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Game-theoretic Analysis of Strategyproofness in Cake-cutting Protocols

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Bachelorarbeit

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Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

Alina Elterman

Abstract

Hier kommt eine ca. einseitige Zusammenfassung der Arbeit rein.

Contents

1 Introduction

Cake Cutting is an interdisciplinairy field which is commonly researched and part of economics, political science, mathematics, operations research, and computer science. Game Theory is fulfilling the same property. Except for this fact, they have hardly something in common. While CC is about fair division of a heterogenous divisible good, game theory is used for defining lifes processes in a mathematical way and analysing them. Example Cost Sharing from Algorithmic GT Noam Nissan chap. 15

2 Preliminaries

2.1 Preliminaries of Game Theory

cooperative vs. noncooperative
algorithmic gt

2.2 Preliminaries of Cake-cutting

2.2.1 Basics

First of all we need to define the components of cake-cutting. Example 1 sketches the problem statement.

Example 1. *It was the year 1922 in London, Alan Mathison Turing was visiting a friend on his birthday. There was aswell Vilfredo Federico Pareto, Karl Popper, Felix Hausdorff und Charles West Churchman.*

Now, what exactly is cake-cutting about? It involves a discrete set of $n \in \mathbb{N}$ agents (or players) $P_n = \{p_1, \cdots, p_n\}$. After the assumption each of them wants to get as much as possible of the good. Only the allocation of a single, divisible and heterogenous good is included in the consideration of this work. It is common to use for the visualization a rectangular cake.

image cake

The division is performed by parallel cuts. The cake X is represented by the unit interval $I = [0, 1] \subseteq \mathbb{R}$. Each subinterval $I' \subseteq I$ or a union of subintervals with $I_m \subseteq I$ is called a boundle (piece). All boundles are disjoint. The boundle of the cake, which the player p_i receives is denoted as X_i. When all boundles of the cake are owned by players, this state is called an ALLOCATION. Each piece has a public length, which can be computed as the sum of all bourderdifferences, and the private value of each player.

Every player $p_i \in P_n$ has a valuation function (valuation) $v_i : \{X' | X' \subseteq X\} \mapsto [0, 1]$ on the cake X with the following properties:

1. Non-negativity[1]: $v_i(C) \geq 0$ forall $C \subseteq [0, 1]$.
2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$.
3. Monotonicity: if $C' \subseteq C$ then $v_i(C') \leq v_i(C)$.
4. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq [0, 1]$.
5. Divisibility: forall $C \subseteq [0, 1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, exist a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.
6. v_i is continuous: if $0 < x < y \leq 1$ with $v_i([0, x]) = \alpha$ and $v_i([0, y]) = \beta$, than for every $\gamma \in [\alpha, \beta]$ there exist a $z \in [x, y]$ so that $v_i([0, z]) = \gamma$.
7. Emptyness of single points: $v_i([x, x]) = 0$ forall $x \in [0, 1]$.

[1] It is common to require positivity: $v_i(C) > 0$ forall $C \subseteq [0, 1]$ and $C \neq \emptyset$

Different Types of Fairness

The goal of the fair division of a heterogeneous, continious good is to allocate the resource in a fair manner. But what is fair? It can be seen as a efficiency criteria of an allocation, which can be normalized and gives a possibility to compare different allocations. We distinguish between the following fairness criteria.

Definition. ***(Proportional or simple fair)***
An allocation is proportional (simple fair), if $v_i(X_i) \geq 1/n$ for each player $p_i \in P_N$.

Definition. ***(Envy-freeness)***
An allocation is envy-free, if $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_N$.

The following conditions can also be used on the proportional case.

Definition. ***(Strong Envy-freeness)***
An allocation is strong-envy-free, if $v_i(X_i) > v_i(X_j)$ for each couple of players $p_i, p_j \in P_N, i \neq j$.

Definition. ***(Super Envy-freeness)***
An allocation is super-envy-free, if $v_i(X_j) \leq 1/n$ for each couple of players $p_i, p_j \in P_N, i \neq j$.

Definition. ***(Strong Super Envy-freeness)***
An allocation is stong-super-envy-free, if $v_i(X_j) < 1/n$ for each couple of players $p_i, p_j \in P_N, i \neq j$.

Hereby is the problem, that for the stronger conditions an allocations cannot exist always, like in the case when all players have equal values on the cake.

Correlation between the Fairness-Properties

Lemma. *Forall allocations [2]:*

1. *Every envy-free allocation is proportional.*
2. *For two players an allocation is envy-free iff it is proportional.*

Definition. ***(Efficiency)***
An allocation is efficient (Pareto optimal) if no other allocation exist, where one player do better, without becomming worse for an other player.

[2]the proof can be found in []

MEHR UEBER EHRLICHKEIT UND STRATEGIESICHERHEIT (mind. eine SEITE)

2.2.2 Strategyproofness

Definition. ***(Thruthfully)*** *An allocation is thruthful if there are no valuations where a player will do better by lying.*

Only such divions are interesting, where without explicit compelling the players to be thruthful, they will be it, because it is their best strategy It is common that by not following the strategy of the protocol the player get into the risk to loose the garanty for their fair share. At the end of an allocation it can be proven whether every player got his fair share. Each protocol

Different Types of Protocols

Since the protocols are going to be analysed in this work, it is very important to understand the types, structure and design of them.

Classes of Protocols

Intuitiv:(Algorithm)
In mathematics, computer science and related subjects, an algorithm is an effective method for solving a problem, hich can be denoted as a finite set of sequiences.

Definition. ***(Protocol (Cake-Cutting-Protocol))***
A Protocol (Cake-Cutting-Protocol) is an adaptive algorithm with a fixed number of players and the following properties:

- *It consists of rules and strategies.*
  Rules are requirements, which has to be followed by the players without knowledge of their valuations.
  Strategies are recommendations, which can be followed for getting the guaranteed fair share.
- *If a player does not follow the strategy of a protocol, he looses his guarantee to get after a fixed number of steps to get a fair piece of cake. His actions does not harm the other players.*
- *Each player has to be possible at each moment indepentent of other players to cut the cake.*
- *The protocol does not own informations about the valuations of the players, except of the ones, it got from the steps before.*

Definition. *A CCP is called proportional, envy-free, strong envy-free a.s.o., if indepentent of the valuations of the players, each allocation is proportional, envy-free, $\cdots$ under the requirement, that all players follows the rules and strategies given by the protocol.*

One of the main goals of cake-cutting is the developement of such protocols.

Definition. ***(finite (discrete)/ continious)***
A finite (discrete) protocol gives a solution after a finite number of queries (valuations, marcs, ...), in comparison has a player to make in a continious protocol infinitely many queries.

Definition. ***(finite bounded/ finite unbounded))***
A finite bounded protocol has an upper bound of steps, which are looked at in the worst case. The number of those steps is only correlated, in some cases, with the number of players. A finite unbounded protocol has no approximated number of steps.

The most desirable protocols are the finite bounded, because they are the easiest to realise in real life applications.

2.3 The Degree of Guaranteed Envy-freeness

In the last sixty years the number of proportional finite bounded protocols have grown for arbitrary n. But there exist none envy-free finite bounded protocol for an arbitrary n. The biggest group we can divide a cake in a fix number of steps in an envy-free way is sadly three. There exist a compromise between envy-freeness and finite boundness and a possibility to valuate proportional cake-cutting protocols between each other, the degree of guaranteed envy-freeness.

Definition. *Sei eine Aufteilung des Kuchens* $X = \bigcup_{i=1}^{n} X_i$ *fur die Menge* $P = \{p_1, \ldots, p_n\}$ *der Spieler gegeben, wobei* v_i *das Mass von* p_i *und* X_i *die Portion von* p_i *ist.*

- *Eine* <u>*Neidrelation*</u>*("envy relation")* $\Vdash$ *ist eine Binarrelation auf* $P(\Vdash, PxP)$: p_i *beneidet* p_j $(p_i \Vdash p_j)$, $1 \leq i, j \leq n$, $i \neq j$, *falls* $v_i(X_i) < v_i(X_j)$.
- *Eine* <u>*Neidfrei-Relation*</u> *("envy-free relation")* $\nVdash$ *ist eine Binarrelation auf* P : p_i *beneidet nicht* p_j $(p_j \nVdash p_j)$,$1 \leq i, j \leq n$,$i \neq j$, *falls* $v_i(X_i) \geq v_i(X_j)$.

<u>Eigenschaften von $\Vdash$ und $\nVdash$:</u>

- $\Vdash$ ist irreflexiv, denn $v_i(X_i) < v_i(X_i)$ gilt nie
- $\nVdash$ ist reflexiv, denn $v_i(X_i) \geq v_i(X_i)$ gilt immer
  Die triviale Beziehung $p_i \nVdash p_i$ zahlt in der Regel nicht mit.
- $\Vdash$ und $\nVdash$ sind nicht transitiv.
  Gilt z.B. $p_i \Vdash p_j$ und $p_j \Vdash p_k$, so kann man daraus nichts uber $v_i(X_k)$ schliessen: $p_i \nVdash p_k$ ist moglich

$\Rightarrow$ Es gibt die folgenden Moglichkeiten:

1. Zwei-Wege-Neid: $p_i \Vdash p_j$ und $p_j \Vdash p_i$
   (Tausch der Portionen macht beide glucklich.)
2. Zwei-Wege-Neidfreiheit: $p_i \nVdash p_j$ und $p_j \nVdash p_i$
   (Alles ist gut.)
3. Ein-Weg-Neid: $p_i \Vdash p_j$ und $p_j \nVdash p_i$
   Ein-Weg-Neidfreiheit: $p_j \Vdash p_i$ und $p_i \nVdash p_j$

Fallerzwungene Neid- bzw. Neidfrei-Relationen: hangen ab von einem Fall geeigneter Masse.
Garantierte Neid- bzw. Neidfrei-Relationen: gelten in jeden Fall (auch im worst case), also unabhangig von den Massen der Spieler.
Anzahl garantierter Neidfrei-Relationen = $\min\limits_{alleFaelle}$ Anzahl der fallerzwungenen Neidfrei-Relationen.

Beispiel. *Aufteilung $X = X_F \cup X_G \cup X_H$ des Kuchens mit*

	$X_F = \begin{array}{ccc} & & \square \\ & \square & \square \\ \square & \square & \square \\ 1 & 2 & 3 \end{array}$	$X_G = \begin{array}{ccc} \square & \square & \square \\ 7 & 8 & 9 \end{array}$	$X_H = \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ 4 & 5 & 6 \end{array}$
F	$\frac{6}{18} = \frac{1}{3}$	$\frac{6}{18} = \frac{1}{3}$	$\frac{6}{18} = \frac{1}{3}$
G	$\frac{9}{18} = \frac{1}{2}$	$\frac{3}{18} = \frac{1}{6}$	$\frac{6}{18} = \frac{1}{3}$
H	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{6}{18} = \frac{1}{3}$

$\Rightarrow$ *nicht proportional wegen* $v_G(X_g) = \frac{1}{6} < \frac{1}{3}$.
Es gibt:

Ein-Weg-Neid von G zu F:
$G \Vdash F$ *wegen* $v_G(X_G) = \frac{1}{6} < v_G(X_F) = \frac{1}{2}$
Gleichzeitig ist dies
$F \nVdash G$ *wegen* $v_F(X_G) = \frac{1}{3} = v_F(X_F)$

Ein-Weg-Neidfreiheit von F zu G

Zwei-Wege-Neidfreiheit zwischen F und H
$F \nVdash H$, *da* $v_F(X_F) = \frac{1}{3} = v_F(X_H)$
$H \nVdash F$, *da* $v_H(X_H = \frac{6}{18} > \frac{5}{18} = v_H(X_F)$

Zwei-Wege-Neid zwischen G und H
$G \Vdash H$, *da* $v_g(X_G) = \frac{1}{6} < \frac{1}{3} = v_G(X_H)$
$H \Vdash G$, *da* $v_H(X_H) = \frac{1}{3} < \frac{7}{18} = v_H(X_G)$

DGEF = Anzahl der Neidfrei-Relationen im worst case

Protocol. *Jorg erhalt den Kuchen.*
DGEF $: n - 1 + (n-1)(n-2) = n - 1 - n^2 - 3n + 2 = n^2 - 2n - 1$

Satz. 1. *Jedes neidfreie CCP fÃ$\frac{1}{4}$r $n \geq 1$ Spieler hat einen **DGEF** von $n(n-1)$.*

2. *Sei $d(n)$ der **DGEF** eines proportionalen CCPs mit $n \geq 2$ Spielern. Dann gilt: $n \leq d(n) \leq n(n-1)$.*

Proof. 1. Da wir $p_i \nVdash p_i$ fur alle i, $1 \leq i \leq n$, ausser 8 lassen, hat jeder der n Spieler zu jedem anderen Spieler eine Neidfreie-Relation, insgesamt also $n(n-1)$.

2. $n = 2$ Offenbar gilt: $d(2) = 2$, denn da das CCP proportional ist, gilt: $v_1(X_1) \geq \frac{1}{2}$ und $v_2(X_2) \geq \frac{1}{2} \Rightarrow v_1(X_1) \geq v_1(X_2)$ und $v_2(X_2) \geq v_2(X_1)$

$n \geq 3$ Da $p_i \nVdash p_i$ fÃ$\frac{1}{4}$r alle i ignoriert wird, gilt $d(n) \leq n(n-1)$.

In einer proportionalen Aufteilung gilt:
$v_i(X_i) \geq \frac{1}{n}$ fÃ$\frac{1}{4}$r $1 \leq i \leq n$.

$\Rightarrow$ Keiner der n Spieler kann gleichzeitig alle anderen Spieler bendeidenl, denn: Angenommen, das ware nicht so. Konkret: $p_1 \not\Vdash p_2$
$\Rightarrow$ $v_1(X_2) > v_1(X_1) \geq \frac{1}{n}$
$\Rightarrow$ $v_1((X - X_1) - X_2) < \frac{n-2}{n}$
$\Rightarrow$ $(X - X_1) - X_2$ kann nicht so in $n - 2$ Portionen aufgeteilt werden, dass $v_i(X_j) \geq \frac{1}{n}$ fÃ$\frac{1}{4}$r alle $j, 3 \leq j \leq n$, gilt.
$\Rightarrow$ es gibt ein $j, 3 \leq j \leq n$, so dass $v_i(X_j) < \frac{1}{n}$, gilt.
$\Rightarrow$ $p_i \not\Vdash p_j$

Also hat jeder der n Spieler mindestens eine garantierte Neidfrei-Relation zu einem anderen Spieler: $n \leq d(n)$

□

Definition (lemma)**.** *Verlangen die Regeln/Strategien eines proportionalen CCPs fÃ$\frac{1}{4}$r $n \geq 2$ Spielern von keinem Spieler, die Portionen der anderen Spieler zu bewerten, dann ist der **DGEF** $= n$.*

Proof. $n = 2$ Proportionalitat $\Rightarrow$ Neidfreiheit
best case = worst case
und wie vorher: **DGEF** $= 2 = n$

$n \geq 3$ Betrachte das folgende Szenario: FÃ$\frac{1}{4}$r eine gegebene Aufteilung $X = \bigcup_{i=1}^{n} X_i$, die proportional ist, aber sonst keinerlei Einschrankungen unterliegt, setzen wir die Masse der Spieler so:
FÃ$\frac{1}{4}$r jedes $i, 1 \leq i \leq n$, bewertet p_i:

- die eigene Portion X_i mit $v_i(X_i) = \frac{1}{n} = \frac{n}{n^2} \Rightarrow$ proportional!
- die Portion X_j eines Spielers $p_j, j \neq i : v_i(X_j) = \frac{2}{n} < \frac{1}{n}$
- jede der $n - 2$ Ã$\frac{1}{4}$brigen Portionen X_k der Spieler $p_k, |i, j, k| = 3, v_i = (X_k) = \frac{n+1}{n^2} > \frac{1}{n}$

Insgesamt gilt dann fÃ$\frac{1}{4}$r jedes $i, 1 \leq i \leq n$:

1. $v_i(X) = v_i(\bigcup_{j=1}^{n} X_j) \overset{\text{Additivitat}}{=} \sum_{j=1}^{n} v_i(X_j) = \frac{1}{n^2}(n + 2 + (n-2)(n+1)) = \frac{1}{n^2}(n + 2 + n^2 + n - 2n - 2) = 1$
2. p_i hat $n - 2$ Neidrelationen und nur eine Neidfrei-Relation
$\Rightarrow$ Insgesamt gibt es n garantierte Neidfrei-Relationen, eine fÃ$\frac{1}{4}$r jeden Spieler.

□

Overview: DGEF in finite bounded CCP

Protocol	DGEF	$n = 4$ (12 EF)
Last Diminisher	$2 + \frac{n\cdot(n-1)}{2}$	8
Lone Chooser	n	4
Lone Divider	$2n - 2$	6
Cut your Own Piece	n	4
Cut your Own Piece (left-right rule)	$2n - 2$	6
Divide & Conquer	$n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1}$	6
Minimal-Envy Divide & Conquer	$n \cdot \lfloor log(n) \rfloor + 2n - 2^{\lfloor log(n) \rfloor + 1}$	6
Rekursive Divide & Choose	n	4

3 Proportional Procedures and their DGEF

Each proportional procedure consist of rules and strategies. Since the rules are compulsory, the strategies of e... . In this chapter the strategies of common used procedures are shown (probably specialised, if possible) and explained with examples. Then their strategyproofness is analysed. During complications the effect on the DGEF is shown. Complete procedures can be found in [].

3.1 The Steinhaus-Kuhn Lone Divider Procedure

3.2 The Banach-Knaster Last-Diminisher Procedure

The generalization of ”I cut, you choose”.

Theorem. *If the valuatons of the players are not equal, there exist a not thruthful strategy for the first player in each round for getting a piece with $v_i(S_i) > 1/N$.*

Proof. Im Schritt 1: Das abgeschnittene St"uck soll den Wert $1/N+$ Rest von S_1 haben. Fallunterscheidung: Entweder Spieler p_1 kriegt dieses St"uck oder Spieler p_i beschneidet es und somit kann Spieler p_1 ein solches St"uck in der darauffolgenden Runde bekommen, oder bei Cut und Choose am Ende. □

Theorem. *Das Last-Diminisher-Protokoll hat einen **DGEF** von* $\frac{n(n-1)}{2} + 2$

Proof. **Runde 1** Sei $\bar{p}_1$ der Spieler, der die erste Portion erhalt. Jeder andere Spieler bewertet diese mit $\leq \frac{1}{n}$, beneidet also $\bar{p}_1$ nicht
$\Rightarrow n - 1$ garantierte Neidfrei-Relationen

Runde $i, 1 < i < n$ Analog zu Runde 1 konnen $n - i$ Neidfrei-Relationen garantiert werden. $\bar{p}_i$, der die ite Portion erhalt, wird von den verbleibenden Spielern nicht beneidet.
$\Rightarrow$ mindestens $\sum\limits_{i=1}^{n} i = \frac{n-1}{2}$ garantierte Neidfrei-Relationen

Letzte Runde
1. Cut & Choose zwischen $\bar{p}_{n-1}$ und $\bar{p}_n$. Keiner dieser beiden beneidet den anderen.
   $\Rightarrow$ eine zusatzliche garantierte Neidfrei-Relation.
2. Da Last-Diminisher proportional ist, gibt es eine weitere garantierte Neidfrei-Relation fÃ$\frac{1}{4}$r $\bar{p}_1$

$\Rightarrow$ **DGEF** $= \frac{(n-1)n}{2} + 2$ □

3.3 The Fink Lone-Chooser Procedure

Theorem. *Das Lone-Chooser-Protokoll hat einen* ***DGEF*** *von n.*

Proof. Kein Spieler bewertet die Portion irgendeines anderen Spielers.
$\overset{\text{Lemma}}{\Longrightarrow}$ $\mathbf{DGEF} = n$ $\square$

3.4 The Cut-Your-Own-Piece Procedure

3.5 The Divide-and-Conquer Procedure

3.6 The Extended Procedure

3.7 The Extension of the Extended Procedure

4 Related Work

5 Conclusions and Open Questions/Problems

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