

N-PERSON ENVY-FREE CHORE DIVISION

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1. INTRODUCTION

In this paper we consider the problem of *chore division*, which is closely related to a classical question, due to Steinhaus [7], of how to cut a cake fairly. We focus on *constructive* solutions, i.e., those obtained via a well-defined procedure or algorithm. Among the many notions of fairness is *envy-freeness*: an envy-free cake division is a set of cuts and an allocation of the pieces that gives each person what she feels is the *largest* piece. Much progress has been made on finding constructive algorithms for achieving envy-free cake divisions; a landmark result was that of Brams and Taylor [1], who gave the first general n -person procedure.

In contrast to cakes, which are desirable, the dual problem of chore division is concerned with dividing an object deemed *undesirable*. Here, each player would like to receive what he considers to be the *smallest* piece, of say, a set of chores. This problem appears to have been first introduced by Martin Gardner in [4]. Oskui (see [6]) referred to it as the *dirty work problem* and gave the first discrete and moving-knife solutions for exact envy-free chore division among 3 people. Peterson and Su [5] gave the first explicit 4-person moving-knife procedure for chore division, adapting ideas of Brams, Taylor, and Zwicker [3] for cake-cutting.

The purpose of this article is to give a general n -person solution to the chore division problem. Su [9] gives an n -person chore division algorithm but it only yields an ϵ -approximate solution after a finite number of steps. Brams and Taylor suggest in [2] how cake-cutting methods could be adapted to chore division without working out the details, and our algorithm owes a great debt to their ideas. But we also show where some new ideas are needed, and why the chore division problem is not exactly a dual or straightforward extension of the cake-cutting problem.

2. SOME KEY IDEAS

We assume throughout this paper that chores are infinitely divisible. This is not unreasonable as a finite set of chores can be partitioned by dividing up each chore (e.g., a lawn to be mowed could be divided just as if it were a cake), or dividing the time spent on them. For ease of expression, we shall call the set to be divided a *cake*, rather than a *set of chores*. Implicit in this is the assumption that the players desire the *smallest, rather than largest, piece of cake*.

We remind the reader that players may have different preferences over pieces of cake (indeed that is what makes the problem interesting). More formally, each player i has a

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measure μ_i that describes what value $\mu_i(A)$ the player assigns to a piece of cake A . (The cake pieces in our construction will always be measurable). Such measures are *additive*, meaning that no value is created or destroyed by cutting or lumping pieces together.

Before addressing the chore division problem, we wish to highlight a couple of key ideas from n -person cake-cutting algorithm of Brams and Taylor, and discuss the analogous ideas in the chore division context.

- **Trimming to Create Ties.** Given a division of cake into several pieces, a player B can create a “tie” in what she considers the largest piece by trimming the largest piece (the trimmings are temporarily set aside). This way, if another person C chooses a piece before her, player B will still have one of her top two choices for largest piece available to choose, so she will not envy C who chose before her.
- **Irrevocable Advantage.** Suppose that the cake has been allocated, except for trimmings, in such a way that player A receives a piece that he believes to be ϵ bigger in his measure than the piece B received. Suppose also that the trimmings are, in A ’s estimation, of size less than ϵ . Then no matter how much of the trimmings are given to B , player A will never envy her. We say that A has an *irrevocable advantage* over B .

For chore division, the idea corresponding to trimming is the idea of “adding back” or *augmentation* from a set of *reserves*. We will also use an analogous concept of an irrevocable advantage. Both ideas are present in the 3-person chore division algorithm of Oskui (see [6]), and we will use them repeatedly in our n -person algorithm.

- **Augmentation with Reserves.** Suppose a player has set aside, in advance, a set of cake we will call her *reserves*. Given a division of cake, a player B can create a tie for *smallest* piece (in her opinion) by augmenting the smallest piece with cake from her reserves. This is to ensure that if another player C chooses a piece before her, she will still have one of her two smallest pieces available to choose from, so that B will not envy C .
- **Irrevocable Advantage.** Suppose that the cake has been allocated, except for unused reserves, in such a way that player A receives a piece that he believes to be ϵ *smaller* in his measure than the piece that B received. Suppose that the unused reserves are, in A ’s estimation, of size less than ϵ . Then no matter how much of the unused reserves are given to A , player A will not envy player B . We say that A has an *irrevocable advantage* over B .

Of course, some issues that we will have to resolve in our algorithm are: (1) how to create enough reserves for players to use and (2) how to deal with unused reserves. Irrevocable advantages can be used for (2) when the reserves are small enough.

3. AN N -PERSON ENVY-FREE CHORE DIVISION PROCEDURE

We now construct our n -person chore-division procedure. Unlike the 4-person moving-knife scheme of Peterson and Su [5], ours is a discrete procedure (involving no continuous evaluations of pieces like moving-knife methods require). And while their 4-person procedure possessed a natural set of reserves due to the initial trimming, for our n -person procedure we need to carefully create enough reserves for use by specific players.

A brief sketch of our procedure runs as follows. Let one player divide the cake and allocate the pieces. As long as there are objections, we shall iterate a procedure that gives an envy-free allocation of part of the cake (Steps 1-9), and also gives a player who objected

an irrevocable advantage over another player with respect to the part of the cake that has not yet been allocated (Steps 10-15). With enough iterations on the leftovers, there will be enough players with irrevocable advantages to allow the allocation of the remainder of the cake (Steps 16-20).

As we noted, our method closely mirrors Brams and Taylor's n -person cake-cutting procedure [1], but differs from theirs in using augmentation (rather than trimming) and the creation of reserves. For ease of comparison with Brams and Taylor's cake-cutting procedure, we include step numbers in our procedure that correspond their step numbers [1]. The significant departures occur in Steps 6.1, 7.1, 7.3, 8, and 10-15.

Following their example, we distinguish *rules* from *strategies* by placing strategies in parentheses. Again, for ease of expression, we refer to the chore set as a cake, bearing in mind that each person wants the piece he/she thinks is *smallest*.

We shall exhibit our procedure for $n = 4$ case. The generalization to more players will be discussed subsequently.

- Step 1.** Let Player 2 cut the cake into 4 pieces (that she considers equal), and then assign one piece to each player.
- Step 2.** Player 2 asks the other 3 players if anyone is envious.
- Step 3.** If no one has envy, then each keeps the piece he was given, and we are done.
- Step 4.** If someone has envy, say Player 1, let Player 1 choose two pieces (that he thinks are not equal in size) and name them A (for the larger piece) and B (for the smaller piece).

Aside. The other pieces are reassembled for allocation later. Note that Player 1 thinks A is larger (hence less desirable) than B but Player 2 thinks they are the same size.

- Step 5.** Let Player 1 name an integer $r \geq 11$ (chosen such that, even if A were divided into r pieces and the 8 smallest pieces of A were removed, he would still prefer B).

Aside. This is possible because the union of the 8 smallest pieces is no larger than 8 times the average size of all r pieces. Hence Player 1 can choose r large enough so that $8\mu(A)/r < \mu(A) - \mu(B)$ where μ is Player 1's measure.

- Step 6.** Player 2 divides each of A and B into r sets (that she considers equal).
- Step 6.1.** From the pieces in B , let Player 3 choose two pieces (he considers largest). This will be used as Player 3's reserves.
- Step 7.** From the remaining pieces, Player 1 chooses (what he thinks are the smallest) 3 sets in B , calling these Y_1 , Y_2 , and Y_3 .
- Step 7.1.** The rest of B is set aside as Player 1's reserves. If necessary, Player 1 uses his reserves to add to two of the Y_i (to make three of the Y_i 's equally sized— the reserves will be enough because they came from at least two pieces in B that Player 1 feels is at least as large as each of the Y_i 's).
- Step 7.2.** If Player 1 considers the three largest pieces in A all strictly larger than these pieces, the three largest pieces in A are identified as Z_1 , Z_2 , and Z_3 .
Otherwise, Player 1 cuts the largest piece in A into three (equal) pieces, calling them Z_1 , Z_2 , and Z_3 (which Player 1 will feel is strictly bigger than each of the Y_i 's).

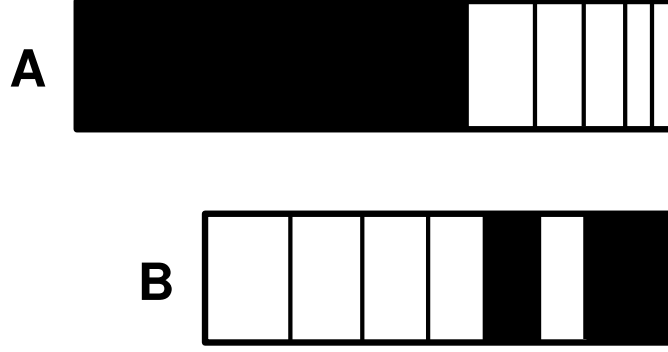


FIGURE 1. Steps 6-8: Player 1's view of the cakes and piece sizes. Player 2 views all pieces as equal. Z_1, Z_2, Z_3 will either be the 3 shaded pieces in A or the largest piece A_1 split into thirds. (The number of pieces r was chosen large enough so that in one of these ways Player 1 would feel the Z_i were strictly larger than the shaded pieces in B .) If A_1 was split to form the Z_i , Player 2 may use the dotted pieces (her reserves) to equalize the Z_i in her opinion. In B , Player 3 chooses, say, the striped pieces as his reserves. After these are chosen Player 1 chooses the smallest 3 remaining (here, the shaded pieces) in B to be Y_1, Y_2, Y_3 , then augments them from his reserves (unshaded pieces of B) to make them equal in his opinion.

Aside. We show why one of the two cases in Step 7.2 must hold. Let μ denote Player 1's measure, and sets A_1, \dots, A_r and B_1, \dots, B_r the pieces of A and B arranged by decreasing μ -size. Since Player 3's reserves were chosen *before* the Y_i 's, the Y_i 's are among the 5 smallest sets B_{r-4}, \dots, B_r . Suppose, to contradict both cases in Step 7.2, that both the following hold: (i) $\mu(B_{r-4}) \geq \mu(A_3)$ and (ii) $\mu(B_{r-4}) \geq \mu(A_1)/3$. From (i),

$$\mu(B_7 \cup \dots \cup B_{r-4}) \geq \mu(A_3 \cup \dots \cup A_{r-8}),$$

because there are $r - 10$ sets in each union and the smallest of the B sets is at least as large as the largest of the A sets. From (ii),

$$\mu(B_1 \cup \dots \cup B_6) \geq \mu(A_1 \cup A_2),$$

since each group of three B sets is at least as large A_1 or A_2 . Taken together, $\mu(B) \geq \mu(A_1 \cup \dots \cup A_{r-8})$, contradicting our choice of r in Step 5.

Step 7.3. The remaining pieces of A are set aside for Player 2's reserves. If necessary, Player 2 uses her reserves to add on to two of the Z_i (to make all three Z_i 's equally sized— this would only be needed in the second case of Step 7.2, and the reserves are enough because they came from pieces in A that were *each* the same size as all three Z_i 's combined).

Aside. At this stage, Player 1 believes $Y_1 = Y_2 = Y_3 < Z_1, Z_2, Z_3$ (note the strict inequality). Player 2 believes $Z_1 = Z_2 = Z_3 \leq Y_1, Y_2, Y_3$. See Figure 1.

Step 8. Player 3 takes this collection of 6 pieces and, if necessary, augments one of the pieces using his reserves (to make a two-way tie for *smallest* piece— Player 3 has enough reserves because the 2nd-smallest piece

cannot be larger than the 2nd-smallest Y_i , and Step 6.1 gave him reserves from two pieces of B , each of which he feels is larger than each Y_i and larger than any possible augmentation of Y_i by Player 1 in Step 7.1).

Step 9. Let the players choose in the order 4-3-2-1, with Player 3 required to take a piece he augmented if it is available. Player 2 must choose one of the Z_i 's, and Player 1 must choose one of the Y_i 's.

Aside. This yields a partition $\{X_1, X_2, X_3, X_4, L_1\}$ of the cake such that the X_i 's are allocated in an envy-free fashion, and L_1 is the leftover piece consisting of all cake not yet allocated. Moreover, note that Player 1 thinks his piece is *strictly smaller* than Player 2's piece, say, by an amount ϵ .

Step 10. Player 1 names s (such that $[\frac{1}{2}\mu(L_1)]^s < \epsilon$, where μ is Player 1's measure).

Aside. Steps 11-14 will result in Player 1 thinking that at least half of L_1 is allocated. Hence s represents the number of times to iterate Steps 11-14 to make the leftover piece smaller than ϵ .

Step 11. Player 1 cuts L_1 into 8 (equal) pieces. Player 2 sets aside (the largest) 2 pieces for her reserves, and Player 3 sets aside a piece from those remaining (that he feels is largest of those remaining) for his reserves.

Step 12. Of the remaining 5 pieces, Player 2 returns part of her reserves to 2 of the pieces, if necessary (to create a 3-way tie for the smallest piece).

Step 13. Player 3 returns, if necessary, some from his reserves to one of the 5 pieces (to create a 2-way tie for the smallest piece— his reserves will be sufficient because his piece is at least as large as the 2nd-smallest piece).

Step 14. Let the players choose in the order 4-3-2-1, with Players 2 and 3 required to take a piece they augmented if one is available. The chosen pieces are combined with the corresponding pieces from the players' earlier envy-free allocation to form a new envy-free partial allocation of cake $\{X'_1, X'_2, X'_3, X'_4\}$, and the non-chosen pieces are lumped together to form a new smaller leftover piece L_2 .

Step 15. Repeat steps 11-14 ($s - 1$) additional times, with each application of these four steps applied to the leftover piece L_2 from the preceding application.

Aside. According to Player 1, the leftover L_2 is now smaller than ϵ , so Player 1 feels that his portion together with L_2 (or any part of L_2) would still be smaller than Player 2's portion. Thus, Player 1 has an *irrevocable advantage* over player 2 with respect to the leftovers. We consider a subset of ordered pairs in $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ called \mathcal{IA} , to keep track of which player pairs have irrevocable advantages with respect to the leftovers, and we place the ordered pair $(1, 2)$ in \mathcal{IA} . Note by definition, any further subdivision and allocation of pieces of L_2 with a smaller set of leftovers will not remove $(1, 2)$ from \mathcal{IA} .

Step 16. Player 2 cuts the remaining leftovers L_2 into 12 pieces (that she feels are the same size).

Step 17. Each player who agrees that all pieces are the same size is placed in the set A . Otherwise, players who disagree are placed in the set D .

Step 18. If $D \times A \subset \mathcal{IA}$, we divide the pieces among the players in A , each receiving the same number of pieces, and we are done.

Aside. Players in A do not envy each other since they agree each of the 12 pieces were the same size. None of the players in D envy those in A by the definition of \mathcal{IA} . Players in D do not envy each other because they have not received any new pieces in this step.

Step 19. Otherwise, we choose the lexicographically least pair (i, j) from $D \times A$ that is not in \mathcal{IA} , and return to step 4, with Player i in place of Player 1, Player j in place of Player 2, and L_2 in place of the cake.

Step 20. Repeat Steps 5-18.

Aside. Since each pass through Step 15 adds a new ordered pair to \mathcal{IA} without removing any ordered pairs, eventually \mathcal{IA} will contain $D \times A$, no matter what $D \times A$ currently is. When this occurs, the algorithm concludes at Step 18 with an envy-free chore division of the entire cake. This ends the procedure.

The extension of this procedure to n players is tedious but straightforward, so we briefly mention the changes that ensue and leave the verification to the reader.

Let $k = 1 + 2 + \dots + (n - 3)$. The 8-piece criterion in Step 5 needs to be increased (to $k^2 + 4k + 3$ pieces) for later use in Step 7. This necessitates an increase in r . In Step 6.1, all players except Players 1, 2, and n will choose, in reverse order, pieces of B to form their reserves. Specifically, player i will choose what she thinks are the $2(n - i)$ largest pieces of what remains in B ; this ensures that these players have enough reserves for Step 8. Thus the total number of pieces chosen for reserves in Step 6.1 is $2k$.

In Step 7, there need to be more Y_i 's and Z_i 's chosen ($k + 2$ of each), so that after the augmentation in Step 8 there are at least two Y_i 's and two Z_i 's untouched. Then Step 9 will allow players to choose in reverse order and still have one of the Z_i 's available for player 2 and one of the Y_i 's available for Player 1. Also, in Step 7.2, if Player 1 feels the biggest $k + 2$ pieces of A are not all larger than the Y_i 's just chosen, then r needs to be large enough ($r \geq k^2 + 5k + 5$) so that the largest piece of A can be split evenly into $k + 2$ pieces all larger than the Y_i 's. (Some care needs to be exercised here, since the reserves are chosen *before* the Y_i 's, so Player 1 may only conclude that the Y_i 's are among the $2k + 1$ smallest pieces of B .) The size of r also guarantees that Player 1 and 2 have enough reserves from the remainders of B and A .

In Step 8, there are enough reserves because each person feels his reserves are enough to cover the correct number of pieces of Y_i and any possible augmentations by Player 1. In Step 9, the players choose pieces in reverse order and players are required to take a piece they augmented if available. Steps 10-15 can be modified analogously (let Player 1 start by cutting the leftovers into $n^2 - 3n + 4$ equal pieces) to accomodate more people in the iterative part of the procedure.

4. REMARKS

We have shown explicitly how cake-cutting algorithms can be translated, with complications, into exact envy-free chore-division algorithms.

There are two important features of this translation. First, the use of augmentation from reserves is very important, and we showed that the creation of such reserves is generally possible. Secondly, the notion of an irrevocable advantage for chores allows one to terminate what might otherwise be an infinite procedure.

Curiously, though our chore-division procedure is more complicated than its cake-cutting counterpart, it may converge faster and require fewer cuts overall. For instance, for $n = 4$, each pass through the iterative part of the procedure (Steps 10-15) guarantees at least half the cake is apportioned rather than only one-fifth. The authors have noted that faster convergence of chore division is also a feature of the ϵ -approximate algorithm of Su [9].

Note that unlike the 4-person moving-knife procedure of Peterson and Su [5] (which required at most 16 cuts), this n -person algorithm may take arbitrarily many cuts (and steps) to resolve, depending on player preferences, even for fixed n . So the number of steps is finite, but not bounded. It remains an open question whether a *bounded* procedure exists for either cake or chore division among n -people.

Given the numerous cuts needed to implement both the Brams-Taylor n -person cake-cutting procedure [2] and our n -person chore-division procedure, one may rightfully question their practicality. There are two possible responses. First, the number of cuts can be reduced if one is willing to accept an ϵ -approximate solution, by rotating players through the roles of Steps 10-15, and quitting when satisfied. Secondly, the construction of an initial solution, however complex, is always the first step towards finding useful simplifications.

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