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Game-theoretic Analysis of Strategyproofness in Cake-cutting Protocols

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Bachelorarbeit

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Erklärung

Hiermit versichere ich, dass ich diese Bachelorarbeit selbstständig verfasst habe. Ich habe dazu keine anderen als die angegebenen Quellen und Hilfsmittel verwendet.

Düsseldorf, den 05. Dezember 2011

Alina Elterman

Abstract

In cake-cutting a protocol instructs the players how to divide a resource between them in a satisfactory manner. A part of those instructions, namely the strategies, are optional and can be examined whether they obtain the best solution for the players. If this is not the case, the players have no intention to follow the protocol and can be overruled. Otherwise it is strategyproof.
Game Theory is designed to determine better strategies. By using a game-theoretic illustration of the cake-cutting problem it is possible to compare the strategies.
The recommended strategy appears to be the best one in the well-known protocols with one exception.

Contents

1 Introduction

1.1 Related Work

In the approaches of [Chen et al., 2010] and [Mossel and Tamuz, 2010], the basic assumptions of cake-cutting is weakened and so only special cases are analysed. [Lindner and Rothe, 2009]
[Brams et al., 2007a][Brams et al., 2008][Thomson, 2006][Brams et al., 2007b][][][][][][][][]

2 Preliminaries

2.1 Preliminaries of Game Theory

In this chapter, some basic concepts from game theory are described and directly applied to the cake-cutting problem. In particular the priority have the possible representations of games. For further reading see [], [] and [Meir, 2009].

2.1.1 Concepts in Game Theory

Definition 1. ***(Game)***
A non-cooperative (strategic) game $\Gamma = (P_n, S, u)$ *consists of the set of players* P_n*, the set of strategies* S *and the set of utility functions (pay-off) of all players* u_i*.*

- *Each player in the set* $P_n = \{p_1, \cdots, p_n\}$ *behaves selfishly and rationally.*
- *Utility is a real-valued quantity measuring an agent's happiness. A utility function is a mapping from outcomes to utilities. The agent is indifferent between outcomes with equal utilities and strictly prefers outcomes with higher utilities.*

Definition 2 (Outcome) An outcome is any end-state of the system.

Definition 2. *Normal form game*
A normal form game is a tabular representation of a game. In the two-dimensional case (i.e. for two agents) the first agent chases the row while the second choses the column. Each cell contains a vector of utility values, one per player.

	$Strategy_a$	$Strategy_b$
$Strategy_I$	$(v_1(X_{1,I,a}), v_2(X_{2,I,a}))$	$(v_1(X_{1,I,b}), v_2(X_{2,I,b}))$
$Strategy_{II}$	$(v_1(X_{1,II,a}), v_2(X_{2,II,a}))$	$(v_1(X_{1,II,b}), v_2(X_{2,II,b}))$

Table 1: Game in normal form

Definition 3. ***(Pure/ Mixed Strategy)***
A

Definition 4. ***(Strategies)***
dominant, dominated, best response
A pure strategy for a normal form game is a single action.
A mixed strategy is a probability distribution over pure strategies.
A best response is a strategy (pure or mixed) which maximizes the agent's expected utility given the strategies of all the other agents.
A dominant strategy is a strategy that is a best response regardless of the strategies of the other players.

Definition 5. ***(Nash-Equilibria)***
A Nash Equilibrium is ct strategy profile where ev- ery agent is playing a best response

to the actions of the others. We can characterize a Nash equilibrium as being a pure-strategy or mixed-strategy equilibrium. It is also useful to distinguish equilibria in dominant strategies.

2.1.2 Classes of Special Games

Information in Games

Definition 6. ***(Perfect/ Imperfect Information)***
A

Definition 33 (Perfect information extensive form game) Perfect information exten- sive form games are games in tree representation where every inner node represents a choice by a single agent, with the out-edges of the node being the possible actions. Each leaf node is an outcome labeled by a vector of utilities (one per agent).

Definition 7. ***(Complete/ Incomplete Information)***
A

Definition 35 (Imperfect information extensive form game) Imperfect information extensive form games are extensive form games where the agent cannot distinguish two or more choice nodes (those nodes are said to belong to an equivalence set).

Definition 37 (Pure strategy (extensive form game)) / pure strategy for an exten- sive form game is a mapping from nodes or equivalence sets to actions analogous to a policy in decision theory.)

Definition 38 (Induced normal form (extensive form game)) We can convert an ex- tensive form game into a normal form game (called the "induced normal form ") where the normal form game's actions are pure strategies in the extensive form game. The payoffs in the normal form game are taken front the leaf-node reached by following those strategies. Because every extensive form game has a corresponding induced normal form game, it must have a Nash equilibrium (by Theorem 27).

Definition 8. ***(Zero-Sum Game)***
A

Cutting a cake is a non-zero-sum game since it allows players to get better off than $1/n$-th. An exception is a homogeneous cake because the valuations over the cake are equal.

Definition 9. ***(Repeated Game)***
A

Definition 10. ***(Bayesian Game)***
A

Often, strategic interactions involve some notion of random events and private information. We model these using Bayesian games. Many card games (for example, poker) are instances of this type of interaction.

Definition 39 (Type) An agent's type is a complete representation of his private information.
Definition 40 (Type profile) A type profile is a vector of types, one per agent. One simple but powerful way to represent Bayesian games is to say that the agents have different information about which normal form game they are actually playing.

Definition (Bayesian Game)
A Bayesian Game is a set of players, each of which has a set of actions and a utility function mapping action and type profiles to utility. A Bayesian game also includes a distribution over type profiles.

Definition 11. ***(Backwards Induction)***
A

Definition 12. ***(Subgame-perfect Equilibria)***
A

A game can be represented in normal or in extended form. The normal form is advantageous for games where players move simultaneous, which is not often the case in cake-cutting. So the presentation in extended form as a game tree will be used in this work.

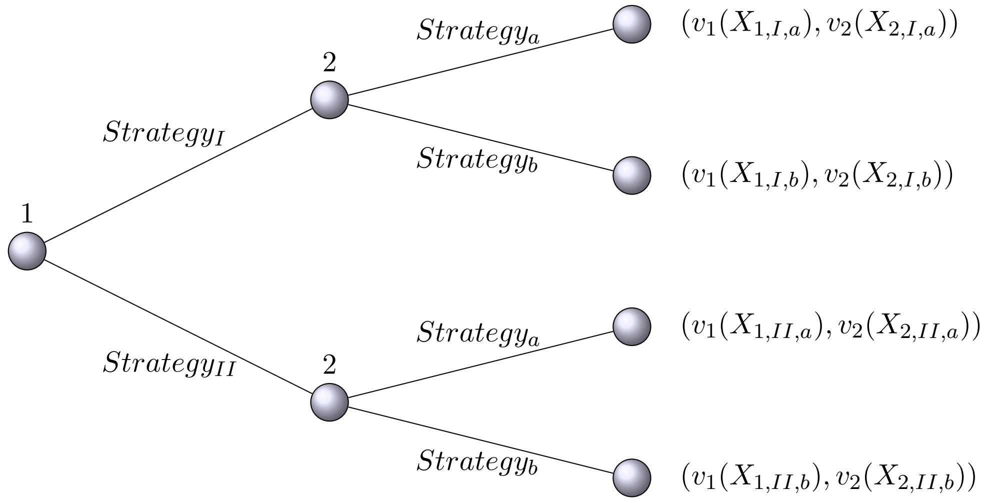

Figure 1: Game in extended form

2.2 Preliminaries of Cake-cutting

2.2.1 Basics

It is necessary to define the components and challenges of cake-cutting. But first, what exactly is cake-cutting about? It involves a set of $n \in \mathbb{N}$ players $P_n = \{p_1, \ldots, p_n\}$. It is assumed that each of them wants to get as much as possible of the divided resource. The goal is to find an allocation of a single, divisible and heterogeneous good between the n players.

Such an allocation has to be of a special kind, so that the involved players are pleased

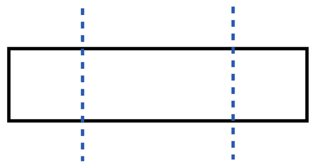

Figure 2: Cake
Example for a visualisation of a cake with two cuts

with the outcome. For the visualization it is common to use a rectangular cake. The division is performed by parallel cuts. The cake X is represented by the unit interval $X = [0, 1] \subseteq \mathbb{R}$. Each subinterval $X' \subseteq X$ or a sequence of disjoint subintervals

$$\bigcup_{m \in \mathbb{N}} X'_m$$

with $X'_m \subseteq X$ is called a *portion (or piece)*. The portion of the cake which is received by player p_i is denoted X_i. The state is called an *allocation*, when all portions of the cake are owned by players. Each piece has a public size, which can be computed as the sum of all border differences, and the private value of each player,(welcher durch die folgend definierte Valuation dargestellt wird) which is constituted by the valuation function.

Every player $p_i \in P_n$ has a *valuation function (valuation)* $v_i : \{X' | X' \subseteq X\} \to [0, 1]$ with the following properties:

1. Non-negativity: $v_i(C) \geq 0$ for all $C \subseteq X$.
2. Normalisation: $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$.
3. Additivity: $v_i(C \cup C') = v_i(C) + v_i(C')$ for disjoint $C, C' \subseteq X$.[1]
4. Divisibility: For all $C \subseteq [0, 1]$ and all $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, there exists a $B \subseteq C$, so that $v_i(B) = \alpha \cdot v_i(C)$.
5. v_i is continuous: If $0 < x < y \leq 1$ with $v_i([0, x]) = \alpha$ and $v_i([0, y]) = \beta$, then for every $\gamma \in [\alpha, \beta]$ there exists a $z \in [x, y]$ so that $v_i([0, z]) = \gamma$.

[1]Monotonicity: If $C' \subseteq C$ then $v_i(C') \leq v_i(C)$. Monotonicity follows from additivity, because for the assumption $C' \subseteq C$ and $A := C \backslash C'$: $v_i(C) = v_i(A \cup C') = v_i(A) + v_i(C') = \underbrace{v_i(C \backslash C')}_{\geq 0} + v_i(C') \geq v_i(C')$.

6. Non-atomic: $v_i([x, x]) = 0$ for all $x \in X$.

After some basics it would be interesting to see how game-theory is applicable to cake-cutting. Example 1 illustrates the problem in a game-theoretic manner.

Example 1.

John Cocke and Tadao Kasami want to divide a chocolate-strawberry-cake. The cake is half chocolate from the left and the right part is strawberry. John Cocke got the first move and is thinking about making three different cuts. After the cuts the two pieces would have the following values:

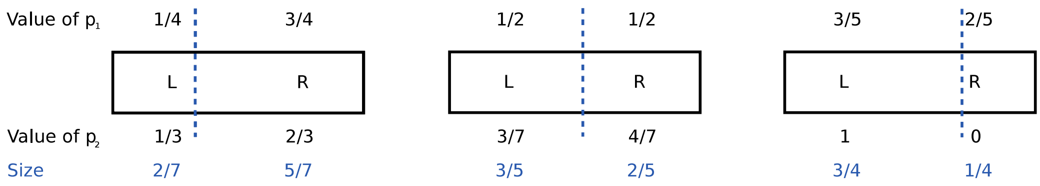

Figure 3: C&K cake game

Before doing so, he analyses his situation via the normal form:

	Leftcut	Middlecut	Rightcut
L	$(3/4, ?)$	$(1/2, ?)$	$(2/5, ?)$
R	$(1/4, ?)$	$(1/2, ?)$	$(3/5, ?)$

Table 2: C&K cake game in normal form

Since the valuation is a private function, he does not know the preferences of his colleague and has to assume that Tadao is indifferent between the two pieces. Tadao is waiting for John's move and will choose his best strategy in the extended game form:

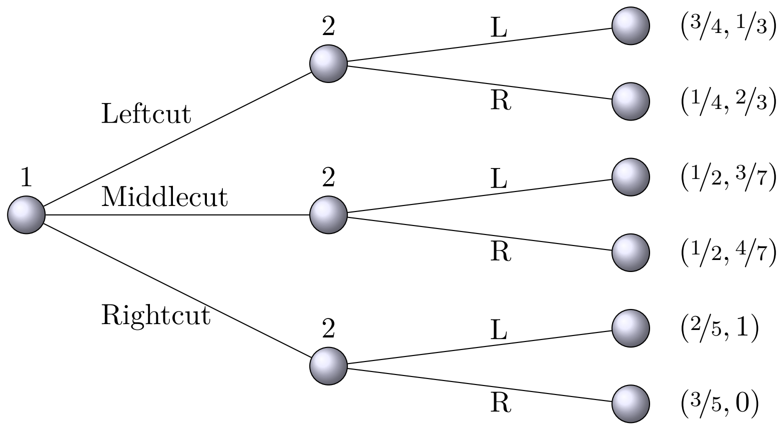

Figure 4: C&K cake game in extended form

For John the graph shows that if he stays secure he would obtain $v_1(X_1) = 1/2$. Otherwise he would get $1/4$ or $2/5$. So to make the middlecut is his best possible move.

2.2.2 Different Types of Fairness

As indicated by the name, fairness plays an important role in fair division. But how is fairness defined? It can be seen as a valuation criterion of an allocation, which can be normalized and gives a possibility to compare different allocations. Usually the fairness criteria are distinguished between the following:

Definition 13. ***(Proportional or Simple Fair)***
An allocation is *proportional (simple fair)* if $v_i(X_i) \geq 1/n$ for each player $p_i \in P_n$.

Definition 14. ***(Envy-Freeness)***
An allocation is *envy-free* if $v_i(X_i) \geq v_i(X_j)$ for each couple of players $p_i, p_j \in P_n$.

Theorem 1 shows the correlation between the two types of fairness.

Theorem 1.

1. *Every envy-free allocation is proportional.*
2. *An allocation between two players is envy-free if and only if it is proportional.*

Proof.

1. Proof by contradiction:
   Assume A is an envy-free allocation, but not proportional. From envy-freeness follows $v_i(X_i) \geq v_i(X_j)$ for each pair of players $p_i, p_j \in P_n$ and so each player has at least an as much valuable piece of cake as each other player in his own valuation. Hereby each player owns in his own valuation at least as much as each of the $(n-1)$ other players. The smallest value for which it is possible that $(n-1)$ players and the considered players own the same value in the valuation of the considered player is $1/((n-1)+1)$. Each player owns in his valuation at least $1/n$. $\lightning$ The allocation A is proportional.
   Therefore, all envy-free allocations are proportional.
2. "$\Leftarrow$" For two players p_1 and p_2 an allocation is proportional if each player has $p_i(X_i) \geq 1/2$ for $i \in \{1,2\}$. The whole cake has the value $v_i(X) = 1$ for $i \in \{1,2\}$. And so for each player $v_i(X - X_i) \leq 1 - 1/2 = 1/2$ for $i \in \{1,2\}$. The value of the piece not obtained by a player is smaller or equal to his obtained piece. So the player p_1 does not envy the other player p_2 and vice versa.
   "$\Rightarrow$" Follows from part 1.

$\square$

A different criterion to valuate the quality of an allocation is efficiency. Further correlations between the fairness criteria and efficiency can be found in [Caragiannis et al., 2009].

Definition 15. ***(Efficiency)***
An allocation

$$A = \{X_1, \ldots, X_n\}$$

is *efficient (Pareto optimal)* if there is no other allocation

$$A' = \{X'_1, \ldots, X'_n\}$$

such that

$$v_i(X_i) \leq v_i(X'_i)$$

for all players $p_i \in P_n$ and for at least one player the inequality is strict.

Theorem 2.

1. *Envy-freeness and proportionality do not imply efficiency.*
2. *Efficiency does not imply proportionality and so envy-freeness.*

Proof.

1. Imagine the following allocation with three players. Each player's portion can consist up to three pieces. The value of the whole portion is (because of the additivity of the valuation) the sum of the pieces :

	$X_1 = X'_1 \cup X''_1$	$X_2 = X'_2 \cup X''_2$
p_1	$1/2 = 1/3 + 1/6$	$1/2 = 1/12 + 11/12$
p_2	$1/3 = 1/6 + 1/6$	$2/3 = 2/3 + 0$

Table 3: Example for envy-freeness does not imply efficiency

This allocation is obviously envy-free, since $v_1(X_1) = v_1(X_2)$ and $v_2(X_2) > v_2(X_1)$. It is not efficient, because if the player p_1 would get p_2's portion X''_2, p_1 would get a more valuable piece of the cake and p_2 would not get a less valuable piece. Since in Theorem 1 was shown that envy-freeness implies proportionality, this example also demonstates that proportionality does not imply efficiency.

2. Allocating the whole cake to one player is efficient, but definitely not proportional and therefore not envy-free.

□

In [Brams and Taylor, 1996] the authors show a general argument that no finite bounded protocol can exist for such an allocation that is both proportional and efficient at the same time.

2.2.3 Different Types of Protocols

It is very important to understand the types, structure and design of protocols, which are analysed in this work.

Informal: (Algorithm)
An *algorithm* is an effective method for solving a problem, which is composed of a finite sequence of instructions.

Definition 16. ***(Cake-Cutting-Protocol)***
A *cake-cutting-protocol* (protocol for short) is an algorithm with a fixed number of players and the following properties:

- A protocol consists of rules and strategies.
  Rules are requirements, which *have to* be followed by the players. Furthermore, it is possible to ensure that the players obey the rules.
  Strategies are recommendations, which *can* be followed for getting the guaranteed fair share. It is impossible to verify whether a player follows the strategy of the protocol.
- Each player should be able to cut the cake at a specific moment independent of other players.
- The protocol has no information about the valuation of the players, except of those it got from the steps before.

Comment: Only such protocols are interesting where the actions of one player does not harm the other players.

Definition 17. ***(Proportional/ Envy-Free Protocol)***
A cake-cutting *protocol* is called *proportional* or *envy-free* if independent of the players' valuations, each allocation is proportional or envy-free provided that all players follow the rules and strategies given by the protocol.

The development of such protocols is one of the main goals of cake-cutting [Robertson and Webb, 1998].

Definition 18. ***(Finite (Discrete) / Continuous (Moving-Knife) Protocol)***
A *finite (discrete) protocol* gives a solution after a finite number of queries (valuations, marks, ...). In a *continuous (moving-knife) protocol* a player has to make up to infinitely many queries.

Definition 19. ***(Finite Bounded / Finite Unbounded Protocol))***
A *finite bounded protocol* has an upper bound of steps for all possible valuations. The number of those steps is correlated with the number of players only in some cases. A *finite unbounded protocol* has no approximated number of steps.

The most desirable protocols are the finite bounded because of the ease of their implementation.

In the last sixty years the number of proportional finite bounded protocols has grown for an arbitrary number of players. But still no envy-free finite bounded protocol for an arbitrary n is known [Chen et al., 2010]. Only for three or less players a cake can be divided in a fixed number of steps, so that it is envy-free. For this reason, only proportional protocols are considered in the further work.

3 Strategyproofness

In the scope of this work it is assumed that players are selfish and try to increase the value of their portion. In order to do so, they may misrepresent their valuation on the cake. The goal is to prevent this.

Definition 20. ***(Non-Truthful (Cheating) / Honest Player)***
Every strategy is called *non-truthful* except of the strategy recommended by the protocol. A player who follows a non-truthful strategy will be called a *non-truthful (cheating) player*. Otherwise the player is called *honest*.

Definition 21. ***(True Value Function)***
A *true value function* provides the value of the piece a player would receive by following the recommended strategy. This value is at least proportional in a proportional protocol.

A strategy S_1 is better for player p_i than another strategy S_2 if the value of the obtained piece by following S_1 is bigger than by following S_2.

Definition 22. ***(Risk Aversion [Brams et al., 2007a])***
A player is *risk averse* if he will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a piece of a guaranteed size.

Definition 23.
(Strategyproofness of a Proportional Protocol [Lindner and Rothe, 2009])
A proportional cake-cutting protocol is said to be *strategyproof for risk averse players* (SPP for short) if a cheating player is no longer guaranteed a proportional share, whereas all other players (provided they play truthful) are still guaranteed to receive their proportional share.

Definition 24. ***(Strategyproofness in the sense of [Brams et al., 2008])***
A protocol is *strategyproof* if no player has a strategy that is assuredly better than his true value function.

The strategyproofness in the sense of Definition 24 will be called weak strategyproofness (WSP for short) since it is always true for a proportional protocol, see also [Hill and Morrison, 2010].

Example 2.

Assume the case when all valuations over the cake are equal, and all players, except of the cheating one follow the strategy provided by the protocol. Each of the honest players will get his proportional share, which the cheating player also values as $1/n$ *or* *more. Sharing a cake with* $(n-1)$ *other players means for the cheater that* $^{(n-1)}/_{n}$ *or a more valuable part of the cake is allocated to other players and so only the value of* $^{1}/_{n}$ *or less remains for him independent of his strategy.*

As depicted in Example 2, there always exists a valuation such that in an allocation a cheating player will never obtain more than a proportional piece. Hence, weak strategyproofness is not significant.
A stronger condition comes from the social choice literature:

Definition 25. ***(Strategyproofness in the sense of [Thomson, 2006])***
A protocol is *strategyproof* if the true value function dominates every other strategy.

In order to prevent misunderstandings, in this work strategyproofness in the sense of Definition 25 will be called strong strategyproofness (SSP for short). It can be shown that none of the known cake-cutting protocols is able to fulfill the strong strategyproofness criteria, if the valuation of the players is not equal. All protocols shown in Chapter 3 work for two players in exactly the same way as Cut & Choose. Example 3 is similar to the one in [Chen et al., 2010].

Example 3.

John Warner Backus and Peter Naur are celebrating and Donald E. Knuth has brought a huge marzipan cake with an enormous cherry on the left side. John loves cherries and hates marzipan, and Peter is just very hungry. The pioneers of computer science apply Cut & Choose (see Chapter 3.2). Peter is the cutter, and his best strategy would be to separate the cake from the cherry. If Peter had full knowledge (which would not violate the preconditions of strategyproofness in [Thomson, 2006]) about the valuations of John, he would benefit from lying. From Table 4 he would know, that John would always take the left piece and so he could easily maximize the value of his portion. Hence, this algorithm is not strongly strategyproof.

	Only Cherry	Middlecut
L	$(^{9}/_{10}, 1)$	$(^{1}/_{2}, 1)$
R	$(^{1}/_{10}, 0)$	$(^{1}/_{2}, 0)$

Table 4: B&N cake game in normal form

In strong strategyproofness a player would never get a more valuable piece by lying independent of the valuation of the other players.

After a counterexample in [Hill and Morrison, 2010] the definition of strategyproofness in [Brams et al., 2008] was restricted to the case with non-equal valuations and for the general case changed to:

Definition 26. ***(Strategyproofness in the sense of [Magid, 2008])***
A protocol is *strategyproof* (SP for short) if no player has a strategy that is sometimes better or is at least as well as his true value function.

Imagine the situation where different sharing processes are happening more than once, and the player is interested in becoming best off in a long term view. A game-theoretical approach leads to the following definition:

Definition 27. ***(Game-Theoretic Strategyproofness)***
A protocol is *game-theoretic strategyproof* (GTSP for short) if no player has a strategy with a higher expected value than the expected value of his true value function.

From results with the expected value it is difficult to conclude something about the other strategyproofness criterion. But it is possible to give a definition of strategyproofness with respect to game-theoretic strategyproofness and which can be compared to the other strategyproofness criterion.

Definition 28. ***(Game-Theoretic Cake-Cutting Strategyproofness)***
A protocol is *game-theoretic cake-cutting strategyproof* (GTCCSP for short) if no player has a strategy that is better in at least one allocation than his true value function and in the other allocations at least as well as his true value function.

3.1 Correlation between Strategyproofness Criteria

Theorem 3. *The relation between different strategyproofness criteria is the following*

$$SPP \overset{1}{\Longrightarrow} SP \overset{2}{\Longrightarrow} GTCCSP \overset{3}{\Longrightarrow} WSP$$

and

$$GTSP \overset{4}{\Longrightarrow} GTCCSP.$$

Proof.

1. If a protocol is SPP, then for every player in each strategy except of the recommended one exists at least one allocation A_c. In A_c the cheating player p_c receives a piece X_c which is not proportional. Since in a proportional protocol all allocations guarantee a proportional share to every player, the value of his piece X_c is less than he would obtain by following the recommended strategy. So the protocol is SP.

2. If a protocol is SP, then for every player in each strategy except of the recommended one exists at least one allocation A_c. In A_c the cheating player p_c receives a piece X_c instead of $X_{\neg c}$, which he had received by following the recommended strategy. The value of X_c is smaller than the value of $X_{\neg c}$. So no non-truthful strategy exist where the cheating player gets in all allocations at least the same valuable piece as in the recommended one. The protocol is $GTCCSP$.

3. If a protocol is $GTCCSP$, then for every player in each strategy except of the recommended one exists at least one allocation A_c. In A_c the cheating player p_c receives a piece X_c instead of $X_{\neg c}$, which he had received by following the recommended strategy. The value of X_c is smaller or equal to the value of $X_{\neg c}$. So no non-truthful strategy exist where the cheating player gets in all allocations a more valuable piece than in the recommended one. The protocol is WSP.

4. **Proof by contradiction:**

   Assumption: If a protocol is $notGTCCSP$ then it is not $notGTSP$.
   If a protocol is $notGTCCSP$, then there is one player with a strategy, which is not the recommended one. And for all allocations A_{S_c} the cheating player p_c receives a piece X_{S_c} instead of $X_{S_{\neg c}}$, which he had received by following the recommended strategy. The value of X_{S_c} is at least equal to the value of $X_{S_{\neg c}}$. In one allocation $A_{S'_c}$ the value of $X_{S'_c}$ is bigger than the value of $X_{S'_{\neg c}}$. Let r be the number of all possible allocations. The expected value of the strategy S_c at least equal with the expected value of the recommended strategy for the $r-1$ allocations since the expected value for the received pieces X_{S_c} is at least equal to the expected value for the pieces $X_{S_{\neg c}}$. The expected value for the piece $X_{S'_c}$ is bigger than the expected value for $X_{S'_{\neg c}}$. The general expected value is $E(S_c) > E(S_{\neg c})$ and is particularly not $E(S_c) \leq E(S_{\neg c})$. ↯ So the protocol is $notGTSP$.
   So $notGTCCSP \Rightarrow notGTSP$ and therefore $GTSP \Rightarrow GTCCSP$.

□

Theorem 4. *The relation between different strategyproofness criteria is the following*

$$SPP \overset{1}{\not\Leftarrow} SP \overset{2}{\not\Leftarrow} GTCCSP \overset{3}{\not\Leftarrow} WSP$$

and

$$GTSP \overset{4}{\not\Leftarrow} GTCCSP.$$

Proof.

Assume four different protocols for two players p_1 and p_2. All of them have two possible allocations A_1 and A_2, which have the same probability. In the recommended strategy $S_\neg c$ and in the non-truthful strategy S_c the player p_2 gets the same values in A_1 and A_2. The values of p_1 are shown in the four tables below:

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	1/2	2/3
$p_1(X_1)$ by S_c	1/2	1/2

Case 1: $SPP \not\Leftarrow SP$

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	7/8	3/4
$p_1(X_1)$ by S_c	7/8	3/4

Case 2: $SP \not\Leftarrow GTCCSP$

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	7/8	3/4
$p_1(X_1)$ by S_c	1	3/4

Case 3: $WSP \not\Leftarrow GTSP$

	in A_1	in A_2
$p_1(X_1)$ by $S_\neg c$	1/2	4/7
$p_1(X_1)$ by S_c	1	3/7

Case 4: $GTSP \not\Leftarrow GTCCSP$

Table 5: Counter-examples for the correlation between the strategyproofness criteria

1 $SPP \not\Leftarrow SP$
In the strategy S_c the player p_1 gets $p_1(X_1) = 1/2$. Since $1/2 < 2/3$ in A_2 the player p_1 would stay honest and this protocol is SP. But $1/2$ is proportional so the protocol is $notSPP$.

2 $SP \not\Leftarrow GTCCSP$
Since $7/8 = 7/8$ in A_1 and $3/4 = 3/4$ in A_2 the player p_1 would stay honest and this protocol is SP. But no value in the non-truthful strategy is smaller than in the recommended strategy, so the protocol is $notSP$.

3 $GTCCSP \not\Leftarrow WSP$
Since $3/4 = 3/4$ in A_2 and so the player p_1 has not a more valuable piece, he would stay honest and this protocol is WSP. But $1 > 7/8$ in A_1 and $3/4 = 3/4$ and the player p_1 has a more valuable piece in one allocation and the same value in the other so the protocol is $notGTCCSP$.

4 $GTSP \not\Leftarrow GTCCSP$
Since $3/7 < 4/7$ in A_2 the player p_1 would stay honest and this protocol is $GTCCSP$. But the expected value for the recommended strategy is $(1/2+4/7)/2 = 15/28$, which is smaller than $(1+3/7)/2 = 20/28$. So the protocol is $notGTSP$.

□

3.2 The Cut & Choose Protocol

Representation is inspired by [Barbanel, 1995]

Cut & Choose for $n = 2$		
Rules	**Player p_1 strategy**	**Player p_2 strategy**
1. Player p_1 partitions the cake X into two pieces $\{X', X - X'\}$	Partition X into two pieces of equal value	
2. Player p_2 chooses one piece		Choose the bigger value
3. Player p_1 gets the remaining piece		

Table 6: Cut & Choose rules and strategies

Theorem 5. *Cut & Choose is strategyproof for proportional protocols.*

Proof.

For the proof a general presentation of Cut & Choose game in extended form is used. W. l. o. g. the non-truthful cut is performed on the right part of the cake. The variables have the following restrictions:

$$0 \leq a \leq 1,\ 0 < \epsilon \leq 1/2,\ 0 \leq \delta \leq 1{-}a$$

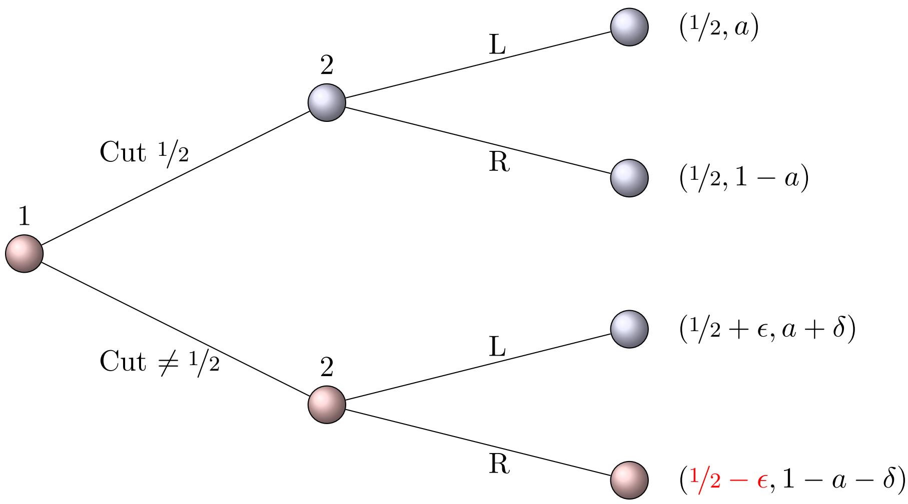

Figure 5: Cut & Choose game in extended form

The red path is the general case for an allocation, where by not following the recommended strategy player p_1 always becomes a less valuable piece than his proportional share. If Player p_2 would not choose the recommended strategy, he has to take a piece less valuable than his proportional share. Both players would stay honest. Thus Cut & Choose is strategyproof for proportional protocols. □

Theorem 6. *Cut & Choose is game-theoretic cake-cutting strategyproof.*

Proof.

Options for not following the recommended strategy:

- Player p_2 takes the smaller piece. This can not be his intention, because then he has a piece with less value.
- Player p_1 cuts the cake into two unequal pieces. The chance to get less in his valuation is equal to the chance to get more in his valuation of the cake. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case:

  $$1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/2$$

  and in the dishonest case:

  $$1/2 \cdot X' + 1/2 \cdot (X - X') = 1/2 \cdot \underbrace{X}_{=1} = 1/2$$

  According to the definition of game-theoretical strategyproofness in the case with equal expected values the player would stay honest. Cut & Choose is game-theoretical strategyproof.

□

Remark 1. *According to Theorem 3, Theorem 5 and Theorem 6 Cut & Choose is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

4 Strategyproofness of Proportional Protocols

The goal in this chapter is to analyse the strategyproofness of well-known protocols. First of all, they are rewritten into game-theoretic manner. Since each player has a truthful and a non-truthful strategy, a protocol with n active players has at least $2n$ strategies. If the obtained value is not equal in different non-truthful strategies they have to be separated. So the amount of strategies would grow and would make the analysis very tedious.
Luckily a protocol consists of a lot of repeats and actually each of the well-known protocols can be simplified to an interaction between two kinds of players. So the analysis of the whole process is unnecessary.
The proceed is as follows, the interactions between two kinds of players are represented in tables. A separation between rules and strategies is given. Afterwards the different strategies of a protocol are represented as an extended form game and the different kinds of strategyproofness are analysed.
An important pre-comment: Proportionality means each player gets $v_j(X_j) = {}^1/_n$ for $1 \leq j \leq n$. So if one player p_n leaves the game with $v_i(X_n) < {}^1/_n$ for $1 \leq i \leq (n-1)$ it still holds for the next round that the cake $X = X - X_n$ is normalized and a proportional piece is ${}^1/_{n-1}$ and not ${}^{1+\epsilon_i}/_n$.
The complete protocols in the standard description as well as the proofs of their proportionality can be found in [Robertson and Webb, 1998].

4.1 The Kuhn à la Dawson Lone-Divider Protocol

Kuhn à la Dawson Lone-Divider protocol for arbitrary n		
Rules	**Player p_1 strategy**	**Players in P_{n-1} strategy**
1. Player p_1 cuts the cake X into n pieces $\{X_1, \ldots, X_n\}$	Cut X into n pieces of equal value	
2. Players in P_{n-1} mark s pieces with $1 \leq s \leq n$		Mark X_j if $v_i(X_j) \geq {}^1/_n$ for $1 \leq j \leq n$ and $2 \leq i \leq n$
3. If an allocation is impossible: Form the non-marked pieces to a new cake and exchange the cutter (last cutter leaves with a non-marked piece)		

Table 7: Lone-Divider rules and strategies

Theorem 7. *Lone-Divider protocol is not game-theoretic cake-cutting strategyproof.*

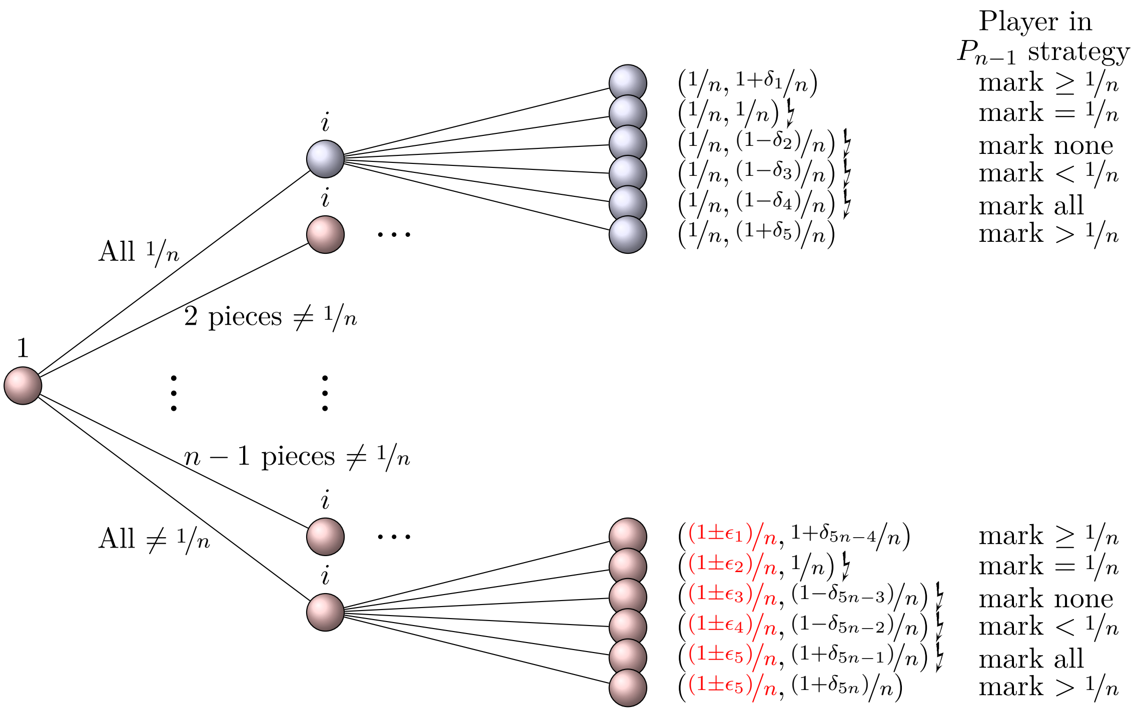

Figure 6: Lone-Divider cake game in extended form

Proof.

For showing that Lone-Divider is not *GTCCSP* it is necessary to have at least one cheating player p_c. This player has a non-truthful strategy with $v_c(X_c) > v_c(X_t)$ in one allocation and $v_c(X_c) \geq v_c(X_t)$ for all other allocations. The proof is divided in two parts. In the first part the case $v_c(X_c) > v_c(X_t)$ can be is illustrated with an example. The second part uses a case distinction to show that this player will never get a less valuable piece. The non-truthful strategy is to mark X_j if $v_c(X_j) > 1/n$ for $1 \leq j \leq n$, or in the only case when all pieces have the same value to mark all of them.

Part I: Example for a successful non-truthful strategy

Imagine the following allocation with three players. The player p_1 is going to cheat.

	X_L	X_M	X_R
p_3 (divider)	$1/3$	$1/3$	$1/3$
p_2 (rank 2)	$3/5$	$2/5$	0
p_1 (rank 1)	$1/2$	$1/3$	$1/6$

Table 8: Example for a successful non-truthful strategy

The acceptable pieces

By following the recommended strategy:

	X_L	X_M	X_R
p_3 (divider)	✓	✓	✓
p_2 (rank 2)	✓	✓	✗
p_1 (rank 1)	✓	✓	✗

By following the non-truthful strategy:

	X_L	X_M	X_R
p_3 (divider)	✓	✓	✓
p_2 (rank 2)	✓	✓	✗
p_1 (rank 1)	✓	✗	✗

Table 9: Acceptable pieces by a successful non-truthful strategy

An allocation is possible since each pieces is acceptable for at least one different player.

If player p_1 chooses the recommended strategy $S_{\neg}c$:
There are no players with just one acceptable piece, so the player with the highest rank can choose first. Player p_2 chooses X_L and player p_1 gets X_M with $p_1(X_M) = 1/3$. The divider gets the last piece.

If player p_1 chooses the non-truthful strategy S_c:
There is one player with just one acceptable piece, so this player chooses first. Player p_1 chooses X_L with $p_1(X_L) = 1/2$ and player p_2 gets X_M. The divider gets the last piece.
The value of player p_1's piece in the non-truthful strategy S_c is $p_1(X_L) = 1/2 > 1/2 = p_1(X_M)$ than in the recommended strategy $S_{\neg}c$. So the player p_1 is at least in one allocation better off with the non-truthful strategy than in the recommended one.

Part II: Case distinction for a successful non-truthful strategy

Case 1:

Case 2:

Case 3:

Case 4:

Case 5:

Case 6:

□

Remark 2. *According to Theorem 3 and Theorem 7 Kuhn à la Dawson Lone-Divider is not strategyproof for proportional protocols, not strategyproof, not game-theoretical strategyproof and not game-theoretical cake-cutting strategyproof. According to Example 2 it is weak strategyproof.*

4.2 The Banach-Knaster Last-Diminisher Protocol

The Last-Diminisher protocol consists of $(n-2)$ rounds.
The players are separated into two groups. In the first group is the player p_i (with i number of the round) and in the second group P_{i+k} are all other players p_j with $i < j \leq n$, $1 \leq k \leq n-1$.
At the end of each round one player gets a piece and leaves the game. If it is player p_i then player p_{i+1} takes his place, otherwise player p_i is player p_{i+1} in the next round $i+1$ and players in group P_{i+k} will be consecutively numbered. In the last round the two remaining players apply Cut & Choose.

The Banach-Knaster Last-Diminisher protocol for arbitrary n		
Rules	**Player p_i strategy**	**Players in P_{i+k} strategy**
1. Player p_i cut a piece I_i	Cut a piece with value $1/n$	
2. Players p_j trim or pass		If $v_j(I_i) > 1/n$ trim so that $v_j(I_i) = 1/n$, else pass
3. Last trimmer take it		

Table 10: Last-Diminisher rules and strategies

Theorem 8. *Last-Diminisher protocol is strategyproof for proportional protocols.*

Proof.

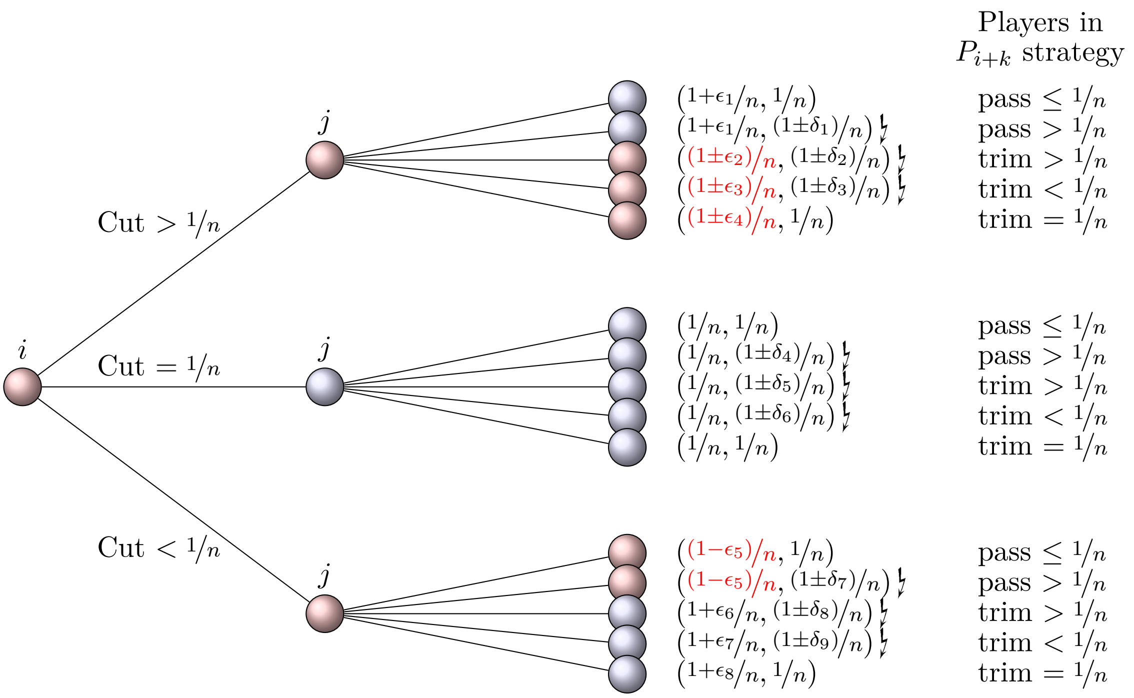

Figure 7: Last-Diminisher cake game in extended form

For the proof a presentation of Last-Diminisher game in extended form is used. The variables have the following restrictions:

$$0 < (1 \pm \epsilon_2), (1 \pm \epsilon_3), (1 \pm \epsilon_4) < n,\ 0 < \epsilon_5 \leq 1,\ 0 < \epsilon_1, \epsilon_6, \epsilon_7, \epsilon_8 \leq (n-1)$$

and :

$$0 \leq (1 \pm \delta_1), (1 \pm \delta_4), (1 \pm \delta_7) \leq n, 0 < (1 \pm \delta_2), (1 \pm \delta_3), (1 \pm \delta_5), (1 \pm \delta_6), (1 \pm \delta_8), (1 \pm \delta_9) < n$$

The consequences for each player p_j in P_{i+k} by choosing the strategy to pass a piece X_i with $v_j(X_i) > 1/n$ or to trim a piece X_i to $v_j(X_i) > 1/n$ are the same as for the player p_i by cutting a piece with $v_i(X_i) > 1/n$. The upper red paths in Figure 7 show that by choosing this non-truthful strategy a player can obtain a less valuable piece. For that to happen, a following player p_s needs to trim the piece s.t. $v_i(X_s) > 1/n$ and $v_j(X_s) > 1/n$ and for the reason that he is the last trimmer to get this piece.

The consequences for each player p_j in P_{i+k} by choosing the strategy trim a piece X_i to $v_j(X_i) < 1/n$ are the same as for the player p_i by cutting a piece with $v_i(X_i) < 1/n$. The lower red paths in Figure 7 show that by choosing this non-truthful strategy a player can obtain a less valuable piece. For that to happen, no following player p_s is allowed to trim the considered piece. So the cheating player will get this undesirable piece.

The only strategy which does not conceal this risk, is for all players the recommended one. □

Remark 3. *According to Theorem 3 and Theorem 8Banach-Knaster Last-Diminisher is strategyproof for proportional protocols, strategyproof,game-theoretical cake-cutting strategyproof and weak strategyproof.*

4.3 The Fink Lone-Chooser Protocol

For the description of this protocol the players will be separated in two groups. In the $(i-1)$ round players in the first group $P_I = \{p_1, \dots, p_i\}$ have already their proportional piece. Assume that player p_1 owns the whole cake before the first round. The performance in each round is the following:

The Fink Lone-Chooser protocol for arbitrary n		
Rules	**Players in P_I strategy**	**Player p_{i+1} strategy**
1. Players in P_I partition their piece X_i into $i+1$ pieces $\{X_{i,1}, \dots, X_{i,i+1}\}$	Partition X_i into $i+1$ pieces of equal value	
2. Player p_{i+1} chooses one piece of each player's cake		Choose the most valuable piece of each player's cake

Table 11: Lone-Chooser rules and strategies

Theorem 9. *Fink Lone-Chooser protocol is game-theoretic strategyproof.*

Proof. Proof by induction:

- Case $i = 2$: Cut & Choose is game-theoretic strategyproof by Theorem 6.

- Case $(i-1) \to i$:

  Options for not following the recommended strategy:

  - Player p_{i+1} takes not the biggest piece. Then he has a less valuable piece than by following the recommended strategy.
  - Players in P_I cut the cake into $i+1$ non-equal pieces. The chance that player p_{i+1} takes a certain piece is ${}^1\!/_{(i+1)}$. In stochastic terms it means, that the expected value at the end of the allocation will be in the honest case (because he had at least ${}^1\!/_i$ before) at least:

    $$i \cdot {}^1\!/_{(i+1)} \cdot {}^1\!/_i = {}^1\!/_{(i+1)}$$

    and in the dishonest case:
    Assume the value of the piece will be distributed on s pieces with $1 \le s \le i+1$. So the expected value will be:

    $${}^1\!/_i \cdot {}^{(i+1-s)}\!/_{(i+1)} + {}^{(s-1)}\!/_{(si)} \cdot {}^s\!/_{(i+1)} = {}^1\!/_{(i+1)}$$

    Even if the value is distributed unequally on the s pieces, it would not affect the value of the piece in the allocation, because the probability of the piece the player p_{i+1} take is uniformly distributed.

According to the definition of game-theoretic strategyproofness in the case with equal expected values the player would stay honest.

□

Theorem 10. *Fink Lone-Chooser protocol is strategyproof for proportional protocols.*

Proof.

For the proof a presentation of Lone-Chooser game in extended form is used. The variables have the following restrictions:

$0 \leq (i \pm \epsilon_1), (i \pm \epsilon_2) \leq (i+1),\ 0 \leq \delta_1, \delta_3 \leq i,\ 0 \leq (i \pm \delta_2), (i \pm \delta_4) \leq (i+1),\ \delta_2 \leq \delta_1,\ \delta_4 \leq \delta_3$

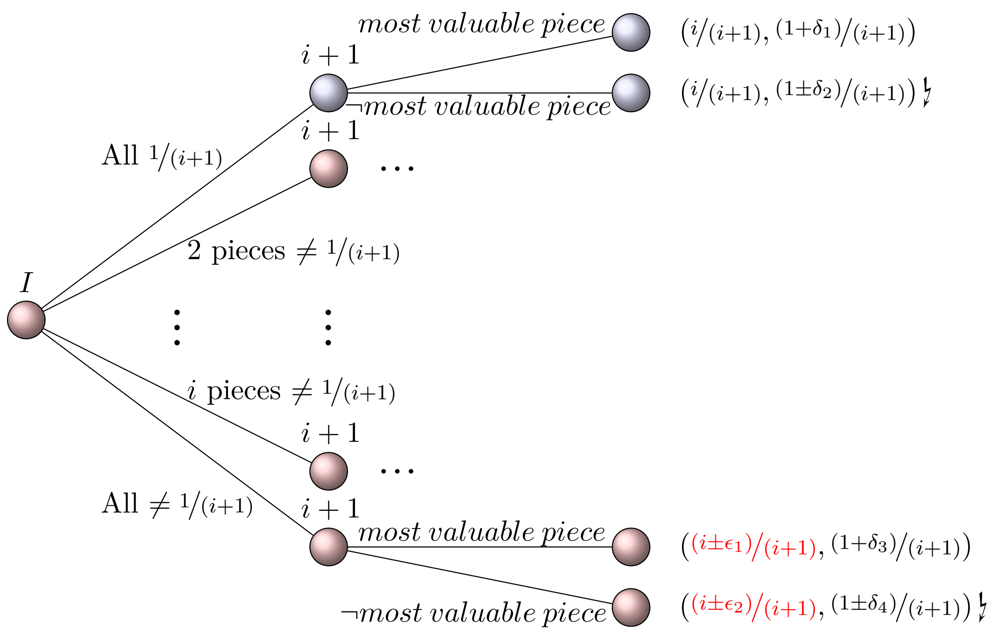

Figure 8: Lone-Chooser cake game in extended form

The red path is the not succesful allocation by following the non-truthful strategy for a player in P_I. Here every player gets less than by following his recommended strategy. Especially it is the case, where the player p_{i+1} takes a piece which a player in P_I values more than $1/(i+1)$.

If Player p_2 would not choose the recommended strategy, he has to take a piece less valuable than in his best possibility. Both kind of players would stay honest. Thus Lone-Chooser is strategyproof for proportional protocols. □

Remark 4. *According to Theorem 3, Theorem 9 and Theorem 10 Fink Lone-Chooser is strategyproof for proportional protocols, strategyproof, game-theoretical strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

4.4 The Even & Paz Divide-and-Conquer Protocol

Divide-and-Conquer protocol for arbitrary n		
Rules	**Players in P_{n-1} strategy**	**Player p_n strategy**
1. Each player in P_{n-1} partitions the cake X into two pieces $\{X', X - X'\}$	Partition X into two pieces in the value-ratio of $\lfloor n/2 \rfloor : \lceil n/2 \rceil$	
2. Player p_n chooses one piece		If $v_n(X') \geq \lfloor n/2 \rfloor / n$ choose X', otherwise $X - X'$
3. Cut in the ratio of the $n/2$-th player from left border and repeat with the new groups separately until each player has his own piece		

Table 12: Divide-&-Conquer rules and strategies

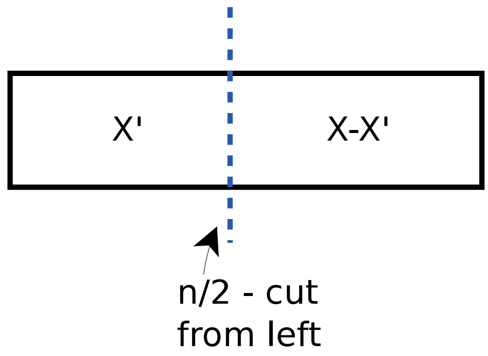

Figure 9: D&C step 3

Theorem 11. *The Divide-and-Conquer protocol is strategyproof for proportional protocols.*

Proof.

The proof can be found in [Brams et al., 2007b]: Consider the case when two players, A and B, must divide a cake, but one may not be truthful. If their truthful $1/2$ points are as shown below, then cutting the cake at $|$ gives each player more than $1/2$:

$$0 - - - - - - - - - - - - - a - | - b - - - - - - - - - - - - - 1$$

But if player A should report that its $1/2$ point is either to the left or right of a, it risks getting less than $1/2$ the cake if $(i)|$ is to the left of a or $(ii)|$ is to the right of b. This argument for the vulnerability of D&C-that it does not guarantee a player a proportional share if it is not truthful-can readily be extended to $n > 2$ players. □

Remark 5. *According to Theorem 3 and Theorem 11 Even & Paz Divide-&-Conquer is strategyproof for proportional protocols, strategyproof, game-theoretical cake-cutting strategyproof and weak strategyproof.*

5 Conclusion

In this work the common proportional cake-cutting protocols have been rewritten in a game-theoretic manner and analysed on whether a non-truthful strategy could yield a more advantageous situation for a non-truthful player. It was possible to approve game-theoretically that the only strategy which promises the best outcome is the strategy recommended by the protocol.
Instead the goal here was to adapt the existing definitions of strategyproofness to the general cake-cutting problem.

Protocol	WSP	GTSP	GTCCSP	SP	SPP	SSP
Cut & Choose	✓	✓	✓	✓	✓	✗
Last-Diminisher	✓	(✓)	✓	✓	✓	✗
Lone-Chooser	✓	✓	✓	✓	✓	✗
Lone-Divider	✓	✗	✗	✗	✗	✗
Divid-&-Conquer	✓	(✓)	✓	✓	✓	✗

Table 13: Overview: Strategyproofness of proportional cake-cutting protocols

6 Open Questions and Future Research

An interesting aspect would be to take a closer look on groupstrategyproofness. Hereby, groups have public valuations for group members.

An other approach about strategyproofness could be a consecutively allocation of several cakes. Then the cake-cutting-game could be interpreted as a repeated game.

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