Equity and Efficiency in a Measure Space with Nonadditive Preferences: The Problem of Cake Division*

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Abstract

This paper considers a classical problem of cake division in a nonatomic finite measure space among finitely many individuals. We investigate a nonadditive continuous preference relation in a Borel σ-field and prove the existence of Pareto optimal envy-free partitions, Pareto optimal α-equitable partitions, and α-Rawls optimal partitions. We also show that Pareto optimal α-equitability is equivalent to α-Rawls optimality, but Pareto optimality does not imply Rawls optimality.

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1 Introduction

The problem of cake division in a finite measure space among finitely many individuals has a long history and is drawing more attentions in recent years. Since the seminal work by Dubins and Spanier (1961), it is commonly assumed in the theory of fair division that preferences of each individual are represented by a nonatomic probability measure. This assumption enables one to apply Lyapunov's theorem, which guarantees the convexity and compactness of the image of the set of measurable partitions under a finite dimensional nonatomic vector measure, to prove the existence of various solutions such as Pareto optimal partitions (Dubins and Spanier 1961), Rawls optimal partitions (Dubins and Spanier 1961), envy-free partitions (Dubins and Spanier 1961), group envy-free partitions (Berliant et al. 1992), super envy-free partitions (Barbanel 1996), and α-fair partitions (Dubins and Spanier 1961).

Representing a preference relation by a probability measure means that the corresponding utility function is countably additive on the σ-field, and consequently assumes a constant marginal utility. This is obviously a severe restriction on the preference relation that is difficult to justify from the economics view point. Some attempts have been made in the theory of fair division to mitigate the additivity of preferences. Stromquist (1980) used a nonadditive continuous utility function on a unit simplex to demonstrate the existence of an envy-free partition. Berliant et al. (1992) formulated the nonadditive integral representation of a utility function satisfying the subadditivity in a Lebesgue measure space to show the existence of α-fair efficient partitions and egalitarian equivalent, efficient partitions. Maccheroni and Marinacci (2003) employed a nonatomic concave capacity on a σ-field to prove the existence of α-fair partitions. The above works, however, are unsatisfactory in that the problem under what conditions for preferences such nonadditive representations are possible is left aside.

In this paper we investigate a not necessarily additive continuous preference relation in a nonatomic finite measure space and prove the existence of Pareto optimal α-equitable partitions, Pareto optimal envy-free partitions, and α-Rawls optimal partitions. Consideration of the nonadditive representation of a preference relation faces with a technical difficulty in the problem of fair division. Because the preference relation is represented by a nonadditive set function on the σ-field, we cannot resort to Lyapunov's theorem in the existence argument. Therefore, some additional structures are needed to ensure the compactness of the utility possibility set, i.e., the image of the set of measurable partitions under the vector-valued set function.

To overcome this difficulty, the following two devices are developed in this paper. First, we identify the Borel σ-field in the nonatomic finite measure space with the set of characteristic functions in L^1. Thus, the Borel σ-field is endowed with the metric induced by L^1-norm. We then introduce the continuity axiom for preferences on the Borel σ-field with L^1-metric topology and show the existence of a continuous utility function representing a continuous preference relation by the standard argument of Debreu (1964). Such an approach for the continuous representation of a preference relation on a Borel σ-field is pursued also by Berliant (1986) and Berliant and Dunz (2004) using different topologies from this paper.

Second, the underlying measure space is supposed to be a Borel measure space of a locally compact topological group with a Haar measure, which is usually unnecessary in the literature. This requirement ensures the compactness of the set of measurable partitions in L^1, but we do not employ explicitly the topological group structure in the existence argument in the problem of fair division. While the weak* topology of L^∞ is employed for the proof of Lyapunov's theorem (see for example Akin 1995), the use of the weak* topology of L^∞ in our setting does not work for the compactness argument because the weak* limit of a sequence of characteristic functions in L^∞ need not be a characteristic function, and consequently the set of measurable partitions lacks the closedness in the weak* topology of L^∞. The additional requirement of topological group structure with a Haar measure is a price for relaxing the additivity of preferences.

With the above additional structures, the utility possibility set is shown to be compact and the set of Pareto optimal partitions is shown to be homeomorphic to the unit simplex. This fact is an analogue of the similar result of Varian (1974) for a finite dimensional commodity space and of Magill (1981) and Mas-Colell (1986) for an infinite dimensional commodity space. This homeomorphism is employed effectively to show the existence of α-equitable partitions and α-Rawls optimal partitions. We also demonstrate that Pareto optimal α-equitability is equivalent to α-Rawls optimality, but Pareto optimality does not imply Rawls optimality. The intersection theorem of Scarf (1967) asserting the nonemptiness of the intersection of a closed covering of a unit simplex is used to show the existence of Pareto optimal envy-free partitions.

2 Continuous Representation of Nonadditive Preferences on a Borel σ-Field

We first introduce a metric on a Borel σ-field in a finite measure space which makes the Borel σ-field a complete metric space. We then show that the Borel σ-field under investigation can be identified with a compact subset of L^1 when the underlying topological space on which the Borel σ-field is defined is a compact subset of a locally compact topological group with a regular Haar measure. With this topological structure, we define the continuity of preferences on the Borel σ-field and demonstrate the existence of a continuous utility function representing the continuous preferences.

2.1 Metric on a Borel σ-Field

Let $(X, \mathscr{B}_X, \mu)$ be a measure space, where X is a topological space, $\mathscr{B}_X$ is the Borel σ-field of X, and μ is a measure on $\mathscr{B}_X$. A measure μ is a *Borel measure* if $\mu(\Omega) < \infty$ for each compact subset Ω of X. Let μ be a Borel measure and $\Omega \in \mathscr{B}_X$ be a compact subset of X. When Ω is endowed with the relative topology from X, the Borel σ-field $\mathscr{B}_\Omega$ of Ω is given by $\mathscr{B}_\Omega = \{E \cap \Omega | \, E \in \mathscr{B}_X\}$ and the restriction μ, which we denote again μ, to the Borel measurable space $(\Omega, \mathscr{B}_\Omega)$ makes $(\Omega, \mathscr{B}_\Omega, \mu)$ a finite Borel measure space. Each element f in $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ is identified with an element $\tilde{f}$ in $L^1(X, \mathscr{B}_X, \mu)$ by the embedding $f \mapsto \tilde{f}$ satisfying $\tilde{f} = f$ on Ω and $\tilde{f} = 0$ on $X \setminus \Omega$. This embedding yields an isometry on $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ into $L^1(X, \mathscr{B}_X, \mu)$ and under this identification $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ is a closed vector subspace of $L^1(X, \mathscr{B}_X, \mu)$. Note that the norm topology on $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ coincides with its relative topology induced by the norm topology on $L^1(X, \mathscr{B}_X, \mu)$.

Two measurable sets A and B in $\mathscr{B}_\Omega$ are *μ-equivalent* if $\mu(A \triangle B) = 0$, where $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B. The μ-equivalence defines an equivalence relation on $\mathscr{B}_\Omega$. We denote the equivalence class of $A \in \mathscr{B}_\Omega$ by $[A]$ and the set of equivalence classes in $\mathscr{B}_\Omega$ by $\mathscr{B}_\Omega[\mu]$. If, for any two μ-equivalence classes $\mathbf{A}$ and $\mathbf{B}$, we define the metric d by $d(\mathbf{A}, \mathbf{B}) = \mu(A \triangle B)$ where A and B are arbitrarily selected elements of $\mathbf{A}$ and $\mathbf{B}$, then $\mathscr{B}_\Omega[\mu]$ becomes a complete metric space (see Dunford and Schwartz, 1958, Lemma III.7.1). Since $\mu(A \triangle B) = \int |\chi_A - \chi_B| d\mu$ where χ_A and χ_B are characteristic functions of A and B respectively, we know that two measurable sets A and B are μ-equivalent if and only if their characteristic functions differ by a μ-null function. Therefore, the mapping $\mathbf{A} \mapsto \chi_A$ where A is an arbitrarily selected element of $\mathbf{A}$ is an isometry on $\mathscr{B}_\Omega[\mu]$ into $L^1(\Omega, \mathscr{B}_\Omega, \mu)$, and hence $\mathscr{B}_\Omega[\mu]$ can be identified with $\mathscr{X}_\Omega =$

$\{\chi_A \in L^1(\Omega, \mathscr{B}_\Omega, \mu) |\, [A] \in \mathscr{B}_\Omega[\mu]\}$.

A topological space X is *locally compact* if each point in X has a neighborhood whose closure is compact. A Borel measure μ on $\mathscr{B}_X$ is *regular* if it is both outer regular and inner regular, that is, $\mu(E) = \inf\{\mu(U) |\, U$ is open and $E \subset U\} = \sup\{\mu(K) |\, K$ is compact and $K \subset E\}$ for each $E \in \mathscr{B}_X$. A *Haar measure* is a Borel measure μ on a locally compact topological group X such that $\mu(U) > 0$ for each nonempty open set U, and $\mu(xE) = \mu(E)$ for each $E \in \mathscr{B}_X$ and $x \in X$. Every locally compact topological group has a regular Haar measure (see Halmos, 1950, Theorem 58.B). When f is a function on X and a is a group element in X, the translate f_a of f denotes the function defined by $x \mapsto f(ax)$.

Theorem 2.1. *Let $(X, \mathscr{B}_X, \mu)$ be a Borel measure space with X a locally compact topological group and μ a regular Haar measure. If Ω is a compact subset of X and $(\Omega, \mathscr{B}_\Omega, \mu)$ is the finite measure space induced by the restriction of $(X, \mathscr{B}_X, \mu)$, then $\mathscr{B}_\Omega[\mu] = \mathscr{X}_\Omega$ is a compact metric space.*

To prove Theorem 2.1, we need the following useful criterion to characterize the compactness in $L^1(X, \mathscr{B}_X, \mu)$, which is a significant generalization of the characterization of the compactness in $L^1(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n}, \mu)$ with μ a Lebesgue measure (see Dunford and Schwartz, 1958, Theorem IV.8.21). For a proof of the next proposition, see Edwards (1965, Theorem 4.20.1).

Proposition 2.1 (Weil). *Let X be a locally compact topological group with a Haar measure μ. A subset $\mathscr{K}$ of $L^1(X, \mathscr{B}_X, \mu)$ is relatively compact if and only if it satisfies the following three conditions:*

(1) *$\mathscr{K}$ is bounded.*

(2) *For any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\int_{X\setminus K} |f| d\mu < \varepsilon$ uniformly in $f \in \mathscr{K}$.*

(3) *For any $\varepsilon > 0$ there exists a neighborhood U of the unit element of X such that $a \in U$ implies $\int |f_a - f| d\mu < \varepsilon$ uniformly in $f \in \mathscr{K}$.*

Proof of Theorem 2.1

Since $\mathscr{X}_\Omega$ is a closed subset of $\mathscr{X} = \{\chi_E \in L^1(X, \mathscr{B}_X, \mu) |\, [E] \in \mathscr{B}_X[\mu]\}$, it suffices to show that $\mathscr{X}$ is compact in $L^1(X, \mathscr{B}_X, \mu)$. It is obvious that $\mathscr{X}$ is bounded and closed. To prove the compactness of $\mathscr{X}$, by virtue of Proposition 2.1, it suffices to show that (i) for any $\varepsilon > 0$, there exists a compact subset K of X such that $\int_{X\setminus K} \chi_E d\mu < \varepsilon$ uniformly in $E \in \mathscr{B}_X$; (ii) for any $\varepsilon > 0$, there exists a neighborhood U of the unit $e \in X$ such that

$a \in U$ implies $\int_X |(\chi_E)_a - \chi_E| d\mu < \varepsilon$ uniformly in $E \in \mathscr{B}_X$, where $(\chi_E)_a$ is a translate of χ_E defined by $x \mapsto \chi_E(ax)$.

Since μ is regular, for any $\varepsilon > 0$ there exists a compact subset K of X such that $\mu(X \setminus K) < \varepsilon$. Therefore, we have $\int_{X\setminus K} \chi_E d\mu \leq \int_{X\setminus K} 1 d\mu = \mu(X \setminus K) < \varepsilon$ for each $E \in \mathscr{B}_X$, which implies (i). To prove (ii) note first that if $f_E(a) = \mu(aE \triangle E)$ for $a \in X$, then f_E is continuous on X for each $E \in \mathscr{B}_X$ (see Halmos, 1950, Theorem XII.A). Define $f(a) = \sup_{E\in\mathscr{B}_X} f_E(a)$. Then f is continuous and $f(e) = 0$ since $f_E(e) = 0$ for each $E \in \mathscr{B}_X$. Therefore, for any $\varepsilon > 0$ there exists a neighborhood U of e such that $a \in U \cap U^{-1}$ implies $f(a^{-1}) < \varepsilon$. This implies $\int |(\chi_E)_a - \chi_E| d\mu = \mu(a^{-1}E \triangle E) \leq f(a^{-1}) < \varepsilon$ for each $E \in \mathscr{B}_X$, which shows (ii). $\square$

A *partition* of Ω is an n-tuple of disjoint elements in $\mathscr{B}_\Omega$ whose union is Ω. We denote the set of partitions of Ω by $\mathscr{P}$. The n-times Cartesian product of $\mathscr{B}_\Omega[\mu]$ is denoted by $\mathscr{B}^n_\Omega[\mu]$.

Lemma 2.1. *For any $(\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{B}^n_\Omega[\mu]$ there exists a partition $(A_1 \dots, A_n)$ satisfying $A_i \in \mathbf{A}_i$ for each $i \in I$ if and only if $\sum_{i\in I} \chi_{B_i} = \chi_\Omega$ for any $B_i \in \mathbf{A}_i$ with $i \in I$.*

Proof. Let $(A_1 \dots, A_n)$ be a partition satisfying $A_i \in \mathbf{A}_i$ for each $i \in I$. Since $(A_1, \dots, A_n)$ is a partition if and only if $\sum_{i\in I} \chi_{A_i}(\omega) = 1$ for any $\omega \in \Omega$, it follows that $\sum_{i\in I} \chi_{B_i}(\omega) = 1$ a.e. $\omega \in \Omega$ for any $B_i \in \mathbf{A}_i$ with $i \in I$ if and only if $\chi_{A_i}(\omega) = \chi_{B_i}(\omega)$ a.e. $\omega \in \Omega$, in view of $\mu(A_i \triangle B_i) = 0$. Conversely, suppose that $\sum_{i\in I} \chi_{B_i} = \chi_\Omega$ for any $B_i \in \mathbf{A}_i$ with $i \in I$. Define $E_{ij} = B_i \cap B_j$ for each $i, j \in I$. Since $\sum_{i\in I} \chi_{B_i}(\omega) = 1$ a.e. $\omega \in \Omega$, we then have $\mu(E_{ij}) = 0$ for $i \neq j$ and $\mu(\Omega \setminus \bigcup_{i\in I} B_i) = 0$. Define $A_i = B_i \setminus \bigcup_{j\in I\setminus\{1,\dots,i\}} E_{ij}$ for $i = 1, \dots, n-1$ and $A_n = B_n \cup (\Omega \setminus \bigcup_{i\in I} B_i)$. By construction, the resulting partition $(A_1, \dots, A_n)$ satisfies $A_i \in \mathbf{A}_i$ for each $i \in I$. $\square$

Define the set of equivalence classes of partitions by

$$\mathscr{P}[\mu] = \left\{ (\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{B}^n_\Omega[\mu] \;\middle|\; \begin{array}{l} \exists (A_1, \dots, A_n) \in \mathscr{P} : \\ A_i \in \mathbf{A}_i \;\; \forall i \in I \end{array} \right\}.$$

Theorem 2.2. *$\mathscr{P}[\mu]$ is a compact metric space.*

Proof. By Theorem 2.1, $\mathscr{X}_\Omega$ is a compact metric space equipped with the L^1-norm metric. Denote the n-times Cartesian product of $\mathscr{X}_\Omega$ by $\mathscr{X}^n_\Omega$. By Lemma 2.1, $(\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{P}[\mu]$ if and only if $\sum_{i\in I} \chi_{A_i} = \chi_\Omega$ for any $A_i \in \mathbf{A}_i$ with $i \in I$. Therefore, $\mathscr{P}[\mu]$ is identified with

$$\mathscr{P}[\mu] = \left\{ (\chi_{A_1}, \dots, \chi_{A_n}) \in \mathscr{X}^n_\Omega \;\middle|\; \sum_{i\in I} \chi_{A_i} = \chi_\Omega \right\}.$$

Since $\mathscr{P}[\mu]$ is a closed subset of $\mathscr{X}^n_\Omega$, it is compact in $\mathscr{X}^n_\Omega$. $\square$

2.2 Existence of a Continuous Utility Function

A *preference relation* $\succsim$ in the finite measure space $(\Omega, \mathscr{B}_\Omega, \mu)$ is a complete transitive binary relation on $\mathscr{B}_\Omega$. The strict preference $A \succ B$ means that $A \succsim B$ and $B \not\succsim A$. The indifference $A \sim B$ means that $A \succsim B$ and $B \succsim A$. A set function u on $\mathscr{B}_\Omega$ *represents* $\succsim$ whenever $A \succsim B$ if and only if $u(A) \geq u(B)$, and such u is called a *utility function*.

Definition 2.1. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is:

(1) *μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $A \succsim B$.

(2) *μ-strictly monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $A \succ B$.

(3) *μ-indifferent* if $\mu(A \triangle B) = 0$ implies $A \sim B$.

The μ-(strict) monotonicity of preferences on the σ-field are analogues of the (strict) monotonicity of preferences on a standard commodity space. If $\succsim$ is μ-indifferent and N is a null set, then $A \sim A \cup N$ for each $A \in \mathscr{B}_\Omega$. Thus the μ-indifference of a preference relation $\succsim$ *induces* a preference relation $\succsim_\mu$ on $\mathscr{B}_\Omega[\mu]$ defined by $\mathbf{A} \succsim_\mu \mathbf{B}$ if and only if there exist $A \in \mathbf{A}$ and $B \in \mathbf{B}$ such that $A \succsim B$. This is equivalent to saying that $\mathbf{A} \succsim_\mu \mathbf{B}$ if and only if $A \succsim B$ for each $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Thus, any utility function u representing $\succsim$ on $\mathscr{B}_\Omega$ induces a utility function u_μ representing $\succsim_\mu$ on $\mathscr{B}_\Omega[\mu]$ by $u_\mu(\mathbf{A}) = u(A)$ with $A \in \mathbf{A}$.

Definition 2.2. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is *μ-continuous* if it is μ-indifferent and for each $\mathbf{A} \in \mathscr{B}_\Omega[\mu]$ both the upper contour set $\{\mathbf{B} \in \mathscr{B}_\Omega[\mu] \mid \mathbf{B} \succsim_\mu \mathbf{A}\}$ and the lower contour set $\{\mathbf{B} \in \mathscr{B}_\Omega[\mu] \mid \mathbf{A} \succsim_\mu \mathbf{B}\}$ are closed in $\mathscr{B}_\Omega[\mu]$.

The μ-continuity of $\succsim$ states that the preference relation $\succsim_\mu$ induced by $\succsim$ satisfies the standard continuity axiom for preferences.

Definition 2.3. A function f on $\mathscr{B}_\Omega$ is:

(1) *μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $f(A) \geq f(B)$.

(2) *μ-strictly monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $f(A) > f(B)$.

(3) *μ-indifferent* if $\mu(A \triangle B) = 0$ implies $f(A) = f(B)$.

(4) *μ-continuous* if it is μ-indifferent and induces a continuous function f_μ on $\mathscr{B}_\Omega[\mu]$.

It is obvious that a utility function representing a μ-indifference preference relation is μ-continuous (resp. μ-strictly monotone, μ-strictly monotone) if and only if the preference relation is μ-continuous (resp. μ-strictly monotone, μ-strictly monotone). The next theorem guarantees the existence of a μ-continuous utility function representing μ-continuous preferences.

Theorem 2.3. *Let $(X, \mathscr{B}_X, \mu)$ be a Borel measure space with X a locally compact topological group and μ a regular Haar measure, and let Ω be a compact subset of X and $(\Omega, \mathscr{B}_\Omega, \mu)$ be the finite measure space induced by the restriction of $(X, \mathscr{B}_X, \mu)$. Then, for any μ-continuous preference relation $\succsim$ on $\mathscr{B}_\Omega$, there exists a μ-continuous utility function representing $\succsim$.*

Proof. Note that $\mathscr{B}_\Omega[\mu]$ is separable by its compactness asserted in Theorem 2.1. Since $\succsim_\mu$ is a continuous preference relation on a separable metric space, by virtue of the celebrated theorem of Debreu (1964), there exists a continuous utility function $\tilde{u}$ on $\mathscr{B}_\Omega[\mu]$ representing $\succsim_\mu$. Define $u(A) = \tilde{u}(\mathbf{A})$ for $A \in \mathbf{A}$. Then u is a μ-continuous utility function on $\mathscr{B}_\Omega$ representing $\succsim$. $\square$

Example 2.1. A typical example of a μ-continuous and μ-monotone (resp. μ-strict monotone) preference relation on $\mathscr{B}_\Omega$ is the one defined by $A \succsim B$ if and only if $f(\mu(A)) \geq f(\mu(B))$, where f is a continuous and increasing (resp. strictly increasing) function on $[0, \mu(\Omega)]$ and μ is a finite measure of a measurable space $(\Omega, \mathscr{B}_\Omega)$. To see this, we show that the function u defined by $u(A) = f(\mu(A))$ is μ-continuous. If $\mu(A \triangle B) = 0$, then $\mu(A \cup B) = \mu(A \cap B) = \mu(A) = \mu(B)$. Thus, u is μ-indifferent and induces a function u_μ on $\mathscr{B}_\Omega[\mu]$ via the formula $u_\mu(\mathbf{A}) = u(A)$ with $A \in \mathbf{A}$. Let $\{\mathbf{A}_n\}$ be a sequence in $\mathscr{B}_\Omega[\mu]$ converging to $\mathbf{A}$. We then have $\lim_n \mu(A_n \cup A) = \lim_n \mu(A_n \cap A)$ for $A_n \in \mathbf{A}_n$ and $A \in \mathbf{A}$. For any $\varepsilon > 0$ there exists an integer n_0 such that $\mu(A_n \cup A) < \mu(A_n \cap A) + \varepsilon$ for each $n \geq n_0$. We thus have $\mu(A_n) \leq \mu(A_n \cup A) < \mu(A_n \cap A) + \varepsilon \leq \mu(A) + \varepsilon$ for each $n \geq n_0$. Thus, $\limsup_n \mu(A_n) < \mu(A) + \varepsilon$. Since ε is arbitrary, we have $\limsup_n \mu(A_n) \leq \mu(A)$. Similarly, since $\mu(A) \leq \mu(A_n \cup A) < \mu(A_n \cap A) + \varepsilon \leq \mu(A_n) + \varepsilon$ for each $n \geq n_0$, we have $\mu(A) \leq \liminf_n \mu(A_n)$. Therefore, $\lim_n u_\mu(\mathbf{A}_n) = \lim_n f(\mu(A_n)) = f(\lim_n \mu(A_n)) = f(\mu(A)) = u_\mu(\mathbf{A})$, and hence u is μ-continuous.

3 Structure of the Pareto Optimal Partitions

This section examines the topological structure of the set of Pareto optimal partitions. The existence of a Pareto optimal partition follows from the continuity of the preferences of each individual and the compactness of the set of partitions in L^1. We construct a homeomorphism between the set of Pareto optimal partitions and the unit simplex under the μ-continuity and μ-strict monotonicity of preferences of each individual. This homeomorphism plays a crucial role for the existence argument in Section 4. For the case of additive preferences with a nonatomic probability measure on a σ-field, Weller (1985) found a useful geometric relation between the set of Pareto optimal partitions and the unit simplex, which did not employ any topological structure. For such a characterization and its extension, see Akin (1995), Barbanel (1999, 2000), and Dall'Aglio (2001).

3.1 Pareto Optimal Partition

Let $(X, \mathscr{B}_X, \mu)$ be a Borel measure space with X a locally compact topological group and μ a nonatomic regular Haar measure. Let Ω be a compact subset of X and $(\Omega, \mathscr{B}_\Omega, \mu)$ be the nonatomic finite measure space induced from $(X, \mathscr{B}_X, \mu)$ as in Section 2.1. A typical example of this structure is the Lebesgue measure space of $\mathbb{R}^n$ with any compact subset of $\mathbb{R}^n$ in which $\mathbb{R}^n$ is locally compact topological Abelian group under the vector addition and the Lebesgue measure is a nonatomic regular Haar measure. Denote the finite set of individuals by $I = \{1, \dots, n\}$. A preference relation of individual $i \in I$ on $\mathscr{B}_\Omega$ is denoted by $\succsim_i$.

By virtue of Theorem 2.3, a preference relation is represented by a μ-continuous and μ-monotone (resp. μ-strictly monotonicity) utility function if and only if the preference relation is μ-continuous and μ-monotone (resp. μ-strictly monotone). The utility function of each individual is employed instead of the preference relation of each individual in the sequel.

Definition 3.1. A partition $(A_1, \dots, A_n)$ is:

(1) *Weakly Pareto optimal* if there exists no partition $(B_1, \dots, B_n)$ such that $u_i(B_i) > u_i(A_i)$ for each $i \in I$.

(2) *Pareto optimal* if there exists no partition $(B_1, \dots, B_n)$ such that $u_i(B_i) \geq u_i(A_i)$ for each $i \in I$ and $u_i(B_i) > u_i(A_i)$ for some $i \in I$.

Theorem 3.1. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then a partition is Pareto optimal if and only if it is weakly Pareto optimal.*

Proof. It is immediate that Pareto optimality implies weak Pareto optimality. We show the converse. Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$, and let $(A_1, \dots, A_n)$ be a weakly Pareto optimal partition. Suppose that $(A_1, \dots, A_n)$ is not Pareto optimal. Then there exists a partition $(B_1, \dots, B_n)$ such that $u_i(B_i) \geq u_i(A_i)$ for each $i \in I$ and

$u_j(B_j) > u_j(A_j)$ for some $j \in I$. The μ-continuity of u_j and the nonatomicity of μ imply that there exists some $C_j \subset B_j$ such that $u_j(C_j) > u_j(A_j)$ and $\mu(B_j \setminus C_j) > 0$. Decompose $B_j \setminus C_j$ into $n-1$ disjoint sets B'_i, $i \in I \setminus \{j\}$, such that $\bigcup_{i \in I \setminus \{j\}} B'_i = B_j \setminus C_j$ and $\mu(B'_i) > 0$ for each $i \in I \setminus \{j\}$. Let $C_i = B_i \cup B'_i$ for each $i \in I \setminus \{j\}$. Then the resulting partition $(C_1, \ldots, C_n)$ satisfies $u_i(C_i) > u_i(A_i)$ for each $i \in I$ by the μ-strict monotonicity of u_i. This contradicts the weak Pareto optimality of $(A_1, \ldots, A_n)$. $\square$

3.2 Topological Properties

Let u_i be μ-continuous for each $i \in I$. Since the range of u_i is bounded for each $i \in I$ by Theorem 2.1, without loss of generality we may assume that $(u_1(A_1), \ldots, u_n(A_n)) \in [0,1]^n$ for any $(A_1, \ldots, A_n) \in \mathscr{P}$, where $[0,1]^n$ is the n-times Cartesian product of the unit interval $[0,1]$.

The *utility possibility set* is defined as follows.

$$U = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists (A_1, \ldots, A_n) \in \mathscr{P} : x_i \le u_i(A_i) \ \forall i \in I\}.$$

Note that if a partition $(A_1, \ldots, A_n)$ is weakly Pareto optimal, then the utility vector $(u_1(A_1), \ldots, u_n(A_n))$ belongs to the boundary of U. Denote the set of weakly Pareto optimal partitions by $\mathscr{P}^*_w$ and define the *weak Pareto frontier* by

$$U^*_w = \{(u_1(A_1), \ldots, u_n(A_n)) \mid (A_1, \ldots, A_n) \in \mathscr{P}^*_w\}.$$

Define the equivalence classes of the set of weakly Pareto optimal partitions by

$$\mathscr{P}^*_w[\mu] = \left\{ (\mathbf{A}_1, \ldots, \mathbf{A}_n) \in \mathscr{B}^n_\Omega[\mu] \;\middle|\; \begin{array}{l} \exists (A_1, \ldots, A_n) \in \mathscr{P}^*_w : \\ A_i \in \mathbf{A}_i \quad \forall i \in I \end{array} \right\}.$$

Lemma 3.1. *If u_i is μ-continuous for each $i \in I$, then U and U^*_w are compact sets in $\mathbb{R}^n$.*

Proof. Let u_i be μ-continuous for each $i \in I$. We first show the compactness of U^*_w. To this end, we claim that $\mathscr{P}^*_w[\mu]$ is a closed subset of $\mathscr{P}[\mu]$. Let $\{(\mathbf{A}^k_1, \ldots, \mathbf{A}^k_n)\}$ be a sequence in $\mathscr{P}^*_w[\mu]$ converging to $(\mathbf{A}_1, \ldots, \mathbf{A}_n) \in \mathscr{B}^n_\Omega[\mu]$. Suppose that $(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ does not belong to $\mathscr{P}^*_w[\mu]$. Then there exists a partition $(B_1, \ldots, B_n)$ such that $u_i(B_i) > u_i(\mathbf{A}_i)$ for each $i \in I$. The μ-continuity of u_i implies that $u_i(B_i) > u_i(\mathbf{A}^k_i)$ for each $i \in I$ for all sufficiently large k. Pick any $(A^k_1, \ldots, A^k_n) \in \mathscr{P}^*_w$ satisfying $A^k_i \in \mathbf{A}^k_i$ for each $i \in I$. The μ-indifference of u_i implies that $u_i(B_i) > u_i(A^k_i)$ for each $i \in I$ and for all sufficiently large k, which contradicts the weak Pareto optimality of

$(A_1^k,\dots,A_n^k)$. Hence, $(\mathbf{A}_1,\dots,\mathbf{A}_n)\in\mathscr{P}_w^*[\mu]$, which asserts the closedness of $\mathscr{P}_w^*[\mu]$. Thus, the compactness of $\mathscr{P}[\mu]$ guaranteed by Theorem 2.2 yields the compactness of $\mathscr{P}_w^*[\mu]$. Since U_w^* is the image of the compact set $\mathscr{P}_w^*[\mu]$ by the continuos function $(\mathbf{A}_1,\dots,\mathbf{A}_n)\mapsto(u_1(\mathbf{A}_1),\dots,u_n(\mathbf{A}_n))$, it follows that U_w^* is also compact in $\mathbb{R}^n$.

We now show the compactness of U. To this end, it suffices to show that U is a closed subset of $[0,1]^n$. Let $\{(x_1^k,\dots,x_n^k)\}$ be a sequence in U converging to $(x_1,\dots,x_n)\in\mathbb{R}^n$. Then there exists a sequence $\{(\mathbf{A}_1^k,\dots,\mathbf{A}_n^k)\}$ in $\mathscr{P}[\mu]$ such that

$$x_i^k\le u_i(\mathbf{A}_i^k)\quad\text{for each } k \text{ and } i\in I.$$

Since $\mathscr{P}[\mu]$ is compact by Theorem 2.2, the sequence $\{(\mathbf{A}_1^k,\dots,\mathbf{A}_n^k)\}$ has a subsequence converging to $(\mathbf{A}_1,\dots,\mathbf{A}_n)\in\mathscr{P}[\mu]$. By the μ-continuity of u_i, taking the limits in the both side of the above inequality along with the subsequences of $\{(x_1^k,\dots,x_n^k)\}$ and $\{(\mathbf{A}_1^k,\dots,\mathbf{A}_n^k)\}$ yields $x_i\le u_i(\mathbf{A}_i)$ for each $i\in I$. The μ-indifference of u_i and the definition of $\mathscr{P}[\mu]$ imply that there exists a partition $(A_1,\dots,A_n)$ such that $A_i\in\mathbf{A}_i$ for each $i\in I$ and $x_i\le u_i(A_i)$ for each $i\in I$. Therefore, $(x_1,\dots,x_n)\in U$. $\square$

Let u_i be μ-strictly monotone for each $i\in I$. Without loss of generality we may assume that $u_i(\emptyset)=0$ and $u_i(\Omega)=1$ for each $i\in I$. We claim that there exists some $\delta>0$ such that $x\in[0,1]^n$ and $\|x\|\le\delta$ imply $x\in U$. To this end, let $(A_1,\dots,A_n)$ be a partition with $\mu(A_i)>0$ for each $i\in I$. We then have $u_i(A_i)>u_i(\emptyset)=0$ for each $i\in I$ by the μ-strict monotonicity of u_i. Now note that the positive real number $\delta=\min\{u_i(A_i)\mid i\in I\}$ satisfies the desired property. Thus, $B_\delta(0)\cap[0,1]^n$ is contained in U and does not intersect with U_w^*, where $B_\delta(0)$ is a δ-neighborhood of the origin in $\mathbb{R}^n$ (see Figure 4.1).

Define the function $\rho:\Delta^{n-1}\to(0,\infty)$ by $\rho(s)=\sup\{\lambda>0\mid\lambda s\in U\}$. Note that the set $\{\lambda>0\mid\lambda s\in U\}$ is nonempty and bounded from above in $\mathbb{R}^n$ in view of $B_\delta(0)\cap[0,1]^n\subset U$ and the compactness of U asserted in Lemma 3.1. Thus, the function ρ is well defined (see Figure 4.1). We also have $\rho(s)s\in\partial U$ for any $s\in\Delta^{n-1}$.

Lemma 3.2. *If u_i is μ-continuous and μ-strictly monotone for each $i\in I$, then ρ is continuous.*

Proof. We show that ρ is both upper semicontinuous and lower semicontinuous. To this end, let $\{s^k\}$ be a sequence in Δ^{n-1} converging to $s\in\Delta^{n-1}$. We first show the upper semicontinuity of ρ. Pick any real number λ satisfying $\rho(s)<\lambda$ and fix some real number λ' satisfying $\rho(s)<\lambda'<\lambda$. Since $s^k\to s$ and $\lambda' s_i<\lambda s_i$ for each $i\in I$ with $s_i>0$, we have $\lambda' s_i\le\lambda s_i^k$ for

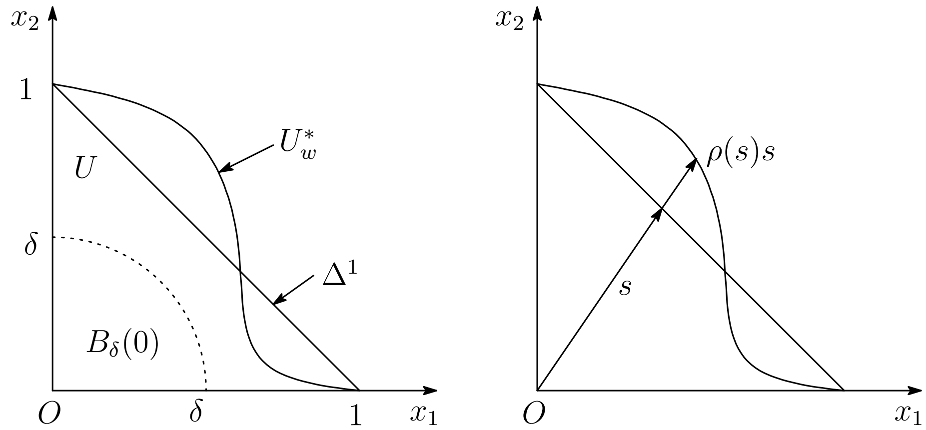

Figure 3.1: The construction of ρ.

each $i \in I$ and all sufficiently large k. Note that if $\lambda s^k \in U$, then $\lambda' s \in U$, contrary to $\rho(s) < \lambda'$. Thus, $\lambda s^k \notin U$, and hence $\rho(s^k) < \lambda$ for all sufficiently large k. This implies $\limsup_k \rho(s^k) \leq \lambda$ for any λ with $\rho(s) < \lambda$. Therefore, $\limsup_k \rho(s^k) \leq \rho(s)$.

We next show the lower semicontinuity of ρ. Take two real numbers λ and λ' satisfying $\lambda s \in U$ and $0 < \lambda' < \lambda$. Then there exists a partition $(A_1, \ldots, A_n)$ such that $u_i(A_i) \geq \lambda s_i$ for each $i \in I$. Thus, $u_i(A_i) \geq \lambda' s_i$ for each $i \in I$ and $u_j(A_j) > \lambda' s_j$ for some $j \in I$ with $s_j > 0$. By the μ-continuity of u_j and the nonatomicity of μ, there exists some $B_j \subset A_j$ such that $\mu(A_j \setminus B_j) > 0$ and $u_j(B_j) > \lambda' s_j$. Decompose $A_j \setminus B_j$ into $n-1$ disjoint sets A'_i, $i \in I \setminus \{j\}$, such that $\bigcup_{i \in I \setminus \{j\}} A'_i = A_j \setminus B_j$ and $\mu(A'_i) > 0$ for each $i \in I \setminus \{j\}$. Let $B_i = A_i \cup A'_i$ for each $i \in I \setminus \{j\}$. Then the resulting partition $(B_1, \ldots, B_n)$ satisfies $u_i(B_i) > u_i(A_i)$ for each $i \in I \setminus \{j\}$ by the μ-strictly monotonicity of u_i. Therefore, $u_i(B_i) > \lambda' s_i$ for each $i \in I$. In view of $s^k \to s$, for all sufficiently large k we have $u_i(B_i) > \lambda' s^k$ for each $i \in I$, which implies $\lambda' s^k \in U$, and hence $\lambda' \leq \rho(s^k)$ for all sufficiently large k. Consequently, $\lambda' \leq \liminf_k \rho(s^k)$ for any $0 < \lambda' < \lambda$ and thus $\lambda \leq \liminf_k \rho(s^k)$ for any $\lambda > 0$ with $\lambda s \in U$. Therefore, $\rho(s) \leq \liminf_k \rho(s^k)$. □

Lemma 3.3. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then for any $s \in \Delta^{n-1}$ there exists a weakly Pareto optimal partition $(A_1, \ldots, A_n)$ satisfying $\rho(s)s = (u_1(A_1), \ldots, u_n(A_n)) \in U_w^*$.*

Proof. Pick any $s \in \Delta^{n-1}$. Since $\rho(s)s \in U$ by the construction of ρ, there exists a partition $(A_1, \ldots, A_n)$ such that $u_i(A_i) \geq \rho(s)s_i$ for each $i \in I$. Suppose that $u_j(A_j) > \rho(s)s_j$ for some $j \in I$. By the μ-continuity of u_j and the nonatomicity of μ, there exists some $B_j \subset A_j$ such that $\mu(A_j \setminus B_j) > 0$ and $u_j(B_j) > \rho(s)s_j$. Decompose $A_j \setminus B_j$ into $n-1$ disjoint sets A'_i, $i \in I \setminus \{j\}$, such that $\bigcup_{i\in I\setminus\{j\}} A'_i = A_j \setminus B_j$ and $\mu(A'_i) > 0$ for each $i \in I \setminus \{j\}$. Let $B_i = A_i \cup A'_i$ for each $i \in I \setminus \{j\}$. Then the resulting partition $(B_1, \dots, B_n)$ satisfies $u_i(B_i) > u_i(A_i)$ for each $i \in I \setminus \{j\}$ by the μ-strict monotonicity of u_i. Therefore, $u_i(B_i) > \rho(s)s_i$ for each $i \in I$. This implies that for any sufficiently small $\varepsilon > 0$ we have $u_i(B_i) > (\rho(s) + \varepsilon)s_i$ for each $i \in I$, and hence $(\rho(s)+\varepsilon)s \in U$, which contradicts the definition of $\rho(s)$. Consequently, we must have $u_i(A_i) = \rho(s)s_i$ for each $i \in I$. The weak Pareto optimality of $(A_1, \dots, A_n)$ follows from the fact that $\rho(s)s$ is in the boundary of U. $\square$

Lemma 3.4. *If u_i is μ-continuous and μ-strictly monotone for each $i \in I$, then U^*_w is homeomorphic to Δ^{n-1} and the homeomorphism is given by $s \mapsto \rho(s)s$.*

Proof. We show that for any $x \in U^*_w$ there exists a unique $s \in \Delta^{n-1}$ satisfying $\rho(s)s = x$. To this end, take any $x = (u_1(A_1), \dots, u_n(A_n)) \in U^*_w$ with $(A_1, \dots, A_n) \in \mathscr{P}^*_w$. Since $x \geq 0$ and $x \neq 0$ by the μ-strict monotonicity of u_i, there exists a unique nonnegative vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n\setminus\{0\}$ such that $\sum_{i\in I} \alpha_i e_i = x$, where $\{e_1, \dots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. Define $s_i = \alpha_i/\|\alpha\|$ for each $i \in I$. We then have $s \in \Delta^{n-1}$ and $\|\alpha\|s = x$. Suppose that $\|\alpha\| \neq \rho(s)$. By the definition of $\rho(s)$ and the fact that $\|\alpha\|s \in U$, we must have $\|\alpha\| < \rho(s)$. Thus, $\|\alpha\|s_i \leq \rho(s)s_i$ for each $i \in I$ and $\|\alpha\|s_i < \rho(s)s_i$ for some $i \in I$ with $s_i > 0$. Since there exists a partition $(B_1, \dots, B_n)$ such that $\rho(s)s = (u_1(B_1), \dots, u_n(B_n))$ by Lemma 3.3, we have $u_i(A_i) \leq u_i(B_i)$ for each $i \in I$ and $u_i(A_i) < u_i(B_i)$ for some $i \in I$. Therefore, $(A_1, \dots, A_n)$ is not Pareto optimal, which is equivalent to the fact that $(A_1, \dots, A_n)$ is not weakly Pareto optimal by Theorem 3.1. We thus have $\|\alpha\| = \rho(s)$, and hence $\rho(s)s = x$. The uniqueness of $s \in \Delta^{n-1}$ follows from its construction.

From the preceding argument we have just shown, for any $x \in U^*_w$ there exists a unique $s \in \Delta^{n-1}$ such that $\rho(s_x)s_x = x$. Thus, we can define the map $f : U^*_w \to \Delta^{n-1}$ by $x \mapsto s_x$. Note that the map $g : \Delta^{n-1} \to U^*_w$ is defined by $g(s) = \rho(s)s$ in view of Lemma 3.3. It is easy to verify that $g \circ f$ is the identity map on Δ^{n-1} and $f \circ g$ is the identity map on U^*_w. Therefore, f is a bijection with $f^{-1} = g$. Since g is continuous by Lemma 3.2, f is a homeomorphism on U^*_w to Δ^{n-1} (see Figure 4.1). $\square$

Denote the set of Pareto optimal partitions by $\mathscr{P}^*$ and define the equivalence classes of the set of Pareto optimal partitions by

$$\mathscr{P}^*[\mu] = \left\{ (\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{B}^n_\Omega[\mu] \;\middle|\; \begin{array}{l} \exists (A_1, \dots, A_n) \in \mathscr{P}^* : \\ A_i \in \mathbf{A}_i \quad \forall i \in I \end{array} \right\}.$$

Theorem 3.2. *If u_i is μ-continuous and μ-strictly monotone for each $i \in I$, then $\mathscr{P}^*[\mu]$ is homeomorphic to Δ^{n-1}.*

Proof. Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. We then have $\mathscr{P}^*[\mu] = \mathscr{P}^*_w[\mu]$ by Theorem 3.1. In view of Lemma 3.4, it suffices to show that U^*_w is homeomorphic to $\mathscr{P}^*_w[\mu]$. To this end, notice that the map $(\mathbf{A}_1, \dots, \mathbf{A}_n) \mapsto (u_1(\mathbf{A}_1), \dots, u_n(\mathbf{A}_n))$ defines a continuous surjection from the compact set $\mathscr{P}^*_w[\mu]$ to the compact set U^*_w and this map is one-to-one by the μ-strict monotonicity of u_i. □

Remark 3.1. It is well known that for a finite dimensional commodity space, the set of Pareto optimal allocations is homeomorphic to the unit simplex if preferences of each individual is continuous, monotone, and strictly convex (see Varian 1974, Corollary). For an infinite dimensional commodity space, the existence of the homeomorphism between the set of Pareto optimal allocations and the unit simplex was demonstrated by Magill (1981) and Mas-Colell (1986). In our setting the key argument is the construction of ρ and its continuity stated in Lemma 3.2. The continuity of ρ follows from the similar argument to that of Mas-Colell (1986), who worked in a commodity space with topological vector lattices. The proof adopted here is a recomposition of Aliprantis et al. (1990, Theorem 3.5.7). Note that Lemma 3.3 is a variant of Magill (1981) and Mas-Colell (1986).

4 Equity and Efficiency

In this section we examine the compatibility between equity and efficiency (Pareto optimality). Three notions of equity are introduced: *Envy-freeness*, *α-equity*, and *α-Rawls optimality*. All these notions have already appeared in the classical paper by Dubins and Spanier (1961) before economists presented an analytical framework in the problem of fair division.

4.1 Envy-Free Partition

The first notion on equity under investigation is envy-freeness along the line of Varian (1974). We prove the existence of a Pareto optimal envy-free partition which is called a 'fair' partition in the literature. For other variants of the notion of envy-freeness, see Barbanel (1996), Berliant et al. (1992), and Ichiishi and Idzik (1999).

Definition 4.1. A partition $(A_1, \dots, A_n)$ is *envy-free* if $u_i(A_i) \geq u_i(A_j)$ for each $i, j \in I$.

Theorem 4.1. *If u_i is μ-continuous and μ-strictly monotone for each $i \in I$, then there exists a Pareto optimal envy-free partition.*

To prove Theorem 4.1, the following simple observation by Varian (1974) is employed.

Lemma 4.1. *For every Pareto optimal partition $(A_1, \dots, A_n)$ there exists some $j \in I$ such that $u_i(A_i) \geq u_i(A_j)$ for each $i \in I$.*

Proof. Let $(A_1, \dots, A_n)$ be a Pareto optimal partition. Suppose to the contrary that for each $j \in I$ there exists some $i_j \in I$ satisfying $u_{i_j}(A_{i_j}) < u_{i_j}(A_j)$. Then for some subset $J = \{i_1, \dots, i_s\}$ of I, it follows that i_1 envies i_2, i_2 envies i_3, ..., and i_s envies i_1: The finite chain $u_{i_1}(A_{i_1}) < u_{i_1}(A_{i_2}), \dots, u_{i_{s-1}}(A_{i_{s-1}}) < u_{i_{s-1}}(A_{i_s})$, and $u_{i_s}(A_{i_s}) < u_{i_s}(A_{i_1})$ exists. Define the partition $(B_1, \dots, B_n)$ by

$$B_i = \begin{cases} A_{i_{j+1}} & \text{if } i = i_j \in J \text{ and } 1 \leq j \leq s-1, \\ A_1 & \text{if } i = i_s, \\ A_i & \text{if } i \notin J. \end{cases}$$

It is obvious that the resulting partition $(B_1, \dots, B_n)$ satisfies $u_i(B_i) > u_i(A_i)$ for each $i \in J$ and $u_i(B_i) = u_i(A_i)$ for each $i \notin J$, which contradicts the Pareto optimality of $(A_1, \dots, A_n)$. $\square$

Denote the $(n-1)$-dimensional *unit simplex* by

$$\Delta^{n-1} = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} \alpha_k = 1 \text{ and } \alpha_k \geq 0,\ k = 1, \dots, n \right\}$$

and let $\Delta_i = \{(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1} | \, \alpha_i = 0\}$ for each $i \in I$. The following theorem due to Scarf (1967, Theorem 3; 1973, Theorem 3.3.1) plays a crucial role to prove Theorem 4.1.

Intersection Theorem (Scarf). If the collection $\{F_1, \dots, F_n\}$ is a closed covering of Δ^{n-1} satisfying $\Delta_i \subset F_i$ for each $i \in I$, then $\bigcap_{i \in I} F_i \neq \emptyset$.

Proof of Theorem 4.1

Let φ be the homeomorphism on $\mathscr{P}^*[\mu]$ to Δ^{n-1} in Theorem 3.2. For each $j \in I$ define the sets

$$\mathbf{D}_j = \{(\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{P}^*[\mu] \mid \mathbf{A}_j = [\emptyset]\}$$

and

$$\mathbf{F}_j = \{(\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{P}^*[\mu] \mid u_i(\mathbf{A}_i) \geq u_i(\mathbf{A}_j) \ \forall i \in I\}.$$

By Lemma 4.1, we have $\bigcup_{j\in I}\mathbf{F}_j = \mathscr{P}^*[\mu]$. Since $\mathbf{F}_j$ is a closed subset of $\mathscr{P}^*[\mu]$ for each $j \in I$ by virtue of the μ-continuity of u_i, the collection $\{\varphi(\mathbf{F}_1),\dots,\varphi(\mathbf{F}_n)\}$ is a closed covering of Δ^{n-1}. Note that $\varphi(\mathbf{D}_j) \subset \varphi(\mathbf{F}_j)$ for each $j \in I$ in view of $\mathbf{D}_j \subset \mathbf{F}_j$ for each $j \in I$. By Lemma 3.4 and Theorem 3.2, it is obvious that φ is obtained from the following scheme of mappings $\mathscr{P}^*[\mu] \to U^* \to \Delta^{n-1}$ defined by

$$(\mathbf{A}_1,\dots,\mathbf{A}_n) \mapsto (u_1(\mathbf{A}_1),\dots,u_n(\mathbf{A}_n)) = \rho(s)s \mapsto s.$$

We thus have $\varphi(\mathbf{D}_j) = \Delta_j$ for each $j \in I$. Therefore, by the intersection theorem of Scarf, there exists some $s \in \bigcap_{j\in I}\varphi(\mathbf{F}_j)$. Then for some $(\mathbf{A}_1,\dots,\mathbf{A}_n) \in \mathscr{P}^*[\mu]$, we have $(u_1(\mathbf{A}_1),\dots,u_n(\mathbf{A}_n)) = \rho(s)s$. Suppose that $(\mathbf{A}_1,\dots,\mathbf{A}_n) \notin \mathbf{F}_j$ for some $j \in I$. Since φ is a bijection, this yields that $s = \varphi(\mathbf{A}_1,\dots,\mathbf{A}_n) \notin \varphi(\mathbf{F}_j)$, a contradiction. Therefore, $(\mathbf{A}_1,\dots,\mathbf{A}_n) \in \bigcap_{j\in I}\mathbf{F}_j$. Take any $(A_1,\dots,A_n) \in \mathscr{P}^*$ satisfying $A_i \in \mathbf{A}_i$ for each $i \in I$. By construction, it is obvious that the partition $(A_1,\dots,A_n)$ is Pareto optimal and envy-free. $\square$

Remark 4.1. The use of the intersection theorem of Scarf (1967) is inspired by Varian (1974), who dealt with a finite dimensional commodity space to prove the existence of a Pareto optimal envy-free allocation. The intersection theorem of Scarf is closely related to Sperner's lemma and has an obvious resemblance to the K-K-M theorem, which provides an alternative set of assumptions guaranteeing the nonemptiness of the intersection of a closed covering of a unit simplex. Evidently, the existence argument in the problem of fair division is closely connected with the results around fixed point theorems. Indeed, for the case of additive preferences with a nonatomic probability measure on a unit simplex, Ichiishi and Idzik (1999) used the K-K-M theorem to demonstrate the existence of an α-equitable partition (in terms of their terminology). For the case of nonadditive continuous preferences on a unit simplex, Stromquist (1980) employed a variant of the K-K-M theorem to show the existence of an envy-free partition. For the case of additive preferences with the Lebesgue measure on the unit interval, Woodall (1980) used Brouwer's fixed point theorem to show the existence of an envy-free partition. For the case of additive preferences with a nonatomic probability measure on a σ-field, Weller (1985) resorted to Kakutani's fixed point theorem to prove the existence of a Pareto optimal envy-free partition.

4.2 α-Equitable and α-Rawls Optimal Partitions

The next notions on equity under investigation are α-equitability and α-Rawls optimality. We prove the existence of a Pareto optimal α-equitable partition and show that Pareto optimal α-equitability is equivalent to α-Rawls optimality. Denote the interior of Δ^{n-1} by

$$\operatorname{int}\Delta^{n-1} = \{(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1} \mid \alpha_k > 0,\ k = 1, \dots, n\}.$$

In this section when mentioning to α, we mean $\alpha = (\alpha_1, \dots, \alpha_n) \in \operatorname{int}\Delta^{n-1}$.

Definition 4.2. A partition $(A_1, \dots, A_n)$ is *α-equitable* if $\alpha_i^{-1}u_i(A_i) = \alpha_j^{-1}u_j(A_j)$ for each $i, j \in I$. An α-equitable partition for $\alpha = (1/n, \dots, 1/n)$ is simply said to be *equitable*.

The following result is immediate from Theorem 3.1 and Lemma 3.3.

Theorem 4.2. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then for any $\alpha \in \operatorname{int}\Delta^{n-1}$ there exists a Pareto optimal α-equitable partition.*

Another important criterion proposed by Dubins and Spanier (1961) is the anticipation of Rawls' criterion of maximizing the welfare of the least well-off individual. To define this formally, consider the following maximization problem:

$$\sup\Big\{\min_{i \in I} \alpha_i^{-1}u_i(A_i) \,\Big|\, (A_1, \dots, A_n) \in \mathscr{P}\Big\}. \tag{R_α}$$

Definition 4.3. A partition $(A_1, \dots, A_n)$ is *α-Rawls optimal* if it is a solution to (R_α). An α-Rawls optimal partition for $\alpha = (1/n, \dots, 1/n)$ is simply said to be *Rawls optimal*.

Theorem 4.3. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then for any $\alpha \in \operatorname{int}\Delta^{n-1}$ every α-Rawls optimal partition is Pareto optimal.*

Proof. Let $(A_1, \dots, A_n)$ be an α-Rawls optimal partition for $\alpha \in \operatorname{int}\Delta^{n-1}$. Suppose to the contrary that $(A_1, \dots, A_n)$ is not Pareto optimal. Since $(A_1, \dots, A_n)$ is not weakly Pareto optimal by Theorem 3.1, there exists a partition $(B_1, \dots, B_n)$ such that $u_i(B_i) > u_i(A_i)$ for each $i \in I$, and hence $\min_{i \in I} \alpha_i^{-1}u_i(B_i) > \min_{i \in I} \alpha_i^{-1}u_i(A_i)$. This obviously contradicts the α-Rawls optimality of $(A_1, \dots, A_n)$. $\square$

As was pointed out by Weller (1985), Rawls optimality is much stronger than Pareto optimality for the case of additive preferences with a nonatomic probability measure. This is also true for the case of nonadditive continuous preferences.

Theorem 4.4. *If u_i be μ-continuous and μ-strictly monotone for each $i \in I$, then there exists a Pareto optimal partition which is not Rawls optimal.*

Proof. Pick an element $s = (s_1, \dots, s_n) \in \Delta^{n-1}$ with $s_1 < \cdots < s_n$. In view of Lemma 3.3, there exists a Pareto optimal partition $(A_1, \dots, A_n)$ with $\rho(s)s = (u_1(A_1), \dots, u_n(A_n))$. We thus have $u_1(A_1) < \cdots < u_n(A_n)$. The μ-continuity of u_i and the nonatomicity of μ imply the existence of $B_2 \subset A_2$ satisfying $\mu(A_2 \setminus B_2) > 0$ and $u_1(A_1 \cup (A_2 \setminus B_2)) < u_2(B_2)$. Define $B_1 = A_1 \cup (A_2 \setminus B_2)$ and $B_i = A_i$ for $i = 3, \dots, n$. By the μ-strict monotonicity of u_i, the resulting partition $(B_1, \dots, B_n)$ satisfies $u_1(A_1) < u_1(B_1) < \cdots < u_n(B_n)$, and hence $\min_{i \in I} u_i(A_i) < \min_{i \in I} u_i(B_i)$. Therefore, $(A_1, \dots, A_n)$ cannot be Rawls optimal. □

However, Pareto optimality implies α-Rawls optimality for some $\alpha \in \operatorname{int} \Delta^{n-1}$. To show this, we introduce a new terminology on the partitions. A partition $(A_1, \dots, A_n)$ is *μ-positive* if $\mu(A_i) > 0$ for each $i \in I$.

Theorem 4.5. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then a μ-positive partition is Pareto optimal if and only if it is α-Rawls optimal for some $\alpha \in \operatorname{int} \Delta^{n-1}$.*

Proof. Let $(A_1, \dots, A_n)$ be a μ-positive Pareto optimal partition. By Theorem 3.1 and Lemma 3.4, there exists some $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ such that $\rho(\alpha)\alpha = (u_1(A_1), \dots, u_n(A_n)) \in U_w^*$. Since $(A_1, \dots, A_n)$ is μ-positive, we have $u_i(A_i) > 0$ for each $i \in I$ by the μ-strict monotonicity of u_i. We thus have $\alpha_i > 0$ for each $i \in I$, and hence $\alpha_i^{-1} u_i(A_i) = \alpha_j^{-1} u_j(A_j) = \rho(\alpha)$ for each $i, j \in I$. We show that $(A_1, \dots, A_n)$ is α-Rawls optimal. Suppose to the contrary that $(A_1, \dots, A_n)$ is not α-Rawls optimal. Then there exists a partition $(B_1, \dots, B_n)$ such that $\min_{i \in I}\{\alpha_i^{-1} u_i(A_i)\} < \min_{i \in I}\{\alpha_i^{-1} u_i(B_i)\}$. This implies $\min_{i \in I}\{\alpha_i^{-1} u_i(A_i)\} = \rho(\alpha) = \alpha_i^{-1} u_i(A_i) < \alpha_i^{-1} u_i(B_i)$ for each $i \in I$, and hence $u_i(A_i) < u_i(B_i)$ for each $i \in I$, which contradicts the Pareto optimality of $(A_1, \dots, A_n)$. The converse conclusion follows from Theorem 4.3. □

Theorem 4.6. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then for any $\alpha \in \operatorname{int} \Delta^{n-1}$ a partition is α-Rawls optimal if and only if it is Pareto optimal and α-equitable.*

Proof. Let $(A_1, \dots, A_n)$ be a Pareto optimal α-equitable partition. Suppose to the contrary that $(A_1, \dots, A_n)$ is not α-Rawls optimal. Then for some partition $(B_1, \dots, B_n)$, we have $\min_{i \in I} \alpha_i^{-1} u_i(B_i) > \min_{i \in I} \alpha_i^{-1} u_i(A_i)$. Since $\alpha_i^{-1} u_i(A_i) = \alpha_j^{-1} u_j(A_j)$ for each $i, j \in I$, we have $\alpha_i^{-1} u_i(B_i) > \alpha_i^{-1} u_i(A_i)$ for each $i \in I$, which contradicts the Pareto optimality of $(A_1, \dots, A_n)$.

Conversely, let $(A_1, \dots, A_n)$ be an α-Rawls optimal partition. Since $(A_1, \dots, A_n)$ is Pareto optimal by Theorem 4.3, it suffices to show that $(A_1, \dots, A_n)$ is α-equitable. Suppose to the contrary that $(A_1, \dots, A_n)$ is not

α-equitable. We then have $\alpha_j^{-1}u_j(A_j) > \alpha_i^{-1}u_i(A_i)$ for some $i, j \in I$. The μ-continuity of u_j and the nonatomicity of μ imply the existence of $B_j \subset A_j$ satisfying $\mu(A_j \setminus B_j) > 0$ and $\alpha_j^{-1}u_j(B_j) > \alpha_i^{-1}u_i(A_i)$. Decompose $A_j \setminus B_j$ into $n-1$ disjoint sets A'_k, $k \in I \setminus \{j\}$, such that $\bigcup_{k\in I\setminus\{j\}} A'_k = A_j \setminus B_j$ and $\mu(A'_k) > 0$ for each $k \in I \setminus \{j\}$. Define $B_k = A_k \cup A'_k$ for each $k \in I \setminus \{j\}$. By the μ-strict monotonicity of u_k, the resulting partition $(B_1, \ldots, B_n)$ satisfies $\alpha_k^{-1}u_k(B_k) > \alpha_k^{-1}u_k(A_k)$ for each $k \in I \setminus \{j\}$, and hence $\min_{k\in I} \alpha_k^{-1}u_k(B_k) > \min_{k\in I} \alpha_k^{-1}u_k(A_k)$, which contradicts the α-Rawls optimality of $(A_1, \ldots, A_n)$. □

For the case of additive preferences with a nonatomic probability measure on a σ-field, Legut and Wilczyńki (1988) used a minimax theorem to show the existence of an α-Rawls optimal partition. The same result is true for our setting. The following are immediate from Theorem 4.2 and Theorem 4.6.

Corollary 4.1. *Let u_i be μ-continuous and μ-strictly monotone for each $i \in I$. Then for any $\alpha \in \operatorname{int} \Delta^{n-1}$ there exists an α-Rawls optimal partition.*

References

Akin, E., (1995). “Vilfred Pareto cuts the cake”, *Journal of Mathematical Economics*, vol. 24, pp. 23–44.

Aliprantis, C. D., Brown, D. J., and O. Burkinshow, (1990). *Existence and Optimality of Competitive Equilibria*, Springer-Verlag, New York.

Barbanel, J. B., (1996). “Super envy-free cake division and independence of measures”, *Journal of Mathematical Analysis and Applications*, vol. 197, pp. 54–60.

Barbanel, J. B., (1999). “Partition ratios, Pareto optimal cake division, and related notions”, *Journal of Mathematical Economics*, vol. 32, pp. 401–428.

Barbanel, J. B., (2000). “On the structure of Pareto optimal cake partitions”, *Journal of Mathematical Economics*, vol. 33, pp. 401–424.

Berliant, M., (1986). “A utility representation for preference relation on a σ-algebra”, *Econometrica*, vol. 54, pp. 359–362.

Berliant, M., and K. Dunz, (2004). “A foundation of location theory: existence of equilibrium, the welfare theorems, and core”, *Journal of Mathematical Economics*, vol. 40, pp. 593–618.

Berliant, M., Thomson, W., and K. Dunz, (1992). "On the fair division of a heterogeneous commodity", *Journal of Mathematical Economics*, vol. 21, pp. 201–216.

Dall'Aglio, M., (2001). "The Dubins-Spanier optimization problems in fair division theory", *Journal of Computational and Applied Mathematics*, vol. 130, pp. 17–40.

Debreu, G., (1964). "Continuity properties of Paretian utility", *International Economic Review*, vol. 5, pp. 285–293.

Dubins, L. E., and E. H. Spanier, (1961). "How to cut a cake fairly", *American Mathematical Monthly*, vol. 68, pp. 1–17.

Dunford, N., and J. T. Schwartz, (1958). *Linear Operators, Part I: General Theory*, John Wiley & Sons, New York.

Edwards, R. E., (1965). *Functional Analysis: Theory and Applications*, Holt, Reinhart and Winston, New York.

Halmos, P. R., (1950). *Measure Theory*, Springer-Verlag, New York.

Ichiishi, T., and A. Idzik, (1999). "Equitable allocation of divisible goods", *Journal of Mathematical Economics*, vol. 32, pp. 389–400.

Legut, J., and M. Wilczyńki, (1988)."Optimal partitioning of a measurable space", *Proceedings of the American Mathematical Society*, vol. 104, pp. 262–264.

Maccheroni F., and M. Marinacci, (2003). "How to cut a pizza fairly: fair division with decreasing marginal evaluations", *Social Choice and Welfare*, vol. 20, pp. 457–465.

Magill, M. J. P, (1981). "An equilibrium existence theorem", *Journal of Mathematical Analysis and Applications*, vol. 84, pp. 162–169.

Mas-Colell, A., (1986). "The price equilibrium existence problem in topological vector lattices", *Econometrica*, vol. 54, pp. 1039–1053.

Scarf, H., (1967). "The approximation of fixed points of a continuous mapping", *SIAM Journal on Applied Mathematics*, vol. 15, pp. 1328–1343.

Scarf, H., (1973). *The Computation of Economic Equilibria*, Yale University Press, New Haven.

Stromquist, W., (1980). "How to cut a cake fairly", *American Mathematical Monthly*, vol. 87, pp. 640–644. Addendum in: *American Mathematical Monthly*, vol. 88, pp. 613–614.

Varian, H. R., (1974): "Equity, envy, and efficiency", *Journal of Economic Theory*, vol. 9, pp. 63–91.

Weller, D., (1985). "Fair division of a measurable space", *Journal of Mathematical Economics*, vol. 14, pp. 5–17.

Woodall, D. R., (1980). "Dividing a cake fairly", *Journal of Mathematical Analysis and Applications*, vol. 73, pp. 233–247.