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Game-theoretic Analysis of Strategyproofness in Cake-cutting Protocols

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Bachelorarbeit

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Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

Alina Elterman

Abstract

Contents

0.1 Preliminaries of Cake-cutting

0.1.1 Basics

It is necessary to define the components and challenges of cake-cutting. But first, what exactly is cake-cutting about? It involves a set of $n \in \mathbb{N}$ players $P_n = \{p_1, \ldots, p_n\}$. It is assumed that each of them wants to get as much as possible of the divided resource. The goal is to find an allocation of a single, divisible and heterogeneous good between the n players.
Such allocation has to be of a special kind, so that the involved players are pleased with

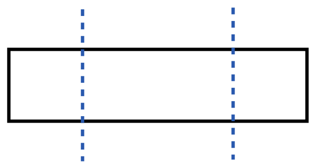

Figure 1: Cake
Example for a visualisation of a cake with two cuts

the outcome. For the visualization it is common to use a rectangular cake. The division is performed by parallel cuts. The cake X is represented by the unit interval $X = [0,1] \subseteq \mathbb{R}$. Each subinterval $X' \subseteq X$ or a sequence of disjoint subintervals

$$\bigcup_{m \in \mathbb{N}} X'_m$$

with $X'_m \subseteq X$ is called a *portion (or piece)*. The portion of the cake, which the player p_i receives is denoted as X_i. The state is called an *allocation*, when all portions of the cake are owned by players. Each piece has a public size, which can be computed as the sum of all border differences, and the private value of each player.

Every player $p_i \in P_n$ has a *valuation function (valuation)* $v_i : \{X' | X' \subseteq X\} \rightarrow [0,1]$ with the following properties:

1. Non-negativity: $v_i(C) \geq 0$ for all $C \subseteq X$.
2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0,1]) = 1$.
3. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq X$.[1]
4. Divisibility: For all $C \subseteq [0,1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, there exists a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.
5. v_i is continuous: If $0 < x < y \leq 1$ with $v_i([0,x]) = \alpha$ and $v_i([0,y]) = \beta$, then for every $\gamma \in [\alpha, \beta]$ there exists a $z \in [x,y]$ so that $v_i([0,z]) = \gamma$.

[1]Monotonicity: If $C' \subseteq C$ then $v_i(C') \leq v_i(C)$. Monotonicity follows from additivity, because for the assumption $C' \subseteq C$ and $A := C \backslash C'$: $v_i(C) = v_i(A \cup C') = v_i(A) + v_i(C') = \underbrace{v_i(C \backslash C')}_{\geq 0} + v_i(C') \geq v_i(C')$.

6. Non-atomic: $v_i([x, x]) = 0$ for all $x \in X$.

After some basics it would be interesting to see how game-theory is applicable to cake-cutting. Example 1 illustrates the problem in a game-theoretic manner.

Example 1.

John Cocke and Tadao Kasami want to divide a chocolate-strawberry-cake. The cake is half chocolate from the left and the right part is strawberry. John Cocke got the first move and is thinking about making three different cuts. After the cuts the two pieces would have the following values:

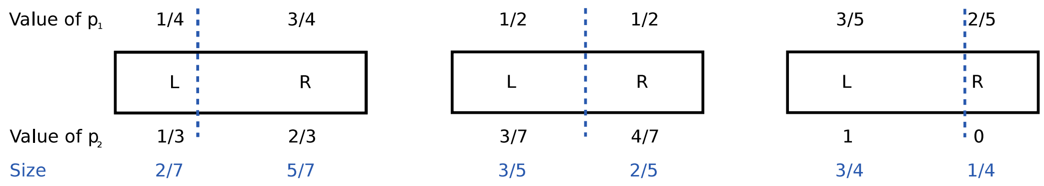

Figure 2: C&K cake game

Before doing so, he analyses his situation via the normal form:

	Leftcut	Middlecut	Rightcut
L	$(3/4, ?)$	$(1/2, ?)$	$(2/5, ?)$
R	$(1/4, ?)$	$(1/2, ?)$	$(3/5, ?)$

Table 1: C&K cake game in normal form

Since the valuation is a private function, he does not know the preferences of his colleague and has to assume that Tadao is indifferent between the two pieces. Tadao is waiting for John's move and will choose his best strategy in the extended game form:

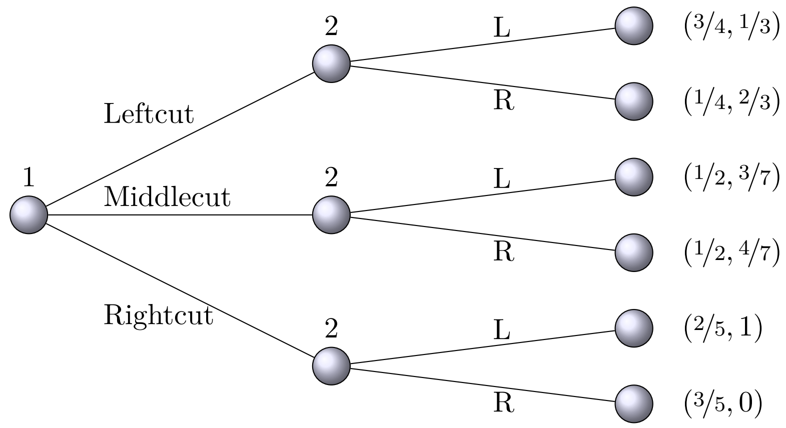

Figure 3: C&K cake game in extended form

For John the graph shows that if he stays secure he would obtain $v_1(X_1) = 1/2$. Otherwise he would get $1/4$ or $2/5$. So to make the middlecut is his best possible move.

0.1.2 Different Types of Fairness

As indicated by the name, fairness plays an important role in fair division. But how is fairness defined? It can be seen as a valuation criterion of an allocation, which can be normalized and gives a possibility to compare different allocations. Usually the fairness criteria are distinguished between the following:

Definition 1. ***(Proportional or Simple Fair)***
An allocation is *proportional (simple fair)* if $v_i(X_i) \geq 1/n$ for each player $p_i \in P_n$.

Definition 2. ***(Envy-Freeness)***
An allocation is *envy-free* if $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_n$.

The following theorem shows the correlation between the two allocations.

Theorem 1.

1. *Every envy-free allocation is proportional.*
2. *An allocation between two players is envy-free if and only if it is proportional.*

Proof.

1. Proof by contradiction:
   Assume A is an envy-free allocation, but not proportional. From envy-freeness follows $v_i(X_i) \geq v_i(X_j)$ for each pair of players $p_i, p_j \in P_n$ and so each player has at least an as much valuable piece of cake as each other player. Hereby each player owns in his own valuation at least as much as $(n-1)$ other players and so at least $1/n$. $\lightning$ The allocation A is proportional.
   Therefore, all envy-free allocations are proportional.
2. "$\Rightarrow$" For two players an allocation is proportional if each player has at least the half of the cake in his valuation. So the first player thinks the second player got at most half of the cake in the valuation of the first player and vice versa. They would not envy each other.
   "$\Leftarrow$" Follows from part 1.

□

A different criterion to valuate the quality of an allocation is efficiency. The correlation between the fairness criterion and efficiency can be found in [**?**].

Definition 3. ***(Efficiency)***
An allocation

$$A = \{X_1, \dots, X_n\}$$

is *efficient (Pareto optimal)* if there is no other allocation

$$A' = \{X'_1, \dots, X'_n\}$$

such that

$$v_i(X_i) \leq v_i(X_i')$$

for all players $p_i \in P_n$ and for at least one player the inequality is strict.

Theorem 2.

1. *Envy-freeness and proportionality does not imply efficiency.*
2. *Efficiency does not ensure proportionality and so envy-freeness.*

Proof.

1. Imagine the following allocation with three players. Each player's portion consists up to three pieces. The value of the whole portion is (because of the additivity of the valuation) the sum of the pieces :

	$X_1 = X_1' \cup X_1''$	$X_2 = X_2' \cup X_2''$	$X_3 = X_3' \cup X_3'' \cup X_3'''$
p_1	$1/2 = 11/12 + 1/12$	$1/3 = 0 + 1/3$	$1/6 = 0 + 1/12 + 1/12$
p_2	$1/3 = 1/6 + 1/6$	$1/3 = 1/3 + 0$	$1/3 = 1/12 + 1/12 + 1/6$
p_3	$0 = 0 + 0$	$7/18 = 6/18 + 1/18$	$11/18 = 6/18 + 3/18 + 2/18$

Table 2: Example for envy-freeness does not imply efficiency

This allocation is obviously envy-free, since $v_i(X_i) \geq v_i(X_j)$ for all $i, j \in \{1, 2, 3\}, i \neq j$. It is not efficient, because if the players p_1 would get p_2's portion X_2'', p_1 would get a more valuable piece of the cake and neither p_2 or p_3 would get less valuable pieces. Since in Theorem 1 was shown that envy-freeness implies proportionality, this example also demonstates that proportionality does not imply efficiency.

2. Allocating the whole cake to one player is efficient, but definitely not proportional and therefore not envy-free.

□

In [?] the authors show a general argument that no finite bounded protocol can exist for such an allocation that is both proportional and efficient at the same time.

0.1.3 Different Types of Protocols

It is very important to understand the types, structure and design of protocols, which are analysed in this paper.

Informal: (Algorithm)
An *algorithm* is an effective method for solving a problem, which is composed of a finite sequence of instructions.

Definition 4. ***(Cake-Cutting-Protocol)***
A *cake-cutting-protocol* (protocol for short) is an adaptive algorithm with a fixed number of players and the following properties:

- A protocol consists of rules and strategies.
  Rules are requirements, which *have to* be followed by the players without knowledge of their valuations and which execution can be verified.
  Strategies are recommendations, which *can* be followed for getting the guaranteed fair share.

- Each player should be able to cut the cake at a specific moment independent of other players.

- The protocol has no information about the valuation of the players, except of those it got from the steps before. It can not prove whether a player follows the strategy of the protocol.

Comment: Only such protocols are interesting where the actions of one player does not harm the other players. If a player does not follow his recommended strategy he may get less then a fair share.

Definition 5. ***(Proportional/ Envy-Free Protocol)***
A cake-cutting *protocol* is called *proportional* or *envy-free* if independent by the players' valuations, each allocation is proportional or envy-free provided that all players follow the rules and strategies given by the protocol.

The development of such protocols is one of the main goals of cake-cutting [?].

Definition 6. ***(Finite (Discrete)/ Continuous (Moving-Knife) Protocol)***
A *finite (discrete) protocol* gives a solution after a finite number of queries (valuations, marks, ...). In a *continuous (a.k.a. moving-knife) protocol* a player has to make up to infinitely many queries.

Definition 7. ***(Finite Bounded/ Finite Unbounded Protocol))***
A *finite bounded protocol* has an upper bound of steps for all possible valuations. The number of those steps is only correlated, in some cases, with the number of players. A *finite unbounded protocol* has no approximated number of steps.

The most desirable protocols are the finite bounded because of the ease of their implementation.
In the last sixty years the number of proportional finite bounded protocols have grown for an arbitrary number of players. But still no envy-free finite bounded protocol for an arbitrary n is known [?]. Only for three or less players a cake can be divided in a fixed number of steps, so that it is envy-free. For this reason only proportional protocols are considered in the further work.

1 Strategyproofness

In practice, players are selfish and try to increase the value of their portion. In order to do so, they may for example report false valuations on parts of the cake. The goal is to prevent this.

Definition 8. ***(Non-Truthful (Cheating) / Honest Player)***
Every strategy is *non-truthful* except of the strategy recommended by the protocol. A player who follows a non-truthful strategy will be called a *non-truthful (cheating) player*. Otherwise the player is called *honest*.

Definition 9. ***(True Value Function)***
A *true value function* provides the value of the piece a player would receive by following the recommended strategy. This value is at least proportional in a proportional protocol.

A strategy is better than an other if the value of the obtained piece is bigger than in the other strategy.

Definition 10. ***(Risk Aversion[?])***
A player is *risk averse* if he or she will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a piece of a guaranteed size.

Definition 11. ***(Strategyproofness of a Proportional Protocol[?])***
A proportional cake-cutting protocol is said to be *strategyproof for risk averse players* (SPP for short) if a cheating player is no longer guaranteed a proportional share, whereas all other players (provided they play truthful) are still guaranteed to receive their proportional share.

Definition 12. ***(Strategyproofness in the sense of [?])***
A protocol is *strategyproof* if no player has a strategy that is assuredly better than his true value function.

The strategyproofness in the sense of [?] will be called weak strategyproofness (WSP for short) since it is always true for a proportional protocol (compare [?]).

Example 2.

Assume the case when all valuations over the cake are equal, and all players, except of the cheating one follow the strategy provided by the protocol. Each of the honest player will get his proportional share, which the cheating player also values as $1/n$ *or more. Sharing a cake with* $(n-1)$ *other players means for the cheater that* $(n-1)/n$ *or a more valuable part of the cake is allocated to other players and so only the value of* $1/n$ *or less remains for him independent of his strategy.*

This characteristic is not significant, since valuations like in Example 2 of the players where the cheater would never get more than a proportional piece exist always.
A controversial point is that with this definition a player with a non-truthful strategy which obtains in one special case the same valuable piece, like the true value function and

in all other strictly more valuable pieces would stay honest.

A stronger condition comes from the social choice literature:

Definition 13. ***(Strategyproofness in the sense of [?])***
A protocol is *strategyproof* if the true value function dominates every other strategy.

In order to prevent misunderstandings, in this paper strategyproofness in the sense of [?] will be called strong strategyproofness (SSP for short). It can be shown that none of the known cake-cutting protocols is able to fulfill the strong strategyproofness criteria, if the valuation of the players is not equal. All protocols shown in Chapter 3 work for two players in exactly the same way as cut & choose. Example 3 is similar to the one in [?].

Example 3.

John Warner Backus and Peter Naur are celebrating and Donald E. Knuth has brought a huge marzipan cake with an enormous cherry on the left side. John loves cherries and hates marzipan, and Peter is just very hungry. The pioneers of computer science apply cut & choose. Peter is the cutter, and his best strategy would be to separate the cake from the cherry. If Peter would have full knowledge (which would not violate the preconditions of strategyproofness in [?]) about the valuations of John, he would always benefit from lying. From table 3 he would know, that John would always take the left piece and so he could easily maximize the value of his portion. Hence, this algorithm is not strongly strategyproof.

	Only Cherry	Middlecut
L	$(9/10, 1)$	$(1/2, 1)$
R	$(1/10, 0)$	$(1/2, 0)$

Table 3: B&N cake game in normal form

In strong strategyproofness a player would never get a more valuable piece by lying independent of the valuation of the other players.

After a counterexample in [?] the definition of strategyproofness in [?] was restricted to the case with non-equal valuations and for the general case changed to:

Definition 14. ***(Strategyproofness in the sense of [?])***
A protocol is *strategyproof* (SP for short) if no player has a strategy that is sometimes better or is at least as well as his true value function.

Imagine the situation where different sharing processes are happening more than once, and the player is interested in becoming best off in a long term view. A game-theoretical approach leads to the following definition:

Definition 15. ***(Game-Theoretic Strategyproofness)***
A protocol is *game-theoretic strategyproof* (GTSP for short) if no player has a strategy with a higher expected value than the expected value of his true value function.

From results with the expected value it is difficult to conclude something about the other strategyproofness criterion. But it is possible to give a definition of strategyproofness with respect to game-theoretic strategyproofness and which can be compared to the other strategyproofness criterion.

Definition 16. ***(Game-Theoretic Cake-Cutting Strategyproofness)***
A protocol is *game-theoretic cake-cutting strategyproof* (GTCCSP for short) if no player has a strategy that is better in at least one allocation than his true value function and in the other allocations at least as well as his true value function.

In Theorem 3 the correlation between the above mentioned definitions of strategyproofness in cake-cutting is shown.

Theorem 3. *The relation between different strategyproofness criteria is the following*

$$SPP \Rightarrow^1 SP \Rightarrow^2 GTCCSP \Rightarrow^3 WSP$$

and

$$GTSP \Rightarrow^4 GTCCSP.$$

Proof.

1. If a protocol is SPP, then for every player in each strategy except of the recommended one exists at least one allocation A_c. In A_c the cheating player p_c receives a piece X_c which is not proportional. Since in a proportional protocol all allocations guarantee a proportional share to every player, the value of his piece X_c is less than he would obtain by following the recommended strategy. So the protocol is SP.

2. If a protocol is SP, then for every player in each strategy except of the recommended one exists at least one allocation A_c. In A_c the cheating player p_c receives a piece X_c instead of $X_{\neg c}$, which he had received by following the recommended strategy. The value of X_c is smaller than the value of $X_{\neg c}$. So no non-truthful strategy exist where the cheating player gets in all allocations at least the same valuable piece as in the recommended one. The protocol is $GTCCSP$.

3. If a protocol is $GTCCSP$, then for every player in each strategy except of the recommended one exists at least one allocation A_c. In A_c the cheating player p_c receives a piece X_c instead of $X_{\neg c}$, which he had received by following the recommended strategy. The value of X_c is smaller or equal to the value of $X_{\neg c}$. So no non-truthful strategy exist where the cheating player gets in all allocations a more valuable piece than in the recommended one. The protocol is WSP.

4. **Proof by contradiction:**

   Assumption: If a protocol is $notGTCCSP$ then it is $GTSP$.
   If a protocol is $notGTCCSP$, then there is one player with a strategy, which is not the recommended one. And for all allocations A_{S_c} the cheating player p_c receives a piece X_{S_c} instead of $X_{S_{\neg c}}$, which he had received by following the recommended strategy. The value of X_{S_c} is at least equal to the value of $X_{S_{\neg c}}$. In one allocation

$A_{S'_c}$ the value of $X_{S'_c}$ is bigger than the value of $X_{S'_{\neg c}}$. Let r be the number of all possible allocations. The expected value of the strategy S_c al least equal with the expected value of the recommended strategy for the $r-1$ allocations since the value of the received pieces X_{S_c} are at least equal to the value of the pieces $X_{S_{\neg c}}$. The expected value for the piece $X_{S'_c}$ is bigger than the expected value for $X_{S'_{\neg c}}$.
The general expected value is $E(S_c) > E(S_{\neg c})$ and is particularly not $E(S_c) \leq E(S_{\neg c})$ and the protocol is not $GTSP$. $\lightning$

So $notGTCCSP \Rightarrow notGTSP$ and especially $GTSP \Rightarrow GTCCSP$.

$\square$

There are two independent types of strategyproofness to show for having a complete quantity of results. The start will be from the left side until one of the criteria is fulfilled, then the other strategyproofness further on the right follow from Theorem 3. Independently a proof for the game-theoretic strategyproofness has to be given, since it cannot be followed from the other.

Theorem 4. *The relation between different strategyproofness criteria is the following*

$$SPP \not\Leftarrow^1 SP \not\Leftarrow^2 GTCCSP \not\Leftarrow^3 WSP$$

and

$$GTSP \not\Leftarrow^4 GTCCSP.$$

Proof.

Assume a protocol with two players p_1 and p_2 and two possible allocations A_1 and A_2, which have the same probability. In the recommended strategy $S_\neg c$ and in the non-truthful strategy S_c the player p_2 gets the same values in A_1 and A_2. The values of p_1 are shown in the four tables below:

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	$1/2$	$2/3$
$p_1(X_1)$ by S_c	$1/2$	$1/2$

Case 1: $SPP \not\Leftarrow SP$

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	$7/8$	$3/4$
$p_1(X_1)$ by S_c	$7/8$	$3/4$

Case 2: $SP \not\Leftarrow GTCCSP$

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	$7/8$	$3/4$
$p_1(X_1)$ by S_c	1	$3/4$

Case 3: $WSP \not\Leftarrow GTSP$

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	$1/2$	$4/7$
$p_1(X_1)$ by S_c	1	$3/7$

Case 4: $GTSP \not\Leftarrow GTCCSP$

Table 4: Counter-examples for the correlation between the strategyproofness criteria

1 $SPP \not\Leftarrow SP$
In the strategy S_c the player p_1 gets $p_1(X_1) = 1/2$. Since $1/2 < 2/3$ in A_2 the player p_1 would stay honest and this protocol is SP. But 1/2 is proportional so the protocol is $notSPP$.

2 $SP \not\Leftarrow GTCCSP$
Since $7/8 = 7/8$ in A_1 and $3/4 = 3/4$ in A_2 the player p_1 would stay honest and this protocol is SP. But no value in the non-truthful strategy is smaller than in the recommended strategy, so the protocol is $notSP$.

3 $GTCCSP \not\Leftarrow WSP$
Since $3/4 = 3/4$ in A_2 and so the player p_1 has not a more valuable piece, he would stay honest and this protocol is WSP. But $1 > 7/8$ in A_1 and $3/4 = 3/4$ and the player p_1 has a more valuable piece in one allocation and the same value in the other so the protocol is $notGTCCSP$.

4 $GTSP \not\Leftarrow GTCCSP$
Since $3/7 < 4/7$ in A_2 the player p_1 would stay honest and this protocol is $GTCCSP$. But the expected value for the recommended strategy is $(1/2+4/7)/2 = 15/28$, which is smaller than $(1+3/7)/2 = 20/28$. So the protocol is $notGTSP$.

□

Example 4. *(Representation is inspired by [?])*

Cut & choose for $n = 2$		
Rules	**Player p_1 strategy**	**Player p_2 strategy**
1. Player p_1 partitions the cake X into two pieces $\{X', X - X'\}$	Partition X into two pieces of equal value	
2. Player p_2 chooses one piece		Choose the bigger value
3. Player p_1 gets the remaining piece		

Table 5: Cut & choose rules and strategies

Theorem 5. *Cut & choose is strategyproof for proportional protocols.*

Proof.

For the proof a general presentation of cut & choose game in extended form is used. W. l. o. g. the non-truthful cut is performed on the right part of the cake. The variables have the following restrictions:

$$0 \leq a \leq 1,\ 0 < \epsilon \leq 1/2,\ 0 \leq \delta \leq 1-a$$

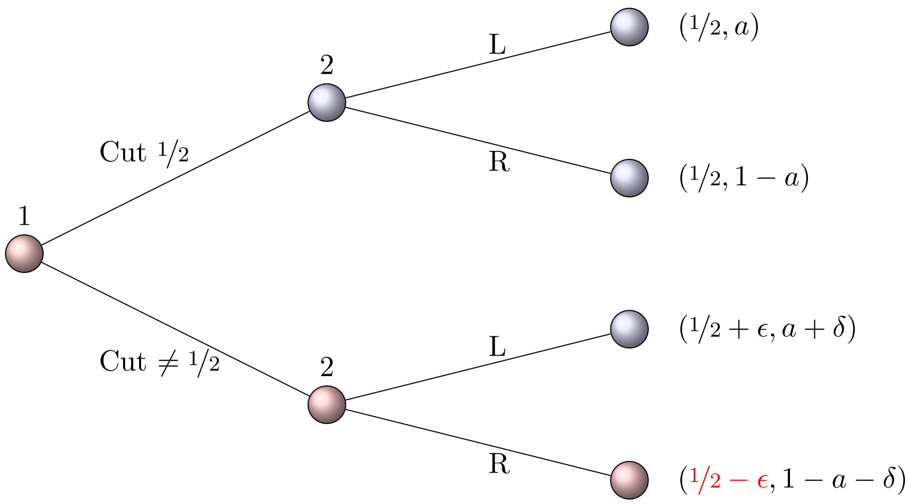

Figure 4: Cut & choose game in extended form

The red path is the general case, where by not following the recommended strategy player p_1 always becomes less than a proportional share. If Player p_2 would not choose the recommended strategy, he has to take a piece smaller than his proportional share. Both players would stay honest. Thus cut & choose is strategyproof for proportional protocols.

□

Theorem 6. *Cut & choose is game-theoretic cake-cutting strategyproof.*

Proof.

Options for not following the recommended strategy:

- Player p_2 takes the smaller piece. This can not be his intention, because then he has a piece with less value.

- Player p_1 cuts the cake into two unequal pieces. The chance to get less in his valuation is equal to the chance to get more in his valuation of the cake. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case:

$$1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/2$$

and in the dishonest case:

$$1/2 \cdot X' + 1/2 \cdot (X - X') = 1/2 \cdot \underbrace{X}_{=1} = 1/2$$

According to the definition of game-theoretical strategyproofness in the case with equal expected values the player would stay honest. Cut & choose is game-theoretical strategyproof.

□

Remark 1. *According to Theorem 3, Theorem 5 and Theorem 6 cut & choose is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

2 Strategyproof Proportional Protocols

The goal in this chapter is to analyse the strategyproofness of well-known protocols. First of all, they are rewritten into game-theoretic manner. Since each player has a truthful and a non-truthful strategy, a protocol with n active players has at least $2n$ strategies. If the obtained value is not equal in different non-truthful strategies they have to be separated. So the amount of strategies would grow and would make the analysis very tediously.
Luckily a protocol consists of a lot of repeats and actually each of the well-known protocols can be simplified to an interaction between two kinds of players. So the analysis of the whole game is unnecessary.
The proceed is as follows, the interactions between two kinds of players are represented in tables. A separation between rules and strategies is given. Afterwards the different strategies of a protocol are represented as an extended form game and the different kinds of strategyproofness are analysed.
An important pre-comment: Proportionality means each player gets $v_j(X_j) = {}^1\!/_n$ for $1 \leq j \leq n$. So if one player p_n leaves the game with $v_i(X_n) < {}^1\!/_n$ for $1 \leq i \leq (n-1)$ it still holds for the next round that the cake $X = X - X_n$ is normalized and a proportional piece is ${}^1\!/_{n-1}$ and not ${}^{1+\epsilon_i}\!/_n$.
The complete protocols in the standard description as well as the proofs of their proportionality can be found in [**?**].

2.1 The Kuhn à la Dawson Lone Divider Protocol

Kuhn à la Dawson Lone Divider protocol for arbitrary n		
Rules	**Player p_1 strategy**	**Players in P_{n-1} strategy**
1. Player p_1 cuts the cake X into n pieces $\{X_1, \ldots, X_n\}$	Cut X into n pieces of equal value	
2. Players in P_{n-1} mark s pieces with $1 \leq s \leq n$		Mark X_j if $v_i(X_j) \geq {}^1\!/_n$ for $1 \leq j \leq n$ and $2 \leq i \leq n$
3. If an allocation is impossible: Form the non-marked pieces to a new cake and exchange the cutter (last cutter leaves with a non-marked piece)		

Table 6: Lone divider rules and strategies

Theorem 7. *Lone divider protocol is not game-theoretic cake-cutting strategyproof.*

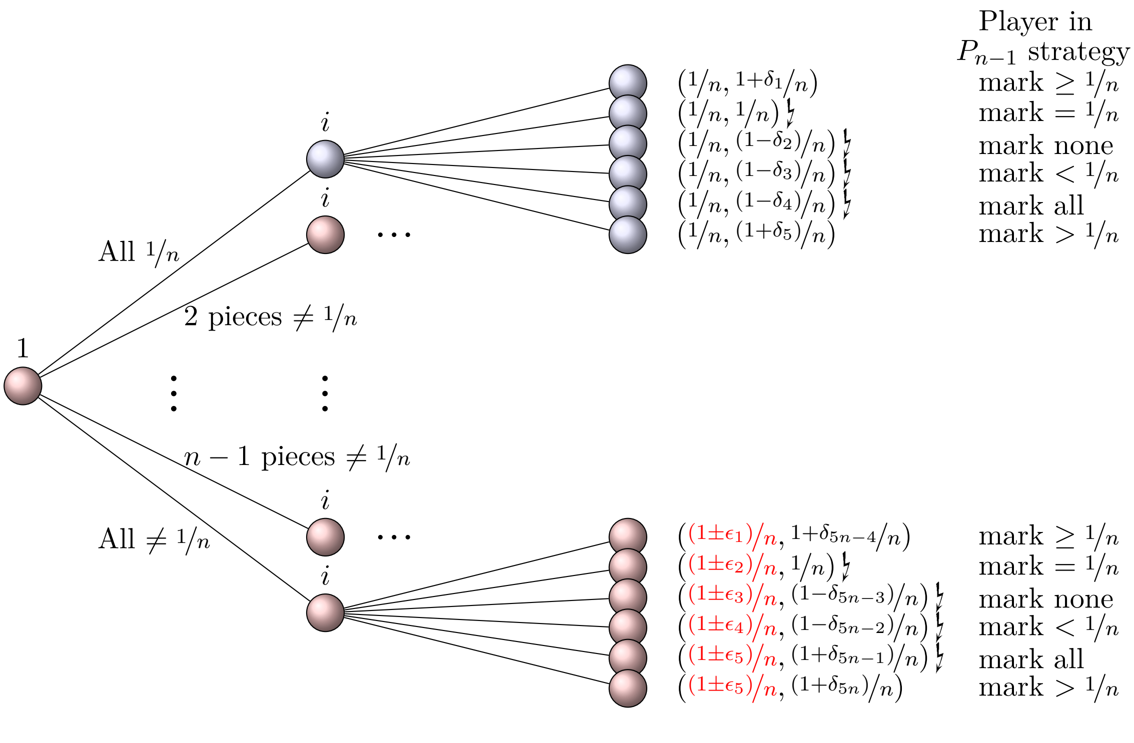

Figure 5: Lone divider cake game in extended form

Proof.

For showing that lone divider is *notGTCCSP* it is necessary to have at least one cheating player p_c. This player has a non-truthful strategy with $v_c(X_c) > v_c(X_t)$ in one allocation and $v_c(X_c) \geq v_c(X_t)$ for all other allocations. The proof is divided in two parts. In the first part the case $v_c(X_c) > v_c(X_t)$ can be is illustrated with an example. And in the second part a case distinction shows that this player is never getting a less valuable piece. The non-truthful strategy is to mark X_j if $v_c(X_j) > 1/n$ for $1 \leq j \leq n$, or in the only case when all pieces have the same value to mark all of them.

Part I: Example for a successful non-truthful strategy

Imagine the following allocation with three players. The player p_1 is going to cheat.

	X_L	X_M	X_R
p_3 (divider)	$1/3$	$1/3$	$1/3$
p_2 (rank 2)	$3/5$	$2/5$	0
p_1 (rank 1)	$1/2$	$1/3$	$1/6$

Table 7: Example for a successful non-truthful strategy

The acceptable pieces

By following the recommended strategy:

	X_L	X_M	X_R
p_3 (divider)	✓	✓	✓
p_2 (rank 2)	✓	✓	✗
p_1 (rank 1)	✓	✓	✗

By following the non-truthful strategy:

	X_L	X_M	X_R
p_3 (divider)	✓	✓	✓
p_2 (rank 2)	✓	✓	✗
p_1 (rank 1)	✓	✗	✗

Table 8: Acceptable pieces by a successful non-truthful strategy

An allocation is possible since each pieces is acceptable for at least one different player.

If player p_1 chooses the recommended strategy $S_{\neg}c$:
There are no players with just one acceptable piece, so the player with the highest rank can choose first. Player p_2 chooses X_L and player p_1 gets X_M with $p_1(X_M) = 1/3$. The divider gets the last piece.

If player p_1 chooses the non-truthful strategy S_c:
There is one players with just one acceptable piece, so this player chooses first. Player p_1 chooses X_L with $p_1(X_L) = 1/2$ and player p_2 gets X_M. The divider gets the last piece. The value of players p_1 piece in the non-truthful strategy S_c is $p_1(X_L) = 1/2 > 1/2 = p_1(X_M)$ than in the recommended strategy $S_{\neg}c$. So the player p_1 is at least in one allocation better off with the non-truthful strategy than in the recommended one.

Part II: Case distinction for a successful non-truthful strategy

Case 1:

Case 2:

Case 3:

Case 4:

Case 5:

Case 6:

□

Remark 2. *According to Theorem 3 and Theorem 7 Kuhn à la Dawson lone divider is not strategyproof for proportional protocols, not strategyproof, not game-theoretical strategyproof and not game-theoretical cake-cutting strategyproof. According to Example 2 it is weak strategyproof.*

2.2 The Banach-Knaster Last-Diminisher Protocol

The last diminisher protocol consists of $(n-2)$ rounds.
The players are separated into two groups. In the first group is the player p_i (with i number of the round) and in the second group P_{i+k} are all other players p_j with $i < j \leq n$, $1 \leq k \leq n-1$.
At the end of each round one player gets a piece and leaves the game. If it is player p_i then player p_{i+1} takes his place, otherwise player p_i is player p_{i+1} in the next round $i+1$ and players in group P_{i+k} will be consecutively numbered. In the last round the two remaining players apply cut & choose.

The Banach-Knaster Last-Diminisher protocol for arbitrary n		
Rules	**Player p_i strategy**	**Players in P_{i+k} strategy**
1. Player p_i cut a piece I_i	Cut a piece with value $1/n$	
2. Players p_j trim or pass		If $v_j(I_i) > 1/n$ trim so that $v_j(I_i) = 1/n$, else pass
3. Last trimmer take it		

Table 9: Last diminisher rules and strategies

Theorem 8. *Last diminisher protocol is strategyproof for proportional protocols.*

Proof.

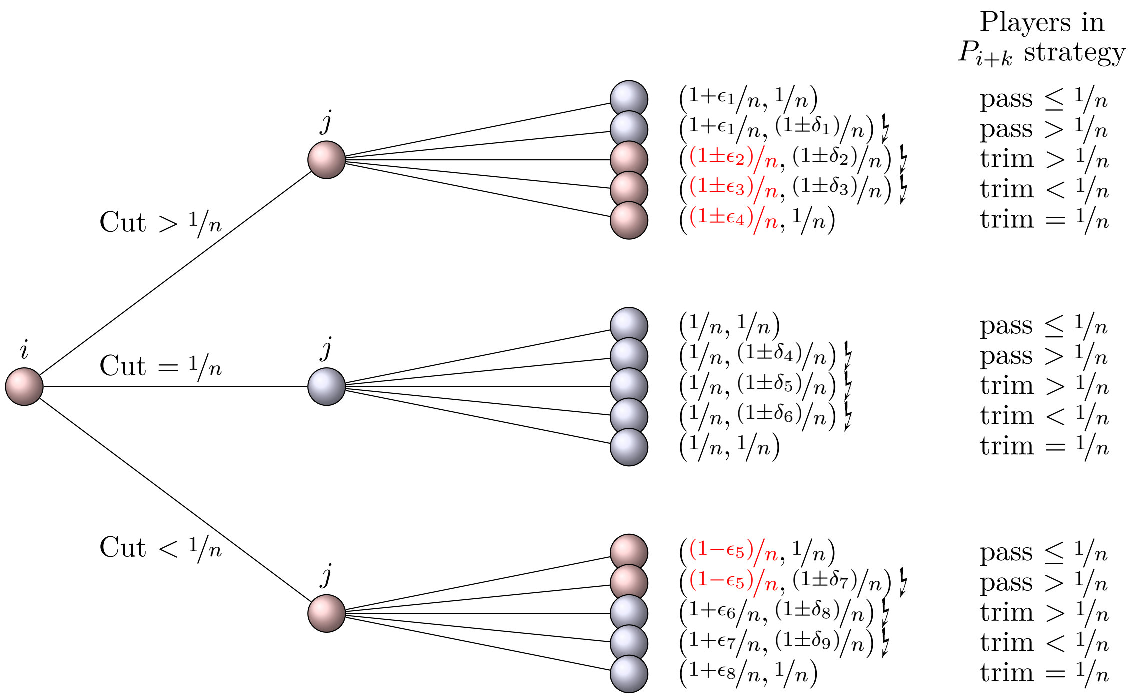

Figure 6: Last diminisher cake game in extended form

The consequences for each player p_j in P_{i+k} by choosing the strategy to pass a piece X_i with $v_j(X_i) > 1/n$ or to trim a piece X_i to $v_j(X_i) > 1/n$ are the same as for the player p_i by cutting a piece with $v_i(X_i) > 1/n$. The upper red path in Figure 6 shows that by choosing this non-truthful strategy a player can obtain a less valuable piece. For that to happen, a following player p_s needs to trim the piece s.t. $v_i(X_s) > 1/n$ and $v_j(X_s) > 1/n$ and for the reason that he is the last trimmer to get this piece.

The consequences for each player p_j in P_{i+k} by choosing the strategy trim a piece X_i to $v_j(X_i) < 1/n$ are the same as for the player p_i by cutting a piece with $v_i(X_i) < 1/n$. The lower red path in Figure 6 shows that by choosing this non-truthful strategy a player can obtain a less valuable piece. For that to happen, no following player p_s is allowed to trim the considered piece. So the cheating player will get this undesirable piece.

The only strategy which does not conceal this risk, is for all players the recommended one. □

Theorem 9. *Last diminisher protocol is game-theoretic strategyproof.*

Proof.

The assumption is that the players, who have to value the cake after the considered player value it with the probability $1/2$ as more valuable than $1/n$ and with the same probability with the value smaller or equal to $1/n$. □

Remark 3. *According to Theorem 3, Theorem 8 and Theorem 9 Kuhn à la Dawnson last diminisher is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

2.3 The Fink Lone-Chooser Protocol

For the description of this protocol the players will be separated in two groups. In the $(i-1)$ round players in the first group $P_I = \{p_1, \dots, p_i\}$ have already their proportional piece. Assume that player p_1 owns the whole cake before the first round. The performance in each round is the following:

The Fink Lone-Chooser protocol for arbitrary n		
Rules	**Players in P_I strategy**	**Player p_{i+1} strategy**
1. Players in P_I partition their piece X_i into $i+1$ pieces $\{X_{i,1}, \dots, X_{i,i+1}\}$	Partition X_i into $i+1$ pieces of equal value	
2. Player p_{i+1} chooses one piece of each player's cake		Choose the most valuable piece of each player's cake

Table 10: Lone chooser rules and strategies

Theorem 10. *Fink Lone-Chooser protocol is game-theoretic strategyproof.*

Proof. Proof by induction:

- Case $i = 2$: Cut & choose is game-theoretic strategyproof by Theorem 6.

- Case $(i-1) \to i$:

  Options for not following the recommended strategy:

  – Player p_{i+1} takes not the biggest piece. Then he has a less valuable piece than by following the recommended strategy.
  – Players in P_I cut the cake into $i+1$ non-equal pieces. The chance that player p_{i+1} takes a certain piece is $1/(i+1)$. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case (because he had at least $1/i$ before) at least:

  $$i \cdot 1/(i+1) \cdot 1/i = 1/(i+1)$$

  and in the dishonest case:
  Assume the value of the piece will be distributed on s pieces with $1 \leq s \leq i+1$. So the expected value will be:

  $$1/i \cdot (i+1-s)/(i+1) + (s-1)/(si) \cdot s/(i+1) = 1/(i+1)$$

  Even if the value is distributed unequally on the s pieces, it would not affect the value of the piece in the allocation, because the probability of the piece the player p_{i+1} take is uniformly distributed.

According to the definition of game-theoretic strategyproofness in the case with equal expected values the player would stay honest.

□

Theorem 11. *Fink Lone-Chooser protocol is strategyproof for proportional protocols.*

Proof.

For the proof a presentation of Lone chooser game in extended form is used. The variables have the following restrictions:

$0 \leq (i \pm \epsilon_1), (i \pm \epsilon_2) \leq (i+1)$, $0 \leq \delta_1, \delta_3 \leq i$, $0 \leq (i \pm \delta_2), (i \pm \delta_4) \leq (i+1)$, $\delta_2 \leq \delta_1$, $\delta_4 \leq \delta_3$

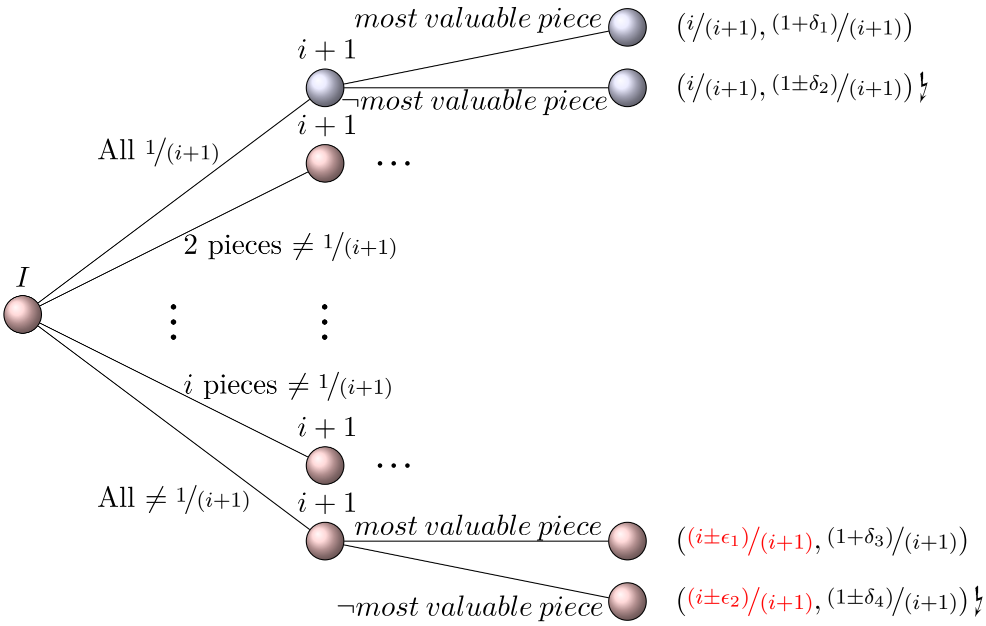

Figure 7: Lone chooser cake game in extended form

The red path is the not succesful allocation by following the non-truthful strategy for a player in P_I. Here every player becomes less than by following his recommended strategy. Especially it is the case, where the player p_{i+1} takes a piece which a player in P_I values more than ${}^{1}/_{(i+1)}$.
If Player p_2 would not choose the recommended strategy, he has to take a piece less valuable than in his best possibility. Both kind of players would stay honest. Thus Lone chooser is strategyproof for proportional protocols. □

Remark 4. *According to Theorem 3, Theorem 10 and Theorem 11 Fink lone chooser is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

2.4 The Even & Paz Divide-and-Conquer Protocol

Divide-and-Conquer protocol for arbitrary n		
Rules	**Players in P_{n-1} strategy**	**Player p_n strategy**
1. Each player in P_{n-1} partitions the cake X into two pieces $\{X', X - X'\}$	Partition X into two pieces in the value-ratio of $\lfloor n/2 \rfloor : \lceil n/2 \rceil$	
2. Player p_n chooses one piece		If $v_n(X') \geq \lfloor n/2 \rfloor / n$ choose X', otherwise $X - X'$
3. Cut in the ratio of the $n/2$-th player from left border and repeat with the new groups separately until each player has his own piece		

Table 11: Divide & conquer rules and strategies

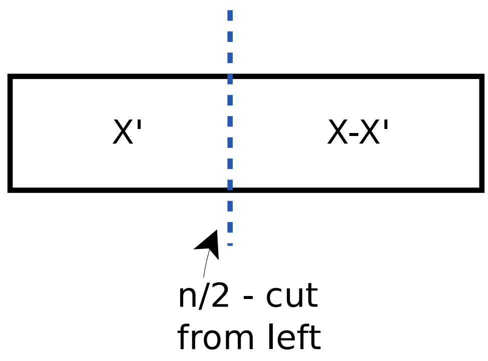

Figure 8: D&C step 3

Theorem 12. *The Divide-and-Conquer protocol is strategyproof for proportional protocols.*

Proof.

The proof can be found in [**?**]: We showed in section 2 that D&C guarantees a player a proportional share if it is truthful. We now consider the case when two players, A and B, must divide a cake, but one may not be truthful. If their truthful $1/2$ points are as shown below, then cutting the cake at | gives each player more than $1/2$:

$$0 - - - - - - - - - - - - - a - | - b - - - - - - - - - - - - - 1$$

But if player A should report that its $1/2$ point is either to the left or right of a, it risks getting less than $1/2$ the cake if $(i)|$ is to the left of a or $(ii)|$ is to the right of b. This argument for the vulnerability of D&C-that it does not guarantee a player a proportional share if it is not truthful-can readily be extended to $n > 2$ players. □

Theorem 13. *The Divide-and-Conquer protocol is game-theoretic strategyproof.*

Proof.

□

Remark 5. *According to Theorem 3, Theorem 12 and Theorem 13 Even & Paz divide-and-conquer is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

3 Conclusion

In this work the common proportional cake-cutting protocols have been rewritten in a game-theoretic manner and analysed on whether a non-truthful strategy could yield a more advantageous situation for a non-truthful player. It was possible to approve game-theoretically that the only strategy which promises the best outcome is the strategy recommended by the protocol.
The approach is very different from [**?**] and [**?**]. In their approaches, the basic assumptions of cake-cutting is weakened and so only special cases are analysed. Instead the goal here was to adapt the existing definitions of truthfulness to the general cake-cutting problem.

Protocol	WSP	GTSP	GTCCSP	SP	SPP	SSP
Cut & Choose	✓	✓	✓	✓	✓	✗
Last Diminisher	✓	✓	✓	✓	✓	✗
Lone Chooser	✓	✓	✓	✓	✓	✗
Lone Divider	✓	✗	✗	✗	✗	✗
Divide & Conquer	✓	✓	✓	✓	✓	✗

Table 12: Overview: Strategyproofness of proportional cake-cutting protocols

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