Rightarrow$ $v_1(X_1) + v_1(X_2) + \cdots + v_1(X_n) = \frac{1}{n} + (n-1)\cdot\frac{a}{n} > 1$ WS!

$\Rightarrow$ It exists at least one $j, 2 \leq j \leq n$, so that $v_i(X_j) < \frac{1}{n}$.

$\Rightarrow p_1 \nVdash p_j$

Each of the n players has at least one guaranteed envy-free-relation to another player: $n \leq d(n)$

□

Theorem. *If the rules/strategies of a proportional cake-cutting protocols with $n \geq 2$ player do not require to value the portion of an other player from a player the* $\boldsymbol{DGEF} = n$.

Proof. $n = 2$ Proportionality $\Leftrightarrow$ Envy-freeness

DGEF $= 2 \cdot 1 = n = n(n-1)$

$n \geq 3$ Imagine the following case: For a given allocation $X = \bigcup\limits_{i=1}^{n} X_i$, which is proportional, but requires no other limitations, let assume the following valuations:

For each $i, 1 \leq i \leq n$, value of p_i:

- the own piece X_i with $v_i(X_i) = \frac{1}{n} = \frac{n}{n^2} \Rightarrow$ proportional!
- the piece X_j of a player $p_j, j \neq i : v_i(X_j) = \frac{2}{n^2} < \frac{n}{n^2} = \frac{1}{n}$

- each of the $n-2$ other pieces X_k of the players $p_k, |i,j,k| = 3, v_i = (X_k) = \frac{n+1}{n^2} > \frac{1}{n}$

In summary for each $i, 1 \leq i \leq n$:

1. $v_i(X) = v_i(\bigcup_{j=1}^{n} X_j) \overset{\text{Additivity}}{=} \sum_{j=1}^{n} v_i(X_j) = \frac{1}{n^2}(n+2+(n-2)(n+1)) = \frac{1}{n^2}(n+2+n^2+n-2n-2) = 1$
2. p_i has $n-2$ envy-relations and only one envy-free-relation
   $\Rightarrow$ There are n guaranteed envy-free-relations, each for every player.

$\square$

Overview: DGEF of proportional cake-cutting protocols for $n \geq 3$[3]

Protocol	DGEF	$n = 5$ (20 EF)
Last Diminisher	$2 + \frac{n \cdot (n-1)}{2}$	12
Lone Chooser	n	5
Lone Divider	$2n-2$	8
Cut your Own Piece	n	5
Cut your Own Piece (left-right rule)	$2n-2$	8
Divide & Conquer	$n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1}$	8
Minimal-Envy Divide & Conquer	$n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1}$	8
Parallelized Last Diminisher	$\lceil \frac{n^2}{2} \rceil + 1$	14

Corollary. *The performance of the protocols have the following order for $n \geq 3$:*
$n < 2n-2 \leq n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1} < 2 + \frac{n(n-1)}{2} < \lceil \frac{n^2}{2} \rceil + 1$

[3]For $n = 1$ the DGEF $= 0$ and for $n = 2$ the DGEF $= 2$. The case $n = 4$ is special, compare [?]

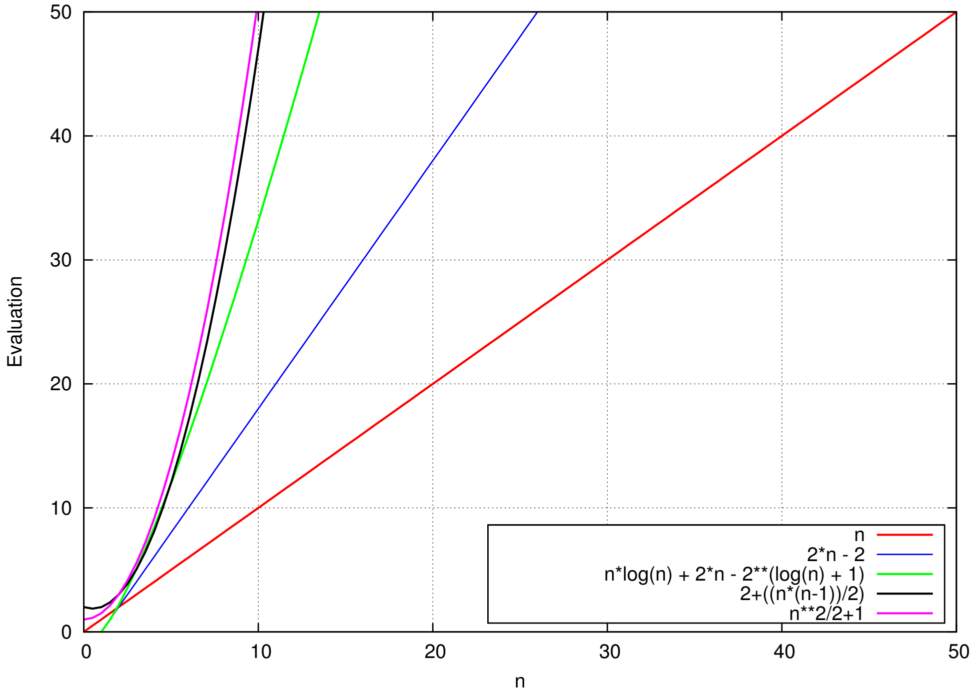

Figure 2: DGEF

3 Proportional Protocols and their DGEF

Each proportional protocol consists of rules and strategies. Since the rules are compulsory, it is optional for the player to follow different strategies.
In this chapter the strategies of commonly used procedures are described. First of all, the protocols are rewritten into the game-theoretic manner. Then their strategyproofness is analysed. Based on appearing complications the effect on the DGEF will be shown. The complete protocols in the standart describtion can be found in [**?**].

3.1 The Steinhaus-Kuhn Lone Divider Procedure

3.2 The Banach-Knaster Last-Diminisher Procedure

Theorem. *If the valuations of the players are not equal, there exists a non truthful strategy for the first player in each round* k *with* $2 \leq k \leq (n-2)$ *of getting a piece with* $v_i(S_i) > 1/n$.

Proof. Assume n is the fixed number of players before the game. W.l.o.g the player p_1 gets the piece X_1. So $v_1(X_1) = 1/n$ and for all p_i with $2 \leq i \leq n$ $v_i(X_1) < 1/n$. W.l.o.g. in the assumption consider the case for player p_2, let $\delta = 1/n - v_2(X_1)$. It is obvious that $\delta > 0$, because otherwise this player would not be in the game anymore.
Step 1: The piece player p_2 cut should have the value $1/n + \epsilon$ with $0 < \epsilon < \delta$.
Destinction of cases: Either player p_2 gets this piece or player p_j with $3 \leq j \leq n$ cuts it and thus player p_2 can get a piece with same properties in the next round or during cut

& choose in the end. So even if player p_2 acts non truthfully in the sence of definition of strategyproofness in [?] he will get more than his proportional share. □

Theorem. *The Last-Diminisher-protocol has a* ***DGEF*** *of* $\frac{n(n-1)}{2} + 2$.

The Banach-Knaster Last-Diminisher Procedure for arbitrary n		
Rules	**Players p_i strategy**	**Player $p_{>i}$ strategy**
i-th round: Player p_i cut a piece I_i	Cut a piece with value $\frac{1}{n}$	
Players p_j trim or pass		If $v_j(I_i) > \frac{1}{n}$ trim ($I_j = I_i - \epsilon$ and $v_j(I_j) = \frac{1}{n}$), else pass
Last trimmer take it		
$n-2$-th round: cut&choose		

Strategies: Better to stay in the game. (Proof?)

3.3 The Fink Lone-Chooser Procedure

Theorem. *The Lone-Chooser procedure has a* ***DGEF*** $= n$.

Proof. No player values a foreign piece $\overset{\text{Theorem}}{\Longrightarrow}$ **DGEF** $= n$ □

For the description of this protocol the players will be separated in two groups. Players in the first group $P_I = \{p_1, \ldots, p_i\}$ in the $i-1$ round have already their proportional piece. Assume that player p_1 owns the whole cake before the first round. The performance in each round is the following:

The Fink Lone-Chooser Procedure for arbitrary n		
Rules	**Players in P_I strategy**	**Player p_{i+1} strategy**
1. Players P_I partition their piece X_i into $i+1$ pieces $\{X_{i,1}, \ldots, X_{i,i+1}\}$	Partition X_i into $i+1$ pieces of equal size	
2. Player p_{i+1} chooses one piece of each player's cake		Choose the biggest piece of each player's cake

Theorem. *Fink Lone-Chooser protocol is strategyproof.*

Proof. **Options for not following the recommended strategy:**

- Player p_{i+1} takes the smaller piece. This can not be his intention, because then he is less satiesfied following the basic assumption.
- Players in P_I cut the cake into $i+1$ not equal pieces. The chance that player p_{i+1} take a certain piece is $\frac{1}{i+1}$. In stochastic terms it means, that the expected value at the end of the allocation will be at least in the honest case: $i \cdot \frac{1}{i+1} \cdot \frac{1}{i} = \frac{1}{i+1}$ (because he had at least $\frac{1}{i}$ before) and in the dishonest case:
  Assume the value of the piece will be distributed on s pieces with $1 \leq s \leq i+1$.

So the expected value will be $\frac{1}{i} \cdot \frac{i+1-s}{i+1} + \frac{s-1}{si} \cdot \frac{s}{i+1} = \frac{1}{i+1}$. Even if the value is distributed inequally on the s pieces, it would not affect the outcome, because the probability of the piece the player p_{i+1} take is uniformly distributed.

In the case with equal outcomes the player would stay honest according to the definition of truthfulness.

□

3.4 The Cut-Your-Own-Piece Procedure

Cut-Your-Own-Piece Procedure for arbitrary n	
Rules	**Players strategy**
1.Make $n-1$ marks	Each piece of size $\frac{1}{n}$
The leftmost (& rightmost) piece get allocated	

3.5 The Divide-and-Conquer Procedure

Divide-and-Conquer Procedure for arbitrary n	
Rules	**Players strategy**
1.Make one mark	Mark in the ratio of $\lfloor \frac{n}{2} \rfloor : \lceil \frac{n}{2} \rceil$
2.Cut in the ratio of the median player and repeat untill each player have its own piece	

Interesting: Since the pay-off can be maximized from collecting players with different valuations, is it possible to use this advantage in this procedure? Or get such information from the sequence?

3.6 The Parallelized Last Diminisher Procedure

4 Conclusion

In this work the common proportional cake-cutting protocols have been rewritten in a game-theoretic manner and analysed on whether a non truthful strategy could yield to a more advantageous situation for a non truthful player. It was possible to approve game-theoretically that the only strategy promissing the best outcome, is the strategy recommended by the protocol.
The approach was very different from [**?**] and [**?**]. They weakened the basic assumptions of cake-cutting and so only analysed special cases. Instead of this, the goal here, was to adapt the existing definitions of truthfulness to the generally cake-cutting problem.

5 Open Questions and Future Research

It would be interesting to take a closer look on the DGEF in the egalitarian point of view. Since a protocol is only fair if every player gets his fair share, it seems to be intuitivly more fair, if $n-1$ player envies one rather than one player envies all of the other players. The definition could be:

MDGEF= minimal guaranteed degree of envy-freeness = $\min_{i\in\mathbb{N}}\{\sum_{j\in\mathbb{N},i\neq j} p_i \nsucceq p_j\}$

Compare Brams, Jones and Klamler (2007) individual envy-relation.

Example. *(Cooperative Cake-Cutting)*
After the win of the election in Turingville the member of President Church coalition want to divide succesfully the power in the country. Some of the members are aware of beeing unsatisfied with the outcome and so cooperate mutually to change the allocation.

An interesting aspect would be groupstrategyproofness. Hereby groups have public valuations for group members.

To study the actions of players if several cakes have to be allocated consecutively and so the cake-cutting-game interpreted as a repeated game can also be observed.

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