

Constructive Control by Adding Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let \mathcal{E} be some voting system.

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES (\mathcal{E} -CCAUC).

Given:

- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$, and
- a distinguished candidate $p \in C$.

Question: Is there a subset D' of D such that p is the unique winner of the \mathcal{E} election $(C \cup D', V)$?

Constructive Control by Adding Candidates

Definition (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let \mathcal{E} be some voting system.

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES (\mathcal{E} -CCAC).

- Given:**
- Disjoint sets C and D of candidates,
 - a list V of votes over $C \cup D$,
 - a distinguished candidate $p \in C$, and
 - a nonnegative integer k .

Question: Is there a subset D' of D such that $\|D'\| \leq k$ and p is the unique winner of the \mathcal{E} election $(C \cup D', V)$?

Constructive Control by Deleting Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let \mathcal{E} be some voting system.

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY DELETING CANDIDATES (\mathcal{E} -CCDC).

Given:

- A set C of candidates,
- a list V of votes over C ,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k .

Question: Is it possible to delete up to k candidates from C such that p is the unique winner of the resulting \mathcal{E} election?

Constructive Control by Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES (\mathcal{E} -CCPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which

- the winners of (C_1, V) surviving the tie-handling rule
- run against all candidates in C_2 ?
- “Ties eliminate” (TE): Only unique winners proceed to final stage.
- “Ties promote” (TP): All winners proceed to final stage.

Constructive Control by Runoff Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY RUNOFF PARTITION OF CANDIDATES (\mathcal{E} -CCRPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which the runoff is between

- the winners of (C_1, V) surviving the tie-handling rule and
 - the winners of (C_2, V) surviving the tie-handling rule?
-
- “Ties eliminate” (TE): Only unique winners proceed to final stage.
 - “Ties promote” (TP): All winners proceed to final stage.

Constructive Control by Adding Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let \mathcal{E} be some voting system.

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY ADDING VOTERS
(\mathcal{E} -CCAV).

Given:

- A set C of candidates,
- a list V of registered votes over C and an additional list W of as yet unregistered votes over C ,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k .

Question: Is there a subset W' of W such that $\|W'\| \leq k$ and p is the unique winner of the \mathcal{E} election $(C, V \cup W')$?

Constructive Control by Deleting Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let \mathcal{E} be some voting system.

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY DELETING VOTERS
(\mathcal{E} -CCDV).

- Given:
- A set C of candidates,
 - a list V of votes over C ,
 - a distinguished candidate $p \in C$, and
 - a nonnegative integer k .

Question: Is it possible to delete up to k voters from V such that p is the unique winner of the resulting \mathcal{E} election?

Constructive Control by Partition of Voters

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: \mathcal{E} -CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS (\mathcal{E} -CCPV).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition V into V_1 and V_2 such that p is the unique winner (with respect to the votes in V) of the final stage of the two-stage election in which the runoff is between

- the winners of (C, V_1) surviving the tie-handling rule and
 - the winners of (C, V_2) surviving the tie-handling rule?
-
- “Ties eliminate” (TE): Only unique winners proceed to final stage.
 - “Ties promote” (TP): All winners proceed to final stage.

Destructive Control

Remark:

- For each constructive control scenario, there is a corresponding **destructive control type** where the chair seeks to block the distinguished candidate's victory:

\mathcal{E} -DCAUC, \mathcal{E} -DCAC, \mathcal{E} -DCDC, \mathcal{E} -DCPC-TE, \mathcal{E} -DCPC-TP, \mathcal{E} -DCRPC-TE, \mathcal{E} -DCRPC-TP, \mathcal{E} -DCAV, \mathcal{E} -DCDV, \mathcal{E} -DCPV-TE, and \mathcal{E} -DCPV-TP.

In \mathcal{E} -DCDC it is not allowed to simply delete the distinguished candidate.

⇒ This sums up to a total of 22 control types (and the corresponding control problems).

- The study of destructive control was initiated by [Hemaspaandra](#), [Hemaspaandra](#), and [Rothe \(2007\)](#).

Immunity and Susceptibility

Definition (Bartholdi, Tovey, and Trick (1992))

Let \mathcal{CT} be a control type.

- 1 We say a voting system is *immune to \mathcal{CT}* if it is impossible for the chair to make the given candidate
 - the unique winner in the constructive case and
 - not a unique winner in the destructive case,respectively, via exerting control of type \mathcal{CT} .
- 2 We say a voting system is *susceptible to \mathcal{CT}* if it is not immune to \mathcal{CT} .

Resistance and Vulnerability

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let \mathcal{CT} be a control type.

A voting system that is susceptible to \mathcal{CT} is said to be

- 1 *vulnerable to \mathcal{CT}* if the control problem corresponding to \mathcal{CT} can be solved in polynomial time, and
- 2 *resistant to \mathcal{CT}* if the control problem corresponding to \mathcal{CT} is NP-hard.

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

- 1 *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
- 2 *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*
- 3 *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*
- 4 *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

- 1 *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
- 2 *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
- 3 *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*
- 4 *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

Links Between Susceptibility Cases

Definition

A voting system is *voiced* if in any election that has exactly one candidate, that candidate is always a (and thus, the unique) winner.

Theorem

- 1 *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
- 2 *Each voiced voting system is susceptible to constructive control by deleting candidates.*
- 3 *Each voiced voting system is susceptible to destructive control by adding candidates.*

Control Complexity of Plurality and Condorcet Voting

	Plurality		Condorcet	
Control by	Constructive	Destructive	Constructive	Destructive
Adding Candidates	R(esistant)	R	I(mmune)	V(ulnerable)
Deleting Candidates	R	R	V	I
Partition of Candidates	TE: R TP: R	TE: R TP: R	V	I
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R	V	I
Adding Voters	V	V	R	V
Deleting Voters	V	V	R	V
Partition of Voters	TE: V TP: R	TE: V TP: R	R	V

Boldface results are due to [Hemaspaandra, Hemaspaandra, and Rothe \(2007\)](#).

Nonboldface results are due to [Bartholdi, Tovey, and Trick \(1992\)](#).

Hitting Set

Definition

Name: HITTING SET.

Given:

- A set $B = \{b_1, b_2, \dots, b_m\}$,
- a family $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets S_i of B , and
- a positive integer k .

Question: Does \mathcal{S} have a hitting set of size at most k ?

That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i , $S_i \cap B' \neq \emptyset$?

Destructive Candidate Control in Plurality Voting

Construction: Given a HITTING SET instance (B, \mathcal{S}, k) , where $B = \{b_1, b_2, \dots, b_m\}$, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$, and $k \leq m$, construct the following election:

- The candidate set is $C = B \cup \{c, w\}$.
- The voter set V is defined as follows:
 - 1 $2(m - k) + 2n(k + 1) + 4$ voters of the form $c \ w \ \dots$, where “ \dots ” means that the remaining candidates follow in an arbitrary order.
 - 2 $2n(k + 1) + 5$ voters of the form $w \ c \ \dots$.
 - 3 For each i , $1 \leq i \leq n$, there are $2(k + 1)$ voters of the form $S_i \ c \ \dots$, where “ S_i ” denotes the elements of S_i in some arbitrary order.
 - 4 For each j , $1 \leq j \leq m$, two voters of the form $b_j \ w \ \dots$.

Destructive Candidate Control in Plurality Voting

Lemma (Hemaspaandra, Hemaspaandra, and Rothe (2007))

If B' is a hitting set of S of size k , then w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Proof: See blackboard.



Destructive Candidate Control in Plurality Voting

Lemma (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $D \subseteq B \cup \{w\}$. If c is not a unique plurality winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

- 1 $D = B' \cup \{w\}$,
- 2 w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$,
and
- 3 B' is a hitting set of S of size less than or equal to k .

Proof: See blackboard.



Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

S has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $\{c, w\}$, spoiler candidates B , distinguished candidate c , and voter set V .

Proof: See blackboard. □

Corollary: Plurality voting is resistant to destructive control by adding candidates.

That is, **Plurality-DCAUC** (and also **Plurality-DCAC**) is NP-hard.

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

S has a hitting set of size at most k if and only if the election with candidate set C , distinguished candidate c , and voter set V can be destructively controlled by deleting at most $m - k$ candidates.

Proof: See blackboard.



Corollary: Plurality voting is resistant to destructive control by deleting candidates. That is, **Plurality-DCDC** is NP-hard.

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

S has a hitting set of size at most k if and only if the election with candidate set C , distinguished candidate c , and voter set V can be destructively controlled by partition of candidates (both in model TE and TP).

Proof: See blackboard. □

Corollary: Plurality voting is resistant to destructive control by partition of candidates (both in model TE and TP).

That is, **Plurality-DCPC-TE** and **Plurality-DCPC-TP** are NP-hard.

Voter Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

- 1 *Plurality voting is vulnerable/certifiably-vulnerable to destructive control both by adding voters and by deleting voters.*

*“Certifiably-vulnerable” means the chair cannot only decide the problems **Plurality-DCAV** and **Plurality-DCDV** in polynomial time, but can even produce in polynomial time a “best possible” control action. Certifiable vulnerability implies vulnerability. (In particular, the “k” may be dropped from the problem instance.)*

- 2 *In model TE, plurality voting is vulnerable/certifiably-vulnerable to constructive and destructive control by partition of voters. That is, **Plurality-CCPV-TE** and **Plurality-DCPV-TE** is in P.*

Destructive Control by Adding Voters in Plurality

Proof:

- 1 (a) Plurality voting is certifiably-vulnerable to destructive control by adding voters: “Smart Greedy”

Given (C, c, V, W) as in DCAV (without k):

- If c already is not a unique plurality winner in (C, V) , adding no voters accomplishes our goal, and we are done.
- Otherwise, sort all $d_i \in C - \{c\}$ by decreasing deficit, i.e., letting $\text{diff}(d_i)$ denote d_i 's deficit of first-place votes needed to tie c , we have

$$\text{diff}(d_1) \leq \text{diff}(d_2) \leq \dots \leq \text{diff}(d_{\|C\|-1}).$$

- For $i = 1, 2, \dots, \|C\| - 1$, if

$$\|\{w \in W \mid w\text{'s first choice is } d_i\}\| \geq \text{diff}(d_i),$$

then add $\text{diff}(d_i)$ of these unregistered voters to ensure that d_i ties c (and c thus is not a unique winner) and halt.

- If no iteration was successful, output “control impossible” and halt.

Destructive Control by Deleting Voters in Plurality Voting

(b) Plurality voting is certifiably-vulnerable to destructive control by deleting voters: “Dumb Greedy”

Given (C, c, V) as in DCDV (without k):

- If $C = \{c\}$, then output “control impossible” and halt;
- else if c already is not a unique plurality winner in (C, V) , deleting no voters accomplishes our goal, and we are done.
- If every candidate other than c gets zero first-place votes, then output “control impossible” and halt.
- Otherwise, let d be the candidate closest to c in first-place votes, and let $\text{diff}(d)$ denote d 's deficit of first-place votes needed to tie c .

Deleting $\text{diff}(d)$ voters whose first choice is c assures that c is not a unique winner, and this is the fewest deletions that can achieve that.

Constructive Control by Partition of Voters (TE) in Plurality Voting

- 2 Plurality voting is certifiably-vulnerable to constructive control by partition of voters:
- Let (C, c, V) be given as in CCPV.
 - For any partition (V_1, V_2) of V , let *Nominees* (C, V_i) , $i \in \{1, 2\}$, denote the set of candidates who are nominated by the subcommittee V_i (with candidates C) for the run-off in model TE.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Consider the following cases (Cases 3 and 5 need not be disjoint):

Case 1: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to $V_2 = \emptyset$.

Case 2: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \{c\}$.

Case 3: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to c and d (and possibly additional other candidates) tying, where $c \neq d$.

Case 4: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \{d\}$, $c \neq d$.

Case 5: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to d and e (and possibly additional other candidates) tying, where $c \neq d \neq e \neq c$.