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Game-theoretic Analysis of Strategyproofness in Cake-cutting Protocols

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Bachelorarbeit

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Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

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Abstract

In cake-cutting a protocol instructs the participants how to divide a resource between them in a satisfactory manner. A part of those instructions, namely the strategies, are optional and can be examined whether they obtain the best solution for the players. If this is not the case, the players have no intention to follow the protocol which can be overruled in this case. Otherwise the protocol is strategyproof.
Game Theory is designed to determine better strategies. By using a game-theoretic illustration of the cake-cutting problem it is possible to compare all strategies.
The strategy recommended by the protocol appears to be the best one in the well-known protocols with one exception.

Contents

1 Introduction

It is Christmas party in the cakes4people agency. Everybody is waiting expectantly on the big promised cake at the end of the party. This cake has been spectacular in the past years. A lot of different cake layers, different fruits on top and even chocolate sprinkles over parts of the cake have been so delicious. So it's no wonder that everyone wants to get as much as possible of this culinary treat, and especially of their individual favourite part. Nevertheless, the people like their colleagues and want still to be as fair as possible to them.
A new employee is also celebrating with the group. Rumors have been told a lot about him, but no one has managed to assess him or his preferences properly. Nevertheless, he also takes part in the big cake division. He even wants to change the allocation procedure. He promises that everyone can keep their wishes private, each of them just needs to make a couple of simple decisions and will get their best possible share.
But the people become suspicious. Different questions occur in their minds: "What if he has a strategy he is not telling us about, which promises him a better piece? What if he is lying about his preferences? Why should we trust him?" The chief sees the mistrust and knows how to reassure the people. He is a game theory enthusiast and promises to show them that the proposed procedure is strategyproof. Hereby, only by taking actions truthful a participant can always get his best possible piece.
Strategyproofness of an allocation procedure ...

1.1 Related Work

Recently, two papers, [?] and [?], with the focus on strategyproofness have been published. In [?] they weakened the assumptions of cake cutting by including the free disposal assumption, which can lead to a not complete allocation of the cake and allow only piecewise uniform valuations. The second restriction is indeed very hard. Their goal was to give a proportional, envy-free, polynomial and strong strategyproof protocol. In [?] the authors invented new procedures including a referee, who has full knowledge. This extension is a restriction of cake-cutting as well. Both papers also researched truthfulness in expectation for protocols with randomness.

In pie-cutting [?] showed that a strategyproof and efficient mechanism must be dictatorial. The definition of strategyproofness in this paper is a much stronger condition. Also pie-cutting slightly differs from cake-cutting, since the pie ist represented as a circular object and the cuts are wedges. The results for pie- and cake-cutting do not carry over to each other, but the definition of strategyproofness does. [?] gave more details in the context of pie-cutting and strategyproofness.

The start of reaserching strategyproofness was [?], where the authors introduced a fitting definition of strategyproofness and proved that two procedures called SP and EP are strategyproof.
A response on their work was a counterexample by [?]. After admitting their mistake, in [?] they restricted their first definition to cases with non-equal valuation functions and introduced a new general definition for strategyproof cake-cutting.
In [?] and in the revisited version of this work [?] the authors focused on the Divide-and-Conquer protocol and showed that it is strategyproof for risk averse players. In the later work they call this property truth-inducing.
[?] parallelyzed the Last-Diminisher and proved that this new protocol is also strategyproof for risk-averse players.

2 Preliminaries

2.1 Basics of Cake-cutting

It is necessary to define the components and challenges of cake-cutting. But first, what exactly is cake-cutting about? It involves a set of $n \in \mathbb{N}$ players $P_n = \{p_1, \dots, p_n\}$. It is assumed that each of them wants to get as much as possible of the divided resource. The goal is to find an allocation of a single, divisible and heterogeneous good between the n players. Such an allocation has to be of a special kind, so that the involved players are

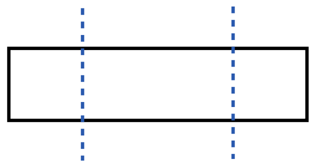

Figure 1: Cake
Example for a visualisation of a cake with two cuts

pleased with the outcome. For the visualization it is common to use a rectangular cake. The division is performed by parallel cuts. The cake X is represented by the unit interval $X = [0, 1] \subseteq \mathbb{R}$. Each subinterval $X' \subseteq X$ or a sequence of disjoint subintervals

$$\bigcup_{m \in \mathbb{N}} X'_m$$

with $X'_m \subseteq X$ is called a *portion (or piece)*. The portion of the cake which is received by player p_i is denoted X_i. The state is called an *allocation*, when all portions of the cake are owned by players. Each piece has a public size, which can be computed as the sum of all border differences, and the private value of each player, which is constituted by the lower defined valuation function.

Every player $p_i \in P_n$ has a *valuation function (valuation)* $v_i : \{X' | X' \subseteq X\} \to [0, 1]$ with the following properties:

1. Non-negativity: $v_i(C) \geq 0$ for all $C \subseteq X$.
2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$.
3. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq X$.[1]
4. Divisibility: For all $C \subseteq [0, 1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, there exists a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.
5. v_i is continuous: If $0 < x < y \leq 1$ with $v_i([0, x]) = \alpha$ and $v_i([0, y]) = \beta$, then for every $\gamma \in [\alpha, \beta]$ there exists a $z \in [x, y]$ so that $v_i([0, z]) = \gamma$.
6. Non-atomic: $v_i([x, x]) = 0$ for all $x \in X$.

[1]Monotonicity: If $C' \subseteq C$ then $v_i(C') \leq v_i(C)$. Monotonicity follows from additivity, because for the assumption $C' \subseteq C$ and $A := C \backslash C'$: $v_i(C) = v_i(A \cup C') = v_i(A) + v_i(C') = \underbrace{v_i(C \backslash C')}_{\geq 0} + v_i(C') \geq v_i(C')$.

2.2 Concepts in Game Theory

Basic concepts from game theory are described and directly applied to the cake-cutting problem. In particular the priority have the possible representations of games. For further reading see [], [] and [**?**].

Definition 1. ***(Game)***
A *non-cooperative game* $\Gamma = (P_n, S, u)$ consists of the set of players P_n, the set of strategies S and the set of utility functions (pay-off) of all players u.

- Each player in the set $P_n = \{p_1, \cdots, p_n\}$ behaves selfish and rational.
- Each player has his own set of strategies. A *pure strategy* is a single action. A *mixed strategy* is a probability distribution over pure strategies.
- Utility is a real-valued quantity measuring an player's happiness. A utility function is a mapping from outcomes to utilities. The agent is indifferent between outcomes with equal utilities and strictly prefers outcomes with higher utilities.

Each game has also an end-state, which is usually called outcome. In cake-cutting an outcome is an allocation. The utility function in cake-cutting is the valuation function. The utility of an allocation for a player is the value of the piece this player obtain in it.

Definition 2. ***(Strategies)***
Assume two strategies S_1 and S_2 for a player p_1 and the value $v_1(X_{1,S_1,i})$ and $v_1(X_{1,S_2,i})$ for $i \in \mathbb{N}$ number of possible different allocations.
The strategy S_1 *dominates* the strategy S_2 if $v_1(X_{1,S_1,i}) \geq v_1(X_{1,S_2,i})$ for all i.

- The strategy S_1 *strictly dominates* the strategy S_2 if $v_1(X_{1,S_1,i}) > v_1(X_{1,S_2,i})$ for all i.
- The strategy S_1 *weakly dominates* the strategy S_2 if $v_1(X_{1,S_1,i}) > v_1(X_{1,S_2,i})$ for at least one i and $v_1(X_{1,S_1,i}) \geq v_1(X_{1,S_2,i})$ for all other i.

The strategy S_1 is *dominated* by the strategy S_2 if $v_1(X_{1,S_1,i}) \leq v_1(X_{1,S_2,i})$ for all i.

- The strategy S_1 is *strictly dominated* by the strategy S_2 if $v_1(X_{1,S_1,i}) < v_1(X_{1,S_2,i})$ for all i.
- The strategy S_1 is *weakly dominated* by the strategy S_2 if $v_1(X_{1,S_1,i}) < v_1(X_{1,S_2,i})$ for at least one i and $v_1(X_{1,S_1,i}) \leq v_1(X_{1,S_2,i})$ for all other i.

The strategy S_1 can be neither dominates nor dominated regarding an other strategy S_2, so $v_1(X_{1,S_1,i}) < v_1(X_{1,S_2,i})$ for at least one i and $v_1(X_{1,S_1,i}) > v_1(X_{1,S_2,i})$ for all other i. Also $v_1(X_{1,S_1,i}) = v_1(X_{1,S_2,i})$ for all i is possible, but in this case the strategies can be seen as equal and the player is indifferent between them.
A game can be rather strategic, where all player move simoultaneous or extensive, where the players move in a fixed or variable order.

Information in Games

Definition 3. ***(Perfect / Imperfect Information)***
Perfect information extensive form games are games in tree representation where every inner node represents a choice by a single agent, with the out-edges of the node being the possible actions. Each leaf node is an outcome labeled by a vector of utilities (one per agent).

Definition 4. ***(Complete / Incomplete Information)***
An imperfect information extensive form games are extensive form games where the agent cannot distinguish two or more choice nodes (those nodes are said to belong to an equivalence set).

Definition 5. ***(Bayesian Game)***
A Bayesian Game is a set of players, each of which has a set of actions and a utility function mapping action and type profiles to utility. A Bayesian game also includes a distribution over type profiles.

Often, strategic interactions involve some notion of random events and private information. We model these using Bayesian games. Many card games (for example, poker) are instances of this type of interaction.

Definition 39 (Type) An agent's type is a complete representation of his private information.
Definition 40 (Type profile) A type profile is a vector of types, one per agent. One simple but powerful way to represent Bayesian games is to say that the agents have different information about which normal form game they are actually playing. For issues in cutting a cake it is especially important that the valuation functions are private and that no player knows each others type or his own position in the game. So for the analysis some assumption have to be made and a stochastic model to be defined.

Definition 6. ***(Probability Model)***
The sample space S of a random phenomenon is the set of all possible outcomes. An event is an outcome or a set of outcomes of a random phenomenon. That is, an event is a subset of the sample space. A *probability model* is a mathematical description of a random phenomenon consisting of two parts: a sample space S and a way of assigning probabilities to events.

Definition 7. ***(Expected Value)***
A

A game can be represented in normal or in extended form. The normal form is advantageous for games where players move simultaneous, which is not often the case in cake-cutting. So rather the presentation in extended form as a game tree will be used in this work.

Definition 8. *(Normal Form Game)*
A *normal form game* is a tabular representation of a game. In the case for two players the first player chases the row while the second choses the column. Each cell contains a vector with the value of the obtained piece, one per player.

	$Strategy_a$	$Strategy_b$
$Strategy_I$	$(v_1(X_{1,I,a}), v_2(X_{2,I,a}))$	$(v_1(X_{1,I,b}), v_2(X_{2,I,b}))$
$Strategy_{II}$	$(v_1(X_{1,II,a}), v_2(X_{2,II,a}))$	$(v_1(X_{1,II,b}), v_2(X_{2,II,b}))$

Table 1: Game in normal form

Definition 9. *(Extended Form Game)*
An *extended form game* is a tree representation of a game. In the case for two players the first player ...

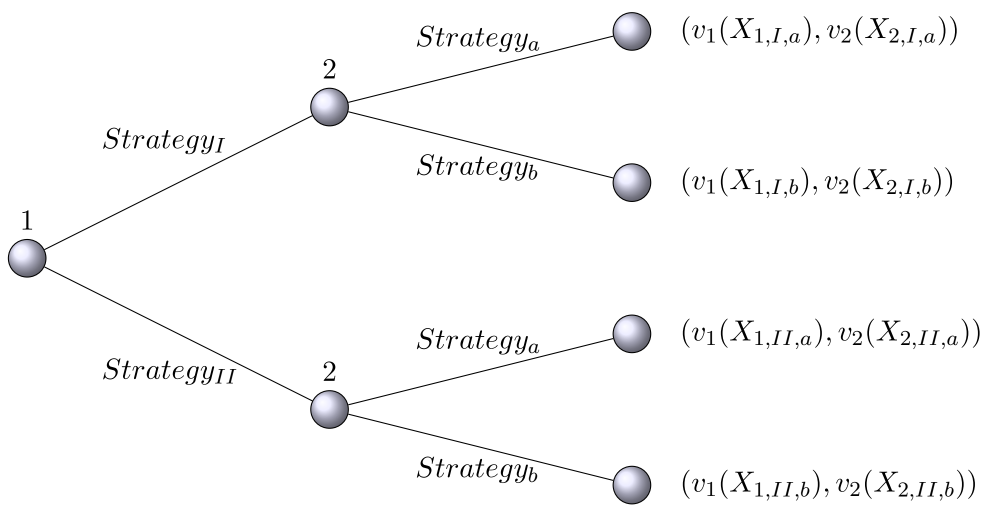

Figure 2: Game in extended form

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