SHARING PROBLEMS AND THE ISSUE OF FAIRNESS

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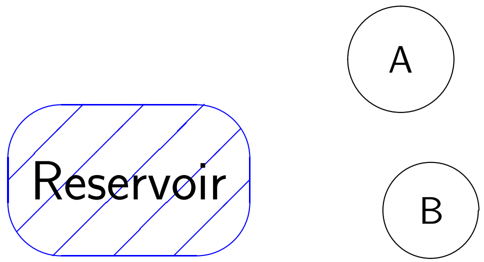
1 Introduction

1.1 Preliminary Example (Young, 1994)

Two towns A and B, with a population of 3.600 and 1.200, respectively, decide to build an aqueduct, using the water from a common reservoir

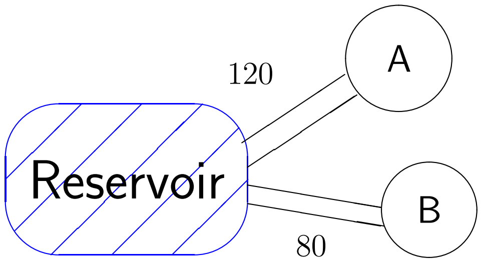
<u>Mathematical programming approach</u>

min *Building costs*
s.t. *A is connected to the reservoir*
Requirements of A are satisfied
B is connected to the reservoir
Requirements of B are satisfied

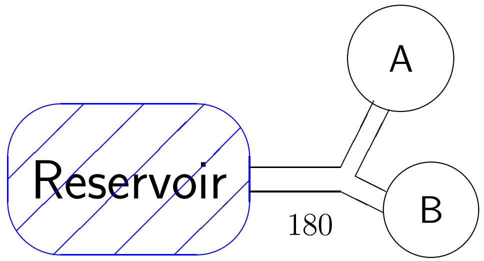


Feasible solutions belongs to two different subsets

Hypothesis 1



Hypothesis 2



Hypothesis 2 corresponds to the optimal solution

This aqueduct requires that the two towns agree on a common infrastructure and on the cost allocation

	Criterion	*A*	*B*
1	*Equal cost allocation per town*	*90*	*90*
2	*Equal cost allocation per inhabitant*	*135*	*45*
3	*Equal savings allocation per town*	*110*	*70*
4	*Equal savings allocation per inhabitant*	*105*	*75*
5	*Proportional allocation of costs (or savings) w.r.t. independent aqueducts*	*108*	*72*

Criterion 1 is the best option for A but it is (rationally) rejected by B

Criterion 2 is the best option for B but it is (rationally) rejected by A

Other criteria cannot be (rationally) rejected by no town

Sharing problems arise when it is necessary to allocate items and/or amounts among different agents, with different needs and different preferences

The setting may be non cooperative or cooperative, depending on the information exchange and on the possibility of subscribing binding agreements
The most usual examples of sharing problems in a non cooperative situation are the fair division problems and auctions
On the other hand, the cooperative approach accounts the individual and collective requirements and preferences, looking for fair allocations

Fairness: satisfaction criteria of the agents

2 Fair Division

Division problem: assignment of one or more items among the agents of the set $N = \{1, ..., n\}$ that have heterogeneous preferences

Perfectly divisible item: its value (utility) does not change if it is divided (money)
Monetary compensations make divisible an indivisible item

2.1 Properties

Definition 2.1 *Each agent i assigns a valuation P_{ij} to the bundle received by agent j, $i, j \in N$*

- *PARETO EFFICIENCY*
  There does not exist a different division, with valuations Q such that $Q_{ii} \geq P_{ii}$ for each $i \in N$ and there exists i^* such that $Q_{i^*i^*} > P_{i^*i^*}$

- *PROPORTIONALITY*
  $P_{ii} \geq \frac{1}{n} \sum_{j \in N} P_{ij}, \forall\ i \in N$

- *EQUITY (EQUITABILITY)*
  $P_{ii} = P_{jj}, \forall\ i, j \in N$, *or* $\dfrac{P_{ii}}{\sum_{k \in N} P_{ik}} = \dfrac{P_{jj}}{\sum_{k \in N} P_{jk}}, \forall\ i, j \in N$

- *ENVY FREENESS*
  There does not exist an agent $i \in N$ such that $P_{ij} > P_{ii}$ for some $j \in N$

- Envy freeness implies proportionality
- In case of two agents also the vice versa holds
- Envy freeness and efficiency may be defined using preferences, without utility functions (this is not possible for proportionality and equity)
  A division is efficient if there does not exist a different division that is weakly preferred by each agent and strictly preferred by at least one agent
  A division is envy free if no agent considers the bundle of another agent preferable to his own

Items are divided using a procedural approach, i.e. the agents perform a sequence of steps

A procedure satisfies a given property if it produces a division for which the same property holds when each agent "correctly" perform his steps

The agents are conscious of the properties of the resulting division including compensations

Example 2.1 (Divide a cake (divisible item)) *A heterogeneous cake has to be shared among two agents with different preferences. The most usual and simplest procedure is the so-called "I cut, you choose":*

- *proportionality - each agent may obtain at least half (of the value) of the cake: the first agent has to cut the cake in two equivalent parts, while the second has to choose his preferred part*
- *envy freeness - no agent prefers the part assigned to the other*
- *no equity - the first agent gets exactly half (of the value) of the cake, while the second may obtain more than half (of the value)*
- *inefficiency - a different division of the cake may be preferred by both agents* ♢

Example 2.2 (Proportionality does not imply envy freeness) *Three agents receive a part of a cake. The first agent has the following valuations* $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$*, the second assigns* $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ *to the parts and the third values them* $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$*. The division is proportional, but the second agent envies the first* ♢

2.2 Procedures for a divisible item

Divide and choose (Dubins-Spanier, 1961)

Moving Knives $n = 2$ (Austin, 1982)

Last Diminisher (Banach-Knaster, 1949 and Steinhaus, 1949)

2.3 Procedure for m indivisible items

Sealed bid (Knaster, 1946 and Steinhaus, 1948)

m indivisible objects $b_1, ..., b_m$ ($M = \{1, ..., m\}$) with additive values have to be assigned to the agents of the set $N = \{1, ..., n\}$

Each object is assigned to just one agent with monetary compensations

Preliminarily, each agent $i \in N$ declares to a mediator (in a *sealed bid*) his valuations $v_{i1}, ..., v_{im}$ of the items; let $E_i = \dfrac{1}{n} \sum_{k \in M} v_{ik}$, i.e. E_i is the initial proportional share according to the valuations of agent i

Then the procedure goes as follows:

Step 1 each object $b_k, k \in M$ is assigned to an agent $j(k)$ that gives the maximal valuation, $j(k) \in argmax\ (v_{ik}, i \in N)$ (if more agents give the same maximal valuation of an object it is assigned randomly);

Step 2 for each agent $i \in N$, let $G_i = \sum_{k:j(k)=i} v_{j(k),k}$; $V_i = E_i + \frac{s}{n}$, where $s = \sum_{j \in N}(G_j - E_j)$;

Step 3 for each agent $i \in N$, if the monetary amount $V_i - G_i$ is positive the player i receives it in addition to his objects;
otherwise player i pays $G_i - V_i$.

Citing Brams and Taylor (1996), the division is efficient and proportional; it is equitable iff all the agents give the same valuation of the whole set of items; it is not envy free; the sum of the compensations is zero (the procedure does not require or produce money); the surplus s is non-negative

2.4 Manipulability and Collusion (Fragnelli-Marina, 2009a)

The misrepresentations of the valuations may provide advantages (or disadvantages), not making use of any statistical information on the valuations of the other agents

A proper subset (coalition) of at least two (completely risk-averse) agents may profit from an information exchange on their true valuations, and a consequent agreement on altering their declarations, getting a safe gain w.r.t. the situation of truthful declarations

2.5 Procedures for m divisible items, with two agents

Adjusted Winner (AW) (Brams-Taylor, 1996)

Proportional Allocation (PA) (Brams-Taylor, 1996)

3 Bankruptcy

3.1 Bankruptcy (or taxation) problem

Allocation of a scarce resource

$$\mathcal{B} = (N, c, E) = (E; c_1, ..., c_n)$$

where	$N = \{1, ..., n\}$	set of claimants
	$c = (c_1, ..., c_n)$	claims
	E	estate, with $E < \sum_{i \in N} c_i = C$

For a taxation problem	$N = \{1, ..., n\}$	set of taxpayers
	$c = (c_1, ..., c_n)$	incomes
	E	government spending

Each feasible ("rational") sharing of the estate, $x = (x_1, ..., x_n)$, has to satisfy:

$$\sum_{i \in N} x_i = E$$

$$0 \le x_i \le c_i, \qquad i \in N$$

3.2 Solutions for a bankruptcy problem (Herrero-Villar, 2001)

- $PROP$ - Each agent receives a quota proportional to his claim:

$$PROP_i = \frac{c_i}{C}E \qquad i \in N$$

- CEA - All the agents receive the same amount, unless the quota is larger than the claim:

$$CEA_i = \min\{\alpha, c_i\} \qquad i \in N$$

where α is the unique positive real number s.t. $\sum_{i \in N} CEA_i = E$

- CEL - All the agents receive the claim diminished of the same amount, unless the quota is negative:

$$CEL_i = \max\{c_i - \beta, 0\} \qquad i \in N$$

where β is the unique positive real number s.t. $\sum_{i \in N} CEL_i = E$

Example 3.1 (Solutions) *Consider the bankruptcy problem* $(22; 4, 16, 24, 44)$.
$C = 88$

$$\begin{array}{lll} \textit{PROP} = (1,4,6,11) & & \\ \textit{CEA} = (4,6,6,6) & & \textit{with } \alpha = 6 \\ \textit{CEL} = (0,0,1,21) & & \textit{with } \beta = 23 \end{array}$$

$\diamondsuit$

- $PROP$ is the most "natural" solution in several situations, CEA takes care of the small claimants, CEL favors the large claimants
- For taxation problems CEA is preferred by richest contributors, CEL is the best option for low income people

4 TU-Games

Agents (*players*) may cooperate in order to improve their utility, when the following hypotheses hold:

- the possibility of agreements, i.e. there do not exist antitrust rules
- the possibility of forcing the respect of the agreements, i.e. there exists a *superpartes* authority, accepted by all the agents

Cooperative games are divided in two classes:

- <u>Cooperative games without transferable utility (NTU-Games)</u>: the players receive the payoff according to the strategy profile they agreed upon
- <u>Cooperative games with transferable utility (TU-Games)</u>: the players may share the total payoff generated by the strategy profile they agreed upon

TU-Games are a special case of NTU-Games

A TU-Game has to satisfy the following three additional hypotheses:

- it is possible to transfer the utility
- there exists a common exchange tool, e.g. the money, which allows transferring the utility (in a material sense)
- the utility functions of the players must be equivalent, e.g. they can be linear in the amount of money

Note that the decision on the sharing of the total payoff in a TU-Game is part of the binding agreement

Definition 4.1 *A TU-Game is a pair* $G = (N, v)$

N *is the* player set*, each subset* $S \subseteq N$ *is a* coalition *and* $S = N$ *it is the* grand coalition

v *is the* characteristic function *of a* n*-person game:*

$$v : 2^N \rightarrow \mathbb{R} \textit{ with } v(\emptyset) = 0$$

$v(S)$ *represents the worth of coalition* S*, i.e. what the agents in* S *may obtain* independently *of the other agents*

If the payoffs of the players are negative it is convenient to represent the game as a *cost game* $G = (N, c)$, where $c = -v$

Example 4.1 (Preliminary Example)

$$N = \{A, B\}$$
$$c(A) = 120; c(B) = 80; c(A, B) = 180$$

$\diamondsuit$

The solutions of a TU Game, i.e. how to share the utility $v(N)$ among all the players, may be divided in two groups:

- *set solutions*: a set of payoff vectors is associated to the player set
- *point solutions*: a single payoff vector is determined

5 Set Solutions for TU Games

5.1 Imputations (von Neumann-Morgenstern, 1944)

The payoff of each player may be obtained equally sharing the value of the game among the players, taking in no account the contribution of each player
A different approach is rooted in the analysis of the role of the players

Definition 5.1 *Given a game* $G = (N, v)$ *an* imputation *is a vector* $x = (x_1, x_2, \ldots, x_n)$ *such that:*

$$\sum_{i \in N} x_i = v(N) \qquad \textit{efficiency}$$

$$x_i \geq v(i) \qquad i = 1, \ldots, n \qquad \textit{individual rationality}$$

For a cost game $G = (N, c)$ *individual rationality is expressed as* $x_i \leq c(i), \forall\, i \in N$

The set of all the imputations is denoted by $E(v)$

An imputation is the first step toward a solution concept that respects the role of the players
For an essential game there exist many imputation vectors, so again we have the problem of choosing a solution: given two different imputations x and y there exists at least one player k such that $x_k > y_k$ and at least one player h such that $x_h < y_h$

5.2 Core

It is the most interesting set solution for many classes of games
It was introduced by Gillies (1953 and 1959)

$$x(S) \geq v(S) \qquad S \subset N \qquad \textit{coalitional rationality}$$

where $x(S) = \sum_{i \in S} x_i$
For a cost game $G = (N, c)$ coalitional rationality is expressed as $x(S) \leq c(S), \forall\ S \subset N$

Definition 5.2 *Given a game $G = (N, v)$, the core is the set:*

$$C(v) = \{x \in E(v) | x(S) \geq v(S), \forall\ S \subset N\}$$

- The core may be empty
- The core is useful to select which solutions should not be chosen (those not belonging to the core) when the core is non-empty. The emptyness of the core does not imply that the grand coalition does not form, but gives clues on its low stability

Example 5.1 (Core of Preliminary Example)

$$C(c) = \{(\alpha, 180 - \alpha)\ s.t.\ 60 \leq \alpha \leq 80\}$$

$\diamondsuit$

6 Point Solutions for TU Games

Widely used as allocation rules

6.1 Shapley Value (1953)

It is rooted in the concept of *marginal contribution*

Definition 6.1 *Given a game $G = (N, v)$, let Π be the set of all the orderings (permutations) of the players; for each ordering $\pi \in \Pi$ let P_π^i be the set of players that precede player i in the ordering π, including player i. The Shapley value is the average marginal contribution of each player w.r.t. all the possible orderings:*

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \left[v(P_\pi^i) - v(P_\pi^i \setminus \{i\}) \right], \; i \in N$$

where n denotes the cardinality of the set of players N

Given a TU-Game, the Shapley value always exists and is unique

6.2 Nucleolus (Schmeidler, 1969)

It is founded on the Rawls principle of fairness: maximize the utility of the worst-off agent

Computationally complex

7 Bankruptcy Games (Aumann-Maschler, 1985 and Curiel-Maschler-Tijs, 1987)

It is possible to define two TU-games, the pessimistic one, (N, v_P), and the optimistic one, (N, v_O), where:

$$v_P(S) = max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \qquad S \subseteq N$$

$$v_O(S) = min\left(E, \sum_{i \in S} c_i\right) \qquad S \subseteq N$$

8 Fixed Tree Games (Megiddo, 1978)

A set of agents is connected to a service via a tree structure; each agent corresponds to a vertex of the tree and the source to the root. The service is paid on the basis of the usage, but maintenance costs have to be allocated. It is possible to define a TU-game where the players are the agents of the set $N = \{1, ..., n\}$ and the characteristic function is:

$$c(S) = \min_{T \supseteq S} \left\{ \sum_{i \in T} c_i \right\} \qquad S \subseteq N$$

where c_i is the maintenance cost of the unique arc entering the vertex associated to agent i and T is the connected component of the tree containing the source

The core of a fixed tree game is the set of the allocations that can be obtained sharing the cost of each arc among the agents in the connected component that does not include the source, after the elimination of the arc

A simple procedure to obtain a core allocation is a *painting story*
Suppose that the maintenance corresponds to paint the tree; then the painting story corresponds to assign a painting speed to each agent and to ask him to paint only arcs that he uses. The quota painted by each player define a core allocation

<u>Shapley value</u>
All the agents have the same painting speed and each agent paints the furthest unpainted arc in the path connecting his vertex with the source

<u>Nucleolus</u>
All the agents have the same painting speed and each agent paints the second closest unpainted arc in the path connecting his vertex with the source, and finally the entering arc

9 Separable Cost Methods

How to share the cost of a joint project among the users, taking into account their roles and needs?
These methods originate from one of the first applications of "Game Theory", the Tennessee Valley Authority problem in the 30s (Ransmeier, 1942 and Straffin-Heaney 1981)
Control the water flow looking at:

- electric energy generation
- flood control
- navigation

Fairness criteria (equivalent if x is efficient):

- *stand-alone cost test:* $x(S) \leq c(S), S \subseteq N$
- *incremental cost test:* $x(S) \geq c(N) - c(N \setminus S), S \subseteq N$

Definition 9.1

- *Given a cost game c, the separable cost of player i is his marginal contribution or marginal cost:*

$$m_i = c(N) - c(N \setminus \{i\})$$

- *If the sum of the separable costs of the players is not larger than the cost of the grand coalition, the non-separable cost is the difference:*

$$g(N) = c(N) - \sum_{i \in N} m_i$$

The methods allocate in different ways the non-separable cost

Information costs

9.1 Equal Charge Allocation (ECA)

$g(N)$ is equally shared

$$ECA_i = m_i + \frac{1}{n}\, g(N)$$

9.2 Alternative Cost Avoided (ACA)

$g(N)$ is shared proportionally to the savings of the separable costs w.r.t. the stand-alone cost of the players

$$ACA_i = m_i + \frac{r_i}{\sum_{j \in N} r_j}\, g(N)$$

where $r_i = c(i) - m_i$

9.3 Cost Gap Allocation (CGA)

$g(N)$ is shared proportionally to the best (minimal) maximal contribution that each player has to pay when he enter a coalition where the other players pay only their separable costs

$$CGA_i = m_i + \frac{g_i}{\sum_{j \in N} g_j}\, g(N)$$

where $g_i = min\ \{g(S) | i \in S\}$ e $g(S) = c(S) - \sum_{i \in S} m_i$

Example 9.1 (Cost Allocation)

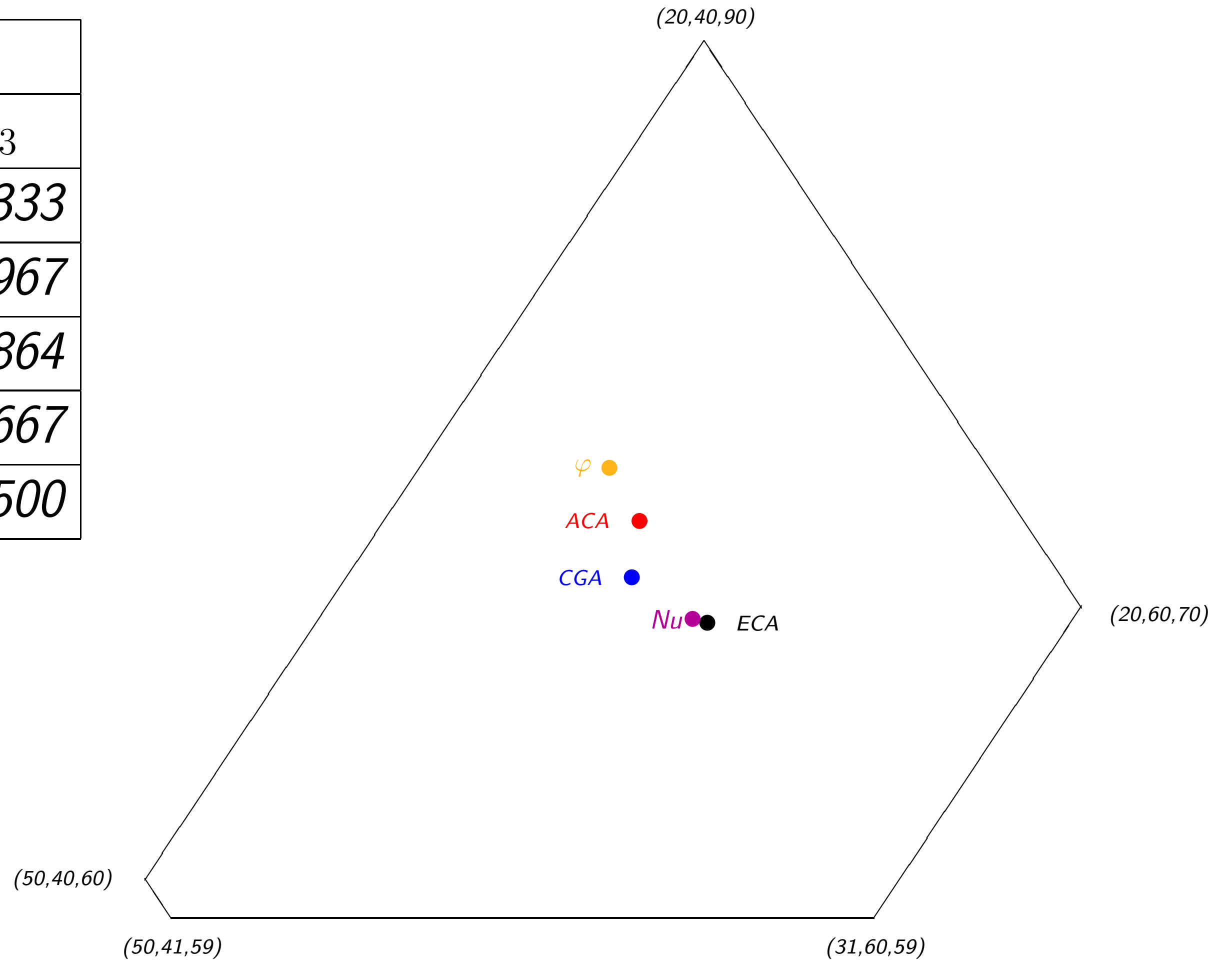
Given the game $G = (N, c)$:

$N = \{1, 2, 3\}$

$c(1) = 50; c(2) = 60; c(3) = 100; c(12) = 91; c(13) = 110; c(23) = 130; c(N) = 150$

	Allocations		
Criterion	x_1	x_2	x_3
ECA	30.333	50.333	69.333
ACA	30.220	46.813	72.967
CGA	31.481	47.654	70.864
Shapley Value	30.167	45.167	74.667
Nucleolus	30.500	50.000	69.500

All the allocations belong to the core



10 Cooperative Cost Allocation Problem

A set of agents $N = \{1, 2, ..., n\}$ has to share the building and/or maintenance cost C of a facility
Agents require the facility at different levels of quality so that each agent $i \in N$ is associated to a cost c_i
Agents are ordered s.t. $C = c_1 \geq c_2 \geq ... \geq c_n > 0$

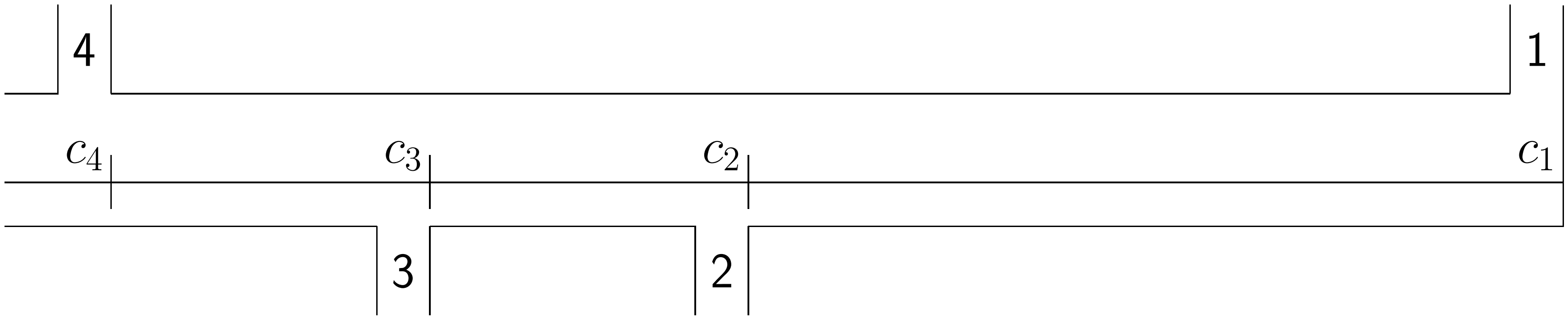
Airport problem (see Littlechild-Thompson, 1977)

- $N = \{1, 2, ..., n\}$ is the set of planes landing in a given time period
- C is the cost of the landing strip to be allocated
- $c_1, c_2, ..., c_n$ are the costs of the landing strip required by the planes

<u>Here</u>

Irrigation problem (Aadland-Kolpin, 2004)

- N is the set of farmers
- C is the cost of the ditch
- $c_1, c_2, ..., c_n$ are the costs of the part required by each farmer



General situation

The agents may use different ordered facilities with the requirement that if an agent asks for using a certain facility he has to use (or at least he has to pay) also for the previous facilities

For instance, a set of independent services that cannot be purchased separately, but sequentially (car market where cars are available with increasing packages plus separate optional extras)

Baker (1965) and Thompson (1971) proposed to share the cost C among the agents, using the so-called Baker-Thompson (BT) rule:

$$BT_i(N, c_1, ..., c_n) = \sum_{j=i,...,n} \frac{c_j - c_{j+1}}{i}, i \in N$$

where $c_{k+1} = 0$

Littlechild and Owen (1973) showed that the BT rule coincides with the Shapley value of the so-called airport game, whose characteristic function is

$$c(S) = \max_{i \in S}\{c_i\}, S \subseteq N$$

Airport game is a special case of fixed tree game where the tree is a chain

10.1 Fairness Criteria for an Allocation Rule (Fragnelli-Marina, 2009b)

An allocation rule for a cooperative cost allocation problem is a map $\mathcal{F}$ that associates with each allocation problem $(N, c_1, ..., c_n)$, a point $\mathcal{F}(N, c_1, ..., c_n) \in \mathbb{R}^n$ s.t. $\sum_{i \in N} \mathcal{F}_i(N, c_1, ..., c_n) = c_1$

Individual equal sharing $(\mathbf{IES})$: all the n agents have the right of usage, each agent $i \in N$ may not ask to pay less then $\frac{c_i}{n}$; an allocation rule $\mathcal{F}$ satisfies $\mathbf{IES}$ if for every allocation problem $(N, c_1, ..., c_n)$:

$$\mathcal{F}_i(N, c_1, ..., c_n) \geq \frac{c_i}{n}, \; i \in N$$

Collective usage right $(\mathbf{CUR})$: each agent of the subset $\{1, ..., i\}$ needs a facility of cost at least c_i and may be required of contributing to the cost of the facility at the level required by agent $i \in N$. The other agents in the subset $\{i+1, ..., n\}$ needing a facility no more expensive than c_i could behave as free riders; so, agent $i \in N$ may ask to pay at most $\frac{c_i}{i}$; an allocation rule $\mathcal{F}$ satisfies $\mathbf{CUR}$ if for every allocation problem $(N, c_1, ..., c_n)$:

$$\mathcal{F}_i(N, c_1, ..., c_n) \leq \frac{c_i}{i}, \; i \in N$$

$\mathbf{IES}$ and $\mathbf{CUR}$ provide a lower bound and an upper bound respectively, on the amount that each agent has to pay

These two criteria do not imply that the amount paid by the agent is monotonic in the individual cost
For instance, consider $N = \{1, 2, 3\}$ with $c_1 = 30, c_2 = 24, c_3 = 21$; in this case $l_1 = 10, l_2 = 8, l_3 = 7, u_1 = 30, u_2 = 12, u_3 = 7$ and a possible allocation is $(11, 12, 7)$

Third criterion
Consistency on last player $(\mathbf{CLAST})$: if agent $n \in N$ withdraws after paying $\mathcal{F}_n(N, c_1, ..., c_n)$ and the remaining agents in $N \setminus \{n\}$ reallocate the remaining amount $c_1 - \mathcal{F}_n(N, c_1, ..., c_n)$ redefining the costs $\hat{c}_i = c_i - \mathcal{F}_n(N, c_1, ..., c_n), i \in N \setminus \{n\}$ the resulting new allocation coincides with the previous one for each agent $i \in N \setminus \{n\}$; an allocation rule $\mathcal{F}$ satisfies $\mathbf{CLAST}$ if for every allocation problem $(N, c_1, ..., c_n)$:

$$\mathcal{F}_i(N, c_1, ..., c_n) = \mathcal{F}_i(N \setminus \{n\}, \hat{c}_1, ..., \hat{c}_{n-1}),\ i \in N \setminus \{n\}$$

Theorem 10.1 *The BT rule is the unique allocation rule that satisfies* $\mathbf{IES}$, $\mathbf{CUR}$ *and* $\mathbf{CLAST}$ *for every allocation problem*

Example 10.1 (Cooperative cost allocation problem) *Consider the problem with* $N = \{1, 2, 3, 4\}$ *and* $c = (176, 84, 48, 12)$

According to **IES** $l = (44, 21, 12, 3)$ *and according to* **CUR** $u = (176, 42, 16, 3)$

The following allocations may be proposed:

N	1	2	3	4
ϕ	125	33	15	3
Nu	120.5	28.5	21	6
ECA	113	21	21	21
$ACA = CGA$	122.95	30.95	17.68	4.42
$Knaster$	68	45	36	27
$PROP$	96.8	46.2	26.4	6.6
$CEA\ (\alpha = 58)$	58	58	48	12
$CEL\ (\beta = 44)$	132	40	4	0

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