Envy Free Division

1 Introduction

Whenever we say something like *Alice has a piece worth 1/2* we mean worth 1/2 TO HER.

Lets say we want Alice, Bob, Carol, to split a cake so that each one thinks they got $\geq 1/3$ and that nobody else got a bigger piece. This is called an *envy-free division*.

2 A 3-Person Discrete Envy Free Protocol

Theorem 2.1 *There is a discrete protocol for three people to achieve an envy-free division.*

Proof:

The algorithm is in two phases.

PHASE ONE:

1. Alice cuts the cake into three pieces. (All equal.)
2. Bob trims a piece or not. Put the trimming aside. (Trim to create a tie for the top two pieces.)
3. There are now three pieces and possibly some Trimming. Call the pieces P_1, P_2, P_3 and the trimming T.
4. Carol takes one of P_1, P_2, P_3. (The most valuable piece.)
5. Bob takes one of the pieces that are left. However, if the trimmed piece is left he must take it. (The biggest piece available.)
6. Alice takes the remaining piece.

Claim: If all players play honestly then the division of P_1, P_2, P_3 is Envy-free. (Note that we still need to deal with T).

Proof of Claim:

If Bob did not trim a piece then he thinks that two pieces are tied for first. If Bob did trim a piece then, since he trimmed it, he thinks that two pieces (now) are tied for first.

Carol will get first pick of P_1, P_2, P_3, so she cannot feel envy. Bob will get second pick but he thinks that two of the pieces were tied, so he cannot feel envy. If there was a trimmed piece then either Carol or Bob got that trimmed piece. Hence Alice will get one of the untrimmed pieces. Since Alice originally cut the pieces equally, and she gets an untrimmed piece, she cannot feel envy.

End of Proof of Claim

If Bob does not trim a piece then Phase one gives us the Envy Free Division and we are done.

If Bob trimmed a piece then we do Phase two where we split the trimming. We assume that Bob got the trimmed piece (the situation where Carol got the trimmed piece is similar).

There is a KEY DIFFERENCE between the situation we have now and what we had in Phase one. Note that Alice thinks that Bob has a piece with some stuff missing. *If Bob gets some or even all of T then Alice cannot be jealous of him.*

PHASE TWO:

1. Carol cuts T into three pieces (equally).
2. Bob picks a piece. (Biggest piece)
3. Alice picks a piece. (Biggest piece left)
4. Carol picks a piece. (Whatever is left)

Claim: If all players play honestly then The division of T of Envy-free.

Proof of Claim:

Bob cannot feel envy since he got the first choice of pieces.

Alice is the interesting case. Alice cannot feel envy towards Bob since WHATEVER Bob gets, Alice just thinks he is making up for having a trimmed piece in the first place. Alice cannot feel envy for Carol since Alice got to pick before Carol.

Carol cannot feel envy since she cut T into three equal pieces.

End of Proof of Claim

Since both Phase one and Phase two are envy free, the entire procedure is envy free.

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Note that the above algorithm took at most five cuts.

Is there a discrete envy free algorithm for four players? five? n? The answer is YES. How many cuts do these take? There is no bound. Here is what we mean: For any number c there is a scenario so that the four-player protocol takes more than c cuts. This protocol is somewhat and will not be in these notes.

3 A 4-Person MK Envy Free Protocol

We present a four person MK envy free protocol. We need a lemma first

Lemma 3.1 *There is a MK protocol for two people to produce four pieces that they both think are worth exactly* $1/4$. *(We assume they are both honest. If they are not then while they may do well in THIS protocol they could do badly later when we use this as a sub-protocol.)*

Proof:

1. Alice holds TWO knives over the cake, one on the left edge. (The other such that the cake between the two knives is 1/2. Note that she must think that inside=outside=1/2.)
2. Alice moves both knives across the cake. (Such that what is in the middle is always 1/2.)
3. Bob yells STOP. (When he thinks inside=outside=1/2.)
4. They cut the cake. They both think that there are two pieces worth 1/2 each.
5. Repeat the procedure on each piece.

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Theorem 3.2 *There is a MK protocol for four people to achieve an envy-free division.*

Proof:

This algorithm is in two phases.

PHASE ONE:

1. Alice and Bob use the protocol from Lemma 3.1 to obtain four pieces they both think are worth exactly 1/4.
2. Carol trims a piece or not. (She wants to create a 2-way tie for best piece). The trimming is put aside for now.
3. Let the pieces be P_1, P_2, P_3, P_4. Let the trimming be T.
4. Donna picks a piece. (Biggest)
5. Carol picks a piece. If the trimmed piece was not already picked then she must pick it. (Biggest)
6. Bob picks a piece. (He thinks that both pieces left are the same size so no instructions needed.)
7. Alice takes the piece that is left.

Claim 1: The division of P_1, P_2, P_3, P_4 is envy free.

Proof:

Donna can't feel envy since she got to pick first.

Carol can't feel envy since she thought there was a tie for the best piece.

Alice and Bob can't feel envy since they thought that all untrimmed pieces were tied for best and each got an untrimmed piece.

End of Proof of Claim 1

We assume that Carol trimmed a piece and later got the trimmed piece (all the other cases are similar). In Phase II we deal with the trimming. KEY: No matter how much Carol gets, neither Alice nor Bob can be envious of her since they just think she is getting back trim from a trimmed piece.

PHASE TWO:

1. Donna and Bob use the protocol from Lemma 3.1 to divide T into four pieces they both think are worth exactly 1/4.

2. Carol takes a piece.

3. Alice takes a piece.

4. Donna takes a piece.

5. Bob takes a piece.

Carol cannot feel envy since she picked first.

Alice is the interesting case. She can't feel envy for Carol since Alice thinks that whatever trim Carol got will only make up for what Carol lost initially by taking the trimmed piece. Alice cannot envy Donna or Bob since she goes before them.

Donna and Bob can't feel envy for anyone since they think that the four pieces are the same.

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4 5 Player Discrete ϵ-Envy Free Protocol

Def 4.1 A division is *ϵ-Envy Free* if every player thinks that, if they have x, then the most someone else can have is $x + \epsilon$.

Theorem 4.2 *For all $\epsilon > 0$ there is a discrete five player ϵ-envy free protocol*

Proof:

The players are Alice, Bob, Carol, Donna, Edgar.

We will use the following as part of our protocol. It will explicitly take as input A, B, C, D, E meaning Alice, Bob, Carol, Donna, Edgar. This is so we can later run it with different people.

PROTOCOL1(A,B,C,D,E)

1. Alice cuts the pie into nine pieces (all equal). YES, its nine not five.

2. Bob trims at most four of the pieces (To create a five-way tie) The trimming is put aside. NOTE:

    - There are at least five untrimmed pieces that Alice likes.

3. Carol trims at most two piece (To create three-way tie). The trimming is put aside. NOTE:
    - There are still at least three untrimmed pieces, which Alice likes.
    - There are still at least three pieces that Bob likes.
4. Donna trims at most one piece (To create a two-way tie). The trimming is put aside. NOTE:
    - There are at least two untrimmed pieces, which Alice likes.
    - There are at least two pieces that Bob likes.
    - There are at least two pieces that Carol likes.
5. Edgar picks a piece (Biggest.)
    - There is at least one untrimmed piece, which Alice likes.
    - There is at least one pieces that Bob likes.
    - There is at least one pieces that Carol likes.
    - There is at least one piece that Donna likes.
6. Donna picks a piece. If the piece she trimmed is still available then she must take it. NOTE:
    - If Donna trimmed a piece then either her or Edgar got that trimmed piece. Nobody else wanted that piece, so still have that everyone likes at least one piece available.
    - If Donna did not trim a piece then before Donna picked a piece everyone liked at least two pieces available. After Donna picks a piece Alice, Bob, and Carol all like some piece that's available.
7. Carol takes a piece. If there is a piece that she trimmed but Donna did not trim then she must take it. (Biggest available.) NOTE: By similar reasoning as with Donna, after Carol picks a piece, Alice and Bob all like some piece that is available.
8. Bob takes a piece. If there is a piece that he trimmed that neither Carol nor Donna trimmed, he must take it. NOTE: After he does this, there must be an untrimmed piece still available.

9. Alice takes the remaining piece.

10. Note that there is another piece, and trimming, that have been put aside.

Claim 1: If Alice, Bob, Carol, Donna, Edgar carry out this protocol honestly then nobody is envious. (Note that there is still part of the cake that nobody got yet.) Also, Alice got 1/9 of the pie. (The rest might have gotten much less- we don't know.)

Proof of Claim 1:

Edgar cannot be envious since, of the 9 pieces, he got first pick.

Donna cannot be envious. When Donna trimmed she created a tie. Hence one of the two pieces left for her to pick is tied for first.

Carol is similar to Donna.

Bob is similar to Carol.

Alice will have available an untrimmed piece. Since she thinks the initially pieces were all worth 1/9 she cannot be envious. Note that she gets 1/9.

End of Proof of Claim 1

We look carefully at what Bob thinks. Let a, b, c, d, e be what Bob thinks Alice, Bob[1] Carol, Donna, Edgar got.

Let L_1 be what Bob thinks the leftovers (trim and the piece not taken) are worth. Note that

$$a + b + c + d + e + L_1 = 1$$

and

$$b \geq a, c, d, e.$$

Note that this implies

$$-a, -c, -d, -e \geq -b.$$

Hence

$$-a - b - c - d - e - \leq -5b.$$

[1] It may seem odd that Bob is saying *Bob thinks Bob's piece is worth b.* However, you may recall that Bob Dole had the habit of talking about himself in the third person.

Thought experiment: If they do the protocol on L_1 BUT this time Bob is the one who cuts it into 9 piece initially then what will happen?

As before we end up with (1) An envy free division of a part of L_1, (2) Bob thinks he has $\frac{1}{9}L_1$.

So what does Bob think he got total?

$$b+\frac{1}{9}L_1 = b+\frac{1}{9}(1-a-b-c-d-e) \geq b \geq \frac{1}{9}(1-5b) \geq b+\frac{1}{9}-\frac{5}{9}b = \frac{1}{9}+\frac{4}{9}b \geq \frac{1}{9}.$$

SO Bob now has at least 1/9.

We now can write PROTOCOL2 which is build on Protocol 1. Its not the final protocol yet. PROTOCOL2 takes as a parameter the object it is going to divide. We will initially run it on the entire cake but will later run it on leftovers.

PROTOCOL2(STUFF). This means we run this protocol on the original cake.

1. Run PROTOCOL1(A,B,C,D,E) on STUFF. Call leftover L_1.
2. Run PROTOCOL1(B,C,D,E,A) on L_1. Call the leftover L_2.
3. Run PROTOCOL1(C,D,E,A,B) on L_2. Call the leftover L_3.
4. Run PROTOCOL1(D,E,A,B,C) on L_3. Call the leftover L_4.
5. Run PROTOCOL1(E,A,B,C,D) on L_4. Call the leftover L_5 which we call R (you will see why later).

By reasoning similar to above one can show that, if PROTOCOL2 is run on the entire cake, then the following happens:

- The division is Envy free (though R_1 was not given to anyone).
- Everyone thinks they have $\geq 1/9$. So everyone thinks that R_1 is worth $\leq 8/9$.

THOUGHT EXPERIMENT: What if they then ran PROTOCOL2 on R_1 ? The division would be envy free. What is left we call R_2 and it would be worth $\leq (8/9)^2$.

PROTOCOL3(n) (n is a parameter that we will pick later.)

1. Run PROTOCOL2(A,B,C,D,E) on the cake. Call leftover R_1.
2. Run PROTOCOL2(A,B,C,D,E) on R_1. Call leftover R_2.
3. Keep on doing this until you get to R_n
4. Give R_n to Alice.

The idea is that the more times we run $PROTOCOL2$ the less is left. We do this until what is left is $\le \epsilon$.

One can show that what is left is $\le (8/9)^n$. Pick n such that $\epsilon \le (8/9)^n$.

So now what do we have? Alice thinks she got the biggest piece. Bob, Carol, Donna, Edgar think:

I have x, Alice has at most $x + \epsilon$, and everyone else has $\le x$.

Clearly this is ϵ-envy free.

We are DONE.

However, you may wonder how much Bob thinks he has. Lets say Bob thinks that Alice has a, Bob has b, Carol has c, Donna has d, Edgar has e. He also thinks

$$a + b + c + d + e = 1$$

$$c, d, e \le b$$

$$-c, -d, -e \ge -b$$

$$a \le b + \epsilon$$

$$-a \ge -b - \epsilon$$

So he thinks

$$b = 1 - a - c - d - e \ge 1 - b - \epsilon - b - b - b = 1 - 4b - \epsilon.$$

$$5b \ge 1 - \epsilon.$$

$$b \geq \frac{1}{5} - \frac{\epsilon}{5}.$$

SO, b thinks he has VERY CLOSE to $\frac{1}{5}$. He may think he has LESS than $\frac{1}{5}$. But that's okay- he thinks he has more than Carol, Donna, Edgar, and just ϵ less than Alice.

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