Manipulation: Strategic Voting

Example

Consider the Borda election with candidates a, b, and c and the following votes:

	Sincere Votes				Strategic Votes		
points :	2	1	0		2	1	0
5 votes :	a	b	c		a	b	c
5 votes :	b	a	c	$\Rightarrow$	b	c	a
1 vote :	c	a	b		c	a	b
	Borda winner a				Borda winner b		

Variants of the Manipulation Problem

Definition (Constructive Coalitional Manipulation)

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE COALITIONAL MANIPULATION ($\mathcal{E}$-CCM).

Given:
- A set C of candidates,
- a list V of nonmanipulative voters over C,
- a list S of manipulative voters (whose votes over C are still unspecified) with $V \cap S = \emptyset$, and
- a distinguished candidate $c \in C$.

Question: Is there a way to set the preferences of the voters in S such that, under election system $\mathcal{E}$, c is a winner of election $(C, V \cup S)$?

Variants of the Manipulation Problem

Remark: Variants:

- $\mathcal{E}$-DESTRUCTIVE COALITIONAL MANIPULATION ($\mathcal{E}$-DCM) is the same with "c is not a winner of $(C, V \cup S)$."
- If $\|S\| = 1$, we obtain the single-manipulator problems:
  - $\mathcal{E}$-CONSTRUCTIVE MANIPULATION ($\mathcal{E}$-CM) and
  - $\mathcal{E}$-DESTRUCTIVE MANIPULATION ($\mathcal{E}$-DM).
- Voters can also be weighted (see next slide).
- These problems can also be defined in the "unique-winner" model.

Variants of the Manipulation Problem

Definition (Constructive Coalitional Weighted Manipulation)

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE (DESTRUCTIVE) COALITIONAL WEIGHTED MANIPULATION ($\mathcal{E}$-CCWM / $\mathcal{E}$-DCWM).

Given:
- A set C of candidates,
- a list V of nonmanipulative voters over C each having a nonnegative integer weight,
- a list of the weights of the manipulators in S (whose votes over C are still unspecified) with $V \cap S = \emptyset$, and
- a distinguished candidate $c \in C$.

Question: Can the preferences of the voters in S be set such that c is a $\mathcal{E}$-winner (is not an $\mathcal{E}$-winner) of $(C, V \cup S)$?

Some Basic Complexity Classes

Definition

1. FP denotes the *class of polynomial-time computable total functions* mapping from Σ^* to Σ^*.
2. P denotes the *class of problems that can be decided in polynomial time* (i.e., via a deterministic polynomial-time Turing machine).
3. NP denotes the *class of problems that can be accepted in polynomial time* (i.e., via a nondeterministic polynomial-time Turing machine).

Some Basic Complexity Classes

Remark:

- Intuitively, FP and P, respectively, capture feasibility/efficiency of computing functions and solving decision problems.
- $A \in \mathrm{NP}$ if and only if there exist a set $B \in \mathrm{P}$ and a polynomial p such that for each $x \in \Sigma^*$,

$$x \in A \quad \Longleftrightarrow \quad (\exists w)\,[|w| \leq p(|x|) \text{ and } (x, w) \in B].$$

  That is, NP is the class of problems whose YES instances can be easily checked.
- Central open question of TCS: $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$
- Examples of problems in NP: SAT, TRAVELING SALESPERSON PROBLEM, VERTEX COVER, CLIQUE, HAMILTON CIRCUIT, ...

NP in Ancient Times

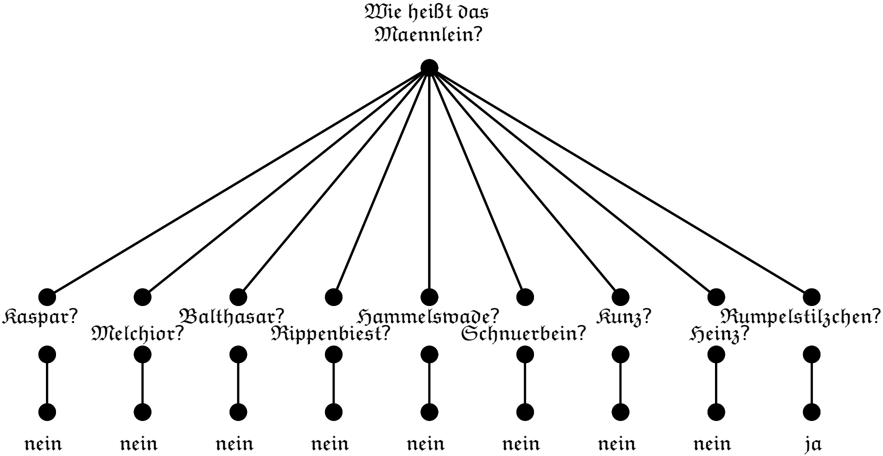

Figure: Nondeterministic Guessing and Deterministic Checking

Pol-Time Many-One Reducibility and Completeness

Definition

Let Σ be an alphabet and $A, B \subseteq \Sigma^*$. Let $\mathcal{C}$ be any complexity class.

1. Define the *polynomial-time many-one reducibility*, denoted by $\leq_m^p$, as follows: $A \leq_m^p B$ if there is a function $f \in \mathrm{FP}$ such that
$$(\forall x \in \Sigma^*)\,[x \in A \iff f(x) \in B].$$
2. A set B is *$\leq_m^p$-hard for $\mathcal{C}$* (or *$\mathcal{C}$-hard*) if $A \leq_m^p B$ for each $A \in \mathcal{C}$.
3. A set B is *$\leq_m^p$-complete for $\mathcal{C}$* (or *$\mathcal{C}$-complete*) if
   1. B is $\leq_m^p$-hard for $\mathcal{C}$ (lower bound) and
   2. $B \in \mathcal{C}$ (upper bound).
4. $\mathcal{C}$ is *closed under the $\leq_m^p$-reducibility* (*$\leq_m^p$-closed*, for short) if
$$(A \leq_m^p B \text{ and } B \in \mathcal{C}) \implies A \in \mathcal{C}.$$

Properties of $\leq_{\mathrm{m}}^{\mathrm{p}}$

1. $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ implies $\overline{A} \leq_{\mathrm{m}}^{\mathrm{p}} \overline{B}$, yet in general it is not true that $A \leq_{\mathrm{m}}^{\mathrm{p}} \overline{A}$.
2. $\leq_{\mathrm{m}}^{\mathrm{p}}$ is a reflexive and transitive, yet not antisymmetric relation.
3. P and NP are $\leq_{\mathrm{m}}^{\mathrm{p}}$-closed.
   That is, upper bounds are inherited downward with respect to $\leq_{\mathrm{m}}^{\mathrm{p}}$.
4. If $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ and A is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for some complexity class $\mathcal{C}$, then B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathcal{C}$.
   That is, lower bounds are inherited upward with respect to $\leq_{\mathrm{m}}^{\mathrm{p}}$.
5. Let $\mathcal{C}$ and $\mathcal{D}$ be any complexity classes. If $\mathcal{C}$ is $\leq_{\mathrm{m}}^{\mathrm{p}}$-closed and B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-complete for $\mathcal{D}$, then $\mathcal{D} \subseteq \mathcal{C} \iff B \in \mathcal{C}$.
   In particular, if B is NP-complete, then
   $$\mathrm{P} = \mathrm{NP} \iff B \in \mathrm{P}.$$

Plurality and Regular Cup are Easy to Manipulate

Theorem (Conitzer, Sandholm, and Lang (2007))

Plurality-CCWM and Regular-Cup-CCWM are in P *(for any number of candidates, in both the unique-winner and nonunique-winner model).*

Proof:

1. For plurality, the manipulators simply check if c wins when they all rank c first.
   - If so, they have found a successful strategy.
   - If not, no strategy can make c win.
2. For the regular cup protocol (given the assignment of candidates to the leaves of the binary balanced tree), see blackboard. ❑

Copeland with three Candidates is Easy to Manipulate

Copeland voting: For each $c, d \in C$, $c \neq d$,

- let $N(c,d)$ be the number of voters who prefer c to d,
- let $C(c,d) = 1$ if $N(c,d) > N(d,c)$ and
- $C(c,d) = 1/2$ if $N(c,d) = N(d,c)$.
- The *Copeland score of c* is $CScore(c) = \sum_{d \neq c} C(c,d)$.
- Whoever has the maximum Copeland score wins.

Theorem (Conitzer, Sandholm, and Lang (2007))

Copeland-CCWM for three candidates is in P
(in both the unique-winner and nonunique-winner model).

Proof: We show that: If Copeland with three candidates has a CCWM, then it has a CCWM where all manipulators vote identically.

And now... see blackboard.

Maximin with three Candidates is Easy to Manipulate

Maximin (a.k.a. Simpson) voting: For each $c, d \in C$, $c \neq d$, let again $N(c, d)$ be the number of voters who prefer c to d.

- The *maximin score of c* is

$$MScore(c) = \min_{d \neq c} N(c, d).$$

- Whoever has the maximum *MScore* wins.

Theorem (Conitzer, Sandholm, and Lang (2007))

Maximin-CCWM for three candidates is in P *(in both the unique-winner and nonunique-winner model).*

Proof: We show that: If Maximin with three candidates has a CCWM, then it has a CCWM where all manipulators vote identically.

And now. . . see blackboard.

Upper bounds are inherited downward w.r.t. $\leq_{\mathrm{m}}^{\mathrm{p}}$

Corollary

All more restrictive variants of the manipulation problem are in P for:

- *plurality (for any number of candidates),*
- *regular cup (for any number of candidates),*
- *Copeland (for at most three candidates), and*
- *maximin (for at most three candidates).*

STV-CM is NP-complete

Single Transferable Vote (STV) for m candidates proceeds in $m-1$ rounds. In each round:

- A candidate with lowest plurality score is eliminated (using some tie-breaking rule if needed) and
- all votes for this candidate transfer to the next remaining candidate in this vote's order.

The last remaining candidate wins.

Theorem (Bartholdi and Orlin (1991))

STV-CONSTRUCTIVE MANIPULATION *is* NP*-complete.*

STV-CM is NP-complete: Reduction from X3C

Proof: Membership in NP is clear.

To prove NP-hardness of STV-CONSTRUCTIVE MANIPULATION, we reduce from the following NP-complete problem:

Name: EXACT COVER BY THREE-SETS (X3C).

Given:
- A set $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m \geq 1$, and
- a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

Question: Is there a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that each element of B occurs in exactly one set in $\mathcal{S}'$?

In other words, does there exist an index set $I \subseteq \{1, 2, \ldots, n\}$ with $\|I\| = m$ such that $\bigcup_{i \in I} S_i = B$?

STV-CM is NP-complete: The Candidates

Given an instance $(B, \mathcal{S})$ of X3C with

$$\begin{aligned} B &= \{b_1, b_2, \ldots, b_{3m}\} \\ \mathcal{S} &= \{S_1, S_2, \ldots, S_n\} \end{aligned}$$

where $m \geq 1$, $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$, construct an election $(C, V \cup \{s\})$ with manipulator s and $5n + 3(m+1)$ candidates:

1. "possible winners": c and w;
2. "first losers": $a_1, a_2, \ldots, a_n$ and $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n$;
3. "w-bloc": $b_0, b_1, \ldots, b_{3m}$;
4. "second line": $d_1, d_2, \ldots, d_n$ and $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n$;
5. "garbage collectors": $g_1, g_2, \ldots, g_n$.

STV-CM is NP-complete: The Properties

Property 1: $a_1, a_2, \ldots, a_n$ and $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n$ are among the first $3n$ candidates to be eliminated.

Property 2: Let $I = \{i \mid \overline{a}_i \text{ is eliminated prior to } a_i\}$. Then

$$c \text{ can be made win } (C, V \cup \{s\}) \iff I \text{ is a 3-cover.}$$

Property 3:

1. For any $I \subseteq \{1, 2, \ldots, n\}$, there is a preference for s such that

$$\overline{a}_i \text{ is eliminated prior to } a_i \iff i \in I.$$

2. Such a preference for s is constructed as follows:
   - If $i \in I$ then place a_i in the ith position of s.
   - If $i \notin I$ then place $\overline{a}_i$ in the ith position of s.

STV-CM is NP-complete: The Nonmanipulative Voters

(1)		$12n$	votes:	c	$\cdots$				
(2)		$12n-1$	votes:	w	c	$\cdots$			
(3)		$10n+2m$	votes:	b_0	w	c	$\cdots$		
(4)	For each $i \in \{1,2,\ldots,3m\}$,	$12n-2$	votes:	b_i	w	c	$\cdots$		
(5)	For each $j \in \{1,2,\ldots,n\}$,	$12n$	votes:	g_j	w	c	$\cdots$		
(6)	For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-5$	votes:	d_j	$\overline{d}_j$	w	c	$\cdots$	
	and if $S_j = \{b_x, b_y, b_z\}$ then	2	votes:	d_j	b_x	w	c	$\cdots$	
		2	votes:	d_j	b_y	w	c	$\cdots$	
		2	votes:	d_j	b_z	w	c	$\cdots$	
(7)	For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-1$	votes:	$\overline{d}_j$	d_j	w	c	$\cdots$	
		2	votes:	$\overline{d}_j$	b_0	w	c	$\cdots$	
(8)	For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-3$	votes:	a_j	g_j	w	c	$\cdots$	
		1	vote:	a_j	d_j	g_j	w	c	$\cdots$
		2	votes:	a_j	$\overline{a}_j$	g_j	w	c	$\cdots$
(9)	For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-3$	votes:	$\overline{a}_j$	g_j	w	c	$\cdots$	
		1	vote:	$\overline{a}_j$	$\overline{d}_j$	g_j	w	c	$\cdots$
		2	votes:	$\overline{a}_j$	a_j	g_j	w	c	$\cdots$

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Lemma (Bartholdi and Orlin (1991))

1. *Exactly one of d_j and $\overline{d}_j$ will be among the first $3n$ candidates to be eliminated.*

2. *Candidate c will win if and only if*

$$J = \{j \mid d_j \text{ is among the first } 3n \text{ candidates to be eliminated}\}$$

is the index set of an exact 3-cover for $\mathcal{S}$.

Proof: See blackboard. ❑

STV-CM is NP-complete: The Manipulor's Preference

Lemma (Bartholdi and Orlin (1991))

Let $I \subseteq \{1, 2, \ldots, n\}$ and consider the strategic preference of manipulator s in which the ith candidate is

- a_i *if* $i \in I$ *and*
- $\overline{a}_i$ *if* $i \notin I$.

Then the order in which the first $3n$ candidates are eliminated is:

1. *The $(3i-2)$nd candidate to be eliminated is*
   - $\overline{a}_i$ *if* $i \in I$ *and*
   - a_i *if* $i \notin I$.
2. *The $(3i-1)$st candidate to be eliminated is*
   - d_i *if* $i \in I$ *and*
   - $\overline{d}_i$ *if* $i \notin I$.
3. *The $3i$th candidate to be eliminated is*
   - a_i *if* $i \in I$ *and*
   - $\overline{a}_i$ *if* $i \notin I$.

{Scoring-Protocols without Plurality}-CCWM

Theorem (Conitzer, Sandholm, and Lang (2007))

{*Scoring-Protocols without Plurality*}-CONSTRUCTIVE COALITIONAL WEIGHTED MANIPULATION *for three candidates is* NP*-complete.*

Remark:

1. For two candidates every scoring protocol is easy to manipulate.
2. Plurality is easy to manipulate for any number of candidates.
3. In particular, Veto-CCWM and Borda-CCWM for three candidates are NP-complete.
4. The above theorem was independently proven by Hemaspaandra & Hemaspaandra (2007) and Procaccia & Rosenschein (2006).

{Scoring-Protocols without Plurality}-CCWM: Reduction from PARTITION

Proof: Membership in NP is clear.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a scoring protocol other than plurality.

To prove NP-hardness of α-CCWM, we reduce from the following NP-complete problem:

Name: PARTITION.

Given: A nonempty sequence $(k_1, k_2, \ldots, k_n)$ of positive integers such that $\sum_{i=1}^{n} k_i$ is an even number.

Question: Does there exist a subset $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} - A} k_i \;?$$

$\{$Scoring-Protocols without Plurality$\}$-CCWM: Reduction from PARTITION

Given an instance $(k_1, k_2, \ldots, k_n)$ of PARTITION with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election $(C, V \cup S)$ with $C = \{a, b, p\}$ and

	Vote Weight	Preference
V :	$(2\alpha_1 - \alpha_2)K - 1$	$a \quad b \quad p$
	$(2\alpha_1 - \alpha_2)K - 1$	$b \quad a \quad p$

S : For each $i \in \{1, 2, \ldots, n\}$, $(\alpha_1 + \alpha_2)k_i$

See blackbord for the proof of:

$(k_1, k_2, \ldots, k_n) \in$ PARTITION $\iff$ p can be made win $(C, V \cup S)$. ❑

Copeland-CCWM for four Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

*Copeland-*CONSTRUCTIVE COALITIONAL WEIGHTED MANIPULATION *for four candidates is* NP*-complete.*

Proof: Membership in NP is clear. To prove NP-hardness of Copeland-CCWM, we again reduce from PARTITION.

Given an instance $(k_1, k_2, \ldots, k_n)$ of PARTITION with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, c, p\}$ and the following votes in $V \cup S$.

Copeland-CCWM for four Candidates is Hard

V :

Vote Weight	Preference
$2K+2$	p a b c
$2K+2$	c p b a
$K+1$	a b c p
$K+1$	b a c p

S : For each $i \in \{1, 2, \ldots, n\}$, k_i

See blackbord for the proof of:

$(k_1, k_2, \ldots, k_n) \in$ PARTITION $\iff$ p can be made win $(C, V \cup S)$. ❑

Maximin-CCWM for four Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

*Maximin-*CONSTRUCTIVE COALITIONAL WEIGHTED MANIPULATION *for four candidates is* NP*-complete.*

Proof: Membership in NP is clear. To prove NP-hardness of Maximin-CCWM, we again reduce from PARTITION.

Given an instance $(k_1, k_2, \ldots, k_n)$ of PARTITION with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, c, p\}$ and the following votes in $V \cup S$.

Maximin-CCWM for four Candidates is Hard

		Vote Weight	Preference
V :		$7K-1$	$a \quad b \quad c \quad p$
		$7K-1$	$b \quad c \quad a \quad p$
		$4K-1$	$c \quad a \quad b \quad p$
		$5K$	$p \quad c \quad a \quad b$
S :	For each $i \in \{1, 2, \ldots, n\}$,	$2k_i$	

See blackbord for the proof of:

$(k_1, k_2, \ldots, k_n) \in \text{PARTITION} \iff p$ can be made win $(C, V \cup S)$. ❑

STV-CCWM for three Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

*STV-*Constructive Coalitional Weighted Manipulation *for three candidates is* NP*-complete.*

Proof: Membership in NP is clear. To prove NP-hardness of STV-CCWM, we again reduce from Partition.

Given an instance $(k_1, k_2, \ldots, k_n)$ of Partition with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, p\}$ and the following votes in $V \cup S$.

STV-CCWM for three Candidates is Hard

		Vote Weight	Preference		
V :		$6K-1$	b	p	a
		$4K$	a	b	p
		$4K$	p	a	b
S :	For each $i \in \{1, 2, \ldots, n\}$,	$2k_i$			

See blackbord for the proof of:

$$(k_1, k_2, \ldots, k_n) \in \text{PARTITION} \iff p \text{ can be made win } (C, V \cup S). \quad \square$$