Constructive Control by Adding Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES ($\mathcal{E}$-CCAUC).

Given:

- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$, and
- a distinguished candidate $p \in C$.

Question: Is there a subset D' of D such that p is the unique winner of the $\mathcal{E}$ election $(C \cup D', V)$?

Constructive Control by Adding Candidates

Definition (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES ($\mathcal{E}$-CCAC).

Given:
- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is there a subset D' of D such that $\|D'\| \leq k$ and p is the unique winner of the $\mathcal{E}$ election $(C \cup D', V)$?

Constructive Control by Deleting Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY DELETING CANDIDATES ($\mathcal{E}$-CCDC).

Given:

- A set C of candidates,
- a list V of votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is it possible to delete up to k candidates from C such that p is the unique winner of the resulting $\mathcal{E}$ election?

Constructive Control by Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES ($\mathcal{E}$-CCPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which

- the winners of (C_1, V) surviving the tie-handling rule
- run against all candidates in C_2?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Constructive Control by Runoff Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY RUNOFF PARTITION OF CANDIDATES ($\mathcal{E}$-CCRPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which the runoff is between

- the winners of (C_1, V) surviving the tie-handling rule and
- the winners of (C_2, V) surviving the tie-handling rule?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Constructive Control by Adding Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING VOTERS ($\mathcal{E}$-CCAV).

Given:
- A set C of candidates,
- a list V of registered votes over C and an additional list W of as yet unregistered votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is there a subset W' of W such that $\|W'\| \leq k$ and p is the unique winner of the $\mathcal{E}$ election $(C, V \cup W')$?

Constructive Control by Deleting Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY DELETING VOTERS ($\mathcal{E}$-CCDV).

Given:
- A set C of candidates,
- a list V of votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is it possible to delete up to k voters from V such that p is the unique winner of the resulting $\mathcal{E}$ election?

Constructive Control by Partition of Voters

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS ($\mathcal{E}$-CCPV).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition V into V_1 and V_2 such that p is the unique winner (with respect to the votes in V) of the final stage of the two-stage election in which the runoff is between

- the winners of (C, V_1) surviving the tie-handling rule and
- the winners of (C, V_2) surviving the tie-handling rule?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Destructive Control

Remark:

- For each constructive control scenario, there is a corresponding destructive control type where the chair seeks to block the distinguished candidate's victory:

  $\mathcal{E}$-DCAUC, $\mathcal{E}$-DCAC, $\mathcal{E}$-DCDC, $\mathcal{E}$-DCPC-TE, $\mathcal{E}$-DCPC-TP, $\mathcal{E}$-DCRPC-TE, $\mathcal{E}$-DCRPC-TP, $\mathcal{E}$-DCAV, $\mathcal{E}$-DCDV, $\mathcal{E}$-DCPV-TE, and $\mathcal{E}$-DCPV-TP.

  In $\mathcal{E}$-DCDC it is not allowed to simply delete the distinguished candidate.

  $\Rightarrow$ This sums up to a total of 22 control types (and the corresponding control problems).

- The study of destructive control was initiated by Hemaspaandra, Hemaspaandra, and Rothe (2007).

Immunity and Susceptibility

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathfrak{CT}$ be a control type.

1. We say a voting system is *immune to $\mathfrak{CT}$* if it is impossible for the chair to make the given candidate
   - the unique winner in the constructive case and
   - not a unique winner in the destructive case,

   respectively, via exerting control of type $\mathfrak{CT}$.

2. We say a voting system is *susceptible to $\mathfrak{CT}$* if it is not immune to $\mathfrak{CT}$.

Resistance and Vulnerability

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $\mathfrak{CT}$ be a control type.

A voting system that is susceptible to $\mathfrak{CT}$ is said to be

1. *vulnerable to $\mathfrak{CT}$* if the control problem corresponding to $\mathfrak{CT}$ can be solved in polynomial time, and
2. *resistant to $\mathfrak{CT}$* if the control problem corresponding to $\mathfrak{CT}$ is NP-hard.

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
2. *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*
3. *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*
4. *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
2. *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
3. *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*
4. *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

Links Between Susceptibility Cases

Definition

A voting system is *voiced* if in any election that has exactly one candidate, that candidate is always a (and thus, the unique) winner.

Theorem

1. *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
2. *Each voiced voting system is susceptible to constructive control by deleting candidates.*
3. *Each voiced voting system is susceptible to destructive control by adding candidates.*

Control Complexity of Plurality and Condorcet Voting

	Plurality		Condorcet	
Control by	Constructive	Destructive	Constructive	Destructive
Adding Candidates	R(esistant)	**R**	I(mmune)	**V(ulnerable)**
Deleting Candidates	R	**R**	V	**I**
Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Run-off Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Adding Voters	V	**V**	R	**V**
Deleting Voters	V	**V**	R	**V**
Partition of Voters	**TE: V** **TP: R**	**TE: V** **TP: R**	R	**V**

Boldface results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).
Nonboldface results are due to Bartholdi, Tovey, and Trick (1992).

Hitting Set

Definition

Name: HITTING SET.

Given:
- A set $B = \{b_1, b_2, \dots, b_m\}$,
- a family $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets S_i of B, and
- a positive integer k.

Question: Does $\mathcal{S}$ have a hitting set of size at most k?
That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Destructive Control Complexity of Plurality Voting

Construction: Given a Hitting Set instance $(B, \mathcal{S}, k)$, where $B = \{b_1, b_2, \ldots, b_m\}$, $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and $k \leq m$, construct the following election:

- The candidate set is $C = B \cup \{c, w\}$.
- The voter set V is defined as follows:
  - $2(m-k) + 2n(k+1) + 4$ voters of the form $c \ w \ \cdots$, where "$\cdots$" means that the remaining candidates follow in an arbitrary order.
  - $2n(k+1) + 5$ voters of the form $w \ c \ \cdots$.
  - For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form $S_i \ c \ \cdots$, where "S_i" denotes the elements of S_i in some arbitrary order.
  - For each j, $1 \leq j \leq m$, two voters of the form $b_j \ w \ \cdots$.

Destructive Control Complexity of Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

If B' is a hitting set of $\mathcal{S}$ of size k, then w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Proof: See blackboard. ❑

Destructive Control Complexity of Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $D \subseteq B \cup \{w\}$. If c is not the unique plurality winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

1. $D = B' \cup \{w\}$,
2. *w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$, and*
3. *B' is a hitting set of $\mathcal{S}$ of size less than or equal to k.*

Proof: See blackboard. ❑

Destructive Control Complexity of Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $\{c, w\}$, spoiler candidates B, distinguished candidate c, and voter set V.

Proof: See blackboard. ❏

Corollary: Plurality voting is resistant to destructive control by adding candidates.

That is, Plurality-DCAUC (and also Plurality-DCAC) is NP-hard.

Destructive Control Complexity of Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set C, distinguished candidate c, and voter set V can be destructively controlled by deleting at most $m - k$ candidates.

Proof: See blackboard. ❑

Corollary: Plurality voting is resistant to destructive control by deleting candidates. That is, Plurality-DCDC is NP-hard.

Destructive Control Complexity of Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set C, distinguished candidate c, and voter set V can be destructively controlled by partition of candidates (both in model TE and TP).

Proof: See blackboard. ❑

Corollary: Plurality voting is resistant to destructive control by partition of candidates (both in model TE and TP).
That is, Plurality-DCPC-TE and Plurality-DCPC-TP are NP-hard.