DETERMINING A FAIR BORDER

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Introduction. Suppose n countries border on a region the ownership of which is in dispute (Fig. 1). Is there a way of partitioning the disputed territory so each country receives a single piece adjacent to itself which it considers at least $1/n$ the total value of the territory, even though different parts of the territory may be valued differently by individual countries? The main purpose of this paper is to show that such fair borders always exist, under the quite natural assumption that each country's value of the territory is nonatomic (i.e., single points have value zero).

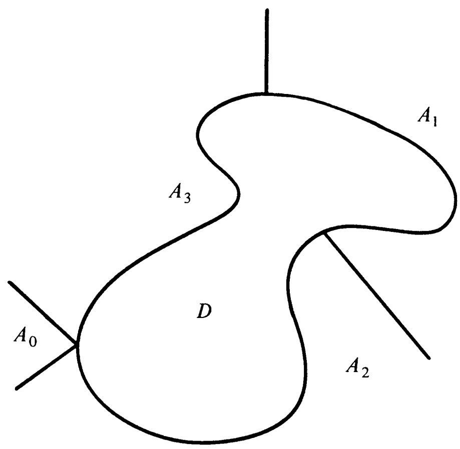

FIG. 1.

This border problem is in some respects closely related to several of the classical fair-division problems such as the "Problem of Cutting a Cake Fairly" of Steinhaus [**2**], [**9**], the "Problem of the Nile" described by Fisher [**4**], [**5**], the "Problem of Similar Regions" of Neyman and Pearson [**8**], and the "Ham Sandwich Problem" of Ulam [**10**].

Steinhaus first raised the question of whether an object (such as a cake) can be divided among n people, who may value different parts of the cake differently, in such a way that each person feels he has received at least $1/n$ the total worth of the cake *according to his own value*. For $n = 2$, the well-known "one cuts, the other chooses" method always yields a solution (although in some respects not an ideal solution, as the second person has an obvious advantage in general). Steinhaus showed that the cake-cutting problem has an affirmative solution for $n = 3$, and then Banach and Knaster solved the problem for general n with the following simple, elegant, and practical solution: pass a long knife parallel to itself slowly over the top of the cake until one of the participants says "stop," cut the cake at that point and give the piece to that participant, and then continue moving the knife. It is easy to see that this procedure guarantees each person at least $1/n$ of the cake, according to his own value, provided that each participant's value of every piece of zero volume is zero.

The author's formal education has included a Bachelor's degree at West Point, a Master's in operations research at Stanford, a year as a Fulbright scholar at Göttingen, and a Ph.D. in mathematics at Berkeley in 1977. He has taught at Washington University and is currently on leave from Georgia Tech as a NATO postdoctoral fellow at the University of Leiden. His main mathematical interests center around stochastic processes (especially abstract gambling theory and optimal stopping theory); this excursion into fair-division problems was largely recreational.

The "Problem of the Nile" [2], [4], [5] involves partitioning a set into k pieces (instead of n, as in the cake-cutting problem) and then evaluating each of n measures on each piece. "Each year the Nile would flood, thereby irrigating or perhaps devastating parts of the agricultural land of a predynastic Egyptian village. The value of the different portions of the land would depend on the height of the flood. In question was the possibility of giving to each of the k residents a piece of land whose value would be $1/k$ of the total land value, no matter what the height of the flood." Feller [3] showed that if there are an infinite number of flood heights possible, the problem need not have a solution, whereas Neyman [7] first proved that a (nonconstructive) solution always exists if there are only a finite number of possible flood heights.

A special case of Neyman and Pearson's "Problem of Similar Regions" [8] which is actually equivalent to the "Problem of the Nile" is the "Bisection Problem": given a set and n probability measures on it, does there always exist a *single* subset having exactly measure one-half with respect to each of the n measures? (More generally, the "Problem of Similar Regions" asks for the existence, for each α in $[0, 1]$, of a single set having exactly measure α with respect to each measure.)

Another classical bisection problem, but one which involves bisection in a particular way (namely by a hyperplane), is that of Ulam's "Ham Sandwich Problem": can an ordinary ham sandwich, consisting of bread, ham, and butter be cut by a plane in such a way that each of the three ingredients is cut exactly in half? More generally, can n objects in euclidean n-space always be simultaneously bisected by a single hyperplane? The answer is affirmative, and the standard proof is an application of the Borsuk-Ulam theorem [1], [2]:

(1) *If f is a continuous map of the sphere in n-dimensional space into $(n-1)$-dimensional space such that $f(-x) = -f(x)$ for every x, then there is some point on the sphere mapped into the origin.*

(For an interesting discussion, historical background, and proofs of the interrelationships among these classical division problems, the reader is referred to [2].)

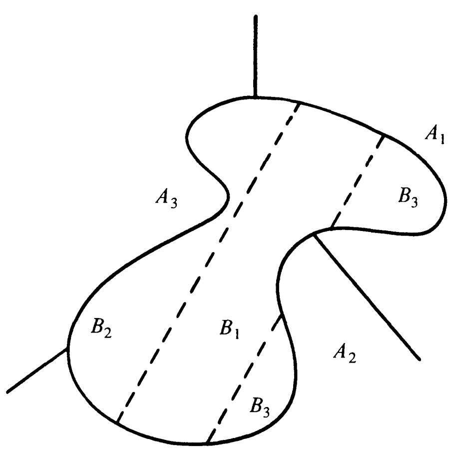

FIG. 2.

Solution of the Border Problem. Solutions of the fair-division problems above are generally not adequate for solution of the border problem for two topological reasons: first, the pieces so determined may be badly disconnected (recall we want each country to receive a *single* piece); and second, the pieces may well not be adjacent to the correct countries (Fig. 2). Moreover, it is not always possible to connect the Banach-Knaster pieces to the correct countries (by thin, nonintersecting roads for example) without destroying either the overall fairness of the division, or the connectivity of the other pieces. However, a (nonconstructive) cake-cutting result of Dubins and Spanier can be used to help settle the border problem affirmatively. Using Lyapounov's convexity theorem [**2**], [**6**],

(2) *The range of every nonatomic, countably additive, finite dimensional, vector-valued measure is convex,*

Dubins and Spanier were able to show there is always a (Borel) partitioning of the cake so that each person receives strictly *more* than his share, provided at least two people value some part of the cake differently.

THEOREM 1 (Dubins and Spanier [**2**], Corollary 1.2). *Let D be a Borel subset of $\mathbb{R}^k$, and suppose $\mu_1, \ldots, \mu_n$ are nonatomic (Borel) probability measures supported on D, with $\mu_i \neq \mu_j$ for some $i \neq j$. Let $p_i > 0$ with $\Sigma^n p_i = 1$. Then there exists a Borel partition $C_1, \ldots, C_n$ of D such that $\mu_i(C_i) > p_i$ for each i.*

(The numbers p_i refer to the minimum proportion of the territory to be given to the ith country; often $p_i \equiv 1/n$ for all i.)

It should be noted that the partition guaranteed by Theorem 1 may also fail to solve the border problem for the same reasons as the Banach-Knaster procedure may; moreover, the elements of the Dubins-Spanier partition can be very complicated Borel sets in general.

Countries (and the disputed territory D) will be identified with *open connected regions* in $\mathbb{R}^2$. For a set $A \subset \mathbb{R}^2$, $\bar{A}$ denotes the closure of A, $\mathring{A}$ the interior of A, and ∂A the boundary ($\bar{A} \setminus \mathring{A}$) of A. By a nonatomic measure is meant a Borel measure which assigns measure zero to each singleton set (single point).

Definition. Open connected subsets A and B of $\mathbb{R}^2$ are *adjacent* if $\partial A \cap \partial B$ contains an open arc α (homeomorphic image of (0, 1)) such that $A \cup B \cup \alpha$ is open and connected.

In Figure 1, D is adjacent to A_1, A_2 and A_3, but not to A_0. Intuitively, two connected regions are adjacent if they "touch" on an interval, that is, if a single country can be formed from the two regions by erasure of some small open arc on their common boundary. That the common boundary just contain an open arc is not enough to insure this merger can always be accomplished (and hence that a fair division in the "connected-adjacent" sense exists), as can be easily seen by looking at such borders as the $\sin(1/x)$ curve describes. (However, it is easy to check in the above definition that if $A \cup B \cup \alpha$ is open, it is automatically connected.) We are now ready to state the main result; recall that p_i represents the minimum proportion of the disputed territory the ith country is to receive.

THEOREM 2. *Suppose $D, A_1, \ldots, A_n$ are open connected regions in $\mathbb{R}^2$ with A_i adjacent to D for each i. If $\mu_1, \ldots, \mu_n$ are nonatomic probability measures on D and $p_i \geq 0$, $\Sigma^n p_i = 1$, then there exist disjoint open connected subsets $B_1, \ldots, B_n$ of D with B_i adjacent to A_i for all i, $\mu_i(B_i) \geq p_i$ for all i, and with $\overline{\cup B_i} = D$.*

Proof of Theorem 2. Without loss of generality, $p_i > 0$ for all i.

Case 1. $\mu_1 = \cdots = \mu_n$. Expand A_1's territory into D continuously, taking care to avoid the boundary of D and to pass continuously through any arcs of positive μ_j measure, until $\mu_1(\bar{B}_1) = \mu_1(B_1) = p_1$ (Fig. 3). (This is possible since μ_1 is nonatomic and D is connected.) Now $D \setminus \bar{B}_1$ is open and connected, and since $\mu_1 = \cdots = \mu_n$, $\mu_j(\overline{D \setminus \bar{B}_1}) = 1 - p_1$, for all $j = 1, \ldots, n$. Continue similarly to find $B_2, \ldots, B_{n-1}$, and let $B_n = D \setminus (\bar{B}_1 \cup \cdots \cup \bar{B}_{n-1})$. This completes Case 1.

Case 2. $\mu_i \neq \mu_j$ for some $i \neq j$. By Theorem 1 there exists a Borel partition $C_1, \ldots, C_n$ of D satisfying $\mu_i(C_i) > p_i$ for all $i = 1, \ldots, n$. Since the $\{\mu_i\}$ are Borel, there exist (disjoint) open balls $E_1, \ldots, E_k$ in D with $\mu_i(\partial E_j) = 0$ for all i and j, and

$$\mu_i\left(\bigcup^k E_j \Delta C_1\right) < \varepsilon = \min_j\{\mu_j(C_j) - p_j\} \text{ for all } i = 1,\dots, n.$$

Since D is open and connected, it is path (piecewise linear) connected and it follows easily that there are (sausage-shaped) open connected subsets of D adjacent to A_1 the closures of which have arbitrarily small μ_i measure for all i, and which contain all the centers of $E_1,\dots, E_k$. Let B_1 be the union of one of these sets with $\cup^k E_j$ which satisfies $\mu_i(B_1 \Delta C_1) < \varepsilon$ and $\mu_i(\partial B_1) = 0$ for all $i = 1,\dots, n$. Then B_1 is open and connected, adjacent to A_1, and satisfies $\mu_1(B_1) > p_1$, and $\mu_j(C_j \setminus \bar{B}_1) > p_j$ for all $j = 2,\dots, n$. (See B_2 in Fig. 3.) Now $D \setminus \overline{B_1}$ is open and connected, so continue in this manner finding $B_2,\dots, B_{n-1}$ and then let $B_n = D\setminus\overline{(B_1 \cup \cdots \cup B_{n-1})}$. It is easy to see that the sets $B_1,\dots, B_n$ satisfy the conclusion of the theorem. □

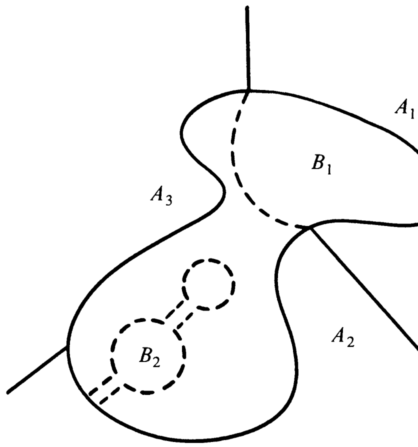

FIG. 3.

If the measures $\mu_1,\dots, \mu_n$ fail to be nonatomic, the conclusion of the theorem may fail: the worst case is that for some point $x \in D$, $\mu_i(\{x\}) = 1$ for all i.

Theorem 2 may easily be strengthened in two respects. First, the conclusion holds also for $\mathbb{R}^k$ (or k-dimensional manifolds) if one generalizes the definition of adjacency as "touching" on $k - 1$ dimensional regions (i.e., $\partial A \cap \partial B$ contains a homeomorphic image of $(0,1)^{k-1}\dots$). And second, the conclusion may also obviously be strengthened to guarantee that the new borders formed are polygonal. If $\mu_i \neq \mu_j$ for some $i \neq j$, the analog of Theorem 2 corresponding to Theorem 1 is also easy to prove: there is always a border which gives each country strictly more than its fair share. As an immediate corollary to Theorem 2, one may drop all adjacency requirements and conclude that a connected cake may be divided so that each person receives a *single* (i.e., connected) piece which is a fair share.

It would perhaps be of interest to find a practical, constructive method for generating the fair border guaranteed by Theorem 2; no such solution is known to the author.

Other Notions of a Fair Share. That each participant receive a piece which he values at least $1/n$ of the total value is certainly not the only criterion for what constitutes a "fair share." Suppose one person receives a piece which he values at exactly $1/n$, whereas others get strictly more than $1/n$ according to his *and* their values. Has the first person received a "fair share"? In the border problem, since D is a nonempty open set, it is easy to see that the solution guaranteed by Theorem 2 is *never* unique, and the question arises of whether or not borders exist which are in some sense optimal, or at least more fair. Dubins and Spanier addressed this question for cake-cutting, and offered several different criteria for determining whether one partition is better than another.

The first criterion suggested is that the partition $\{A_1, A_2, \ldots, A_n\}$ is better than the partition $\{B_1, \ldots, B_n\}$ if $\Sigma^n \mu_i(A_i) \geqslant \Sigma^n \mu_i(B_i)$, and for cake-cutting they proved [**2**, Theorem 2] that optimal partitions in this sense always exist. On the other hand, for the border problem it may be seen from Fig. 2, with μ_i uniformly distributed on B_i for each i, that optimal partitions in this first sense do not always exist (although clearly "ε-optimal" ones always exist, as can be shown by a slight modification in the proof of Theorem 2 using [**2**, Theorem 2] in place of Theorem 1).

A second notion of optimality Dubins and Spanier suggested was the following: "Find a partition that maximizes the amount received by the person who gets the least, and, among all such partitions, find one which maximizes the amount received by the person who gets next to the least, etc." (for a formal definition, see [**2**, p. 8]). Again, for cake-cutting, optimal partitions in this second sense always exist [**2**, Cor. 6.10] but for the border problem they do not exist in general as can be seen from Fig. 2 again with the uniform distributions given above. However, "ε-optimal" borders in this second sense also always exist; this time use [**2**, Cor. 6.10] in the modification of the proof of Theorem 2.

A third notion of optimal partition suggested in [**2**] is one in which $\mu_i(A_j) = p_j$ for all i and all j, that is, all participants agree that each person received *exactly* the correct amount. Proving an assertion of Steinhaus [**9**], Dubins and Spanier showed that for cake-cutting, optimal partitions in this third sense also always exist [**2**, Cor. 1.1], and again it is easy to use their result to modify the proof of Theorem 2 and conclude that ε-optimal (i.e., $|\mu_i(A_j) - p_j| < \varepsilon$ for all i and j) solutions to the border problem always exist. But whether or not *optimal* borders (in this third sense) always exist is not known to the author, even if D is bounded. The techniques above do not seem to work; simply "taking limits" in general destroys both the connectivity and the correct adjacency of the pieces.

One last criterion for comparing borders which are already fair in one of the above measure-theoretic senses, a criterion suggested by L. Karlowitz, is that of comparing total lengths of borders, and looking for one with minimal length (although some countries with large armies, for example, may well prefer long borders). Another look at Fig. 2 shows that fair borders of minimal length do not exist in general, but if D is bounded, there are always fair borders of nearly minimal length.

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