MATH 239 Notes

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1 Lecture 1

1.1 Definition: Set, Element

A set S is a collection of dinstinct objects. These objects are the elements of S. The size or cardinality of S, denoted $|S|$, is the number of elements in S
If S and T are disjoint and finite, then $|S| + |T| = |S \cup T|$

1.2 Definition: Cartesian Porduct

Given two sets S and T, their cartesian product $S \times T$ is the set $\{(s,t) : s \in S, t \in T\}$
If S, T are finite, $|S \times T| = |S| \cdot |T|$

1.3 Definition: Subset

Given S and T, we say S is a subset of T, denoted $S \subseteq T$, if every element of S is also an element of T

1.4 Theorem

For every $n \geq 0$, the number of subsets of an n-element set is 2^n

1.5 Definition: List, Permutations

A list of a set S is an ordered list of elements of S exactly once each. When $S = \{1, \cdots, n\}$ for some n, then the lists of S are called permutations of n

1.6 Theorem

Let S be a set with $|S| = n$ for some $n \in \mathbb{N}$. Then the number of distinct lists of S is

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

1.7 Definition: Partial List

A partial list of length k of a set S is an ordered list of k of the elements of S, exactly once each

1.8 Theorem

For $n, k \geq 0$, the number of partial lists of length k of an n-element set is

$$n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

1.9 Theorem

For $n \geq k \geq 0$, the number of k-element subsets of an n-element set S is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\binom{n}{k} = 0$ if $k > n$ or $k < 0$

2 Lecture 2

2.1 Theorem

How many paths from $(0,0)$ to (k,l) use only north & east steps?

$$\binom{k+l}{l} = \binom{k+l}{k}$$

2.2 Pascal's Triangle

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

2.3 Definition: Multiset

A multiset of size n with element of t types is a sequence $(m_1, \cdots, m_t)$ of non-negative integers such that $m_1 + \cdots + m_t = n$

2.4 Theorem

For $n \geq 0$ and $t \geq 1$, the number of multisets of size n with t types is $\binom{n+t-1}{t-1}$

2.5 Definition: Function

$$f : X \to Y$$

where X is the domain / input, Y is the range / output
This is used to prove that f is well-defined

2.6 Definition: Injective

f is injective if for all $x, x' \in X$ with $x \neq x'$, we have $f(x) \neq f(x')$

2.7 Definition: Surjective

f is surjective if for every $y \in Y$ there exists $x \in X$ with $f(x) = y$

2.8 Definition: Bijective

f is bijective if f is both injective and surjective

2.9 Definition: Bijective, Inverse, Mutually Inverse

If $f : X \to Y$ and $g : Y \to X$ are functions such that

- for all $x \in X$, $g(f(x)) = x$ and
- for all $y \in Y$, $f(g(y)) = y$

then f is bijective, g is the inverse of f, and f & g are mutually inverse bijections

2.10 Theorem

$\binom{n}{k} = \binom{n}{n-k}$

3 Lecture 3

3.1 Definition: Formal Power Series

A formal power series is an object of the form

$$A(x) = \sum_{n \geq 0} a_n x^n$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. For $A(x)$, we write $[x^n]A(x)$ for the coefficient a_n, the square brackets denote coefficient extraction
We denote the ring of formal power series by $\mathbb{C}[[x]]$

3.2 Theorem

Given $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$ in $\mathbb{C}[[x]]$, we define addition as

$$A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n$$

and multiplication as

$$A(x) \cdot B(x) = \sum_{n \geq 0} \sum_{k=0}^{n} a_k b_{n-k} x^n$$

3.3 Definition: Geometric Series

The geometric series is

$$G(x) = 1 + x + x^2 + \cdots = \sum_{n \geq 0} x^n$$

and

$$G(x) = \frac{1}{1-x}$$

3.4 Definition: Concatenation

Let $A(x) = \sum_{i \geq 0} a_i x_i$. Then, if the following is a FPS, we define $A(B(x)) = \sum_{i \geq 0} a_i (B(x))^i$
Works if $[x^0]B(x) = 0$ or A has finitely many terms

3.5 Definition: Weight Function, Generating Series

Given a set S, a weight function on S is a function $w : S \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, then set $\{s \in S : w(s) = i\}$ is finite
Associated with set and its weight function is the generating series

$$\Phi_S^w(x) = \sum_{s \in S} x^{w(x)}$$

3.6 Definition: Weight-Preserving Bijection

A weight-preserving bijection from S with weight function w_S to T with weight function w_T is a bijection $f : S \to T$ such that $w_S(x) = w_T(f(s))$ for all $s \in S$

3.7 Theorem

If there is a weight-preserving bijection f between T, w_T and S, w_S, then $\Phi_T^{w_T}(x) = \Phi_S^{w_S}(x)$

4 Lecture 4

5 Lecture 5

5.1 Binomial Theorem

For all $n \in \mathbb{N}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

5.2 Negative Binomial Theorem

For all positive integers t

$$\frac{1}{(1-x)^t} = \sum_{n\geq 0} \binom{n+t-1}{t-1} x^n$$

6 Lecture 6

6.1 Definition: Union

The union of S, w and S', w' is defined if $S \cap S' = \emptyset$, and then it is $S \cup S'$ with weight function $w \cup w'$ where $(w \cup w')(s)$ is defined to be $w(s)$ if $s \in S$ and $w'(s)$ if $s \in S'$

6.2 Lemma: Sum Lemma

Given sets with weight function S, w and S', w' with $S \cap S' = \emptyset$, we have

$$\Phi_{S\cup S'}^{w\cup w'}(x) = \Phi_S^w(x) + \Phi_{S'}^{w'}(x)$$

6.3 Definition: Product

The product of S, w and S', w' is defined as $S \times S'$ with weight function w, w' where $(w \times w')((s, s')) = w(s) + w'(s')$ for $s \in S$ and $s' \in S'$

6.4 Lemma: Product Lemma

Given sets with weight functions S, w and S', w', we have

$$\Phi_{S\times S'}^{w\times w'}(x) = \Phi_S^w(x) \cdot \Phi_{S'}^{w'}(x)$$

7 Lecture 7

7.1 Definition: Strings Over (Alphabet) S

Given $k \in \mathbb{N}$ (including 0), we write S^k, w^k for the set

$$S^k = \{(s_1, \cdots, s_k) : s_1, \cdots, s_k \in S\}$$

with weight function

$$w^k((s_1, \cdots, s_k)) = w(s_1) + \cdots + w(s_k)$$

Suppose now that $w(s) > 0$ for all $s \in S$. Then there is a well-defined set with weight function S^*, w^*, called strings over (alphabet) S, given

$$S^* = \bigcup_{k\geq 0} S^k$$

and

$$w^* = \bigcup_{k\geq 0} w^k$$

7.2 Lemma: String Lemma

Given a set with weight function S, w with $w(s) > 0$ for all $s \in S$, we have

$$\Phi_{S^*}^{w^*} = \frac{1}{1 - \Phi_S^w(x)}$$

7.3 Definition: Binary Strings

Binary strings are sequences $(b_1, \cdots, b_k)$ (usually written as $b_1, \cdots, b_k$) for $k \in \mathbb{N}$ such that each bit b_i is in $\{0, 1\}$. In other words, defining $w(0) = w(1) = 1$, the set of all binary strings is $\{0, 1\}^*$ with weight function w^*, where $w^*((b_1 \cdots b_k)) = k$ is the length of a binary string

7.4 Definition: Composition

A composition is a finite sequence $\gamma = (c_1, \cdots, c_k)$ of positive integers. The c_i are its parts, and its length is the number k of parts. The size of a composition, $|\gamma|$, is defined as $c_1 + \cdots + c_k$
The empty composition ϵ with no integers is also allowed as the unique composition with length and size 0

8 Lecture 8

8.1 Definition: Concatenation, Concatenation Product

Let S be a set, and let $R, T \subseteq S^*$. Then we define the concatenation of $r = (r_1 \cdots r_k) \in \mathbb{R}$ and $t = (t_1, \cdots, t_l) \in T$ as $rt = (r_1, \cdots, r_k, t_1, \cdots, t_l)$. We also define the concatenation product $RT = \{rt : (r, t) \in R \times T\}$

8.2 Definition: $*$

If S is a set of strings, then $S^* = \bigcup_{k \geq 0} s^{(k)}$

9 Lecture 9

9.1 Definition: Regular Expression

A regular expression is defined recursively, as follows:

- All of $\epsilon, 0$, and 1 are regular expressions
- If R and S are regular expressions, then so is $R \smile S$
- If R and S are regular expressions, then so is RS
  For any finite $k \in \mathbb{N}$ we also use R^k for the k-fold concatenation of R: that is $R^2 = RR$ and $R^3 = RRR$, and so on
- If R is regular expression, then so is R^*

9.2 Definition: Ambiguity

A regular expression is unambiguous if it doesn't produce the same string in two different ways

- $R \smile T$ is unambiguous if:
  - R, T are unambiguous
  - R produces $\mathfrak{R}$, T produces $\mathfrak{T}$, and $\mathfrak{R}$ is disjoint from $\mathfrak{T}$
- RT is unambiguous if :
  - R, T are unambiguous
  - R produces $\mathfrak{R}$, T produces $\mathfrak{T}$, and $f : \mathfrak{R} \times \mathfrak{T} \to \mathfrak{R}\mathfrak{T}$, $f((r,t)) = rt$ is bijection
- R^* is unambiguous if:
  - R is unambiguous
  - R produces $\mathfrak{R}$, and for $k \in \mathbb{N}$, $f : \mathfrak{R}^n \to \mathfrak{R}^{\mathfrak{k}}$, $f((r_1, \cdots, r_k)) = r_1 \cdots r_k$ is a bijection and
  - $\bigcup\limits_{k \geq 0} \mathfrak{R}^{(k)}$ is disjoint union

9.3 Translation into Generating Series

A regular expression leads to a rational function; this is defined recursively, as follows. Assume that R and S are regular expressions that lead to $R(x)$ and $S(x)$, respectively:

- ϵ leads to 1, 0 leads to x, 1 leads to x
- $R \smile T$ leads to $R(x) + S(x)$
- RS leads to $R(x) \cdot S(x)$
- R^* leads to $\dfrac{1}{1 - R(x)}$

9.4 Theorem

Let R be regular expression producing rational language $\mathcal{R}$ and lead to rational function $R(x)$. If R is unambiguous expression for $\mathcal{R}$ then $R(x) = \Phi_{\mathcal{R}}(x)$, the generating series for $\mathcal{R}$ with respect to length

10 Lecture 10

10.1 Definition: Substring, Block, Empty String

For a string $s = s_1 \cdots s_j$, a substring of s is either ε or a string of the form $s_i \cdots s_{i'}$ for some $i \leq i'$ with $i, i' \in \{1, \cdots, j\}$. Let us say that a block of a string $s = s_1 \cdots s_j$ is a non-empty maximal substring $s_i \cdots s_{i'}$ of s such that $s_i = \cdots = s_{i'}$. We write ε for the empty string

10.2 Block Decomposition

break down regular expression into blocks which contains consecutive bits of same elements

10.3 Set Operations Unambiguous

- A is unambiguous
- AB is unambiguous if $f : A \times B \to AB$, $f((a,b)) = ab$ is a bijection
- $A \cup B$ is unambiguous if $A \cap B = \emptyset$
- A^* is unambiguous if $\bigcup\limits_{k \geq 0} A^{(k)}$ is a disjoint union, and for all k, $f : A^k \to A^{(k)}$, $f((a_1, \cdots, a_k)) = a_1 \cdots a_k$ is a bijection.

11 Lecture 11

11.1 Definition: Prefix, Suffix

Let us say a substring t' of t is a prefix of t if $t = t't''$ for some string t'', and t' is a suffix of t if $t = t''t'$ for some string t''

12 Lecture 12

12.1 Prefix Decomposition

break down regular expression into blocks every time you see 1

13 Lecture 13

13.1 Theorem

Let $\mathcal{K} \in \{0,1\}^*$ be a nonempty string of length n, $\mathcal{A} = \mathcal{A}_{||}$ be the set of binary strings that avoid $\mathcal{K}$. Let $\mathcal{C}$ be the set of all nonempty suffixes γ of $\mathcal{K}$ such that $\mathcal{K}\gamma = \eta\mathcal{K}$ for some nonempty prefix η of $\mathcal{K}$. Let $C(x) = \sum_{\gamma \in \mathcal{C}} x^{l(\gamma)}$

$$A(x) = \frac{1 + C(x)}{(1 - 2x)(1 + C(x)) + x^n}$$

14 Lecture 14

14.1 Homogeneous Lineawr Recurrence Relation

Let $g = (g_0, g_1, g_2, \cdots)$ be an infinite sequence of complex numbers. Let $a_1, a_2, \cdots, a_d$ be in $\mathbb{C}$, and let $N \geq d$ be an integer. We say that g satisfies a homogeneous linear recurrence relation provided that

$$g_n + a_1 g_{n-1} + a_2 g_{n-2} + \cdots a_d g_{n-d} = 0$$

for all $n \geq N$. The values $g_0, g_1, \cdots, g_{N-1}$ are the initial conditions of the recurrence. The relation is linear bc the LHS is a linear combination of the entries of the sequence g; it is homogeneous bc the RHS of the equation is zero

15 Lecture 15

15.1 Partial Fraction

Let $G(x) = P(x)/Q(x)$ be a rational function in which $deg(P) < deg(Q)$ and the constant term of $Q(x)$ is 1. Factor the denominator to obtain its inverse roots

$$Q(x) = (1 - \lambda_1 x)^{d_1}(1 - \lambda_2 x)^{d_2} \cdots (1 - \lambda_s x)^{d_s}$$

in which $\lambda_1, \cdots, \lambda_s$ are distinct nonzero complex numbers and $d_1 + \cdots + d_s = d = deg(Q)$. Then there are d complex numbers

$$C_1^{(1)}, C_1^{(2)}, \cdots, C_1^{(d_1)}; \cdots; C_s^{(1)}, \cdots, C_s^{(d_s)}$$

such that (uniquely determine)

$$G(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{d_s} \frac{C_i^{(j)}}{(1 - \lambda_i x)^j}$$

16 Lecture 16

16.1 Theorem

Let $g = (g_0, g_1, g_2)$ be a sequence of complex numbers, and let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the corresponding generating series. Assume

$$G(x) = R(x) + \frac{P(x)}{Q(x)}$$

for some polynomial $P(x)$, $Q(x)$ and $R(x)$ with $\deg P(x) < \deg Q(x)$ and $Q(0) = 1$. Factor $Q(x)$ to obtain its inverse roots and their multiplicities, then there are polynomial $p_i(n)$ for $1 \leq i \leq s$ with $\deg p_i(n) < d_i$, st for all $n > \deg R(x)$,

$$g_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \cdots + p_s(n)\lambda_s^n$$

17 Lecture 17

17.1 Graph

A graph G is a finite nonempty set $V(G)$, of objects, called vertices, together with a set, $E(G)$, of unordered pairs of distinct vertices. The elements of $E(G)$ called edges

18 Lecture 18

18.1 Isomorphism

Two graph G_1 and G_2 are isomorphic if there exists a bijection $f : V(G_1) \to V(G_2)$ such that vertices $f(u)$ and $f(v)$ are adjacent in G_2 iff u and v are adjacent in G_1

18.2 Degree

The number of edges incident with a vertex v is called degree of v, denoted by $\deg(v)$

18.3 Theorem

Any graph G we have

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

18.4 Corollary

The number of vertices of odd degree in a graph is even

18.5 Corollary

The average degree of a vertex in graph G is

$$\frac{2|E(G)|}{|V(G)|}$$

18.6 Complete Graph

A complete graph is one which all pairs of distinct vertices are adjacent. The complete graph with p vertices is denoted by K_p, $p \geq 1$

19 Lecture 19

19.1 Bipartie Graph

A graph in which all edges join a vertex in A to a vertex in B, is called a bipartie graph, with bipartition (A, B)

19.2 n-cube

For $n \geq 0$, the $n-$cube is the graph whose vertices are the $\{0, 1\}-$strings of length n, and two strings are adjacent iff they differ in exactly one position

20 Lecture 20

20.1 Adjacency Matrix

The adjacency matrix of a graph G having vertices $v_1, v_2, \cdots, v_p$ is the $p \times p$ matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

20.2 Incidence Matrix

The incidence matrix of a graph G with vertices $v_1, \vdots, v_p$ and edges $e_1, \cdots, e_q$ is a $p \times q$ matrix $B = [b_i j]$ where

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with } e_j \\ 0, & \text{otherwise} \end{cases}$$

21 Lecture 21

21.1 Subgraph

A subgraph of a graph G is a graph whose vertex set is a subset U of $V(G)$ and whose edge set is a subset of those edges of G that have both vertices in U
If $V(H) = V(G)$, say H is spanning subgraph of G
If H is subgraph of G and H is not equal to G, say H is a proper subgraph of G

21.2 Walk, Length, Closed

A walk in a graph G from v_0 to v_n is an alternating sequence of vertices and edges of G

$$v_0 e_1 v_1 e_2 \cdots e_n v_n$$

Call it v_0, v_n-walk.
Length of a walk is number of edges in it.
A walk is closed if $v_0 = v_n$

21.3 Path

A path is a walk in chich all vertices are distinct

21.4 Theorem

If there is a walk from vertex x to vertex y in G, then there is a path from x to y in G

21.5 Corollary

Let x, y, z be vertices of G. If there is a path from x to y in G and path from y to z in G, then there is a path from x to z in G

21.6 Theorem

If every vertex in G has degree at least 2, then G contains a cycle

21.7 Grith

The grith of graph G is the length of the shortest cycle in G, and denoted as $g(G)$. If no cycle, then $g(G)$ is infinite

21.8 Hamilton Cycle

A spanning cycle in a graph is known as a Hamilton cycle

21.9 Connected

A graph G is connected if, for every two vertices, there is a path connect the two

21.10 Component

A component of G is subgraph C of G s.t.

- C is connected
- no subgraph of G that properly contains C is connected

21.11 Theorem

A graph G is not connected iff there exists a proper nonempty subset X of $V(G)$ s.t. the cut induced by X is empty

22 Lecture 22

22.1 Eulerian Circuit

An Eulerian circuit of graph G is a closed walk that contains every edge of G exactly once

22.2 Theorem

Let G be connected graph. G has an Eulerian circuit iff every vertex has even degree

22.3 Bridges

Edge e of G is a bridge if $G - e$ has more components thatn G

22.4 Lemma

If $e = \{x, y\}$ is a bridge of connected graph G, then $G - e$ has precisely two components, x and y are in different components

22.5 Theorem

Edge e is bridge of graph G iff it is not contained in any cycle of G

22.6 Corollary

If there are two distinct paths from u to vertex v in G, G contains a cycle

23 Lecture 23

23.1 Tree

A tree is a connected graph with no cycle

23.2 Forest

A forest is graph with no cycles

23.3 Lemma

If u and v are vertices in tree T, then there is a unique $u, v-$ path in T

23.4 Lemma

Every edge of a tree T is a bridge

23.5 Lemma

If T is a tree, then $|E(T)| = |V(T)| - 1$

23.6 Corollary

If G is a forest with k components, then $|E(G)| = |V(G)| - k$

23.7 Leaf

A leaf in tree is a vertex of degree 1

23.8 Theorem

A tree with at least two vertices has at least two leaves

24 Lecture 24

24.1 Spanning Tree

A spanning subgraph which is also a tree is called spanning tree

24.2 Theorem

A graph G is connected iff it has a spanning tree

24.3 Corollary

If G is connected, with p vertices and $q = p - 1$ edges, then G is a tree

24.4 Theorem

If T is spanning tree of G and e is edge not in T, then $T+e$ contains exactly one cycle C. Moreover, if e' is any edge on C, then $T + e - e'$ is also spanning tree of G

24.5 Theorem

If T is spanning tree of G and e is edge in T, then $T - e$ has 2 components. If e' is in the cut induced by one of the components, then $T - e + e'$ is also spanning tree of G

25 Lecture 25

25.1 Odd Cycle

An odd cycle is a cycle on an odd number of vertices

25.2 Lemma

An odd cycle is not bipartite

25.3 Theorem

A graph is bipartite iff it has no odd cycle

26 Lecture 26

26.1 Planar

A graph G is planar if it has a drawing in the plane so that its edges intersect only at their ends, and so that no two vertices coincide.
The actual drawing is called a planar embedding of G, or a planar map

26.2 Faces

A planar embedding partitions the plane into connected regions called faces

26.3 Boundary

The subgraph formed by the vertices and edges in a face is called boundary of the face
Two faces are adjacent if they are incident with a common edge

26.4 Theorem

If we have a planar embedding of a connected graph G with faces $f_1, \cdots, f_s$, then

$$\sum_{i=1}^{s} deg(f_i) = 2|E(G)|$$

26.5 Corollary

If the connected graph G has a panar embedding with f faces, the average degree of a face in the embedding is $\frac{2|E(G)|}{f}$

27 Lecture 27

27.1 Euler's Formula

Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then

$$p - q + f = 2$$

28 Lecture 28

28.1 Platonic Solid

The polyhedra that the faces have the same degree, vertices have the same degree are call platonic solids
There are five platonic solids: tetrahedron, cube, octahedron, dodecahedron, icosahedron
We call a graph platonic if it admits a planar embedding in which each vertex have the same degree $d \geq 3$, each face has the same degree $d^* \geq 3$

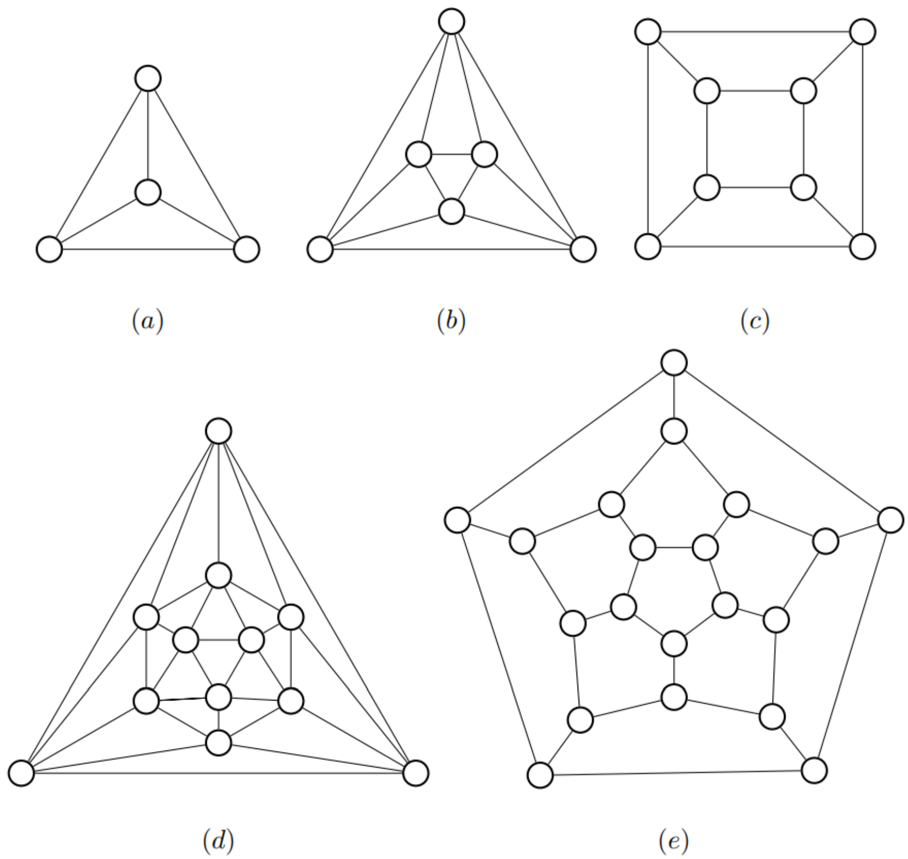

Figure 7.9: (a) the tetrahedron; (b) the octahedron; (c) the cube;(d) the icosahedron; (e) the dodecahedron

28.2 Lemma

Let G be a planar embedding with p vertices, q edges and s faces, in which each vertex has degree $d \geq 3$ and each face has degree $d^* \geq 3$. Then (d, d^*) is one of the five pairs

$$\{(3,3),(3,4),(4,3),(3,5),(3,5)\}$$

28.3 Lemma

If G is a platonic graph with p vertices, q edges and f faces, where each vertex has degree d and each face degree d^*, then

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

and $p = 2q/d$ and $f = 2q/d^*$

29 Lecture 29

29.1 Lemma

If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle

29.2 Lemma

Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d^*, then $(d^* - 2)q \leq d^*(p - 2)$

29.3 Theorem

In a planar graph G with $p \geq 3$ vertices and q edges, we have

$$q \leq 3p - 6$$

29.4 Corollary

K_5 is not planar

29.5 Corollary

A planar graph has a vertex of degrees at most five

29.6 Theorem

In a bipartite planar graph G with $p \geq 3$ vertices and q edges, we have

$$q \leq 2p - 4$$

29.7 Lemma

$K_{3,3}$ is not planar

30 Lecture 30

30.1 Edge Subdivision

An edge subdivision of a graph G is obtained by applying the following operations, independently, to each edge of G

- replace the edge by a path of length 1 or more
- if the path has length $m > 1$, then there are $m - 1$ new vertices and $m - 1$ new edges created
- if the path has length $m = 1$, then the edge is unchanged

30.2 Kuratowski's Theorem

A graph is not planar iff it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$

31 Lecutre 31

31.1 Colouring and Planar Graph

A k-colouring of graph G is function from $V(G)$ to set of size k. Adjacent vertices always have different colours.

Theorem

A garph is 2-colourable iff it is bipartite

Theorem

K_n is n-colourable, and not k-colourable for any $k < n$

Theorem

Every planar graph is 6-colourable

Definition

G be graph and $e = \{x, y\}$ be an edge of G. Graph G/e obtained from G by contracting edge e is graph with vertex set $V(G)\backslash\{x, y\} \cup \{z\}$, where z is new vertex, and edge set $\{\{u, v\} \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{\{u, z\} : u \notin \{x, y\}, \{u, w\} \in E(G) \text{ for some } w \in \{x, y\}\}$

Theorem

Every planar graph is 5-colourable

Theorem

Every planar graph is 4-colourable

32 Lecture 32

32.1 Matching

Matching

A matching in a graph G is a set of edges of G st no two dges in M have common end

Saturated

A vertex v of G is saturated by M, if v in incident with an edge in M

Perfect Matching

A special kind of maximum matching is one having size $p/2$, that is, one that saturates every vertex

Alternating Path

Say a path $v_0 v_1 \cdots v_n$ is alternating path with respect to M if one of the following holds

- $\{v_i, v_{i+1}\} \in M$ if i is even and $\{v_i, v_{i+1}\} \notin M$ if i is odd
- $\{v_i, v_{i+1}\} \notin M$ if i is even and $\{v_i, v_{i+1}\} \in M$ if i is odd

Augmenting Path

An augmenting path with respect to M is an alterting oath joining two distinct vertices neither of which is satuated by M

Lemma

If M has augmenting path, it is not a max matching

33 Lecture 33

33.1 Covers

Cover

A cover of graph G is a set C of vertices st every edge of G has at least one end in C

Lemma

If M is matching of G, C is cover of G, $|M| \leq |C|$

Lemma

If M is matching and C is cover and $|M| = |C|$, then M is max matching and C is min cover

33.2 Konig's Theorem

Konig's Theorem

In a bipartite graph the max size of a matching is min size of cover

Lemma

Let M be matching of bipartite graph G with bipartition A, B, and let X and Y defined as

- Z denote set of vertices in G that joined by a vertex in set of vertices in A not saturated by M by an alternating path
- $X = A \cap Z$
- $Y = B \cap Z$

Then

- there is no edge of G from X to $B \backslash Y$
- $C = Y \cup (A \backslash X)$ is cover of G
- no edge of M from Y to $A \backslash X$
- $|M| = |C| - |U|$ where U is set of unsaturated vertiices in Y
- there is an augmenting path to each vertex in U

Bipartite Matching Algorithm

Step 1: let M be any matching of G

Step 2: set $\hat{X} = \{v \in A : v \text{ is unsaturated}\}$, set $\hat{Y} = \emptyset$, set $pr(v)$ to be undefined for all $v \in V(G)$

Step 3: for each vertex $v \in B \backslash \hat{Y}$ such that there is an edge $\{u, v\}$ with $u \in \hat{X}$, add v to $\hat{Y}$, and set $pr(v) = u$

Step 4: if step 2 added no vertex to $\hat{Y}$, return the max matching M and min cover $C = \hat{Y} \cup (A \backslash \hat{X})$, and stop

Step 5: if step 2 added unsaturated vertex v to $\hat{Y}$, us pr values to trace an augmenting path from v to an unsaturated element of $\hat{X}$, use the path to produce a larger matching M', replace M by M', and back to step 1

Step 6: for each vertex $v \in A \backslash \hat{X}$ st there is an edge $\{u, v\} \in M$ with $u \in \hat{Y}$, add v to $\hat{X}$ and set $pr(v) = u$, back to step 2

34 Lecture 34

34.1 Application of Konig's Theorem

Neighbour Set

Let neighbour set $N(D)$ of D be $\{v \in V(G) : \text{there exists } u \in D \text{ with } \{u, v\} \in E(G)\}$

Hall's Theorem

A bipartite graph G with bipartition A, B has matching saturating every vertex in A, iff every subset D of A satisfies $|N(D)| \geq |D|$

35 Lecture 35

35.1 Perfect Matchings in Bipartite Graphs

Corollary

Bipartite graph G with bipartition A, B has perfect matching iff $|A| = |B|$ and subset D of A satisfies

$$|N(D)| \geq |D|$$

Theorem

If G is $k-$regular bipartite graph with $k \geq 1$, then G has perfect matching

36 Lecture 36

36.1 Edge-Colouring

Edge k-colouring

An edge $k-$colouring of graph G is function from $E(G)$ to set of size k st no two edges incident with same vertex have same colour

Theorem

Bipartite graph with max degree $\triangle$ has an edge $\triangle-$colouring

Lemma

G be a bipartite graph having at least one edge. Then G has a matching saturating each vertex of max degree