# Math 137 Notes

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## Contents

# 1 Sequence and Convergence

## 1.1 Absolute Value and Distance

**Definition 1**

For a real number $x$, the absolute value of $x$, denoted $|x|$ is defined by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

**Proposition 2**

For all real number $x$, $|x| = |-x|$

**Proposition 3**

For all real nubers $u$ and $v$, the distance from $u$ and $v$ is $|u - v|$

**Theorem 5 (Triangle Inequality I)**

For all real numbers, $x$ $y$ and $z$, $|x - z| \leq |x - y| + |y - z|$

**Theorem 7 (Triangle Inequality II)**

For all real numbers $a$ and $b$, $|a + b| \leq |a| + |b|$

**Definition 9**

The symbol $\cup$ means "union" and roughly correspond to "or"
The symbol $\cap$ means "intersect" and roughly represent "and"

**Fact 11**

If $a < b$, then $a \leq b$
$a < b$, $c > 0$, $ac < bc$
$a < b$, $c < 0$, $ac > bc$
$a < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$
$|ac| = |a||c|$, $|ac + bc| = |c||a + b|$

## 1.2 Sequence

**Definition 13**

An infinite sequence is an infinite ordered list of numbers

**Notation**

$\{a_n\}_{n=1}^{\infty}$ the term indexing starts at 1
$\{a_n\}_{n=5}^{\infty}$ $a_n = ln(n-4)$
$\{a_n\}$ the starting point either doesn't matter or is clear from context
$\{a_1, a_2, a_3, a_4, \cdots\} \iff (a_1, a_2, a_3, a_4, \cdots)$

**Definition 16**

Let $A = (a_1, a_2, a_3, a_4, \cdots)$ be a sequence

1. If $n_1, n_2, n_3, \cdots$ is a sequence of positive integers, then $a_{n_1}, a_{n_2}, a_{n_3}, \cdots$ is a subsequence of $A$, $(n_1 < n_2 < n_3 < \cdots)$
2. A tail of $A$ is a subsequence of the form $a_k, a_{k+1}, a_{k+2}, \cdots$

**Definition 18**

The sequence $A = (a_1, a_2, a_3, a_4, \cdots)$converges to $L$ if for any error (positive number), there is a tail of the sequence, each term of which within that error of $L$

**Definition 19**

Let $A = \{a_n\}$ be s sequence. We say taht $A$ converges to $L$ and write $\lim_{n\to\infty} a_n = L$ of for every number $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| = \epsilon$, $a_n \in (L - \epsilon, L + \epsilon)$

**Proposition 20**

The harmonic sequence converges, eg: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

**Proposition 21**

The sequence $(-1, 1, -1, 1, \cdots)$ does not converge $a_n = (-1)^n$

**Example 22**

Show that $\{a_n\}$ with $a_n = \dfrac{n+1}{2n+3}$ converges and finds the limit
Guess $L = \frac{1}{2}$, want $|a_n - \frac{1}{2}| < \epsilon$
$|a_n - \frac{1}{2}| = |\frac{n+1}{2n+3} - \frac{1}{2}| = |\frac{-1}{4n+6}| = \frac{1}{4n+6} < \frac{1}{4n}$
$|a_n - \frac{1}{2}| < \frac{1}{4n} < \epsilon$, choose $N$ such that $\frac{1}{4N} < \epsilon \iff \frac{1}{4\epsilon} < N$
$n \geq N$, $\frac{1}{4n} \leq \frac{1}{4N}$
Proof:
Let $\epsilon > 0$ be arbitrary and set $N = \lceil \frac{1}{4\epsilon} \rceil + 1$ so that $N > \frac{1}{4\epsilon}$ which implies $\frac{1}{4N} < \epsilon$
For any $n \geq N$, $\frac{1}{4n} \leq \frac{1}{4N}$
$|a_n - \frac{1}{2}| = |\frac{n+1}{2n+3} - \frac{1}{2}| = \frac{1}{4n+6} < \frac{1}{4n} \leq \frac{1}{4N} < \epsilon$
so $\displaystyle\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

**Theorem 23**

A sequence has at most one limit

**Definition 24**

A sequence is said to diverge if it does not converge

**Definition 25**

Let $\{a_n\}$ be a sequence. We say that $\{a_n\}$ diverges to infinity and write $\displaystyle\lim_{n\to\infty} a_n = \infty$ if for every real number $M > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, $a_n > M$
$*$ $\displaystyle\lim_{n\to\infty} a_n = \infty$ DOES NOT mean the sequence "converge to $\infty$"

**Theorem 27 (Arithmetic of limits)**

Let $\{a_n\}$ and $\{b_n\}$ be sequence with limits $L$ and $M$ respectively

1. For any $c \in \mathbb{R}$, if $a_n = c$ for all $n$, $L = c$
2. For any $c \in \mathbb{R}$, $\displaystyle\lim_{n\to\infty} c\ a_n = cL$
3. $\displaystyle\lim_{n\to\infty} a_n + b_n = L + M$

4. $\lim_{n\to\infty} a_n b_n = LM$

5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}, M \neq 0$

6. If $a_n > 0$ for all $n$ and $a > 0$, $\lim_{n\to\infty} a_n^x = L^x$

7. For any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$

8. If $a > 0$, then $\lim_{n\to\infty} n^a = \infty$

9. If $a < 0$, then $\lim_{n\to\infty} n^a = 0$

**Theorem 30**

Suppose $\lim_{n\to\infty} b_n = 0$ and $b_n \neq 0$ for all $n$. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then $\lim_{n\to\infty} a_n = 0$

**Fact**

If $a_n \geq 0$ for all $n$ and $\lim_{n\to\infty} a_n = L$, then $L \geq 0$

**Theorem 33 (Squeeze theorem for sequence)**

Suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, and $a_n \leq b_n \leq c_n$ for all but finite $n$, then $\lim_{n\to\infty} b_n = L$

**Definition 37**

We say $\{a_n\}$ is

1. increasing if $a_n < a_{n+1}$ for all n

2. non-decreasing if $a_n \leq a_{n+1}$ for all n

3. decreasing if $a_n > a_{n+1}$ for all n

4. non-increasing if $a_n \geq a_{n+1}$ for all n

5. monotonic if either non-increasing or non-decreasing

## 1.3 Monotone Convergence Theorem

**Definition 38**

Let $S \subseteq \mathbb{R}$, $S$ is

1. bounded above if $\exists \alpha \in \mathbb{R}$ such that $\forall a \in S$, $a \leq \alpha$, called "upper bound"
2. bounded below if $\exists \beta \in \mathbb{R}$ such that $\forall a \in S$, $a \geq \beta$, called "lower bound"

We say $S$ is bounded if $S$ is bounded above AND bounded below

**Fact**

Any finite subset of $\mathbb{R}$ is bounded

**Definition 39**

Let $S \subseteq \mathbb{R}$ then $\alpha$ is a lowest upper bound if:

1. $\alpha$ is a upper bound
2. $\alpha \leq k$ for every $k \in \mathbb{R}$ that is also an upper bound

**Definition 40**

Greatest Lower Bound (GLB) is called the $inf(S)$ (unique)
Lowest Upper Bound (LUB) is called the $sup(S)$ (unique)

**Definition 41**

Let $\{a_n\}_{n=0}^{\infty}$ be a non-decreasing sequence in $\mathbb{R}$, then

1. if $\{a_n\}_{n=0}^{\infty}$ is bounded above, then $\lim\limits_{n \to \infty} a_n = sup(A)$
2. if $\{a_n\}_{n=0}^{\infty}$ is not bounded above, then $\lim\limits_{n \to \infty} a_n = \infty$

# 2 Limits and Continuity

## 2.1 Limits of Functions

**Definition 48**

Let $f(x)$ be a function and $a$ be a real number. We say that the limits as $x$ approaches $a$ of $f(x)$ is $L$ and write $\lim\limits_{x\to a} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$

**Fact**

The value and existence of $\lim\limits_{x\to a} f(x)$ has nothing to do with $f(a)$. In fact, $f(a)$ need not exist to talk about $\lim\limits_{x\to a} f(x)$

**Example**

Prove $\lim\limits_{x\to 3} 3x + 1 = 10$
$|3x + 1 - 10| < \epsilon$
$|3x - 9| < \epsilon$
$3|x - 3| < \epsilon$
$|x - 3| < \frac{\epsilon}{3}$
$0 < |x - 3| < \delta$
So $\delta = \frac{\epsilon}{3}$
Proof:
Let $\epsilon > 0$ be arbitrary and choose $\delta = \frac{\epsilon}{3}$. Suppose $0 < |x - 3| < \delta$. Then $|3x + 1 - 10| = |3x - 9| = 3|x - 3| < 3\delta = \epsilon$
$\therefore \lim\limits_{x\to 3} 3x + 1 = 10$

**Theorem 53**

Suppose $f(x)$ is defined on some open interval containing $a$, but possibly not at $a$. The following are equivalent:

1. $\lim\limits_{x\to a} f(x) = L$
2. For any sequence $\{a_n\}_{n=1}^{\infty}$ with $\lim\limits_{n\to\infty} x_n = a$ and $x_n = a$ for all $n$, $\lim\limits_{n\to\infty} f(x_n) = L$

**Sequential Characterization of Limits**

Useful for showing that limits do not exist if you can construct $\{x_n\}$ and $\{y_n\}$ so that $x_n \neq a$ and $y_n \neq a$ for all $n$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = a$, but $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$, then $\lim\limits_{x\to a} f(x)$ DNE

**Theorem 57**

Let $f(x)$ be a function and $a \in \mathbb{R}$. If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} f(x) = M$, then $L = M$

**Theorem 58 (arithmetic of limits)**

Let $f(x)$ and $g(x)$ be functions and $a \in \mathbb{R}$. Assume $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$

1. If $f(x) = c$ for all $x \in \mathbb{R}$, then $c = L$ ($\lim\limits_{x\to a} c = c$)
2. For any $c \in \mathbb{R}$, $\lim\limits_{x\to a} cf(x) = cL$
3. $\lim\limits_{x\to a}(f(x) + g(x)) = L + M$
4. $\lim\limits_{x\to a} f(x)g(x) = LM$
5. If $M \neq 0$, $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$
6. If $L > 0$, then $\lim\limits_{x\to a}(f(x))^\alpha = L^\alpha$

**Theorem 59**

Suppose $\lim\limits_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ exists, then $\lim\limits_{x\to a} f(x) = 0$

**Theorem 60**

If $p(x)$ is a polynomial then $\lim\limits_{x\to a} p(x) = p(a)$

**Theorem 61**

If $p(x)$ and $q(x)$ are polynomials with $q(a) \neq 0$, then $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$

**Definition 63**

Let $f(x)$ be a function and $a \in \mathbb{R}$. We say that the limit as $x$ approaches $a$ from the left of $f(x)$ equals $L$ and write $\lim\limits_{x \to a^-} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < a - x < \delta$ then $|f(x) - L| < \epsilon$.
The "limit" from the right, denoted $\lim\limits_{x \to a^+} f(x) = L$ is defined similarly

**Fact**

The arithmetic rule for limits apply to one sided limit

**Theorem 67 (The sequeeze theorem for functions)**

Let $f(x), g(x)$, and $h(x)$ be functions, $a \in \mathbb{R}$, and $I$ be an open interval containing $a$. Suppose $f, g$ and $h$ are defined on $I$ except possibly at $a$. Further suppose the following:

1. $f(x) \leq g(x) \leq h(x)$ for all $x \in I$ (except possibly $a$)
2. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L \in \mathbb{R}$

Then $\lim\limits_{x \to a} g(x) = L$ as well
This holds for one-sided limits as well

**Theorem 69**

The fundamental trig limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

## 2.2 Infinite limits

**Definition 71**

Let $f(x)$ be a function and $L \in \mathbb{R}$. We say that the limit as $x$ approaches $\infty$ of $f(x)$ equals $L$ and write $\lim\limits_{x \to \infty} f(x) = L$ if for every $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $x > N$ implies $|f(x) - L| < \epsilon$
Similarly, we can define $\lim\limits_{x \to -\infty} f(x) = L$

**Definition 72**

Let $f(x)$ be a function and $L \in \mathbb{R}$. We say that the line with equation $y = L$ is a horizontal asymptote of $f(x)$ if $\lim\limits_{x \to \infty} f(x) = L$ or $\lim\limits_{x \to -\infty} f(x) = L$

**Definition 74**

Let $f(x)$ be a function. We say that $f(x)$ approaches infinity as $x$ approaches $\infty$ if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that if $x > N$ then $f(x) > M$. We write $\lim\limits_{x \to \infty} f(x) = \infty$
Similarly for $\lim\limits_{x \to \infty} f(x) = -\infty$, $\lim\limits_{x \to -\infty} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = -\infty$

**Fact**

1. $\lim\limits_{x \to \infty} x^\alpha = \infty$ if $\alpha > 0$ and equals 0 if $\alpha < 0$

2. Suppose $p(x)$ and $q(x)$ are polynomials of degree $m$ and $n$ respectively

   - if $n > m$, then $\lim\limits_{x \to \infty} \dfrac{p(x)}{g(x)} = \lim\limits_{x \to -\infty} \dfrac{p(x)}{g(x)} = 0$
   - if $m < n$, then $\lim\limits_{x \to \infty} \dfrac{p(x)}{g(x)} = \pm\infty$, $\lim\limits_{x \to -\infty} \dfrac{p(x)}{g(x)} = \pm\infty$

     To determine the sign, you need to consider the signs of the leading coefficients
   - if $m = n$, then $\lim\limits_{x \to \infty} \dfrac{p(x)}{g(x)}$ and $\lim\limits_{x \to -\infty} \dfrac{p(x)}{g(x)}$ are both equal to the ratio of the leading coefficients

**Theorem 78 Fundamental log limit**

$\lim_{x\to\infty} \frac{\ln x}{x} = 0$, $\ln x < x$

**Fact**

1. $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for all $p > 0$
2. $\lim_{x\to\infty} \frac{\ln x^p}{x} = 0$ for all $p \in \mathbb{R}$

**Definition 82**

Let $f(x)$ be a function and $a \in \mathbb{R}$

1. We say that $f$ approaches infinity as $x$ approaches $a$ from the right and write $\lim_{x\to a^+} f(x) = \infty$ if for every $M > 0$, there exists $\epsilon > 0$ such taht if $0 < x - a < \epsilon$, then $f(x) > M$
2. We can similarly define $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = -\infty$
3. We say that $\lim_{x\to a} f(x) = \infty$ if both $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to a^-} f(x) = \infty$
4. Similarly, we define $\lim_{x\to a} f(x) = -\infty$

**Definition 83**

We say that $f(x)$ has a vertical asymptote at $x = a$ if any of $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, and $\lim_{x\to a^-} f(x) = \pm\infty$ are true

## 2.3 Continuity

**Definition 87**

Let $f(x)$ be a function and $a \in \mathbb{R}$ such that $f(a)$ is defined
We say that $f$ is continuous at $x = a$ if

1. $\lim_{x \to a} f(x)$ exists

2. $\lim_{x \to a} f(x) = f(a)$

**Definition 88**

Let $f(x)$ be a function and $a \in \mathbb{R}$ such that $f(a)$ is defined. We say that $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$

**Theorem 89 (Sequential Characterization of Continuity)**

Let $f(x)$ be a function and $a \in \mathbb{R}$. $f(x)$ is continuous at $x = a$ if and only if for every sequence $\{x_n\}$ with $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = f(a)$

**Theorem 90**

Suppose $f(x)$ and $g(x)$ are continuous at $x = a$

1. $f(x) + g(x)$ is continuous at $x = a$

2. $f(x)g(x)$ is continuous at $x = a$

3. If $g(x) \neq 0$, then $\dfrac{f(x)}{g(x)}$ is continuous at $x = a$

**Theorem 91**

The following are continuous at $x = a$ for all $a$ in the domain

1. polynomial

2. rational function

3. $\sin x$ and $\cos x$

4. $e^x$ and $\ln x$

**Fact**

$f(x)$ is continuous at $x = a$ iff $\lim_{h \to 0} f(a + h) = f(a)$

**Theorem 93**

Suppose $g(x)$ is the inverse of $f(x)$ and that $f$ is continuous at $x = a$. Then $g(x)$ is continuous at $x = f(a)$

**Theorem 94**

If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$

**Definition 95**

We say that $f$ is continuous on the open interval $I$ if $f$ is continuous at $x = a$ for every $a \in I$. If $I \in \mathbb{R}$, we might sometimes say "$f$ is continuous"

**Definition 97**

We say that $f(x)$ is continuous on $[a, b]$ if

1. $f(x)$ is continuous on $(a, b)$
2. $\lim\limits_{x \to a^+} f(x) = f(a)$
3. $\lim\limits_{x \to b^-} f(x) = f(b)$

**Theorem 99 (The Intermediate Value Theorem)**

Suppose $f(x)$ is a function that is continuous on a closed interval $[a, b]$. If there is $\alpha \in \mathbb{R}$ such that $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = \alpha$

# 3 Derivatives

## 3.1 Extreme Value Theorem

**Definition 103**

Suppose $f(x)$ is a function that is defined on an interval $I$

1. We say that $f(x)$ has a global max on $I$ at $c \in I$ if $f(x) \leq f(c)$ for all $x \in I$

2. We say that $f(x)$ has a global min on $I$ at $c \in I$ if $f(x) \geq f(c)$ for all $x \in I$

3. We say that $f(x)$ has a global extremum on $I$ at $c \in I$ if $f$ has a global max or min at c

**Theorem 104 (Extreme Value Theorem)**

Let $f(x)$ be function that is continuous on the closed interval $[a, b]$. Then $f$ has a global min and a global max on $[a, b]$. In symbols, there exists $c_1, c_2 \in [a, b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a, b]$

## 3.2 Instantaneous Velocity

Imagine some object has position $f(t)$ at time $t$

We can compute the average velocity from time $t_1$ to time $t_2$, as $\dfrac{f(t_2) - f(t_1)}{t_2 - t_1}$

**Definition 110**

Let $f(x)$ be a function and $a \in \mathbb{R}$. We say $f(x)$ is differentiable at $x = a$ if $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. If it exists, we denote $f'(x)$ and call it the derivatives of $f$ at $a$

**Theorem 111**

$f(x)$ is differentiable at $x = a$ iff $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists. If it does exist, then the limit equals $f'(a)$

**Definition 113**

Suppose $f(x)$ is differentiable at $x = a$. We define the tangent line to the (graph of) $f(x)$ at $(a, f(a))$ to be the line with equation $y = f'(a)(x - a) + f(a)$. This is precisely the line through $(a, f(a))$ with slope $f'(a)$

## 3.3 Differentiability vs. Continuity

**Theorem 116**

If $f(x)$ is differentiable at $x = a$ then it is continuous at $x = a$

**Definition 117**

Let $f(x)$ be a function defined on an open interval $I$. We say that $f(x)$ is differentiable on $I$ if it is differentiable at $x = a$ for each $a \in I$. In this case, we define a function $f'(x)$ called derivative of $f(x)$ by $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

## 3.4 Higher Derivatives and some Basic Derivatives

**Definition 118**

If $f(x)$ is differentiable at $x = a$ for $a \in \mathbb{R}$, we say that $f$ is differentiable

**Leibniz Notation**

Sometimes we denote $f'(x)$ by $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$

**Definition 119**

Let $f(x)$ be a function that is differentiable on an open interval $I$

1. If $f'(x)$ is differentiable on $I$, then its derivative is called the second derivative of $f(x)$ on $I$. It is denoted $f''(x), f^(2)(x), \frac{d}{dx}f(x), \frac{d^2 f}{d^2 x}$
2. Inductively, if the $(n-1)$ derivative of $f(x)$ is differentiable on $I$, the $n$ derivative of $f(x)$ is the derivative of the $(n-1)$ derivative, $\frac{d}{dx}f^{(n-1)}(x) = f^{(n)}(x)$

**Some common derivatives**

1. Let $c \in \mathbb{R}$ and $f(x) = c$ be a constant function. $f'(x)$ is the constant 0 function

2. Let $m, b \in \mathbb{R}$ with $m \neq 0$ and set $f(x) = mx+b$. $f'(x)$ of a non-vertical line is always its slope

3. Power rule: $\frac{d}{dx}x^n = nx^{n-1}$

4. Let $f(x) = \sin x$, then $f'(x) = \cos x$

## 3.5 More derivatives and differentiation Rules

**Fact**

1. With $f(x) = \cos x$, $f'(x) = -\sin x$

2. If $a > 0$, $\frac{d}{dx}a^x = \ln(a)a^x$. In particular, if $a = e$, $\ln e = 1$, so $\frac{d}{dx}e^x = e^x$

**Fact**

- $f(x)$ is differentiable at $x = 0$, and hence differentiable everywhere

- The derivative is a scalar multiple of $f(x)$ and that scale is $f'(0)$ which happens to be $\ln a$

**Theorem 128 (Differentiation Rule)**

Suppose $f(x)$ and $g(x)$ are differentiable at $x = a$

1. Let $w(x) = cf(x)$ for some $c \in \mathbb{R}$. Then $w(x)$ is differentiable at $x = a$ and $w'(a) = cf'(a)$

2. Let $w(x) = f(x) + g(x)$. Then $w$ is differentiable at $x = a$ and $w'(a) = f'(a) + g'(a)$

3. Product Rule: Let $w(x) = f(x)g(x)$. Then $w$ is differentiable at $x = a$ and $w'(a) = f'(a)g(a) + f(a)g'(a)$

4. Let $w(x) = \dfrac{1}{f(x)}$. Then if $f(a) \neq 0$, then $w$ is differentiable at $x = a$ and $w'(a) = \dfrac{-f'(a)}{(f(a))^2}$

5. Quotient Rule: Let $w(x) = \dfrac{f(x)}{g(x)}$. Then if $g(a) \neq 0$, $w$ is differentiable at $x = a$ and $w'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$

## 3.6 More derivatives and the chain rule

Using the power rule when $n \in \mathbb{R}$

### Fact

1. Polynomials and rational functions are differentiable on their domains. In particular, if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, then $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 + a_1$

2. $\tan x$ is differentiable on its domain, which is all $x \in \mathbb{R}$ except $\{\frac{\pi}{2}+2\pi k : k \in \mathbb{Z}\}$. $\frac{d}{dx}\tan x = \sec^2 x$

3. $\sec x$ is differentiable on its domain, which is the same as $\tan x$. $\frac{d}{dx}\sec x = \sec x \cdot \tan x$

### Chain Rule

Suppose $f(x)$ is differentiable at $x = a$ and $g(x)$ is differentiable at $x = f(a)$. Then the composition $gof$ is differentiable at $x = a$ with $[gof]'(a) = g'(f(a))f'(a)$

### Logarithmic Differentiation

Let $g(x) = \ln(f(x))$ where $f(x)$ is some positive function. Then $g'(x) = \dfrac{f'(x)}{f(x)}$

## 3.7 Linear Approximation

$\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$ where $|x-a|$ is small $\dfrac{f(x) - f(a)}{x - a} \approx f'(a) \Rightarrow f(x) - f(a) \approx f'(a)(x-a)$ if $|x - a|$ is small, then $f(x) \approx f'(a)(x-a) + f(a)$

**Definition 137**

Let $f(x)$ be differentiable at $x = a$. The linear approximation or linearization of $f(x)$ at $x = a$ is the line with equation $L_a^f(x) = f'(a)(x - a) + f(a)$

- a tangent line
- superscript $f$ will often be omitted

**Definition 140**

Let $f(x)$ be a function that is differentiable at $x = a$ and $L_a(x)$ be its linearization at $x = a$. The error in approximating $f(x)$ be $L_a(x)$ at $x$ is $|f(x) - L_a(x)|$

**Theorem 141 (Error in linear approximation)**

Suppose $f(x)$ is twice differentiable on some interval $I$, $a \in I$, and $L_a(x)$ is the linearization of $f$ at $x = a$. If $M$ is a constant such that $|f''(x)| \leq M$ for all $x \in I$, then $|f(x) - L_a(x)| \leq \frac{M}{a}(x - a)^2$

## 3.8 Newton's Method

Need a formula for the root of $L_a(x)$

$$\begin{aligned} 0 &= f'(a)(x - a) + f(a) \\ -f(a) &= f'(a)(x - a) \\ x &= a - \frac{f(a)}{f'(a)} \end{aligned}$$

## 3.9 Algorithm (Newton-RHapson Method for approximating root)

Given $f(x)$ differentiable on an interval $I$ and have a root on $I$

1. choose $x_0$ in $I$ as an initial estimate
2. recursively compute for $n \geq 0$

a new estimate $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

## 3.10 Inverse Functions and their derivatives

**Theorem 146 (Inverse function theorem)**

Suppose $f(x)$ is continuous and inversible on $[c, d]$ (this is the domain). Let $g(x)$ be its inverse and suppose $a \in [c, d]$ is such that $f$ is differentiable at $x = a$. Then $g(x)$ is differentiable at $b = f(a)$ and $g'(b) = \frac{1}{f'(a)}$. Moreover, $(L_a^f(x))^{-1} = L_b^g(x)$

## 3.11 Implicit Differentiation

**Example 153**

Find the slope of tangent to curve with equation $x^2 + y^3 + 2xy + x + y = 0$ at $(0, 0)$
Differentiate both side: $y = y(x)$

$$\begin{aligned} 2x + 3y^2y' + 2y + 2xy' + 1 + y' = 0 \\ y'(3y^2 + 2x + 1) &= -2x - 2y - 1 \\ y' &= \frac{-2x - 2y - 1}{3y^2 + 2x + 1} \\ x &= y = 0 \\ y' &= -1 \end{aligned}$$

## 3.12 Extreme Value

**Theorem 155**

Suppose $f(x)$ is a positive differentiable function. Then $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ (logarithmic differentiation)

**Definition 157**

Suppose $f(x)$ is a function with domain $D$. We say that $f$ has a local min/max at $a = c$ if there exists an open interval $(a, b)$ such that $c \in a, b \subseteq D$ satisfying $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all $x \in (a, b)$. If $f$ has a local min/max at $x = c$, then we say it has a local extrema at $x = c$

**Fact**

Suppose $f(x)$ is defined on $[a, b]$ and has a globa max at $c$ with $c \in (a, b)$. Then $f$ has a local max at $x = c$. Similar for global/local min

**Theorem 159**

Suppose $f(x)$ is a function with local extrema at $x = c$. If $f'(c)$ exists, then $f'(c) = 0$

**Definition 162**

Let $f(x)$ be a function. We say that a point $c$ in the domain of $f$ is a critical point if $f'(c) = 0$ or $f'(c)$ DNE

**Fact**

Suppose $f(x)$ is a function that is continuous on $[a, b]$. If $f(x)$ has an extreme value at $x = c$, then $c = a$, $c = b$ or $c$ is a critical point

# 4 The Mean Value Theorem

**Theorem 166 (Rolle's Theorem)**

Suppose $f(x)$ is continuous on $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b) = 0$. Then there is $c \in (a, b)$ such that $f'(c) = 0$

**Theorem 167 (Mean Value Theorem)**

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there is $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

**Theorem 168 (Constant Function Theorem)**

Suppose $f(x)$ is differentiable on an open interval $I$ and that $f'(x) = 0$ for all $x \in I$. Then $f$ is a constant on $I$

**Definition 170**

Suppose $f(x)$ and $F(x)$ are defined on interval $I$. We say that $F(x)$ is an anti-derivatives of $f(x)$ on $I$ if $F'(x) = f(x)$ for all $x \in I$

**Theorem 171 (The Anti-Derivative Theorem)**

Suppose $f'(x) = g'(x)$ for all $x \in I$ where $I$ is open interval. Then there exists $c \in \mathbb{R}$ such that $f(x) = g'(x) + c$ for all $x \in I$

**Definition 172**

Suppose $f(x)$ is defined on an interval $I$

1. if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$, we say $f(x)$ is increasing on $I$

2. if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$, we say $f(x)$ is non-decreasing on $I$

3. if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$, we say $f(x)$ is decreasing on $I$

4. if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$, we say $f(x)$ is non-increasing on $I$

**Theorem 173**

Suppose $f(x)$ is differentiable on an open interval $I$

1. if $f'(x) > 0$ for all $x \in I$, then $f$ is increasing on $I$

2. if $f'(x) \geq 0$ for all $x \in I$, then $f$ is non-decreasing on $I$

3. if $f'(x) < 0$ for all $x \in I$, then $f$ is decreasing on $I$

4. if $f'(x) \leq 0$ for all $x \in I$, then $f$ is non-increasing on $I$

## 4.1 Functions of bounded derivative

**Theorem 175 (Bounded Derivative Theorem)**

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also suppose there are $m, M \in \mathbb{R}$ such that $m \leq f'(x) \leq M$ for all $x \in (a, b)$. Then, $f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$ for all $x \in (a, b)$

**Theorem 177**

Suppose $f$ and $g$ are continuous at $x = a$ and $f(a) = g(a)$

1. If $f$ and $g$ are differentiable on $(a, \infty)$ and $f'(x) \leq g'(x)$ for all $x \in (a, \infty)$, then $f(x) \leq g(x)$ for all $x \in (a, \infty)$

2. If $f$ and $g$ are differentiable on $(-\infty, a)$ and $f'(x) \leq g'(x)$ for all $x \in (-\infty, a)$, then $f(x) \geq g(x)$ for all $x \in (-\infty, a)$

## 4.2 L'Hôpital's Rule

**Theorem 179 (L'Hôpital's Rule)**

Suppose $f(x)$ and $g(x)$ differentiable in some open interval containing $a$ and that $g'(x) \neq 0$ for all $x$ in the interval, with possible exception of $x = a$. If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists or equal $\pm\infty$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$. This works when $a = \infty$, for one-sided limits

## 4.3 The Second Derivatives

**Definition 185**

Suppose $f(x)$ is defined on some interval $I$. We say that

1. $f$ is concave up on $I$ if for any $a, b \in I$, the line segement connecting $a, f(a)$ to $(b, f(b))$ lies above the graph of $f$ on $(a, b)$

2. $f$ is concave down on $I$ if for any $a, b \in I$, the line segement connecting $a, f(a)$ to $(b, f(b))$ lies below the graph of $f$ on $(a, b)$

**Theorem 187 (Second Derivative Test for Concavity)**

Suppose $f(x)$ is twice differentiable on an open interval $I$

1. if $f''(x) < 0$ for all $x \in I$, then $f$ is concave down on $I$
2. if $f''(x) > 0$ for all $x \in I$, then $f$ is concave up on $I$

**Definition 190**

Suppose $f(x)$ is continuous on $x = c$. The point $(c, f(c))$ is called inflection point for $f$ if the concavity of $f$ change at $x = c$

**Theorem 191**

If $f''(x)$ is continuous at $x = c$ and $(c, f(c))$ is an inflection popint for $f$, then $f''(c) = 0$

## 4.4 Curve Sketching

**Theorem 194**

Assume that $f$ has a cv at $x = c$

1. If there is an interval $(a, b)$ containing $c$ such that $f'(x) < 0$ on $(a, c)$ and $f'(x) > 0$ on $(c, b)$, then $c$ has a local min at $x = c$
2. similar for local max

**Theorem 195**

Suppose $f'(c) = 0$ and that $f''(x)$ is continuous at $x = c$

1. If $f''(c) < 0$, then $f$ has a local max at $x = c$
2. If $f''(c) > 0$, then $f$ has a local min at $x = c$

**Curve Sketching Checklist**

1. Domain
2. Intercepts (x and y)
3. Asymptotes (vertical and horizontal)
4. The first derivatives, critical values, local extrema
5. The second derivatives, inflection points
6. Intervals of increase/decrease, concavity
7. label stuff from 2, 3, 4, 5

# 5 Taylor Polynomials and Taylor's Theorem

**Fact**

The $n^{th}$ degree Taylor polynomial for $e^x$ centered at 0 is

$$\begin{aligned} T_{n,0}(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} \\ &= \sum_{k=0}^{n} \frac{x^k}{k!} \end{aligned}$$

The $n^{th}$ degree Taylor polynomial of $\ln x$ centered at $x = 1$ is

$$T_{n,1}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n}$$

**Definition 202**

Suppose $f(x)$ is $n$ times differentiable at $x = a$. The $n^{th}$ degree Taylor polynomial centered at $x = a$ is

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^(3)(a)}{3!}(x-a)^3 + \cdots$$
$$+ \frac{f^(k)(a)}{k!}(x-a)^k + \cdots + \frac{f^(n)(a)}{n!}(x-a)^n$$

or

$$T_{n,a}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n \text{ where } a_k = \frac{f^(k)(a)}{k!}$$

**Definition 206**

Suppose $f(x)$ is $n$ times differentiable on an interval $I$ containing $a$. The $n^{th}$ degree Taylor polynomial remainder centered at $a$ is $R_{n,a}(x) = f(x) - T_{n,a}(x)$, $x \in I$

**Theorem 207 (Taylor's Remainder Theorem)**

Suppose $f(x)$ is $n+1$ times differentiable on an interval $I$ containing $a$. For any $x \in I$, there exists $c$ between $x$ and $a$ such that

$$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$$

**Theorem 208**

Suppose $f(x)$ is $n+1$ times differentiable on an interval $I$ containing $a$. Suppose $f^(n+1)(x)$ is continuous on $I$. For any $x_0 \in I$, if $M$ is a constant/real satisfying $|f^(n+1)(x)| \leq M$ for all $x$ between $x$, and $a$, then

$$|R_{n,a}(x_a)| \leq \frac{M}{(n+1)!}|x_0 - a|^{n+1}$$