Journal of Artificial Intelligence Research 1 (2010) 1–31 Submitted 12/10; published ??/??

Control Complexity of Bucklin and Fallback Voting*

Gábor Erdélyi ERDELYI@CS.UNI-DUESSELDORF.DE
Lena Piras LENA.PIRAS@UNI-DUESSELDORF.DE
Jörg Rothe ROTHE@CS.UNI-DUESSELDORF.DE
Heinrich-Heine-Universität Düsseldorf
Institut für Informatik
40225 Düsseldorf
Germany

Abstract

Electoral control models ways of changing the outcome of an election via such actions as adding/deleting/partitioning either candidates or voters. These actions modify an election's participation structure and aim at either making a favorite candidate win ("constructive control") or prevent a despised candidate from winning ("destructive control"). To protect elections from such control attempts, computational complexity has been used to show that electoral control, though not impossible, is computationally prohibitive.

We show that fallback voting, an election system proposed by Brams and Sanver (2009) to combine Bucklin with approval voting, is resistant to each of the common types of control except to destructive control by adding or deleting voters. Thus fallback voting displays the broadest control resistance currently known to hold among natural election systems with a polynomial-time winner problem. We also study the control complexity of Bucklin voting itself and show that it behaves almost as good (possibly even as good) as fallback voting in terms of control resistance. As Bucklin voting is a special case of fallback voting, each resistance shown for Bucklin voting strengthens the corresponding resistance for fallback voting.

1. Introduction

Since the seminal paper of Bartholdi, Tovey, and Trick (1992), the complexity of *electoral control*—changing the outcome of an election via such actions as adding/deleting/partitioning either candidates or voters—has been studied for a variety of voting systems. Unlike *manipulation* (Bartholdi, Tovey, & Trick, 1989; Bartholdi & Orlin, 1991; Conitzer, Sandholm, & Lang, 2007; Faliszewski, Hemaspaandra, Hemaspaandra, & Rothe, 2009c), which models attempts of strategic voters to influence the outcome of an election via casting insincere votes, control models ways of an external actor, the "chair," to tamper with an election's participation structure so as to alter its outcome. Another way of tampering with the outcome of elections is *bribery* (Faliszewski, Hemaspaandra, & Hemaspaandra, 2009a; Faliszewski, Hemaspaandra, Hemaspaandra, & Rothe, 2009b), which shares with manipulation the feature that votes are being changed, and with control the aspect that an external actor tries to change the outcome of the election. Faliszewski et al. (2009c) survey known

*. Preliminary versions of this paper appear in the proceedings of *Computing: the 16th Australasian Theory Symposium* (Erdélyi & Rothe, 2010) and as the technical reports (Erdélyi, Piras, & Rothe, 2010a, 2010b). This work was supported in part by DFG grants RO 1202/11-1 and RO 1202/12-1 (within the European Science Foundation's EUROCORES program LogICCC). This work was done in part while the first author was visiting Universität Trier, while the third author was visiting the University of Rochester, and while the first and third authors were visiting NICTA, Sydney, and the University of Newcastle.

©2010 AI Access Foundation. All rights reserved.

complexity results for control, manipulation, and bribery in various voting systems, and Baumeister et al. (2010) do so with a particular emphasis on approval voting and its variants.

Elections have been used for preference aggregation not only in the context of politics and human societies, but also in artificial intelligence, especially in multiagent systems, and other topics in computer science (see, e.g., (Ephrati & Rosenschein, 1997; Ghosh, Mundhe, Hernandez, & Sen, 1999; Dwork, Kumar, Naor, & Sivakumar, 2001)). That is why it is important to study the computational properties of voting systems. In particular, complexity can be used to protect elections against tampering attempts in control, manipulation, and bribery attacks by showing that such attacks, though not impossible in principle, are computationally prohibitive.

Regarding control, a central question is to find voting systems that are computationally resistant to as many of the common 22 control types as possible, where resistance means the corresponding control problem is NP-hard. Each control type is either constructive (the chair seeking to make some candidate win) or destructive (the chair seeking to make some candidate end up not winning).

We study the control complexity of fallback voting, an election system introduced by Brams and Sanver (2009) as a way of combining Bucklin and approval voting. We prove that fallback voting is resistant (i.e., the corresponding control problem is NP-hard) to each of the common types of control except two, and we show it is vulnerable (i.e., the corresponding control problem is in P) to these two control types (namely, to destructive control by adding or deleting voters). With these twenty control resistances, fallback voting displays the broadest control resistance currently known to hold among natural election systems with a polynomial-time winner problem. In particular, fallback voting is fully resistant to constructive control and it is fully resistant to candidate control.

We also study the control complexity of Bucklin voting itself and show that it has no more than one control resistance fewer than fallback voting (namely, possibly, regarding destructive control to partition of voters in the tie-handling model TP, see Section 2.2 for the definition). In particular, Bucklin voting is also fully resistant to constructive control and fully resistant to candidate control. Since Bucklin voting is a special case of fallback voting, each resistance result for Bucklin strengthens the corresponding resistance result for fallback voting.

Related work: The study of electoral control was initiated by Bartholdi et al. (1992) who introduced a number of constructive control types and investigated plurality and Condorcet voting in this regard. The common types of destructive control were proposed by Hemaspaandra, Hemaspaandra, and Rothe (2007) who studied destructive control for plurality and Condorcet voting and constructive and destructive control for approval voting. Plurality voting was the first system found to be fully resistant to candidate control.

Faliszewski et al. (2009b) studied the control complexity of the whole family of Llull/Copeland voting systems and were the first to find a natural voting system (among those having a polynomial-time winner problem) that is resistant to all common types of constructive control.[1] Another such voting system with full resistance to constructive control is sincere-strategy preference-based approval voting (SP-AV), as shown by Erdélyi, Nowak, and Rothe (2009). Prior to this paper, SP-AV was the voting system displaying the broadest control resistance among natural systems with a polynomial-time winner problem. However, SP-AV (as modified by Erdélyi et al. (2009)) is ar-

1. Hemaspaandra, Hemaspaandra, and Rothe (2009) provide a system that, via "hybridization," shows perfect control resistance. However, this system is not natural and not suitable to be applied in practice, and it was not designed for that purpose. Rather, it was designed to prove the impossibility of a certain impossibility theorem.

guably less natural a system than fallback voting.[2] Note also that plurality has fewer resistances to voter control and Copeland voting has fewer resistances to destructive control than fallback voting.

Organization: This paper is organized as follows. In Section 2, we recall some notions from voting theory, define the commonly studied types of control, and explain Bucklin voting and the fallback voting procedure of Brams and Sanver (2009) in detail. Our results on the control complexity of Bucklin voting and fallback voting are presented in Section 3. Finally, Section 4 provides some conclusions and open questions.

2. Preliminaries

2.1 Elections and Voting Systems

An *election* (C,V) is given by a finite set C of candidates and a finite list V of votes over C. A *voting system* is a rule that specifies how to determine the winner(s) of any given election. The two voting systems considered in this paper are Bucklin voting and fallback voting.

In *Bucklin voting*, votes are represented as (strict) linear orders over C, i.e., each voter ranks all candidates according to his or her preferences. For example, if $C = \{a,b,c,d\}$ then a vote might look like $c\ d\ a\ b,$ i.e., this voter (strictly) prefers c to d, d to a, and a to b. Given an election (C,V) and a candidate $c \in C$, define the *level i score of c in* (C,V) (denoted by $score^i_{(C,V)}(c)$) as the number of votes in V that rank c among their top i positions. Denoting the *strict majority threshold for a list V of voters* by $maj(V) = \lfloor \|V\|/2 \rfloor + 1$, the *Bucklin score of c in* (C,V) is the smallest i such that $score^i_{(C,V)}(c) \geq maj(V)$. All candidates with a smallest Bucklin score, say k, and a largest level k score are the *Bucklin winners (BV winners, for short) in* (C,V). If some candidate becomes a Bucklin winner on level k, we call him or her a *level k BV winner in* (C,V). Note that a level 1 BV winner must be unique, but there may be more level k BV winners than one for $k > 1$, i.e., an election may have more than one Bucklin winner in general.

Brams and Sanver (2009) proposed fallback voting as a hybrid voting system that combines Bucklin with approval voting. In *approval voting*, votes are represented by approval vectors in $\{0,1\}^{\|C\|}$ (with respect to a fixed order of the candidates in C), where 0 stands for disapproval and 1 stands for approval. Given an election (C,V) and a candidate $c \in C$, define the *approval score of c in* (C,V) (denoted by $score_{(C,V)}(c)$) as the number of c's approvals in (C,V), and all candidates with a largest approval score are the *approval winners in* (C,V). Note that an election may have more than one approval winner.

Fallback voting combines Bucklin with approval voting as follows. Each voter provides both an approval vector and a linear ordering of all approved candidates. The subset of candidates approved of by a voter is also called his or her *approval strategy*. For simplicity, we will omit the disapproved

2. SP-AV is (a variant of) another hybrid system combining approval and preference-based voting that was also proposed by Brams and Sanver (see (Brams & Sanver, 2006) for the original system and (Erdélyi et al., 2009) for its modification SP-AV). The reason we said SP-AV is less natural than fallback voting is that, in order to preserve "admissibility" of votes (as required by Brams and Sanver (2006) to preclude trivial approval strategies), SP-AV employs an additional rule to (re-)coerce admissibility (in particular, if in the course of a control action an originally admissible vote becomes inadmissible). This point has been discussed in detail by Baumeister et al. (2010). In a nutshell, this rule, if applied, changes the approval strategies of the votes originally cast by the voters. The effect of this rule is that SP-AV can be seen as a hybrid between approval and plurality voting, and it indeed possesses each resistance either of these two systems has. In contrast, here we study the original fallback voting system (Brams & Sanver, 2009), in which votes, once cast, do not change.

candidates in each vote.[3] For example, if $C = \{a,b,c,d\}$ and a voter approves of a, c, and d but disapproves of b, and prefers c to d and d to a, then this vote will be written as: $c \;\; d \;\; a$. We will always explicitly state the candidate set, so it will always be clear which candidates participate in an election and which of them are disapproved by which voter (namely those not occurring in his or her vote). Given an election (C,V) and a candidate $c \in C$, the notions of *level i score of c in* (C,V) and *level k fallback voting winner (level k FV winner, for short) in* (C,V) are defined analogously to the case of Bucklin voting, and if there exists a level k FV winner for some $k \leq \|C\|$, he or she is called a *fallback winner (FV winner, for short) in* (C,V). However, unlike in Bucklin voting, in fallback voting it may happen that no candidate reaches a strict majority for any level, due to voters being allowed to disapprove of (any number of) candidates, so it may happen that for no $k \leq \|C\|$ a level k FV winner exists. In such a case, every candidate with a largest (approval) score is an *FV winner in* (C,V). Note that Bucklin voting is the special case of fallback voting where each voter approves of all candidates.

As a notation, when a vote contains a subset of the candidate set, such as $c \;\; D \;\; a$ for a subset $D \subseteq C$, this is a shorthand for $c \;\; d_1 \;\; \cdots \;\; d_\ell \;\; a$, where the elements of $D = \{d_1,\ldots,d_\ell\}$ are ranked with respect to some (tacitly assumed) fixed ordering of all candidates in C. For example, if $C = \{a,b,c,d\}$ is assumed to be ordered lexicographically and $D = \{b,d\}$ then "$c \;\; D \;\; a$" is a shorthand for the vote $c \;\; b \;\; d \;\; a$.

2.2 Types of Electoral Control

There are eleven types of electoral control, each coming in two variants. In *constructive control* (Bartholdi et al., 1992), the chair tries to make his or her favorite candidate win; in *destructive control* (Hemaspaandra et al., 2007), the chair tries to prevent a despised candidate's victory. We refrain from giving a detailed discussion of natural, real-life scenarios for each of these 22 standard control types that motivate them; these can be found in, e.g., (Bartholdi et al., 1992; Hemaspaandra et al., 2007; Faliszewski et al., 2009b; Hemaspaandra et al., 2009; Erdélyi et al., 2009). However, we stress that every control type is motivated by an appropriate real-life scenario, and we will briefly point some of them out below.

When we now formally define our 22 standard control types as decision problems, we assume that each election or subelection in these control problems will be conducted with the voting system at hand (i.e., either Bucklin or fallback voting) and that each vote will be represented as required by the corresponding voting system. We also assume that the chair has complete knowledge of the voters' preferences and/or approval strategies. This assumption may be considered to be unrealistic in certain settings, but is reasonable and natural in certain others, including small-scale elections among humans and even large-scale elections among software agents. More to the point, assuming the chair to have complete information makes sense for our results, as most of our results are NP-hardness lower bounds showing resistance of a voting system against specific control attempts and complexity lower bounds in the complete-information model are inherited by any natural partial-information model (see (Hemaspaandra et al., 2007) for a more detailed discussion of this point).

3. Erdélyi and Rothe (2010) use a slightly different notation by separating the approved candidates from the disapproved candidates by a line, where all candidates to the left of this approval line are ranked and all candidates to the right of this approval line are unranked.

2.2.1 CONTROL BY ADDING CANDIDATES

We formally state our control problems in the common instance/question format. We start with the four problems modeling control by adding candidates. In these control scenarios, the chair seeks to make his or her favorite candidate win (in the constructive cases) or prevent a victory of his or her despised candidate (in the destructive cases) via introducing new candidates from a given pool of spoiler candidates into the election. Faliszewski et al. (2009b) formalize this problem as follows.

Name: CONSTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES.
Instance: An election $(C \cup D, V)$, $C \cap D = \emptyset$, a distinguished candidate $c \in C$, and a nonnegative integer k. (C is the set of originally qualified candidates and D is the set of spoiler candidates that may be added.)
Question: Does there exist a subset $D' \subseteq D$ such that $\|D'\| \leq k$ and c is the unique winner (under the election system at hand) of election $(C \cup D', V)$?

CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES, the problem variant originally proposed by Bartholdi et al. (1992), is the same except there is no limit k on the number of spoiler candidates that may be added. Faliszewski et al. (2009b) discuss in detail the reasons of why it makes sense to also consider the limited version of the problem. Although the difference in the definitions may appear to be negligible, note that the complexity of these problems differs significantly in some cases, e.g., in Llull's voting system (Faliszewski et al., 2009b).

The destructive variants of both problems defined above are obtained by asking whether c is *not* a unique winner of $(C \cup D', V)$.

2.2.2 CONTROL BY DELETING CANDIDATES

This control problem is defined analogously to control by adding a limited number of candidates, except that the chair now seeks to make a distinguished candidate c win by deleting up to k candidates from the given election.[4] This control scenario models candidate suppression. For example, by deleting certain candidates other than c the chair may hope that their voters swing to now support c.

Name: CONSTRUCTIVE CONTROL BY DELETING CANDIDATES.
Instance: An election (C, V), a distinguished candidate $c \in C$, and a nonnegative integer k.
Question: Does there exist a subset $C' \subseteq C$ such that $\|C'\| \leq k$ and c is the unique winner (under the election system at hand) of election $(C - C', V)$?

The destructive version of this problem is the same except that the chair now wants to preclude c from being a unique winner (and, to prevent the problem from being trivial, simply deleting c is not allowed).

2.2.3 CONTROL BY PARTITION OR RUN-OFF PARTITION OF CANDIDATES

Both CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES and CONSTRUCTIVE CONTROL BY RUN-OFF PARTITION OF CANDIDATES take as input an election (C, V) and a candidate $c \in C$ and ask whether c can be made a unique winner in a certain two-stage election consisting of

4. No unlimited version has been considered previously for this control type or for the types of control by adding or deleting voters to be defined below.

one (in the partition case) or two (in the run-off partition case) first-round subelection(s) and a final round. In both variants, following Hemaspaandra et al. (2007), we consider two tie-handling rules, TP ("ties promote") and TE ("ties eliminate"), that enter into force when more candidates than one are tied for winner in any of the first-round subelections:

Name: CONSTRUCTIVE CONTROL BY RUN-OFF PARTITION OF CANDIDATES (TP).
Instance: An election (C,V) and a distinguished candidate $c \in C$.
Question: Is it possible to partition C into C_1 and C_2 such that c is the unique winner (under the election system at hand) of election $(W_1 \cup W_2, V)$, where W_i, $i \in \{1,2\}$, is the set of winners of subelection (C_i,V)?

Name: CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES (TP).
Instance: An election (C,V) and a distinguished candidate $c \in C$.
Question: Is it possible to partition C into C_1 and C_2 such that c is the unique winner (under the election system at hand) of election $(W_1 \cup C_2, V)$, where W_1 is the set of winners of subelection (C_1,V)?

In both cases, when the TE rule is used, none of multiple, tied first-round subelection winners is promoted to the final round. For example, if we have a run-off and $\|W_2\| \geq 2$ then the final-round election collapses to (W_1,V); only a unique first-round subelection winner is promoted to the final round in TE.

It is obvious how to obtain the destructive variants of these four problems formalizing control by candidate partition. Summing up, we now have defined 14 candidate control problems.

The following example gives a real-life scenario of control by partition of candidates.

Example 2.1 *In the Eurovision Song Contest, which has been broadcast annually since 1956 on live television in Europe and other parts of the world (e.g., in more than 130 countries in 2006), each participating country submits a song (that has previously been selected in a national competition) and casts votes for the other countries' songs. The active member countries of the European Broadcasting Union (EBU) are eligible to participate in this competition. Since 2000, however, four EBU member countries have a privileged status because they are the four biggest financial contributors to the EBU: France, Germany, Spain, and the United Kingdom—the "Big Four"—are automatically qualified for the final round of the Eurovision Song Contest, whereas the other candidates have to participate in the semi-finals first to determine who among them enters the final round. Formalized as a* CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES *instance, the participating countries are partitioned into $C = C_1 \cup C_2$, where C_1 contains the semi-finalists and C_2 contains the Big Four, such that all winners of the semi-finals (as modeled by the TP rule) move forward to the final round to run against the Big Four.*

Other real-life examples include sports tournaments in which certain teams (such as last year's champion and the team hosting this year's championship) are given an exemption from qualification.

2.2.4 CONTROL BY ADDING VOTERS

Turning now to the voter control problems, we start with control by adding voters. This control scenario models attempts by the chair to influence the outcome of elections via introducing new voters. There are many ways of introducing new voters into an election—think, for example, of "get-out-the-vote" drives, or of lowering the age-limit for the right to vote, or of attracting new voters with certain promises or even small gifts.

Name: CONSTRUCTIVE CONTROL BY ADDING VOTERS.
Instance: An election $(C, V \cup V')$, $V \cap V' = \emptyset$, where V is a list of registered voters and V' a pool of as yet unregistered voters that can be added, a distinguished candidate $c \in C$, and a nonnegative integer k.
Question: Does there exist a sublist $V'' \subseteq V'$ of size at most k such that c is the unique winner (under the election system at hand) of election $(C, V \cup V'')$?

The destructive variant of this problem is the same except that the chair now wants to preclude c from being a unique winner.

2.2.5 CONTROL BY DELETING VOTERS

Disenfranchisement and other means of voter suppression is modeled as control by deleting voters.

Name: CONSTRUCTIVE CONTROL BY DELETING VOTERS.
Instance: An election (C, V), a distinguished candidate $c \in C$, and a nonnegative integer k.
Question: Does there exist a sublist $V' \subseteq V$ such that $\|V'\| \leq k$ and c is the unique winner (under the election system at hand) of election $(C, V - V')$?

Again, the destructive variant of this problem is the same except that the chair now wants to preclude c from being a unique winner.

2.2.6 CONTROL BY PARTITION OF VOTERS

Name: CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS (TP).
Instance: A set C of candidates, a list V of votes over C, and a distinguished candidate $c \in C$.
Question: Is it possible to partition V into V_1 and V_2 such that c is the unique winner (under the election system at hand) of election $(W_1 \cup W_2, V)$, where W_i, $i \in \{1, 2\}$, is the set of winners of subelection (C, V_i)?

The destructive variant of this problem is defined analogously, except it asks whether c is *not* a unique winner of this two-stage election. In both variants, if one uses the tie-handling model TE instead of TP in the two first-stage subelections, a winner w of (C, V_1) or (C, V_2) proceeds to the final stage if and only if w is the only winner of his or her subelection. Each of the four problems just defined models "two-district gerrymandering."

Summing up, we now have defined eight voter control problems and thus a total of 22 control problems.

2.3 Immunity, Susceptibility, Resistance, and Vulnerability

Let $\mathfrak{CT}$ be a control type; for example, $\mathfrak{CT}$ might stand for "constructive control by partition of voters in model TP" or any of the other types of control defined in the previous section. We say a voting system is *immune to* $\mathfrak{CT}$ if it is impossible for the chair to make the given candidate the unique winner in the constructive case (not a unique winner in the destructive case) via exerting control of type $\mathfrak{CT}$. We say a voting system is *susceptible to* $\mathfrak{CT}$ if it is not immune to $\mathfrak{CT}$. A voting

	Fallback Voting		Bucklin Voting		SP-AV		Approval	
Control by	Const.	Dest.	Const.	Dest.	Const.	Dest.	Const.	Dest.
Adding Candidates (unlimited)	**R**	**R**	**R**	**R**	R	R	I	V
Adding Candidates (limited)	**R**	**R**	**R**	**R**	R	R	I	V
Deleting Candidates	**R**	**R**	**R**	**R**	R	R	V	I
Partition of Candidates	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: R TP: R	TE: R TP: R	TE: V TP: I	TE: I TP: I
Run-off Partition of Candidates	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: R TP: R	TE: R TP: R	TE: V TP: I	TE: I TP: I
Adding Voters	**R**	**V**	**R**	**V**	R	V	R	V
Deleting Voters	**R**	**V**	**R**	**V**	R	V	R	V
Partition of Voters	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: S**	TE: R TP: R	TE: V TP: R	TE: R TP: R	TE: V TP: V

Table 1: Overview of results. Key: I = immune, S = susceptible, R = resistant, V = vulnerable, TE = ties eliminate, and TP = ties promote. Results new to this paper are in boldface.

system that is susceptible to $\mathfrak{CT}$ is said to be *vulnerable to* $\mathfrak{CT}$ if the control problem corresponding to $\mathfrak{CT}$ can be solved in polynomial time, and is said to be *resistant to* $\mathfrak{CT}$ if the control problem corresponding to $\mathfrak{CT}$ is NP-hard. These notions are due to Bartholdi et al. (1992) (except that we follow the now more common approach of Hemaspaandra et al. (2009) who define *resistant* to mean "susceptible and NP-hard" rather than "susceptible and NP-complete").

3. Results

3.1 Overview

Table 1 shows in boldface our results on the control complexity of Bucklin voting and fallback voting for all 22 standard control types. For comparison, this table also shows the results for approval voting that are due to Hemaspaandra et al. (2007) and for SP-AV that are due to Erdélyi et al. (2009).

3.2 Susceptibility

If an election system $\mathscr{E}$ satisfies the "unique" variant of the Weak Axiom of Revealed Preference[5] (Unique-WARP, for short), then $\mathscr{E}$ is immune to constructive control by adding candidates (no matter whether a limited or an unlimited number of candidates is being added), and this observation has been applied to approval voting (Bartholdi et al., 1992; Hemaspaandra et al., 2007). Unlike approval voting, however, Bucklin voting and fallback voting do not satisfy Unique-WARP.

Proposition 3.1 *Neither Bucklin voting nor fallback voting satisfies Unique-WARP.*

Proof. We show this result for Bucklin voting only; the proof for fallback voting follows immediately. Consider the election (C,V) with candidate set $C = \{a,b,c,d\}$ and voter collection

5. This variant of the axiom says that the unique winner w of any election is also the unique winner of every subelection including w.

$V = (v_1, v_2, \ldots, v_6)$:

$$\begin{array}{r|c} & (C,V) \\ \hline v_1 = v_2 = v_3: & a\ c\ b\ d \\ v_4 = v_5: & b\ d\ c\ a \\ v_6: & d\ a\ c\ b \end{array}$$

Candidate a is the unique Bucklin winner of the election (C,V), reaching the strict majority threshold on level 2 with $score^2_{(C,V)}(a) = 4$. By removing candidate b from the election, we get the subelection (C',V) with $C' = \{a,c,d\}$. There is no candidate on level 1 who passes the strict majority threshold. However, there are two candidates on the second level with a strict majority, namely candidates a and c. Since $score^2_{(C',V)}(c) = 5 > 4 = score^2_{(C',V)}(a)$, the unique Bucklin winner of the subelection (C',V) is candidate c. Thus, Bucklin voting does not satisfy Unique-WARP. ❑

Indeed, as we will now show, Bucklin voting and fallback voting are susceptible to each of our 22 control types. Our proofs make use of the results of Hemaspaandra et al. (2007) that provide general proofs of and links between certain susceptibility cases. For the sake of self-containment, we state their results, as Theorems A.1, A.2, and A.3, in the appendix.

We start with susceptibility to candidate control for Bucklin voting.

Lemma 3.2 *Bucklin voting is susceptible to constructive and destructive control by adding candidates (in both the “limited” and the “unlimited” case), by deleting candidates, and by partition of candidates (with or without run-off and for each in both model TE and model TP).*

Proof. From Theorem A.1 and the fact that Bucklin voting is a voiced voting system,[6] it follows that Bucklin voting is susceptible to constructive control by deleting candidates, and to destructive control by adding candidates (in both the “limited” and the “unlimited” case).

Now, consider the election (C,V) given in the proof of Proposition 3.1. The unique Bucklin winner of the election is candidate a. Partition C into $C_1 = \{a,c,d\}$ and $C_2 = \{b\}$. The unique Bucklin winner of subelection (C_1,V) is candidate c, as shown in the proof of Proposition 3.1. In both partition and run-off partition of candidates and for each in both tie-handling models, TE and TP, candidate b runs against candidate c in the final stage of the election. The unique Bucklin winner is in each case candidate c. Thus, Bucklin voting is susceptible to destructive control by partition of candidates (with or without run-off and for each in both model TE and model TP).

By Theorem A.3, Bucklin voting is also susceptible to destructive control by deleting candidates. By Theorem A.2, Bucklin voting is also susceptible to constructive control by adding candidates (in both the “limited” and the “unlimited” case).

Now, changing the roles of a and c makes c our distinguished candidate. In election (C,V), c loses against candidate a. By partitioning the candidates as described above, c becomes the unique Bucklin winner of the election. Thus, Bucklin voting is susceptible to constructive control by partition of candidates (with or without run-off and for each in both tie-handling models, TE and TP). ❑

We now turn to susceptibility to voter control for Bucklin voting.

Lemma 3.3 *Bucklin voting is susceptible to constructive and destructive control by adding voters, by deleting voters, and by partition of voters (in both model TE and model TP).*

6. An election system is said to be *voiced* if the single candidate in any one-candidate election always wins.

Proof. Consider the election (C,V), where $C = \{a,b,c,d\}$ is the set of candidates and $V = (v_1, v_2, v_3, v_4)$ is the collection of voters with the following preferences:

	(C,V)
v_1 :	$a\ c\ b\ d$
v_2 :	$d\ c\ a\ b$
v_3 :	$b\ a\ c\ d$
v_4 :	$b\ a\ c\ d$

We partition V into $V_1 = (v_1, v_2)$ and $V_2 = (v_3, v_4)$. Thus we split (C,V) into two subelections:

	(C,V_1)	and	(C,V_2)
v_1 :	$a\ c\ b\ d$		
v_2 :	$d\ c\ a\ b$		
v_3 :			$b\ a\ c\ d$
v_4 :			$b\ a\ c\ d$

Clearly, candidate a is the unique Bucklin winner of (C,V). However, c is the unique Bucklin winner of (C,V_1) and b is the unique Bucklin winner of (C,V_2), and so a is not promoted to the final stage. Thus, Bucklin voting is susceptible to destructive control by partition of voters in both tie-handling models, TE and TP.

By Theorem A.1 and the fact that Bucklin voting is a voiced voting system, Bucklin voting is susceptible to destructive control by deleting voters. By Theorem A.2, Bucklin voting is also susceptible to constructive control by adding voters.

By changing the roles of a and c again, we can see that Bucklin voting is susceptible to constructive control by partition of voters in both model TE and model TP. By Theorem A.3, Bucklin voting is also susceptible to constructive control by deleting voters. Finally, again by Theorem A.2, Bucklin voting is susceptible to destructive control by adding voters. ❑

Since Bucklin voting is a special case of fallback voting, fallback voting is also susceptible to all 22 common types of control.

Corollary 3.4 *Fallback voting is susceptible to each of the* 22 *control types defined in Section 2.2.*

3.3 Candidate Control

Theorem 3.5 *Bucklin voting is resistant to each of the* 14 *standard types of candidate control.*

For the hardness proofs showing Theorem 3.5, we use a restricted version of the NP-complete problem HITTING SET (see, e.g., (Garey & Johnson, 1979)), which is defined as follows:

Name: RESTRICTED HITTING SET.

Instance: A set $B = \{b_1, b_2, \ldots, b_m\}$, a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of nonempty subsets $S_i \subseteq B$ such that $n > m$, and a positive integer k with $1 < k < m$.

Question: Does $\mathscr{S}$ have a hitting set of size at most k, i.e., is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Note that by dropping the requirement "$n > m > k > 1$," we obtain the (unrestricted) HITTING SET problem. We first need to show that RESTRICTED HITTING SET is NP-complete as well.

Lemma 3.6 RESTRICTED HITTING SET *is* NP-*complete.*

Proof. It is immediate that RESTRICTED HITTING SET is in NP. To show NP-hardness, we reduce the (general) HITTING SET problem to RESTRICTED HITTING SET. Let $(\hat{B},\hat{\mathscr{S}},\hat{k})$ be a given instance of HITTING SET, where $\hat{B} = \{b_1,b_2,\ldots,b_{\hat{m}}\}$ is a set, $\hat{\mathscr{S}} = \{S_1,S_2,\ldots,S_{\hat{n}}\}$ is a collection of nonempty subsets of $\hat{B}$, and $\hat{k} \leq \hat{m}$ is a positive integer. If $\hat{k} = \hat{m}$ or $\hat{k} = 1$, $(\hat{B},\hat{\mathscr{S}},\hat{k})$ is trivially in HITTING SET, so we may assume that $1 < \hat{k} < \hat{m}$.

Define the following instance $(B,\mathscr{S},k)$ of RESTRICTED HITTING SET:

$$
\begin{aligned}
B &= \begin{cases} \hat{B} \cup \{a\} & \text{if } \hat{n} \leq \hat{m} \\ \hat{B} & \text{if } \hat{n} > \hat{m}, \end{cases} \\
\mathscr{S} &= \begin{cases} \hat{\mathscr{S}} \cup \{S_{\hat{n}+1},S_{\hat{n}+2},\ldots,S_{\hat{m}+2}\} & \text{if } \hat{n} \leq \hat{m} \\ \hat{\mathscr{S}} & \text{if } \hat{n} > \hat{m}, \end{cases} \\
k &= \begin{cases} \hat{k}+1 & \text{if } \hat{n} \leq \hat{m} \\ \hat{k} & \text{if } \hat{n} > \hat{m}, \end{cases}
\end{aligned}
$$

where $S_{\hat{n}+1} = S_{\hat{n}+2} = \cdots = S_{\hat{m}+2} = \{a\}$. Since $1 < \hat{k} < \hat{m}$, we have $1 < k < m$.

Let n be the number of members of $\mathscr{S}$ and m be the number of elements of B. Note that if $\hat{n} > \hat{m}$ then $(B,\mathscr{S},k) = (\hat{B},\hat{\mathscr{S}},\hat{k})$, so $n = \hat{n} > \hat{m} = m$; and if $\hat{n} \leq \hat{m}$ then $n = \hat{m}+2 > \hat{m}+1 = m$. Thus, in both cases $(B,\mathscr{S},k)$ fulfills the restriction of RESTRICTED HITTING SET.

It is easy to see that $\hat{\mathscr{S}}$ has a hitting set of size at most $\hat{k}$ if and only if $\mathscr{S}$ has a hitting set of size at most k. In particular, assuming $\hat{n} \leq \hat{m}$, if $\hat{\mathscr{S}}$ has a hitting set B' of size at most $\hat{k}$ then $B' \cup \{a\}$ is a hitting set of size at most $k = \hat{k}+1$ for $\mathscr{S}$; and if $\hat{\mathscr{S}}$ has no hitting set of size at most $\hat{k}$ then $\mathscr{S}$ can have no hitting set of size at most $k = \hat{k}+1$ (because $a \notin \hat{B}$, so $\{a\} \cap S_i = \emptyset$ for each i, $1 \leq i \leq \hat{n}$). ❑

In this section, all reductions except one (namely that for constructive control by deleting candidates, see Lemma 3.7) will apply Construction 3.8 below. We first handle this one exception.

Lemma 3.7 *Bucklin voting is resistant to constructive control by deleting candidates.*

Proof. Susceptibility holds by Lemma 3.2. To prove NP-hardness we give a reduction from RESTRICTED HITTING SET. Let $(B,\mathscr{S},k)$ be a RESTRICTED HITTING SET instance with $B = \{b_1,b_2,\ldots,b_m\}$ a set, $\mathscr{S} = \{S_1,S_2,\ldots,S_n\}$ a collection of nonempty subsets $S_i \subseteq B$, and k a positive integer satisfying $k < m < n$. Let $s_i = n+k-\|S_i\|$, $1 \leq i \leq n$, and $s = \sum_{i=1}^{n} s_i$. Note that all s_i are positive, since $m < n$.

Define election (C,V) with candidate set $C = B \cup C' \cup D \cup E \cup F \cup \{w\}$, where

$$
\begin{aligned}
C' &= \{c_1,c_2,\ldots,c_{k+1}\}, \\
D &= \{d_1,d_2,\ldots,d_s\}, \\
E &= \{e_1,e_2,\ldots,e_n\}, \\
F &= \{f_1,\ldots,f_{n+k}\},
\end{aligned}
$$

and let w be the distinguished candidate. Note that the number of candidates in D is

$$s = n^2 + kn - \sum_{i=1}^{n} \|S_i\|.$$

For each i, $1 \leq i \leq n$, let

$$D_i = \{d_{1+\sum_{j=1}^{i-1} s_j}, \ldots, d_{\sum_{j=1}^{i} s_j}\},$$

so $\|D_i\| = s_i$. Define V to be the following collection of $2(n+k+1)+1$ voters:

#	For each ...	number of voters	ranking of candidates
1	$i \in \{1,\ldots,n\}$	1	$S_i\ D_i\ w\ C'\ E\ (D-D_i)\ (B-S_i)\ F$
2	$j \in \{1,\ldots,k+1\}$	1	$E\ (C'-\{c_j\})\ c_j\ B\ D\ w\ F$
3		$k+1$	$w\ F\ C'\ E\ B\ D$
4		n	$C'\ D\ F\ B\ w\ E$
5		1	$C'\ w\ D\ F\ E\ B$

There is no unique BV winner in election (C,V), since the candidates in C' and candidate w are level $n+k+1$ BV winners.

We claim that $\mathscr{S}$ has a hitting set of size k if and only if w can be made the unique BV winner by deleting at most k candidates.

From left to right: Suppose $\mathscr{S}$ has a hitting set B' of size k. Delete the corresponding candidates. Now, w is the unique level $n+k$ BV winner of the resulting election.

From right to left: Suppose w can be made the unique BV winner by deleting at most k candidates. Since $k+1$ candidates other than w have a strict majority on level $n+k+1$ in election (C,V), after deleting at most k candidates, there is still at least one candidate other than w with a strict majority of approvals on level $n+k+1$. However, since w was made the unique BV winner by deleting at most k candidates, w must be the unique BV winner on a level lower than or equal to $n+k$. This is possible only if in all n votes of the first voter group w moves forward by at least one position. This, however, is possible only if $\mathscr{S}$ has a hitting set B' of size k. ❑

Construction 3.8 will be applied to prove the remaining 13 cases of candidate control stated in Theorem 3.5.

Construction 3.8 *Let $(B,\mathscr{S},k)$ be a given instance of* RESTRICTED HITTING SET, *where $B = \{b_1,b_2,\ldots,b_m\}$ is a set, $\mathscr{S} = \{S_1,S_2,\ldots,S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and $k < m$ is a positive integer. (Thus, $n > m > k \geq 1$.)*

Define election (C,V), where $C = B \cup \{c,d,w\}$ is the candidate set and where V consists of the following $6n(k+1)+4m+11$ voters:

#	*For each ...*	*number of voters*	*ranking of candidates*
1		$2m+1$	$c\ d\ B\ w$
2		$2n+2k(n-1)+3$	$c\ w\ d\ B$
3		$2n(k+1)+5$	$w\ c\ d\ B$
4	$i \in \{1,\ldots,n\}$	$2(k+1)$	$d\ S_i\ c\ w\ (B-S_i)$
5	$j \in \{1,\ldots,m\}$	2	$d\ b_j\ w\ c\ (B-\{b_j\})$
6		$2(k+1)$	$d\ w\ c\ B$

We now prove Theorem 3.5 (except for the case of constructive control by deleting candidates, which has already been handled separately in Lemma 3.7) via Construction 3.8 in Lemmas 3.10, 3.11, and 3.12. The proofs of these lemmas in turn will make use of Lemma 3.9 below.

Lemma 3.9 *Consider the election (C,V) constructed according to Construction 3.8 from a* RESTRICTED HITTING SET *instance $(B,\mathscr{S},k)$.*

1. *c is the unique level 2 BV winner of $(\{c,d,w\},V)$.*
2. *If $\mathscr{S}$ has a hitting set B' of size k, then w is the unique BV winner of election $(B'\cup\{c,d,w\},V)$.*
3. *Let $D\subseteq B\cup\{d,w\}$. If c is not a unique BV winner of election $(D\cup\{c\},V)$, then there exists a set $B'\subseteq B$ such that*
   - (a) *$D=B'\cup\{d,w\}$,*
   - (b) *w is a level 2 BV winner of election $(B'\cup\{c,d,w\},V)$, and*
   - (c) *B' is a hitting set for $\mathscr{S}$ of size at most k.*

Proof. For the first part, note that there is no level 1 BV winner in election $(\{c,d,w\},V)$ and we have the following level 2 scores in this election:

$$\begin{aligned}
score^2_{(\{c,d,w\},V)}(c) &= 6n(k+1)+2(m-k)+9,\\
score^2_{(\{c,d,w\},V)}(d) &= 2n(k+1)+4m+2k+3,\\
score^2_{(\{c,d,w\},V)}(w) &= 4n(k+1)+2m+10.
\end{aligned}$$

Since $n>m$ (which implies $n>k$), we have:

$$\begin{aligned}
score^2_{(\{c,d,w\},V)}(c)-score^2_{(\{c,d,w\},V)}(d) &= 4n(k+1)-(2m+4k)+6>0,\\
score^2_{(\{c,d,w\},V)}(c)-score^2_{(\{c,d,w\},V)}(w) &= 2n(k+1)-(2k+1)>0.
\end{aligned}$$

Thus, c is the unique level 2 BV winner of $(\{c,d,w\},V)$.

For the second part, suppose that B' is a hitting set for $\mathscr{S}$ of size k. Then there is no level 1 BV winner in election $(B'\cup\{c,d,w\},V)$, and we have the following level 2 scores:

$$\begin{aligned}
score^2_{(B'\cup\{c,d,w\},V)}(c) &= 4n(k+1)+2(m-k)+9,\\
score^2_{(B'\cup\{c,d,w\},V)}(d) &= 2n(k+1)+4m+2k+3,\\
score^2_{(B'\cup\{c,d,w\},V)}(w) &= 4n(k+1)+2(m-k)+10,\\
score^2_{(B'\cup\{c,d,w\},V)}(b_j) &\leq 2n(k+1)+2 \quad \text{for all } b_j\in B'.
\end{aligned}$$

It follows that w is the unique level 2 BV winner of election $(B'\cup\{c,d,w\},V)$.

For the third part, let $D\subseteq B\cup\{d,w\}$. Suppose c is not a unique BV winner of election $(D\cup\{c\},V)$.

(3a) Other than c, only w has a strict majority of votes on the second level and only w can tie or beat c in $(D\cup\{c\},V)$. Thus, since c is not a unique BV winner of election $(D\cup\{c\},V)$, w is clearly in D. In $(D\cup\{c\},V)$, candidate w has no level 1 strict majority, and candidate c has already on level 2 a strict majority. Thus, w must tie or beat c on level 2. For a contradiction, suppose $d \notin D$. Then

$$\begin{aligned} score^2_{(D\cup\{c\},V)}(c) &\geq 4n(k+1)+2m+11; \\ score^2_{(D\cup\{c\},V)}(w) &= 4n(k+1)+2m+10, \end{aligned}$$

which contradicts the above observation that w ties or beats c on level 2. Thus, $D = B' \cup \{d,w\}$, where $B' \subseteq B$.

(3b) This part follows immediately from the proof of part (3a).

(3c) Let ℓ be the number of sets in $\mathscr{S}$ not hit by B'. We have that

$$\begin{aligned} score^2_{(B'\cup\{c,d,w\},V)}(w) &= 4n(k+1)+10+2(m-\|B'\|), \\ score^2_{(B'\cup\{c,d,w\},V)}(c) &= 2(m-k)+4n(k+1)+9+2(k+1)\ell. \end{aligned}$$

From part (3b) we know that

$$score^2_{(B'\cup\{c,d,w\},V)}(w) \geq score^2_{(B'\cup\{c,d,w\},V)}(c),$$

so

$$4n(k+1)+10+2(m-\|B'\|) \quad \geq \quad 2(m-k)+4n(k+1)+9+2(k+1)\ell.$$

The above inequality implies

$$1 > \frac{1}{2} \geq \|B'\| - k + (k+1)\ell.$$

Since $T = \|B'\| - k + (k+1)\ell$ is an integer, we have $T \leq 0$. If $T = 0$ then $\ell = 0$ and $\|B'\| = k$. Now assume $T < 0$. If $\ell = 0$, B' is a hitting set with $\|B'\| < k$, and if $\ell > 0$ then $(k+1)\ell > k$, which contradicts $T = \|B'\| - k + (k+1)\ell < 0$. In each possible case, we have a hitting set (as $\ell = 0$) of size at most k. This completes the proof of Lemma 3.9. ❑

Lemma 3.10 *Bucklin voting is resistant to constructive and destructive control by adding candidates (both in the limited and the unlimited version of the problem).*

Proof. Susceptibility holds by Lemma 3.2. NP-hardness follows immediately from Lemmas 3.6 and 3.9, via mapping the RESTRICTED HITTING SET instance $(B, \mathscr{S}, k)$ to the instance

1. $((\{c,d,w\} \cup B, V), w, k)$ of CONSTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES,

2. $((\{c,d,w\} \cup B, V), c, k)$ of DESTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES,

3. $((\{c,d,w\} \cup B, V), w)$ of CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES, and

4. $((\{c,d,w\} \cup B, V), c)$ of DESTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES.

where in each case c, d, and w are the qualified candidates and B is the set of spoiler candidates. ❑

Lemma 3.11 *Bucklin voting is resistant to destructive control by deleting candidates.*

Proof. Susceptibility holds by Lemma 3.2. To show the problem NP-hard, let (C,V) be the election resulting from a RESTRICTED HITTING SET instance $(B, \mathscr{S}, k)$ according to Construction 3.8, and let c be the distinguished candidate.

We claim that $\mathscr{S}$ has a hitting set of size at most k if and only if c can be prevented from being a unique BV winner by deleting at most $m-k$ candidates.

From left to right: Suppose $\mathscr{S}$ has a hitting set B' of size k. Delete the $m-k$ candidates $B-B'$. Now, both candidates c and w have a strict majority on level 2, but

$$\begin{aligned} score^2_{(\{c,d,w\} \cup B', V)}(c) &= 4n(k+1) + 2(m-k) + 9, \\ score^2_{(\{c,d,w\} \cup B', V)}(w) &= 4n(k+1) + 2(m-k) + 10, \end{aligned}$$

so w is the unique level 2 BV winner of this election.

From right to left: Suppose that c can be prevented from being a unique BV winner by deleting at most $m-k$ candidates. Let $D' \subseteq B \cup \{d,w\}$ be the set of deleted candidates (so $c \notin D'$) and $D = (C - D') - \{c\}$. It follows immediately from Lemma 3.9 that $D = B' \cup \{d,w\}$, where B' is a hitting set for $\mathscr{S}$ of size at most k. ❑

Lemma 3.12 *Bucklin voting is resistant to constructive and destructive control by partition of candidates and by run-off partition of candidates (for each in both tie-handling models, TE and TP).*

Proof. Susceptibility holds by Lemma 3.2, so it remains to show NP-hardness. For the constructive cases, map the given RESTRICTED HITTING SET instance $(B, \mathscr{S}, k)$ to the election (C,V) from Construction 3.8 with distinguished candidate w.

We claim that $\mathscr{S}$ has a hitting set of size at most k if and only if w can be made the unique BV winner by exerting control via any of our four control scenarios (partition of candidates with or without run-off, and for each in either tie-handling model, TE and TP).

From left to right: Suppose $\mathscr{S}$ has a hitting set $B' \subseteq B$ of size k. Partition the set of candidates into the two subsets $C_1 = B' \cup \{c,d,w\}$ and $C_2 = C - C_1$. According to Lemma 3.9, w is the unique level 2 BV winner of subelection $(C_1, V) = (B' \cup \{c,d,w\}, V)$. No matter whether we have a run-off or not, and regardless of the tie-handling rule used, the opponents of w in the final stage (if there are any opponents at all) each are candidates from B. Since $n > m$, w has a majority in the final stage on the first level with a score of $4n(k+1)+9$. Thus, w is the unique BV winner of the resulting election.

From right to left: Suppose w can be made the unique BV winner via any of our four control scenarios. Since c is not a BV winner of the election, there is a subset $D \subseteq B \cup \{d,w\}$ of candidates such that c is not a unique BV winner of the election $(D \cup \{c\}, V)$. By Lemma 3.9, there exists a hitting set for $\mathscr{S}$ of size at most k.

For the four destructive cases, we simply change the roles of c and w in the above argument. ❑

Since Bucklin voting is a special case of fallback voting, we have the following corollary.

Corollary 3.13 *Fallback voting is resistant to each of the* 14 *standard types of candidate control.*

3.4 Voter Control

We now turn to voter control for Bucklin voting and fallback voting, starting with the resistance proofs.

3.4.1 RESISTANCE PROOFS

Most of our reductions showing resistance of voter control problems are from the NP-complete problem EXACT COVER BY THREE-SETS (X3C, for short), which is defined as follows (see, e.g., (Garey & Johnson, 1979)):

Name: EXACT COVER BY THREE-SETS (X3C).
Instance: A set $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m \geq 1$, and a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.
Question: Is there a subcollection $\mathscr{S}' \subseteq \mathscr{S}$ such that each element of B occurs in exactly one set in $\mathscr{S}'$?

Theorem 3.14 *Bucklin voting is resistant to constructive control by adding voters.*

Proof. Susceptibility holds by Lemma 3.3. To show NP-hardness we reduce X3C to our control problem. Let $(B, \mathscr{S})$ be an X3C instance, where $B = \{b_1, b_2, \ldots, b_{3m}\}$ is a set with $m > 1$ and $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ is a collection of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$. (Note that X3C is trivial to solve for $m = 1$.)

Define the election $(C, V \cup V')$, where $C = B \cup \{w\} \cup D$ with $D = \{d_1, \ldots, d_{n(3m-4)}\}$ is the set of candidates, w is the distinguished candidate, and $V \cup V'$ is the following collection of $n+m-2$ voters:

1. V consists of $m-2$ registered voters of the form: $B \;\; D \;\; w$.

2. V' consists of unregistered voters, where for each i, $1 \leq i \leq n$, there is one voter of the form: $D_i \;\; S_i \;\; w \;\; (D - D_i) \;\; (B - S_i)$, where $D_i = \{d_{(i-1)(3m-4)+1}, \ldots, d_{i(3m-4)}\}$.

Since $b_1 \in B$ has a majority already on the first level, w is not a unique BV winner in (C, V).

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made a unique BV winner by adding at most m voters from V'.

From left to right: Suppose $\mathscr{S}$ contains an exact cover for B. Let V'' contain the corresponding voters from V' (i.e., voters of the form $D_i \;\; S_i \;\; w \;\; (D - D_i) \;\; (B - S_i)$, for each S_i in the exact cover) and add V'' to the election. It follows that

$$
\begin{aligned}
score^{3m+1}_{(C,V \cup V'')}(d_j) &= m-1 \quad \text{for all } d_j \in D, \\
score^{3m+1}_{(C,V \cup V'')}(b_j) &= m-1 \quad \text{for all } b_j \in B, \text{ and} \\
score^{3m+1}_{(C,V \cup V'')}(w) &= m.
\end{aligned}
$$

Thus, only w has a strict majority up to the $(3m+1)$st level and so w is the unique level $3m+1$ BV winner of the election.

From right to left: Let $V'' \subseteq V'$ be such that $\|V''\| \leq m$ and w is the unique winner of election $(C, V \cup V'')$. Since w must in particular beat every $b_j \in B$ at or before the $(3m+1)$st level, it follows that $\|V''\| = m$ and each $b_j \in B$ can gain only one additional point. Thus, the m voters in V'' correspond to an exact cover for B. ❑

Theorem 3.15 *Bucklin voting is resistant to constructive control by deleting voters.*

Proof. Susceptibility holds by Lemma 3.3. To show NP-hardness we reduce X3C to our control problem. Let $(B, \mathscr{S})$ be an X3C instance as above. Define the election (C,V), where $C = B \cup \{c,w\} \cup D \cup F \cup G$ is the set of candidates with $D = \{d_1, d_2, \dots, d_{3nm}\}$, $F = \{f_1, f_2, \dots, f_{3n(m-1)}\}$, and $G = \{g_1, g_2, \dots, g_{3m(m-1)}\}$, and where w is the distinguished candidate. For each j, $1 \leq j \leq 3m$, define $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$, and for each $i, 1 \leq i \leq n$, define[7]

$$
\begin{array}{rcl}
B_i & = & \{b_j \in B \mid i \leq n - \ell_j\}, \\
D_i & = & \{d_{(i-1)3m+1}, \dots, d_{3im-\|B_i\|}\}, \text{ and} \\
F_i & = & \{f_{(i-1)(3m-3)+1}, \dots, f_{i(3m-3)}\}.
\end{array}
$$

Also, for each k, $1 \leq k \leq m-1$, define $G_k = \{g_{3m(k-1)+1}, \dots, g_{3mk}\}$. Let V consist of the following $2n+m-1$ voters:

#	For each ...	number of voters	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$S_i\ c\ F_i\ D\ (B - S_i)\ G\ (F - F_i)\ w$
2	$i \in \{1, \dots, n\}$	1	$B_i\ D_i\ w\ F\ (D - D_i)\ (B - B_i)\ G\ c$
3	$k \in \{1, \dots, m-1\}$	1	$c\ G_k\ F\ D\ (G - G_k)\ B\ w$

Candidate c is the unique level 4 BV winner in the election (C,V).

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made the unique BV winner by deleting at most m voters.

From left to right: Suppose $\mathscr{S}$ contains an exact cover for B. By deleting the corresponding voters from the first voter group, we have the following level $3m+1$ scores in the resulting election (C,V'):

$$
\begin{array}{rcl}
score^{3m+1}_{(C,V')}(w) & = & n, \\
score^{3m+1}_{(C,V')}(b_i) & = & score^{3m+1}_{(C,V')}(c) = n-1 \quad \text{for all } i,\ 1 \leq i \leq 3m, \\
score^{3m+1}_{(C,V')}(d_j) & = & 1 \quad \text{or} \quad score^{3m+1}_{(C,V')}(d_j) = 0 \quad \text{for all } d_j \in D, \text{ and} \\
score^{3m+1}_{(C,V')}(f_j) & = & score^{3m+1}_{(C,V')}(g_k) = 1 \quad \text{for all } f_j \in F \text{ and all } g_k \in G.
\end{array}
$$

Since now there are $2n-1$ voters in the election, candidate w is the first candidate having a strict majority, so w is the unique BV winner of election (C,V').

From right to left: Suppose w can be made the unique BV winner by deleting at most m voters. Since w doesn't score any points on any of the first $3m$ levels (see Footnote 7), neither c nor any of the b_i can have a strict majority on any of these levels. In particular, candidate c must have lost

7. Note that $D_i = \emptyset$ if $\|B_i\| = 3m$ and that w is always ranked at or after the $(3m+1)$st position.

exactly m points (up to the $(3m+1)$st level) after deletion, and this is possible only if the m deleted voters are all from the first or third voter group. On the other hand, each $b_i \in B$ must have lost at least one point (up to the $(3m+1)$st level) after deletion, and this is possible only if exactly m voters were deleted from the first voter group. These m voters correspond to an exact cover for B. ❑

Theorem 3.16 *Bucklin voting is resistant to constructive control by partition of voters in both tie-handling models, TE and TP.*

Proof. Susceptibility holds by Lemma 3.3. To show NP-hardness we reduce X3C to our control problems. Let $(B, \mathscr{S})$ be an X3C instance with $B = \{b_1, b_2, \dots, b_{3m}\}$, $m \geq 1$, and a collection $\mathscr{S} = \{S_1, S_2, \dots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$. We define the election (C,V), where $C = B \cup \{c,w,x\} \cup D \cup E \cup F \cup G$ is the set of candidates with $D = \{d_1, \dots, d_{3nm}\}$, $E = \{e_1, \dots, e_{(3m-1)(m+1)}\}$, $F = \{f_1, \dots, f_{(3m+1)(m-1)}\}$, and $G = \{g_1, \dots, g_{n(3m-3)}\}$, and where w is the distinguished candidate. For each j, $1 \leq j \leq 3m$, define $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$, and for each $i, 1 \leq i \leq n$, define

$$
\begin{aligned}
B_i &= \{b_j \in B \mid i \leq n - \ell_j\}, \\
D_i &= \{d_{(i-1)3m+1}, \dots, d_{3im-\|B_i\|}\}, \text{ and} \\
G_i &= \{g_{(i-1)(3m-3)+1}, \dots, g_{i(3m-3)}\}.
\end{aligned}
$$

Also, for each k, $1 \leq k \leq m+1$, define $E_k = \{e_{(3m-1)(k-1)+1}, \dots, e_{(3m-1)k}\}$, and for each l, $1 \leq l \leq m-1$, define $F_l = \{f_{(3m+1)(l-1)+1}, \dots, f_{(3m+1)l}\}$. Let V consist of the following $2n+2m$ voters:

#	For each ...	number of voters	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$c\ S_i\ G_i\ (G - G_i)\ F\ D\ E\ (B - S_i)\ w\ x$
2	$i \in \{1, \dots, n\}$	1	$B_i\ D_i\ w\ G\ E\ (D - D_i)\ F\ (B - B_i)\ c\ x$
3	$k \in \{1, \dots, m+1\}$	1	$x\ c\ E_k\ F\ (E - E_k)\ G\ D\ B\ w$
4	$l \in \{1, \dots, m-1\}$	1	$F_l\ c\ (F - F_l)\ G\ D\ E\ B\ w\ x$

In this election, candidate c is the unique level 2 BV winner with a level 2 score of $n+m+1$.

We claim that $\mathscr{S}$ has an exact cover $\mathscr{S}'$ for B if and only if w can be made the unique BV winner of the resulting election by partition of voters (regardless of the tie-handling model used).

From left to right: Suppose $\mathscr{S}$ has an exact cover $\mathscr{S}'$ for B. Partition V as follows. Let V_1 consist of:

- the m voters of the first group that correspond to the exact cover (i.e., those m voters of the form $c\ S_i\ G_i\ (G - G_i)\ F\ D\ E\ (B - S_i)\ w\ x$ for which $S_i \in \mathscr{S}'$) and
- the $m+1$ voters of the third group (i.e., all voters of the form $x\ c\ E_k\ F\ (E - E_k)\ G\ D\ B\ w$).

Let $V_2 = V - V_1$. In subelection (C,V_1), candidate x is the unique level 1 BV winner. In subelection (C,V_2), candidate w is the first candidate who has a strict majority and moves on to the final round of the election. Thus there are w and x in the final run-off, which w wins with a strict majority on the first level. Since both subelections, (C,V_1) and (C,V_2), have unique BV winners, candidate w can be made the unique BV winner by partition of voters, regardless of the tie-handling model used.

From right to left: Suppose that w can be made the unique BV winner by exerting control by partition of voters (for concreteness, say in TP). Let (V_1,V_2) be such a successful partition. Since w wins the resulting two-stage election, w has to win at least one of the subelections (say, w wins (C,V_1)). If candidate c participates in the final round, he or she wins the election with a strict majority no later than on the second level, no matter which other candidates move forward to the final election. That means that in both subelections, (C,V_1) and (C,V_2), c must not be a BV winner. Only in the second voter group candidate w (who has to be a BV winner in (C,V_1)) gets points earlier than on the second-to-last level. So w has to be a level $3m+1$ BV winner in (C,V_1) via votes from the second voter group in V_1. As c scores already on the first two levels in voter groups 1 and 3, only x and the candidates in B can prevent c from winning in (C,V_2). However, since voters from the second voter group have to be in V_1 (as stated above), in subelection (C,V_2) only candidate x can prevent c from moving forward to the final round. Since x is always placed behind c in all votes except those votes from the third voter group, x has to be a level 1 BV winner in (C,V_2). In (C,V_1) candidate w gains all the points on exactly the $(3m+1)$st level, whereas the other candidates scoring more than one point up to this level receive their points on either earlier or later levels, so no candidate can tie with w on the $(3m+1)$st level and w is the unique level $3m+1$ BV winner in (C,V_1). As both subelections, (C,V_1) and (C,V_2), have unique BV winners other than c, the construction works in model TE as well.

It remains to show that $\mathscr{S}$ has an exact cover $\mathscr{S}'$ for B. Since w has to win (C,V_1) with the votes from the second voter group, not all voters from the first voter group can be in V_1 (otherwise c would have n points already on the first level). On the other hand, there can be at most m voters from the first voter group in V_2 because otherwise x would not be a level 1 BV winner in (C,V_2). To ensure that no candidate contained in B has the same score as w, namely n points, and gets these points on an earlier level than w in (C,V_1), there have to be exactly m voters from the first group in V_2 and these voters correspond to an exact cover for B. ❑

Since Bucklin voting is a special case of fallback voting, fallback voting inherits the NP-hardness lower bounds from Bucklin voting stated in Theorems 3.14, 3.15, and 3.16.

Corollary 3.17 *Fallback voting is resistant to constructive control by adding voters, by deleting voters, and by partition of voters in model TE and model TP.*

We now turn to destructive control by partition of voters in model TE where we show resistance for Bucklin and fallback voting via a reduction from the dominating set problem, one of the standard NP-complete problems (see, e.g., (Garey & Johnson, 1979)). First we need to define some basic graph-theoretic notions.

Definition 3.18 *Let $G=(B,A)$ be an undirected graph without loops or multiple edges. We say that two distinct vertices b_i and b_j are* adjacent in G *if and only if there is an edge $\{b_i,b_j\}\in A$. Adjacent vertices are called* neighbors in G.

The neighborhood of a vertex $b_i\in B$ *is defined by* $\mathrm{N}(b_i)=\{b_j\in B \mid \{b_i,b_j\}\in A\}$. *The* closed neighborhood of $b_i\in B$ *is defined by* $\mathrm{N}[b_i]=\mathrm{N}(b_i)\cup\{b_i\}$. *For a subset $S\subseteq B$, the* neighborhood of S *is defined as* $\mathrm{N}(S)=\bigcup_{b_i\in S}\mathrm{N}(b_i)$ *and the* closed neighborhood of S *is defined as* $\mathrm{N}[S]=\bigcup_{b_i\in S}\mathrm{N}[b_i]$.

Name: DOMINATING SET (DS).
Instance: A graph $G=(B,A)$ and a positive integer $k\le\|B\|$.

Question: Is there a subset $B' \subseteq B$ with $\|B'\| \leq k$ such that for each $b_i \in B - B'$ there is a $b_j \in B'$ for which $\{b_i, b_j\} \in A$?

In other words, the dominating set problem tests, given a graph $G = (B, A)$ and an integer k, whether there is a subset $B' \subseteq B$ of size at most k such that $B = \mathrm{N}[B']$.

The following construction will be used in the proof of Theorem 3.21.

Construction 3.19 *Let $((B,A),k)$ be a given instance of* DOMINATING SET *with $B = \{b_1, b_2, \ldots, b_n\}$ and $n \geq 1$. Define election (C,V) with candidate set*

$$C = B \cup D \cup E \cup F \cup H \cup \{c, u, v, w, x, y\},$$

where

$$\begin{array}{rclcrcl} D & = & \{d_1, d_2, \ldots, d_{(k-1)(n+4)}\}, & & E & = & \{e_1, e_2, \ldots, e_{2(k+n)}\}, \\ F & = & \{f_1, f_2, \ldots, f_{3n}\}, \text{ and} & & H & = & \{h_1, h_2, \ldots, h_{n^2}\}. \end{array}$$

Let c be the distinguished candidate. V consists of the following $2k+2n$ votes over C that can be arranged in four groups:

1. *For each $i, 1 \leq i \leq n$, there is one voter of the form:*

$$F_i \;\; (B - \mathrm{N}[b_i]) \;\; H_i \;\; y \;\; w \;\; (\mathrm{N}[b_i] \cup D \cup E \cup (F - F_i) \cup (H - H_i)) \;\; u \;\; v \;\; c \;\; x,$$

   where $F_i = \{f_{3(i-1)+1}, f_{3(i-1)+2}, f_{3i}\}$ and $H_i = \{h_{(i-1)n+1}, \ldots, h_{(i-1)n+\|\mathrm{N}[b_i]\|}\}$. Note that $\|H_i\| = \|\mathrm{N}[b_i]\|$ and $\|F_i\| = 3$, so candidate w is always placed on the $(n+5)$th position.

2. *There is one voter of the form:*

$$x \;\; w \;\; c \;\; B \;\; u \;\; v \;\; (D \cup E \cup F \cup H) \;\; y.$$

3. *For each i, $1 \leq i \leq k-1$, there is one voter of the form:*

$$x \;\; D_i \;\; (B \cup (D - D_i) \cup E \cup F \cup H) \;\; u \;\; v \;\; y \;\; w \;\; c,$$

   where $D_i = \{d_{(i-1)(n+4)+1}, \ldots, d_{i(n+4)}\}$, so $\|D_i\| = n+4$.

4. *For each i, $1 \leq i \leq k+n$, there is one voter of the form:*

$$c \;\; E_i \;\; x \;\; y \;\; (B \cup D \cup (E - E_i) \cup F \cup H) \;\; u \;\; v \;\; w,$$

   where $E_i = \{e_{2i-1}, e_{2i}\}$, so $\|E_i\| = 2$.

Table 2 shows the scores of c, w, and x on the first three levels. None of the other candidates scores more than one point up to the third level. Note that c reaches a strict majority on this level and thus is the unique level 3 BV winner in this election.

Lemma 3.20 *In election (C,V) from Construction 3.19, for every partition of V into V_1 and V_2, candidate c is a unique BV winner of at least one of the subelections, (C,V_1) and (C,V_2).*

	c	w	x
$score^1$	$k+n$	0	k
$score^2$	$k+n$	1	k
$score^3$	$\underline{k+n+1}$	1	k

Table 2: Level i scores of c, w, and x in (C,V) for $i \in \{1,2,3\}$.

Proof. For a contradiction, we assume that in both subelections, (C,V_1) and (C,V_2), candidate c is not a unique BV winner. Table 2 shows that half of the voters in V place c already on the first level. So the following must hold:

- Both $\|V_i\|$ must be even numbers for $i \in \{1,2\}$, and
- $score^1_{(C,V_i)}(c) = \|V_i\|/2$ for $i \in \{1,2\}$.

Because of the voter in the second voter group, candidate c will get a strict majority on the third level in one of the subelections, let us say in (C,V_1). So there has to be a candidate beating or tieing with candidate c on the second or third level in (C,V_1). The candidates in B, D, E, F, and H and the candidates u, v, w, and y do not score more than one point up to the third level. Thus only candidate x can possibly beat or tie with c on the second or third level in (C,V_1). However, since x does not score more than k points in total until the fourth level, c is the unique level 3 BV winner in (C,V_1), a contradiction. It follows that c is a unique BV winner of at least one of the subelections. ❑

Theorem 3.21 *Bucklin voting is resistant to destructive control by partition of voters in model TE.*

Proof. Susceptibility follows from Lemma 3.3. To prove NP-hardness, we provide a reduction from the NP-complete problem DOMINATING SET. Given a DOMINATING SET instance $((B,A),k)$, construct a Bucklin election (C,V) according to Construction 3.19.

We claim that $G=(B,A)$ has a dominating set B' of size k if and only if candidate c can be prevented from being a unique BV winner by partition of voters in model TE.

From left to right: Let B' be a dominating set for G of size k. Partition V into V_1 and V_2 as follows. Let V_1 consist of the following $2k$ voters:

- The voters of the first voter group corresponding to the dominating set, i.e., the k voters of the form:
$$F_i \;\; (B-\mathrm{N}[b_i]) \;\; H_i \;\; y \;\; w \;\; (\mathrm{N}[b_i] \cup D \cup E \cup (F-F_i) \cup (H-H_i)) \;\; u \;\; v \;\; c \;\; x$$
for those i for which $b_i \in B'$,
- the one voter from the second group:
$$x \;\; w \;\; c \;\; B \;\; u \;\; v \;\; (D \cup E \cup F \cup H) \;\; y,$$
and
- the entire third voter group, i.e., the $k-1$ voters of the form:
$$x \;\; D_i \;\; (B \cup (D-D_i) \cup E \cup F \cup H \cup) \;\; u \;\; v \;\; y \;\; w \;\; c,$$
where $1 \leq i \leq k-1$.

Let $V_2 = V - V_1$. Note that $maj(V_1) = k+1$. Again, since the candidates in D, E, F, and H do not score more than one point up to level $n+5$, their level $n+5$ scores are not shown in Table 3. The level $n+5$ scores of the remaining candidates are shown in this table. Note that w reaches a strict majority of $k+1$ on this level (and no other candidate reaches a strict majority on this or an earlier level). Hence, w is the unique level $n+5$ BV winner in subelection (C, V_1) and thus participates in the final round.

	c	w	x	y	$b_i \in B$
$score^{n+5}$	1	$\underline{k+1}$	k	k	$\leq k$

Table 3: Level $n+5$ scores in (C, V_1).

From Lemma 3.20 it follows that candidate c is the unique winner in subelection (C, V_2). So the final-stage election is $(\{c, w\}, V)$ and we have the following scores on the first two levels:

$$\begin{aligned} score^1_{(\{c,w\},V)}(c) = score^1_{(\{c,w\},V)}(w) &= k+n, \\ score^2_{(\{c,w\},V)}(c) = score^2_{(\{c,w\},V)}(w) &= 2k+2n. \end{aligned}$$

Since none of c and w have a strict majority on the first level, both candidates are level 2 BV winners in this two-candidate final-stage election. Hence, c has been prevented from being a unique BV winner by partition of voters in model TE.

From right to left: Assume that c can be prevented from being a unique BV winner by partition of voters in model TE. From Lemma 3.20 we know that candidate c must participate in the final-stage election. Since we are in model TE, at most two candidates participate in the final run-off. To prevent c from being a unique BV winner of the final election, there must be another finalist and this other candidate has to beat or tie with c. Since w is the only candidate that can beat or tie with c in a two-candidate election, w has to move on to the final round to run against c. Let us say that c is the unique winner of subelection (C, V_1) and w is the unique winner of subelection (C, V_2). For w to be the unique winner of subeletion (C, V_2), V_2 has to contain voters from the first voter group and w can win only on the $(n+5)$th level.[8]

Let $I \subseteq \{1, \ldots, n\}$ be the set of indices i such that first-group voter

$$F_i \;\; (B - \mathrm{N}[b_i]) \;\; H_i \;\; y \;\; w \;\; (\mathrm{N}[b_i] \cup D \cup E \cup (F - F_i) \cup (H - H_i)) \;\; u \;\; v \;\; c \;\; x$$

belongs to V_2. Let $\ell = \|I\|$. Since w is the unique level $n+5$ BV winner of (C, V_2) but y is placed before w in every vote in the first group, the one voter from the second group (which is the only voter who prefers w to y) must belong to V_2. Thus we know that

$$score^{n+5}_{(C,V_2)}(w) = \ell + 1 \quad \text{and} \quad score^{n+4}_{(C,V_2)}(y) = score^{n+5}_{(C,V_2)}(y) = \ell.$$

For the candidates in B, we have

$$score^{n+4}_{(C,V_2)}(b_j) = score^{n+5}_{(C,V_2)}(b_j) = 1 + \|\{b_i \mid i \in I \text{ and } b_j \notin \mathrm{N}[b_i]\}\|,$$

8. In particular, x is placed before w in all voter groups except the first, so w can win in (C, V_2) only via voters from the first voter group participating in (C, V_2). Moreover, since w is placed in last or second-to-last position in all voters from the third and fourth groups, and since there is only one voter in the second group, w can win only on the $(n+5)$th level (which is w's position in the votes from the first voter group).

since each b_j scores one point up to the $(n+4)$th level from the voter in the second group and one point from the first group for every b_i with $i \in I$ such that $b_j \notin \mathrm{N}[b_i]$ in graph G. Again, since w is the unique level $n+5$ BV winner of (C,V_2), no $b_j \in B$ can score a point in *each* of the ℓ votes from the first voter group that belong to V_2. This implies that for each $b_j \in B$ there has to be at least one b_i with $i \in I$ that is adjacent to b_j in G. Thus, the set B' of candidates b_i with $i \in I$ corresponds to a dominating set in G.

Recall that $score^{n+5}_{(C,V_2)}(w) = \ell+1$ and $score^{n+4}_{(C,V_2)}(y) = \ell$. Note also that $score^{n+4}_{(C,V_2)}(b_j) \leq \ell$ for $1 \leq j \leq n$. Since w needs a strict majority to be a BV winner in (C,V_2), it must hold that $maj(V_2) \leq \ell+1$. Since y and the $b_j \in B$ have a score of ℓ already one level earlier than w, it must hold that $maj(V_2) = \ell+1$, which implies $\|V_2\| = 2\ell$ or $\|V_2\| = 2\ell+1$. To ensure this cardinality of V_2, other votes have to be added. Since y must not gain additional points from these votes up to the $(n+5)$th level, they cannot come from the fourth voter group. The remaining votes from the third voter group total up to $k-1$. Thus, since w is the unique BV winner in subelection (C,V_2), it must hold that $\ell \leq k$. So $\|B'\| = \ell \leq k$ and this means that there exists a dominating set of size at most k. ❑

The resistance for fallback voting to this control type now follows immediately.

Corollary 3.22 *Fallback voting is resistant to destructive control by partition of voters in model TE.*

The following construction will be used to handle the case of destructive control by partition of voters in model TP (see Theorem 3.25 below). Construction 3.23 starts from an instance of the NP-complete problem RESTRICTED HITTING SET defined in Section 3.3.

Construction 3.23 *Let $(B,\mathscr{S},k)$ be a given instance of* RESTRICTED HITTING SET, *where $B = \{b_1,b_2,\dots,b_m\}$ is a set, $\mathscr{S} = \{S_1,S_2,\dots,S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and k is an integer with $1 < k < m$.*

Define the election (C,V), where

$$C = B \cup D \cup E \cup \{c,w\}$$

is the candidate set with $D = \{d_1,\dots,d_{2(m+1)}\}$ and $E = \{e_1,\dots,e_{2(m-1)}\}$ and where V consists of the following $2n(k+1)+4m+2mk$ voters:

#	*For each ...*	*number of voters*	*ranking of approved candidates*
1	$i \in \{1,\dots,n\}$	$k+1$	$w\ S_i\ c$
2	$j \in \{1,\dots,m\}$	1	$c\ b_j\ w$
3	$j \in \{1,\dots,m\}$	$k-1$	b_j
4	$p \in \{1,\dots,m+1\}$	1	$d_{2(p-1)+1}\ d_{2p}\ w$
5	$r \in \{1,\dots,2(m-1)\}$	1	e_r
6		$n(k+1)+m-k+1$	c
7		$mk+k-1$ *voters*	$c\ w$
8		1	$w\ c$

	c	w	$b_j \in B$	$d_p \in D$	$e_r \in E$
score[1]	$n(k+1)+2m+mk$	$n(k+1)+1$	$k-1$	≤ 1	1
score[2]	$n(k+1)+2m+mk+1$	$n(k+1)+mk+k$	$\leq k+n(k+1)$	1	1
score^{m+2}	$2n(k+1)+2m+mk+1$	$n(k+1)+2m+mk+k+1$	$\leq k+n(k+1)$	1	1

Table 4: Level i scores for $i \in \{1,2,m+2\}$ in the election (C,V) from Construction 3.23.

The strict majority threshold for V is $maj(V) = n(k+1)+2m+mk+1$. In election (C,V), only the two candidates c and w reach a strict majority, w on the third level and c on the second level (see Table 4). Thus c is the unique level 2 FV winner of election (C,V).

Lemma 3.24 will be used in the proof of Theorem 3.25.

Lemma 3.24 *In election (C,V) from Construction 3.23, for every partition of V into V_1 and V_2, candidate c is an FV winner of either (C,V_1) or (C,V_2).*

Proof. For a contradiction, suppose that in both subelections, (C,V_1) and (C,V_2), candidate c is not an FV winner. Since $score^1_{(C,V)}(c) = \|V\|/2$, the two subelections must satisfy that both $\|V_1\|$ and $\|V_2\|$ are even numbers, and that $score^1_{(C,V_1)}(c) = \|V_1\|/2$ and $score^1_{(C,V_2)}(c) = \|V_2\|/2$. Otherwise, c would have a strict majority already on the first level in one of the subelections and would win that subelection. For each $i \in \{1,2\}$, c already on the first level has only one point less than the strict majority threshold $maj(V_i)$ in subelection (C,V_i), and c will get a strict majority in (C,V_i) no later than on the $(m+2)$nd level. Thus, for both $i=1$ and $i=2$, there must be candidates whose level $m+2$ scores in (C,V_i) are higher than the level $m+2$ score of c in (C,V_i). Table 4 shows the level $m+2$ scores of all candidates in (C,V). Only w and some $b_j \in B$ have a chance to beat c on that level in (C,V_i), $i \in \{1,2\}$.

Suppose that c is defeated in both subelections by two distinct candidates from B (say, b_x defeats c in (C,V_1) and b_y defeats c in (C,V_2)). Thus the following must hold:[9]

$$\begin{aligned}
score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(b_y) &\geq score^{m+2}_{(C,V)}(c) + 2 \\
2n(k+1) + 2k - n(k+1) &\geq 2n(k+1) + mk + 2m + 3 \\
2k &\geq n(k+1) + mk + 2m + 3,
\end{aligned}$$

which by our basic assumption $m > k > 1$ implies the following contradiction:

$$0 \geq n(k+1) + (m-2)k + 2m + 3 > n(k+1) + (k-2)k + 2k + 3 = n(k+1) + k^2 + 3 > 0.$$

Thus the only possibility for c to not win any of the two subelections is that c is defeated in one subelection, say (C,V_1), by a candidate from B, say b_x, and in the other subelection, (C,V_2), by candidate w. Then it must hold that:[9]

$$\begin{aligned}
score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(w) &\geq score^{m+2}_{(C,V)}(c) + 2 \\
2n(k+1) + 2k + 2m + mk + 1 - n(k+1) - 1 &\geq 2n(k+1) + mk + 2m + 3 \\
2k &\geq n(k+1) + 3.
\end{aligned}$$

9. For the left-hand sides of the inequalities, note that each vote occurs in only one of the two subelections. To avoid double-counting those votes that give points to both candidates, we first sum up the overall number of points each candidate scores and then substract the double-counted points.

Since $n > 1$, this cannot hold, so c must be an FV winner in one of the two subelections. ❑

Theorem 3.25 *Fallback voting is resistant to destructive control by partition of voters in model TP.*

Proof. Susceptibility holds by Corollary 3.4. To prove NP-hardness, we reduce RESTRICTED HITTING SET to our control problem. Consider the election (C,V) constructed according to Construction 3.23 from a given RESTRICTED HITTING SET instance $(B,\mathscr{S},k)$, where $B = \{b_1,\ldots,b_m\}$ is a set, $\mathscr{S} = \{S_1,\ldots,S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$, and k is an integer with $1 < k < m < n$.

We claim that $\mathscr{S}$ has a hitting set $B' \subseteq B$ of size k if and only if c can be prevented from being a unique FV winner by partition of voters in model TP.

From left to right: Suppose, $B' \subseteq B$ is a hitting set of size k for $\mathscr{S}$. Partition V into V_1 and V_2 the following way. Let V_1 consist of those voters of the second group where $b_j \in B'$ and of those voters of the third group where $b_j \in B'$. Let $V_2 = V - V_1$. In (C,V_1), no candidate reaches a strict majority (see Table 5), where $maj(V_1) = \lfloor k^2/2 \rfloor + 1$, and candidates c, w, and each $b_j \in B'$ win the election with an approval score of k.

	c	w	$b_j \in B'$	$b_j \notin B'$
score^1	k	0	$k-1$	0
score^2	k	0	k	0
score^3	k	k	k	0

Table 5: Level i scores in (C,V_1) for $i \in \{1,2,3\}$ and all candidates in $B \cup \{c,w\}$.

	c	w	$b_j \notin B'$	$b_j \in B'$
score^1	$n(k+1)+2m-k+mk$	$n(k+1)+1$	$k-1$	0
score^2	$n(k+1)+2m-k+mk+1$	$n(k+1)+mk+k$	$\leq k+n(k+1)$	$\leq n(k+1)$
score^3	$\geq n(k+1)+2m-k+mk+1$	$n(k+1)+mk+2m+1$	$\leq k+n(k+1)$	$\leq n(k+1)$

Table 6: Level i scores in (C,V_2) for $i \in \{1,2,3\}$ and all candidates in $B \cup \{c,w\}$.

The level i scores in election (C,V_2) for $i \in \{1,2,3\}$ and all candidates in $B \cup \{c,w\}$ are shown in Table 6. Since in (C,V_2) no candidate from B wins, the candidates participating in the final round are $B' \cup \{c,w\}$. The scores in the final election $(B' \cup \{c,w\},V)$ can be seen in Table 7. Since candidates c and w with the same level 2 scores are both level 2 FV winners, candidate c has been prevented from being a unique FV winner by partition of voters in model TP.

	c	w	$b_j \in B'$
score^1	$n(k+1)+2m+mk$	$n(k+1)+m+2$	$k-1$
score^2	$n(k+1)+2m+mk+1$	$n(k+1)+2m+mk+1$	$\leq k+n(k+1)$

Table 7: Level i scores in the final-stage election $(B' \cup \{c,w\},V)$ for $i \in \{1,2\}$.

From right to left: Suppose candidate c can be prevented from being a unique FV winner by partition of voters in model TP. From Lemma 3.24 it follows that candidate c participates in the final round. Since c has a strict majority of approvals, c has to be tied with or lose against another candidate by a strict majority at some level. Only candidate w has a strict majority of approvals, so w has to tie or beat c at some level in the final round. Because of the low scores of the candidates in D and E we may assume that only candidates from B are participating in the final round besides c and w. Let $B' \subseteq B$ be the set of candidates who also participate in the final round. Let ℓ be the number of sets in $\mathscr{S}$ not hit by B'. As w cannot reach a strict majority of approvals on the first level, we consider the level 2 scores of c and w:

$$\begin{aligned} score^2_{(B' \cup \{c,w\},V)}(c) &= n(k+1)+2m+mk+1+\ell(k+1), \\ score^2_{(B' \cup \{c,w\},V)}(w) &= n(k+1)+2m+mk+k-\|B'\|+1. \end{aligned}$$

Since c has a strict majority already on the second level, w must tie or beat c on this level, so the following must hold:

$$\begin{aligned} score^2_{(B' \cup \{c,w\},V)}(c) - score^2_{(B' \cup \{c,w\},V)}(w) &\leq 0 \\ n(k+1)+2m+mk+1+\ell(k+1)-n(k+1)-2m-mk-k+\|B'\|-1 &\leq 0 \\ \|B'\|-k+\ell(k+1) &\leq 0. \end{aligned}$$

This is possible only if $\ell = 0$ (i.e., all sets in $\mathscr{S}$ are hit by B'), which implies $\|B'\| \leq k$. Thus $\mathscr{S}$ has a hitting set of size at most k. ❑

3.4.2 VULNERABILITY PROOFS

In contrast to the cases of constructive voter control stated in Corollary 3.17, fallback voting is vulnerable to destructive control by adding voters and to destructive control by deleting voters. In fact, the proof of Theorem 3.26 shows something slightly stronger: FV is what Hemaspaandra et al. (2007) call "certifiably vulnerable" to these two destructive voter-control cases, i.e., the algorithm we present in this proof for destructive control by adding voters even computes a successful control action if one exists (instead of only solving the corresponding decision problem).[10]

Theorem 3.26 *Fallback voting is vulnerable to destructive control by adding voters and by deleting voters.*

Proof. Susceptibility holds by Lemma 3.3 in both cases. We present a polynomial-time algorithm for solving the destructive control by adding voters case. We will make use of the following notation. Given an election (C,V), recall that $maj(V) = \lfloor \|V\|/2 \rfloor + 1$ denotes the strict majority threshold for V, and define the deficit of candidate $d \in C$ for reaching a strict majority in (C,V) on level i, $1 \leq i \leq \|C\|$, by

$$def^i_{(C,V)}(d) = maj(V) - score^i_{(C,V)}(d).$$

The input to our algorithm is an election $(C, V \cup V')$ (where C is the set of candidates, V is the collection of registered voters, and V' is the collection of unregistered voters), a distinguished candidate $c \in C$, and an integer ℓ (the number of voters allowed to be added). The algorithm either

10. And the same holds for the algorithm showing that FV is vulnerable to destructive control by deleting voters, which is not presented here, as it works in an analogous fashion.

outputs a sublist $V'' \subseteq V'$, $\|V''\| \leq \ell$, that describes a successful control action (if any exists), or indicates that control is impossible for this input.

We give a high-level description of the algorithm. We assume that c is initially the unique FV winner of election (C,V); otherwise, the algorithm simply outputs $V'' = \emptyset$ and halts, since there is no need to add any voters from V'.

Let n be the largest number of candidates any voter in $V \cup V'$ approves of. Clearly, $n \leq \|C\|$. The algorithm proceeds in at most $n+1$ stages, where the last stage is the *approval stage* and checks whether c can be dethroned as a unique FV winner by approval score via adding at most ℓ voters from V', and all preceding stages are *majority stages* that check whether a candidate $d \neq c$ can tie or beat c on level i via adding at most ℓ voters from V'. Since the first majority stage is slightly different from the subsequent majority stages, we describe both cases separately.

Majority Stage 1: For each candidate $d \in C - \{c\}$, check whether d can tie or beat c on the first level via adding at most ℓ voters from V'. To this end, first check whether

$$def^1_{(C,V)}(d) \leq \frac{\ell}{2}; \tag{1}$$

$$score^1_{(C,V)}(d) \geq score^1_{(C,V)}(c) - \ell. \tag{2}$$

If (1) or (2) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (1) and (2) hold, find a list $V'_d \subseteq V'$ of largest cardinality such that $\|V'_d\| \leq \ell$ and all voters in V'_d approve of d on the first level but disapprove of c on the first level. Check whether

$$score^1_{(C,V \cup V'_d)}(d) \geq score^1_{(C,V \cup V'_d)}(c). \tag{3}$$

If (3) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (3) holds, check whether d has a strict majority in $(C, V \cup V'_d)$ on the first level, and if so, output $V'' = V'_d$ and halt. Otherwise, this d is hopeless, so go to the next candidate (or stage).

Majority Stage i, $1 < i \leq n$: This stage is entered only if it was not possible to find a successful control action in majority stages $1, \ldots, i-1$. For each candidate $d \in C - \{c\}$, check whether d can tie or beat c up to the ith level via adding at most ℓ voters from V'. To this end, first check whether

$$def^i_{(C,V)}(d) \leq \frac{\ell}{2}; \tag{4}$$

$$score^i_{(C,V)}(d) \geq score^i_{(C,V)}(c) - \ell. \tag{5}$$

If (4) or (5) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (4) and (5) hold, find a list $V'_d \subseteq V'$ of largest cardinality such that $\|V'_d\| \leq \ell$ and all voters in V'_d approve of d up to the ith level but disapprove of c up to the ith level. Check whether

$$score^i_{(C,V \cup V'_d)}(d) \geq score^i_{(C,V \cup V'_d)}(c). \tag{6}$$

If (6) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (6) holds, check whether d has a strict majority in $(C, V \cup V'_d)$ on the ith level, and if so, check whether

$$score^{i-1}_{(C,V \cup V'_d)}(c) \geq maj(V \cup V'_d). \tag{7}$$

If (7) fails to hold, output $V'' = V'_d$ and halt. Otherwise (i.e., if (7) holds), though d might dethrone c by adding V'_d on the ith level, it was not quick enough: c has already won earlier. In that case, find a largest list $V'_{cd} \subseteq V'$ such that

1. $\|V'_d \cup V'_{cd}\| \leq \ell$,
2. all voters in V'_{cd} approve of both c and d up to the ith level, and
3. the voters in V'_{cd} are chosen such that c is approved of as late as possible by them (i.e., at levels with a largest possible number, where ties may be broken arbitrarily).

Now, check whether

$$score^{i-1}_{(C,V\cup V'_d\cup V'_{cd})}(c) \quad \geq \quad maj(V \cup V'_d \cup V'_{cd}). \tag{8}$$

If (8) holds, then d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). Otherwise (i.e., if (8) fails to hold), check whether $\|V'_{cd}\| \geq def^i_{(C,V\cup V'_d)}(d)$. If so (note that d has now a strict majority on level i), output $V'' = V'_d \cup V'_{cd}$ and halt. Note that, by choice of V'_{cd}, (6) implies that

$$score^i_{(C,V\cup V'_d\cup V'_{cd})}(d) \geq score^i_{(C,V\cup V'_d\cup V'_{cd})}(c).$$

Thus, in $(C, V \cup V'_d \cup V'_{cd})$, d ties or beats c and has a strict majority on the ith level (and now, we are sure that d was not too late). Otherwise (i.e., if $\|V'_{cd}\| < def^i_{(C,V\cup V'_d)}(d)$), this d is hopeless, so go to the next candidate (or stage).

Approval Stage: This stage is entered only if it was not possible to find a successful control action in majority stages $1, 2, \ldots, n$.

First, check if

$$score_{(C,V)}(c) < \left\lfloor \frac{\|V\| + \ell}{2} \right\rfloor + 1. \tag{9}$$

If (9) fails to hold, output "control impossible" and halt, since we have found no candidate in the majority stages who could tie or beat c and would have a strict majority when adding at most ℓ voters from V', so adding any choice of at most ℓ voters from V' would c still leave a strict majority. If (9) holds, looping over all candidates $d \in C - \{c\}$, check whether there are $score_{(C,V)}(c) - score_{(C,V)}(d) \leq \ell$ voters in V' who approve of d and disapprove of c. If this is not the case, move on to the next candidate, since d could never catch up on c via adding at most ℓ voters from V'. If it is the case for some $d \in C - \{c\}$, however, add this list of voters (call it V'_d) and check whether

$$score_{(C,V\cup V'_d)}(c) \quad < \quad maj(V \cup V'_d). \tag{10}$$

If (10) holds, output $V'' = V'_d$ and halt. Otherwise (i.e., if (10) fails to hold), check whether

$$\begin{aligned} \ell - \|V'_d\| \quad &\geq \quad \|V'_\emptyset\| \\ &\geq \quad 2\left(score_{(C,V\cup V'_d)}(c) - \frac{\|V \cup V'_d\|}{2}\right), \end{aligned} \tag{11}$$

where $V'_\emptyset$ consists of those voters in V' who disapprove of both candidates, c and d. If (11) does not hold, move on to the next candidate, since after adding these voters c would still have a strict majority. Otherwise (i.e., if (11) holds), add exactly $2\left(score_{(C,V\cup V'_d)}(c) - \|V\cup V'_d\|/2\right)$ voters from $V'_\emptyset$ (denoted by $V'_{\emptyset,+}$). Output $V'' = V'_d \cup V'_{\emptyset,+}$ and halt.

If we have entered the approval stage (because we were not successful in any of the majority stages), but couldn't find any candidate here who was able to dethrone c by adding at most ℓ voters from V', we output "control impossible" and halt.

The correctness of the algorithm follows from the remarks made above. Crucially, note that the algorithm proceeds in the "safest way possible": If there is any successful control action then our algorithm finds some successful control action. It is also easy to see that this algorithm runs in polynomial time. (Note that we didn't optimize it in terms of running time; rather, we described it in a way to make it easier to check its correctness.)

The deleting-voters case follows by a similar algorithm. ❑

Since Bucklin voting is a special case of fallback voting, Bucklin voting inherits vulnerability from fallback voting in these control scenarios.

Corollary 3.27 *Bucklin voting is vulnerable to destructive control by adding voters and by deleting voters.*

4. Conclusions and Open Questions

We have shown that, among natural election systems with a polynomial-time winner problem, fallback voting displays the broadest control resistance currently known to hold. We have also shown that Bucklin voting shows an almost as good (possibly even as good) behavior as fallback voting in terms of control resistance. In particular, both voting systems are resistant to all standard types of candidate control and to all standard types of constructive control. In total, fallback voting has twenty resistances and two vulnerabilities and Bucklin voting possesses at least 19 (possibly even twenty) resistances and has at least two (and can have no more than three) vulnerabilities. One case remains open: destructive control by partition of voters in the tie-handling model "ties promote" for Bucklin voting. For comparison, recall from Table 1 that, for destructive control by partition of voters, approval voting is vulnerable both in model TE and TP and SP-AV is vulnerable in model TE but resistant in model TP.

The ultimate goal of finding a natural voting system (whose winners can be determined in polynomial time) that is resistant to all 22 common types of electoral control remains open. As another interesting task we propose to study other natural control scenarios than those common control types. It would also be interesting and challenging to complement our worst-case hardness results by theoretical and empirical typical-case studies of these problems.

References

Bartholdi, III, J., & Orlin, J. (1991). Single transferable vote resists strategic voting. *Social Choice and Welfare*, *8*(4), 341–354.

Bartholdi, III, J., Tovey, C., & Trick, M. (1989). The computational difficulty of manipulating an election. *Social Choice and Welfare*, *6*(3), 227–241.

Bartholdi, III, J., Tovey, C., & Trick, M. (1992). How hard is it to control an election? *Mathematical Comput. Modelling*, *16*(8/9), 27–40.

Baumeister, D., Erdélyi, G., Hemaspaandra, E., Hemaspaandra, L., & Rothe, J. (2010). Computational aspects of approval voting. In Laslier, J., & Sanver, R. (Eds.), *Handbook on Approval Voting*, chap. 10, pp. 199–251. Springer.

Brams, S., & Sanver, R. (2006). Critical strategies under approval voting: Who gets ruled in and ruled out. *Electoral Studies*, *25*(2), 287–305.

Brams, S., & Sanver, R. (2009). Voting systems that combine approval and preference. In Brams, S., Gehrlein, W., & Roberts, F. (Eds.), *The Mathematics of Preference, Choice, and Order: Essays in Honor of Peter C. Fishburn*, pp. 215–237. Springer.

Conitzer, V., Sandholm, T., & Lang, J. (2007). When are elections with few candidates hard to manipulate? *Journal of the ACM*, *54*(3), Article 14.

Dwork, C., Kumar, R., Naor, M., & Sivakumar, D. (2001). Rank aggregation methods for the web. In *Proc. WWW'01*, pp. 613–622. ACM Press.

Ephrati, E., & Rosenschein, J. (1997). A heuristic technique for multi-agent planning. *Annals of Mathematics and Artificial Intelligence*, *20*(1–4), 13–67.

Erdélyi, G., Nowak, M., & Rothe, J. (2009). Sincere-strategy preference-based approval voting fully resists constructive control and broadly resists destructive control. *Mathematical Logic Quarterly*, *55*(4), 425–443.

Erdélyi, G., Piras, L., & Rothe, J. (2010a). Bucklin voting is broadly resistant to control. Tech. rep. arXiv: 1005.4115v1 [cs.GT], ACM Computing Research Repository (CoRR).

Erdélyi, G., Piras, L., & Rothe, J. (2010b). Control complexity in fallback voting. Tech. rep. arXiv: 1004.3398v1 [cs.GT], ACM Computing Research Repository (CoRR). A precursor appeared as (Erdélyi & Rothe, 2010).

Erdélyi, G., & Rothe, J. (2010). Control complexity in fallback voting. In *Proceedings of Computing: the 16th Australasian Theory Symposium*, pp. 39–48. Australian Computer Society *Conferences in Research and Practice in Information Technology Series*, vol. 32, no. 8.

Faliszewski, P., Hemaspaandra, E., & Hemaspaandra, L. (2009a). How hard is bribery in elections? *Journal of Artificial Intelligence Research*, *35*, 485–532.

Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L., & Rothe, J. (2009b). Llull and Copeland voting computationally resist bribery and constructive control. *Journal of Artificial Intelligence Research*, *35*, 275–341.

Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L., & Rothe, J. (2009c). A richer understanding of the complexity of election systems. In Ravi, S., & Shukla, S. (Eds.), *Fundamental Problems in Computing: Essays in Honor of Professor Daniel J. Rosenkrantz*, chap. 14, pp. 375–406. Springer.

Garey, M., & Johnson, D. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company.

Ghosh, S., Mundhe, M., Hernandez, K., & Sen, S. (1999). Voting for movies: The anatomy of recommender systems. In *Proceedings of the 3rd Annual Conference on Autonomous Agents*, pp. 434–435. ACM Press.

Hemaspaandra, E., Hemaspaandra, L., & Rothe, J. (2007). Anyone but him: The complexity of precluding an alternative. *Artificial Intelligence*, *171*(5–6), 255–285.

Hemaspaandra, E., Hemaspaandra, L., & Rothe, J. (2009). Hybrid elections broaden complexity-theoretic resistance to control. *Mathematical Logic Quarterly*, *55*(4), 397–424.

Appendix A. Some Results of Hemaspaandra et al. (2007) Used in Section 3.2

Theorem A.1 ((Hemaspaandra et al., 2007))

1. *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
2. *Each voiced voting system is susceptible to constructive control by deleting candidates.*
3. *Each voiced voting system is susceptible to destructive control by adding candidates.*[11]

Theorem A.2 ((Hemaspaandra et al., 2007))

1. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
2. *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*
3. *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*
4. *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Theorem A.3 ((Hemaspaandra et al., 2007))

1. *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
2. *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
3. *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*
4. *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

11. Following Bartholdi et al. (1992), Hemaspaandra et al. (2007) considered only the case of control by adding a limited number of candidates—the “unlimited” case was introduced only in (the conference precursors of) (Faliszewski et al., 2009b). However, it is easy to see that all results about control by adding candidates stated in Theorems A.1, A.2, and A.3 hold true in both the limited and the unlimited case.