Mathematisches Institut
der Heinrich-Heine-Universität
Düsseldorf

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Introduction to Algebraic Number Theory **WS 2010/11**

Homework 4

1. (a) Show that $\mathbb{Z}[\sqrt{2}\,]$ and $\mathbb{Z}[\sqrt{3}\,]$ are Euclidean domains.

   (b) Show that $\mathbb{Z}\left[\dfrac{1+\sqrt{-7}}{2}\right]$ is a Euclidean domain, but $\mathbb{Z}[1+\sqrt{-7}\,] = \mathbb{Z}[\sqrt{-7}\,]$ is not.

2. Let R be a Euclidean domain with a Euclidean function f. Show that R can be endowed with a Euclidean function $\overline{f}$ that satisfies the following additional property:

$$\overline{f}(a) \leq \overline{f}(ab) \quad \text{for all} \quad a, b \in R \smallsetminus \{0\} \tag{*}$$

   Hint. You can define $\overline{f}$ as follows. For $a \in R \smallsetminus \{0\}$, let

$$\overline{f}(a) = \min_{x \in R \smallsetminus \{0\}} f(ax).$$

   Show that $\overline{f}$ is a Euclidean function that satisfies property (*).

3. Use the Euclidean algorithm to compute the following greatest common divisors in $\mathbb{Z}$:

   (a) $\gcd(408, 552)$

   (b) $\gcd(333, 707)$

   (c) $\gcd(12345, 67890)$

   (d) $\gcd(20785, 44350)$

   In (d), write also the $\gcd(20785, 44350)$ as a linear combination of 20785 and 44350.

4. (a) For the polynomials

$$f = X^4 + X^3 - 5X^2 + 18X - 15 \quad \text{and} \quad g = X^4 + 16X^2 - 17$$

   find the $\gcd(f, g)$ in $\mathbb{Q}[X\,]$ and write it in the form

$$h_1 f + h_2 g$$

   for some $h_1, h_2 \in \mathbb{Q}[X\,]$.

   (b) Find $\gcd\,(2^{250} - 1, 2^{100} - 1)$.

Due at 2:00 pm on Thursday, November 11, 2010