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CONSENSUS-HALVING VIA THEOREMS OF BORSUK-ULAM AND TUCKER

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Abstract. In this paper we show how theorems of Borsuk-Ulam and Tucker can be used to construct a *consensus-halving*: a division of an object into two portions so that each of n people believe the portions are equally split. Moreover, the division takes at most n cuts, which is best possible. This extends prior work using methods from combinatorial topology to solve fair division problems. Several applications of consensus-halving are discussed.

1. Introduction

The study of *fair division* problems is concerned with finding ways to divide an object among several parties according to some notion of fairness. The cake-cutting problem of Steinhaus [13] is perhaps the best known example. Aside from the division of *goods*, other fair-division problems address the division of *burdens* (e.g., the *chore-division* problem [7, 10]) and the division of *mixtures* of goods and burdens (e.g., the *rent-partitioning* problem: how to split the rent so that housemates are satisfied by different rooms).

Recently, ideas from combinatorial topology have provided new and constructive methods for obtaining solutions to fair-division problems. In [14], Su discusses a cake-cutting procedure of Simmons that can be extended to obtain envy-free solutions for chore division and rent-partitioning using variants of a result known as Sperner's lemma, which is the combinatorial equivalent of the Brouwer fixed point theorem of topology.

In this paper, we demonstrate how a result known as Tucker's lemma, which is the combinatorial equivalent of the Borsuk-Ulam theorem of topology, can be used to solve a different kind of fair-division problem: is it possible to divide a mixture into 2 portions so that each of n people believes both portions are the same size (a *consensus-halving*)? Moreover, a constructive proof of Tucker's lemma yields an efficient procedure for constructing an approximate solution using a minimal number of cuts.

As an application, a consensus-halving procedure could allow two families to split a piece of land into two regions such that *every* member of both families believes the land is nearly equally divided. We discuss potential applications to the Law of the Sea Treaty [5] and the *necklace-splitting* problem of Alon [1]. Another application solves a *team-splitting* problem: given a territory and a pair each of zoologists, botanists, and archaeologists, is it possible to divide the territory into

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two portions in such a way that members of any given pair will prefer to explore different portions? Thus the group could be split in an *envy-free* fashion— into two teams with one member of each specialty among them. We explain how a consensus-halving method can be adapted for this purpose near the end of this article.

2. Consensus-Halving

To be precise, we assume that any *object* A to be divided is a measurable bounded set in $\mathbf{R}^2$ or $\mathbf{R}^3$, infinitely divisible, and that each player i has a bounded continuous measure μ_i on (measurable) subsets of A which describes the (positive or negative) value that she assigns to that subset. The (absolute) continuity of the measures (with respect to Lebesgue measure) forbids the existence of "point masses" — zero volume subsets with non-zero worth.

Although we model player preferences with measures, we remark that none of our proofs will require the μ_i to be additive over subsets. (The μ_i may as well be continuous set functions defined on the Borel σ-algebra and satisfying all the properties of measures except countable additivity.) Thus we do *not* need to require that player valuations over subsets of A be additively separable.

We show the following theorem.

Theorem 1 (Consensus Halving)**.** *Given an object A and n persons whose preferences are modeled by continuous measures $\{\mu_1, ..., \mu_n\}$, there exists a partition of A into two portions A_1 and A_2 such that each of n persons thinks that A_1 and A_2 are exactly equal, i.e., $\mu_i(A_1) = \mu_i(A_2)$ for all $i \in \{1, 2, \ldots, n\}$. Using cuts by parallel planes, n cuts are sufficient to achieve the division, and in some cases best possible. An algorithm exists for locating an arbitrarily close approximation to a solution.*

Note that the object can be a mixture of desirable and undesirable parts (the players may in fact disagree on which parts are desirable and undesirable).

Non-constructive versions of this result have already been obtained; for instance, see Goldberg and West [8] and Alon and West [2]. The latter uses the Borsuk-Ulam theorem but in a fashion that requires additivity of the measures μ_i. Alon [1] proves a generalization that produces k equal portions according n probability measures; it yields our result when $k = 2$, but it is also non-constructive and based on a topological result of Bárány-Shlosman-Szücs [3]. Another approach to produce the existence of the sets A_1 and A_2 is to use Lyapunov's theorem (see Barbanel [4]); however, it is even less constructive because it does not even say how many cuts are required or what the sets A_1 and A_2 might look like.

By constrast, our proof is constructive and based on a combinatorial result of Tucker. It does not require additivity nor positivity of the measures, nor must the measures satisfy $\mu_i(A) = 1$ (but they should be bounded for the conclusion to make sense). A constructive proof yields a simplicial algorithm that guarantees *approximate* solutions up to a pre-specified tolerance for error. In some sense this is the best one can hope for; Robertson and Webb [11, p.104] have shown that there is no finite discrete procedure that will produce an *exact* equal division. Robertson and Webb also propose an procedure for approximate division into ratios (see [11, p.128]); however, it involves a large number of cuts which grows as ε decreases, whereas our approach uses at most n cuts (hence does not decimate the object). In addition, our procedure handles mixtures easily.

3. Tucker's Combinatorial Lemma and the Borsuk-Ulam Theorem

Recall that an *n-simplex* in $\mathbf{R}^m$ is the convex hull of $n+1$ affinely independent points (*vertices*) in $\mathbf{R}^m$. A *k-face* of an n-simplex is the k-simplex spanned by any subset of $k+1$ vertices. A *triangulation* of a set X is a collection of (distinct) n-simplices whose union is X, with the property that any two of them intersect in a face common to both, or not at all.

Represent the n-ball B^n by the set of all points $\mathbf{x} = (x_i) \in \mathbf{R}^n$ such that $|x_1| + \ldots + |x_n| \leq 1$. The boundary of this "octahedral" ball is the set of all points in $\mathbf{R}^n$ satisfying $|x_1| + \ldots + |x_n| = 1$ and may be thought of an $(n-1)$-sphere S^{n-1}. A *centrally symmetric triangulation* of S^{n-1} is one such that if σ is any face of the triangulation, then $-\sigma$ also is.

The following combinatorial theorem of Tucker [16] was proved in 1945 for the case $n = 2$. The proof for general n may be found in Lefschetz [9].

Tucker's Lemma. *Let T be a centrally symmetric triangulation of S^n whose vertices are assigned labels from $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that labels of antipodal vertices sum to zero, i.e., the labelling function l satisfies $l(-\mathbf{x}) = -l(\mathbf{x})$ for any vertex $\mathbf{x}$. Then there exist adjacent vertices in the triangulation whose labels sum to zero.*

This result is often stated for a triangulation of a ball, but for our purposes later, we have cast it for a triangulation of a sphere (obtained by gluing two n-balls along their boundaries.) Tucker's lemma is equivalent (see [6]) to the following famous theorem from topology:

The Borsuk-Ulam Theorem. *For any continuous function $f : S^n \to \mathbf{R}^n$, there exist antipodal points $\mathbf{x}, -\mathbf{x} \in S^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.*

The equivalence is valuable because Tucker's lemma has a constructive proof, while the Borsuk-Ulam theorem can be used to prove fair division theorems. For instance, the Ham Sandwich Theorem, which says that that there exists a hyperplane that perfectly bisects n sets of positive measure in $\mathbf{R}^n$, is well-known to be a consequence of the Borsuk-Ulam theorem [12, p. 413] and therefore Tucker's lemma can be used to find such a hyperplane. However, as a fair-division theorem, the Ham Sandwich Theorem is of little practical value when the dimension of the sets is greater than 3, and even in dimension 3, it is unneeded if one allows several cuts.

We seek somewhat more practical applications of the Borsuk-Ulam theorem, such as the consensus-halving result of Theorem 1. In fact, a constructive proof of Tucker's lemma yields an algorithm for finding a consensus-halving to any desired accuracy. The dimension of our set A is immaterial because we achieve our division by parallel planes.

Proof of Theorem 1. For ease of expression, we refer to the object A to be divided as "cake" even though players may find certain subsets undesirable.

Place A in a coordinate system aligned with the cardinal directions of the compass. Assume without loss of generality that the (east/west) width of A is one unit. Suppose further that A is to be divided by vertical, parallel north-south planes.

Each point $(x_1, ..., x_{n+1})$ of S^n corresponds to a set of cuts of the cake (called a *cut-set*) obtained by making north/south cuts so that (from west to east) the pieces have widths of $|x_1|, |x_2|, \ldots, |x_{n+1}|$. Use the respective signs of $x_1, x_2, \ldots x_{n+1}$ to determine which portion of the division gets the corresponding piece: collect all the pieces for which x_i is positive, lump them together, and call this the portion A_1. The other pieces will be lumped together to create portion A_2.

The existence of a division such that A_1 and A_2 are deemed exactly equal by all players follows easily from the Borsuk-Ulam theorem; consider the function $f : S^n \to \mathbf{R}^n$ such that the i-th coordinate function $f_i(\mathbf{x}) = \mu_i(A_1)$, player i's measure of the "value" of A_1. This is a continuous function of $\mathbf{x}$ (because of the continuity assumption on the measures), hence by the Borsuk-Ulam theorem there exists a point $\mathbf{x}$ such that $f(\mathbf{x}) = f(-\mathbf{x})$. But since antipodal points on S^{n-1} correspond to the same division with the roles of A_1 and A_2 interchanged, the Borsuk-Ulam point $\mathbf{x}$ corresponds to a set of (at most) n cuts (and fewer if the components of A_i are adjacent) such that $\mu_i(A_1) = \mu_i(A_2)$ for all i, i.e., the pieces are deemed equal by all players. One may see that n cuts are also necessary in the case in which A is a line segment and the player measures have support in n disjoint subintervals of A. This shows the existence of a solution to the consensus halving problem.

To construct an approximate solution (to any pre-specified error tolerance ε), use Tucker's lemma. Recall that every point in S^n corresponds to a cut-set. Given $\varepsilon > 0$, choose a triangulation of S^n with mesh size so small that the in the cut-sets corresponding to any two adjacent vertices, the portions A_1 and A_2 differ by no more than ε in any of the player measures.

We now assign to every vertex a label in the set $\{+1, -1, \dots, +n, -n\}$ which consists of a number and a sign. The *number* assigned to a vertex will be the number of the player who is believes the difference between A_1 and A_2 is greatest for the cut-set corresponding to that vertex. (In there are players equally distressed about the difference, choose the smallest-numbered player.) The *sign* assigned to a vertex will signify the piece that the "most distressed" player prefers in cut-set corresponding to that vertex: if piece A_1 (resp. A_2) is preferred, the sign assigned is $+$ (resp. $-$). (In case that player prefers both pieces equally, choose the portion containing the west edge of the cake).

Note that this gives an anti-symmetric labelling l in which $l(-\mathbf{x}) = -l(\mathbf{x})$ at every vertex, because when the roles of A_1 and A_2 are reversed, the same player is most distressed but her preference is reversed. (Thus moving to the antipodal vertex leaves the label number the same but flips the sign.)

Applying Tucker's lemma, there exists a pair of adjacent vertices in the triangulation with the same label number but opposite signs. Either of these vertices corresponds to a cut-set that is an *approximate* consensus-halving, since at these two nearby cut-sets, the maximally distressed player prefers different portions. For this player, both portions are within ε of each other in value, and since this player's distress was maximal, no other player will dispute this assessment by more than ε.

These adjacent vertices may be found efficiently using the algorithm of Freund and Todd [6], or more recent methods found in [17]. These are *simplicial algorithms* that follow paths of simplices in the triangulation; we do not review them here for lack of space. However, we do emphasize the important feature of such algorithms is the fact that they use vertex labels to determine a path that finds the desired adjacent vertices. In our setting, each vertex corresponds to a cut-set, so the vertex labels can be determined on the fly by moving from vertex to vertex and interactively polling the players for their preferences at the cut-sets along the path.

□

We remark that the proof of Theorem 1 can be modified to address preference measures on a measurable set of any dimension as long as it can be mapped onto a bounded real interval such that the image measures (of the players' measures) are absolutely continuous. In this case, the inverse images of the cut sets of the interval yield the cut sets of the object.

4. Remarks on Implementation

In an actual implementation, the algorithm for consensus-halving can be coded so that a computer could proceed through the algorithm and interactively ask players at each step which portion they would prefer and their perceived difference in size between the portions. (See [14] for a similar fair division procedure based on Sperner's lemma.) Convergence to a solution can be enhanced by existent homotopy algorithms in which ϵ need not be specified in advance and generally decreases with the run time. See Todd [15] or Yang [17] for a survey of such methods applied to fixed point problems.

We state some features of our approach:

1. It gives an constructive algorithm to locate an approximate solution for a problem where it is impossible to use a finite exact procedure.
2. Previous methods (e.g., [11, p.128]) decimate the object by cutting it into a very large number of pieces and reassembling them.
3. Because the algorithm is interactive, players do not have to reveal their a priori preferences (which may in general be very hard to describe). Moreover, during the procedure they do not need to reveal their preferences over all possible cut-sets, but only for cut-sets near a path followed by the simplicial algorithm.
4. On the other hand, if players are able to fully describe their preferences beforehand, the algorithm can be run from the initial data alone. This may possible with sufficiently nice preferences, or if the measures are describing some objective data, such as in the "necklace-splitting" problem below.
5. Mixtures are treated by this method exactly the same as goods or burdens.

We remark that issues such as strategic manipulability of the algorithm and Pareto-efficiency of the outcome are not meaningful in the context of the consensus-halving problem if players are not assigned either of the pieces that result. In this case all players desire the same goal: to get agreement by all n people that two portions of the cake are nearly equal in size. For example, if the players are parents wishing to halve their estate between two children, the parents desire that both portions be equal. (Efficiency and manipulability are only important issues when each player seeks *different* goals, such as maximizing different pieces.)

Even in applications where players (or groups of players) are assigned to one of the pieces, we can specify that the assignment be made only after the halving has already been decided. Thus, no player would have any assurance that she would be assigned to a piece that she tried to fatten up, and strategic play might backfire. On the other hand, stating her true intentions will guarantee her an approximate consensus-halving.

5. The Law of the Sea Treaty and Necklace Splitting

We remarked earlier that a consensus-halving procedure could help families to split a piece of land into two regions in such a way that *every* member of both families believes the land is nearly equally divided.

An analogous situation arises in the 1994 Convention of the Law of the Sea [5, p.10], which uses a divide-and-choose procedure to protect the interests of developing countries when an industrialized nation wants to mine a portion of the seabed in international waters. An agency representing the developing countries chooses one of the two halves to reserve it for future mining by less-developed nations. If a consensus-halving procedure were used instead of divide-and-choose, it would yield a division into two portions such that *every nation* agreeed both portions were almost equally valuable.

Our algorithm also provides a constructive solution to the discrete problem of "splitting necklaces" [1]. Imagine a necklace of jewels of n different colors but an even number of identical jewels of each color. (The position of each jewel is fixed relative to the other jewels.) Using a minimal number of cuts, we desire a division of the necklace into two portions such that for any color, both portions have the same number of jewels of that color. (We imagine the necklace laid out along a straight line, with cuts made perpendicular to this line.) Theorem 1 can be applied in this context by assigning each player a jewel color, and replacing each player's subjective measure by a precise count of the number of jewels of her assigned color in each portion. In this case, since the measures are completely known from initial data, a simplicial algorithm can be adapted to compute an *exact* solution: if ϵ is chosen to be 1 jewel, then the simplicial algorithm will find adjacent vertices possessing opposite labels, and a little thought reveals that one of these vertices must represent cuts that divide *all* the jewel colors in half. (Otherwise the same player could not be "most distressed" at both vertices and still change preferences.)

6. team-splitting

Each consensus-halving result corresponds to a related *envy-free* division problem for twice the number of people, by averaging measures. For instance, our consensus-halving result can be used to address the following "team-splitting" problem.

Suppose among the $2n$ explorers on an expedition there are two of each specialty: two zoologists, two botanists, two archeologists, etc. They want to know the fairest way to split both their team and their territory. In other words, they want to split into two teams in such a way that each specialty is represented on each team, and such that each team member is satisfied that she is on the team with the best half of the territory to explore.

Theorem 2 (Team-Splitting)**.** *Given a territory and such a collection of $2n$ explorers, there exists a way to divide the territory and the people into two teams of n explorers (one of each type) such that each explorer is satisfied with his/her territory.*

This result assumes there are no coalitions (sets of people who desire to be on the same team) and that the players have continuous (though not necessarily additive) measures over the territory.

Proof. The territory is the object A that will be divided by consensus-halving. Consider the i-th pair of scientists by specialty, with measures λ_i and λ_i'. Let

$\mu_i = \lambda_i + \lambda'_i$. These form a collection of n measures with which to apply consensus-halving, obtaining two portions A_1 and A_2 for which

$$\lambda_i(A_1) + \lambda'_i(A_1) = \lambda_i(A_2) + \lambda'_i(A_2)$$

for all i. If $\lambda_i(A_1) = \lambda_i(A_2)$ and $\lambda'_i(A_1) = \lambda'_i(A_2)$ then both scientists of the i-th pair are indifferent between the portions; we can flip a coin to make assignments. Otherwise we conclude that one member of the i-th pair believes A_1 is more valuable than A_2, and the other believes the opposite. In this case assign each scientist of the i-th pair the portion of the territory that she prefers. □

From the consensus-halving theorem, we see that the team-splitting solution is even reasonably practical— it would never involve more than n straight cuts through the territory.

7. Open problems

We close with some open problems.

1. *Consensus-splitting in an arbitrary ratio.* Suppose we desired a division of cake into two portions so that each of n people agreed the split was some other ratio, say two-to-one? Under what conditions can this be achieved constructively using a minimal number of cuts?
2. *Consensus-$1/k$-division.* Is there a constructive method for obtaining a division into k portions such that each of n people believe all k portions are equal in size? Such a method could be used to divide an estate among k children such that each of n people (parents, children, and others) agreed that all children received equal portions.
3. *A generalized Tucker's lemma.* It seems quite likely that the above problem could be addressed by proving some generalization of Tucker's lemma. What is the appropriate combinatorial generalization, and is there a constructive proof?

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