CO 250 Notes / Definition

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1 Module 1: Modelling Optimization Problems

1.1 Week 1 Linear Programs

1.1.1 Definition: affine/linear

A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$. It is linear if, in addition, $\beta = 0$

1.1.2 Definition: linear program

The optimization problem

$$min\{f(x) : \ g_i(x) \leq b_i, \ \forall 1 \leq i \leq m, \ x \in \mathbb{R}^n\}$$

is called a linear program if f is affine and $g_1, \cdots, g_m$ is finite number of linear functions

Comment

- insted of set notation, we often write LPS more verbosely
- Often give non-negativity constraints seperately
- May use max instead of min
- Sometimes replace subject to by s.t.
- Often write $x \geq 0$ as a short form for all variables are non-negative
- Dividing by variables is not allowed
- No strict inequalities
- Must have finite number of constraints

$$\begin{aligned} \text{max} \quad & -2x_1 + 7x_3 \\ \text{subject to} \quad & x_1 + 7x_2 \leq 3 \\ & 3x_2 + 4x_3 \leq 2 \\ & x_1, x_3 \geq 0 \end{aligned}$$

1.2 Week 2: Integer Programs / Optimization on Graphs

1.2.1 Integer Program

An integer program is a linear program with added integrality constraints for some / all of the variables

Comments

- Call an IP mixed if there are integer and fractional variables, and pure otherwise
- The difference between LPs and IPs is subtle (LPs are easy to solve, IPs are not)

1.2.2 Running Time

The running time of an algorithm is the number of steps that an algorithm takes

Comments

- It is stated as a function of n: $f(n)$ measures the largest number of steps an algorithm takes on an instance of size n
- Algorithm is efficient if its running time, $f(n)$, is a polynomial of n

1.2.3 Binary Variable

It is very useful for modeling logical constraints of the form:

$$\text{Condition (A or B) and C} \rightarrow D$$

Variables that can take only a value of 0 or 1 are called binary

1.2.4 Definition: Matching

A collection $M \subseteq E$ is a matching if no two edges $ij, i'j' \in M$ $(ij \neq i'j')$ share an endpoint, $\{i, j\} \cap \{i', j'\} = \emptyset$

1.2.5 Definition: Perfect

A matching M is perfect if every vertex v in the graph is incident to an edge in M

1.2.6 Definition: Perfect Matching

Given $G = (V, E)$, $M \subseteq E$ is a perfect matching iff $M \cap \delta(v)$ contains a single edge for all $v \in V$
V = vertices, E = edges

1.3 Week 3: Shortest Paths and Nonlinear Models

1.3.1 Definition: $\delta(S)$

For $S \subseteq V$, we let $\delta(S)$ be the set of edges with exactly one endpoint in S

$$\delta(S) = \{uv \in E : \ u \in S, v \notin S\}$$

1.3.2 Definition: s,t-cut

$\delta(S)$ is an s,t-cut if $s \in S$ and $t \notin S$

1.3.3 IP for Shortest Paths

Variable: we have one binary variabl x_e for each edge $e \in E$, want

$$x_e = \begin{cases} 1 : e \in P \\ 0 : otherwise \end{cases}$$

Constraints: we have one constraint for each s,t-cut $\delta(U)$, forcing P to have an edge from $\delta(S)$

$$\sum(x_e : e \in \delta(U)) \geq 1$$

for all s,t-cuts $\delta(U)$
Objective: $\sum(c_e x_e : e \in E)$

Note

- P is an s,t-path if it is of the form

$$v_1v_2, v_2v_3, \cdots, v_{k-1}v_k$$

  and

  - $v_iv_{i+1} \in E$ for all $i \in \{1, \cdots, k-1\}$
  - $v_i \neq v_j$ for all $i \neq j$
  - $v_1 = s$ and $v_k = t$

- If P is an s,t-path and $\delta(S)$ is an s,t-cut, then P must have an edge from $\delta(S)$
- If $S \subseteq E$ contains at least one edge from every s,t-cut, then S contains an s,t-path

1.3.4 Nonlinear Program

A nonlinear program (NLP) is of the form

$$\begin{aligned} \min & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & \cdots \\ & g_m(x) \leq 0 \end{aligned}$$

where

- $x \in \mathbb{R}^n$
- $f : \mathbb{R}^n \to \mathbb{R}$, and
- $g_i : \mathbb{R}^n \to \mathbb{R}$

Linear programs are NLPs

2 Module 2: Solving Linear Programs

2.1 Week 4: Possible Outcomes / Certificates / Standard Equality Forms

2.1.1 Feasible Solution

An aassignment of values to each of the variables is a feasible solution if all the constraints are satisfied

2.1.2 Feasible / Infeasible

An optimization problem is feasible if it has at least one feasible solution. It is infeasible otherwise

2.1.3 Optimal Solution

- For a maximization problem, an optimal solution is a feasible solution that maximizes the objective function
- For a minimization problem, an optimal solution is a feasible solution that minimizes the objective function

2.1.4 Unbounded

- A maximization problem is unbounded if for every value M, there exists a feasible solution with objective value greater than M
- A minimization problem is unbounded if for every value M, there exists a feasible solution with objective value smaller than M

2.1.5 Fundamental Theorem of Linear Programming

For any linear program, EXACTLY ONE of the following holds

- It has an optimal solution
- It is infeasible
- It is unbounded

2.1.6 Proposition

There is no solution to $Ax = b$, $x \geq 0$, if there exists y where

$$y^T A \geq 0^T \quad \text{and} \quad y^T b < 0$$

2.1.7 Farka's Lemma

If there is no solution to $Ax = b$, $x \geq 0$, then there exists y where

$$y^T A \geq 0^T \quad \text{and} \quad y^T b < 0$$

2.1.8 Claims to show unboundness

- Claim 1: $x(t)$ is feasible for all $t > 0$
- Claim 2: $z \to \infty$ when $t \to \infty$

2.1.9 Proposition

The linear program

$$\max\{c^T x : Ax = b, x \geq 0\}$$

is unbounded if we can find $\overline{x}$ and r such that

$$\overline{x} \geq 0, r \geq 0, A\overline{x} = b, Ar = 0, \text{ and } c^T r > 0$$

2.1.10 Definition: Standard Equality Form (SEF)

A linear program (LP) is in standard equality form (SEF) if

1. it is a maximization problem
2. for every variable x_j, we have the constraint $x_j \geq 0$
3. all other constraints are equality constraints

2.1.11 What to do if LP is not in SEF?

1. find an "equivalent" LP in SEF
2. solve the "equivalent" LP using simplex
3. use the solution of "equivalent" LP to get the solution of the original LP

2.1.12 Definition: Equivalent LP

A pair of LPs are equivalent if they behave in the same way
Linear programs (P) and (Q) are equivalent if

- (P) infeasible $\iff$ (Q) infeasible
- (P) unbounded $\iff$ (Q) unbounded
- can construct optimal solution of (P) from optimal solution of (Q)
- can construct optimal solution of (Q) from optimal solution of (P)

2.1.13 Theorem

Every LP is equivalent to an LP in SEF

Tips

Set free variable $x_3 := a - b$ where $a, b \geq 0$

2.2 Week 5: Basis, Canonical Form, Formalizing the Simplex

2.2.1 Notation

Let B be a subset of column indices
Then A_B is a solumn sub-matrix of A indexed by set B
A_j denotes column j of A

2.2.2 Definition: Basis

Let B be a subset of column indices. B is a basis if

1. A_B is a square matrix
2. A_B is non-singular (columns are independent)

2.2.3 Theorem

Max number of independent columns = max number of independent rows

2.2.4 Definition: Basic solution

x is a basic solution for basis B if

1. $Ax = b$
2. $x_j = 0$ whenever $j \notin B$

2.2.5 Proposition

Consider $Ax = b$ and a basis of A. Then there exists a unique basic solution x for B

2.2.6 Definition: Basic solution

Consider $Ax = b$ with independent rows. Vector x is a basic solution if it si a basic solution for some basis B

2.2.7 Definition

A basic solution x of $Ax = b$ is feasible if $x \geq 0$

2.2.8 Definition: Canonical form

$$\max\{c^T x : Ax = b, x \geq 0\}$$

Let B be a basis of A. Then (P) is in canonical form for B if

P1 $A_b = I$

P2 $c_j = 0$ for all $j \in B$

2.2.9 Proposition

For any basis B, there exists (P') in canonical form for B such that

1. (P) and (P') have the same feasible region
2. feasible solutions have the same objective value for (P) and (P')

2.2.10 Proposition

Consider A with basis B, P :

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq \vec{0} \end{aligned}$$

P' :

$$\begin{aligned} \max \quad & [c^T - y^T A]x + y^T b \\ \text{s.t.} \quad & A_B^{-1} A x = A_B^{-1} b \\ & x \geq \vec{0} \end{aligned}$$

where $y = A_B^{-T}$

1. (P') is in canonical form for basis B, $\bar{c}_B = 0$ and $A'_B = I$
2. (P) and (P') have the same feasible region
3. feasible solutions have the same objective value for (P) and (P')

2.2.11 The Simplex Algorithm

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Input: a feasible basis B
Output: an optimal solution or it detects that the LP is unbounded

1. rewrite in canonical form for basis B
2. find a better basis B or get required outcome

Try to find a better basis

$$\begin{aligned} \max \quad & z = c_N^T x_N + \bar{z} \\ \text{s.t.} \quad & x_B + A_N x_N = b \\ & x \geq 0 \end{aligned}$$

2.2.12 Proposition

Simplex tells the truth:

- If it claims the LP is unbounded, it is unbounded
- if it claims the solution is optimal, it is optimal

2.2.13 Definition: Bland's Rule

Bland's rule is as follows:

- if we have a choice for the element entering the basis, pick the smallest one
- if we have a choice for the element leaving the basis, pick the smallest one

2.2.14 Theorem

If we use Bland's rule, then the Simplex algorithm always terminates

2.3 Week 6: Feasible Solution, Half Spaces and Convexity, Extreme Points

2.3.1 Feasible Solution

$$\max \ \{c^T : Ax = b, x \geq 0\}$$

Algorithm 1:

Input: A, b, c and a feasible solution

Output: Optimal solution / detect LP unbounded

Algorithm 2:

Input: A and b

Output: Feasible solution / detect there is none

Proposition

Can use Algorithm 1 to get Algorithm 2

2.3.2 Definition: Feasible Region

For an optimication problem, the

$$\text{feasible region} \ = \ \text{set of all feasible solutions}$$

2.3.3 Definition: Polyhedron

$P \subseteq \mathbb{R}^n$ is a polyhedron if there exists a matrix A and a vector b such that

$$P = \{x : Ax \leq b\}$$

2.3.4 Proposition

The feasible region of an LP is a polyhedron

2.3.5 Definition: Hyperplane, Halfspace

Let $a \neq 0$ be a vector and β a read number

1. $\{x : a^T x = \beta\}$ is a hyperplane
2. $\{x : a^T x \leq \beta\}$ is a halfspace

A hyperplane is the set of solutions to a single linear equation
A halfspace is the set of solutions to a single linear inequality

2.3.6 Definition: Translate

Let $S, S' \subseteq \mathbb{R}^n$. Then S' is a translate of S if there exists $p \in \mathbb{R}^n$ and

$$S' = \{s + p : s \in S\}$$

2.3.7 Proposition

Let $a \neq 0$ be a vector and β a real number and let

$$H := \{x : a^T x = \beta\} and\ H_0 := \{x : a^T x = 0\}$$

It follows that H is a translate of H_0

2.3.8 Proposition

Let $a \neq 0$ be a vector and β a real number and let

$$F := \{x : a^T x \leq \beta\} and\ F_0 := \{x : a^T x \leq 0\}$$

It follows that F is a translate of F_0

2.3.9 Proposition

The dimension of a hyperplane in $\mathbb{R}^n$ is $n - 1$

2.3.10 Definition: Line Through

Let $x_1, x_2 \in \mathbb{R}^n$. The line through x_1 and x_2 is defined as

$$L = \{x = \lambda x_1 + (1 - \lambda) x_2 : \lambda \in \mathbb{R}\}$$

2.3.11 Definition: Line Segement

Let $x_1, x_2 \in \mathbb{R}^n$. The line segement x_1 and x_2 is defined as

$$S = \{x = \lambda x_1 + (1 - \lambda) x_2 : \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1\}$$

2.3.12 Definition: Convex

A set $S \subseteq \mathbb{R}^n$ is convex if, for any pair of points $x_1, x_2 \in S$, the line segement between x_1 and x_2 is in S

2.3.13 Proposition

Let $H = \{x : a^T x \leq \beta\}$ be a halfspace. It follows that H is convex

2.3.14 Corollary

If P is a polyhedron, then P is convex

2.3.15 Definition: Properly Contained

Point $x \in \mathbb{R}^n$ is properly contained in the line segement L if

- $x \in L$
- x is distinct from the endpoints of L

2.3.16 Definition: Extreme Point

Let S be a convex set and $\overline{x} \in S$
It follows that $\overline{x}$ is NOT an exteme point if there exists a line segement $L \subseteq S$ where L properly contains $\overline{x}$

Remark

A convex set may have an infinite number of extreme points
A convex set may have NO extreme points

2.3.17 Definition: Tight

Let $P = \{x : Ax \leq b\}$ be a polyhedron and let $x \in P$

- A constraint is tight for x if it is satisfied with equality
- the set of all tight constraints is denoted $\overline{A}x \leq \overline{b}$

2.3.18 Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and let $\overline{x} \in P$

- if $\text{rank}(\overline{A}) = n$, then $\overline{x}$ is an extreme point
- if $\text{rank}(\overline{A}) < n$, then $\overline{x}$ is NOT an extreme point

3 Module 3: Duality Through Examples

3.1 Week 7: Duality Through Examples / Weak Duality

3.1.1 Definition

A width assignment $\{y_U : \delta(U)\ s, t - cut\}$ is feasible if, for every edge $e \in E$, the total width of all cuts containing e is no more than c_e

3.1.2 Proposition

If y is a feasible width assignment, then any s,t-path must have length at least

$$\sum(y_U : U \ s,t-cut)$$

3.1.3 Theorem: Weak Duality

If $\overline{x}$ is feasible for (P) and $\overline{y}$ is feasible for (D), then $b^T\overline{y} \leq c^T\overline{x}$

3.1.4 Theorem: Strong Duality

Let P and D be a pair of primal-dual LPs. Then

- if there exists an optimal solution $\overline{x}$ of P, then there exists an optimal solution $\overline{y}$ of D
- the value of $\overline{x}$ in P equals the value of $\overline{y}$ in D

3.2 Week 8: Shortest Path Algorithm and Correctness

3.2.1 Definition: Slack

Let y be a feasible dual solution. The slack of an edge $e \in E$ is defined as

$$\text{slack}_y(e) = c_e - \sum(y_U : \delta(U) \ s,t - \text{cut}, e \in \delta(U))$$

3.2.2 Shortest Path Algorithm

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$
Output: A shortest st-path P

1. $y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
2. while $t \notin U$ do
3. Let ab be an edge in $\delta(U)$ of smallest slack for y where $a \in U$, $b \notin U$
4. $y_U := \text{slack}_y(ab)$
5. $U := U \cup \{b\}$
6. change edge ab into an arc $\overrightarrow{ab}$
7. end while
8. return A directed st-path P

3.2.3 Proposition

Let y be a feasible dual solution, and P and s,t-path. P is a shortest if

- all edges on P are equality edges, and
- every active cut $\delta(U)$ has exactly one edge of P

3.2.4 Proposition: Correctness of the Shortest Path Algorithm

The Shortest Path Algorithm maintains throughout its execution if:

- y is a feasible dual
- arcs are equality arcs (have 0 slack)
- no active cut $\delta(U)$ has an entering arc: an arc wu with $w \notin U$, and $u \in U$
- for every $u \in U$ there is a directed s,u-path, and
- arcs have both ends in U

4 Module 4: Duality Theory

4.1 Week 9: Duality Theory

4.1.1 Primal-Dual Pairs

The following table shows constraints and variables in primal and dual LPs correspond

(P_{max})		(P_{min})	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \;?\; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \;?\; c$
$x \;?\; \mathbb{0}$	free variable	$=$ constraint	$y \;?\; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

4.1.2 Theorem: Weak Duality

Let (P_{max}) and (P_{min}) represent the above. If $\overline{x}$ and $\overline{y}$ are feasible for two LPs, then

$$c^T\overline{x} \leq b^T\overline{y}$$

If $c^T\overline{x} = b^T\overline{y}$, then $\overline{x}$ is optimal for (P_{max}) and $\overline{y}$ is optimal for (P_{min})

4.1.3 Consequence of Weak Duality

- (P_{max}) is unbounded $\rightarrow$ (P_{min}) infeasible
- (P_{min}) is unbounded $\rightarrow$ (P_{max}) infeasible
- (P_{max}) and (P_{min}) feasible $\rightarrow$ both must have optimal solutions

4.1.4 Theorem: Strong Duality

If (P_{max}) has an optimal solution $\overline{x}$, then (P_{min}) has an optimal solution $\overline{y}$ such that $c^T\overline{x} = b^T\overline{y}$
Let (P) and (D) be a primal-dual pairs of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal

4.1.5 Strong Duality Theorem - Feasibility Version

Let (P) and (D) be primal-dual pairs of LPs. If both are feasible, then both optimal solutions of the same objective value

4.1.6 Complementary Slackness - special case

Let $\overline{x}$ and $\overline{y}$ be feasible for (P) and (D)
Then $\overline{x}$ and $\overline{y}$ are optimal if and only if

- $\overline{y}_i = 0$, or
- the ith constraint of (P) is tight for $\overline{x}$, for every row index i

4.1.7 Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\overline{x}$ and $\overline{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold

4.1.8 Definition

Let $a^{(1)}, \cdots, a^{(k)}$ be vectors in $\mathbb{R}^n$. The cone generated by these vectors is given by

$$C = \{\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \cdots + \lambda_k a^{(k)} : \lambda \geq 0\}$$

4.1.9 Theorem

Let $\overline{x}$ be a feasible solution to

$$\max\{c^T x : Ax \leq b\}$$

Then $\overline{x}$ is optimal if and only if c is in the cone of tight constraints for $\overline{x}$

5 Module 5: Solving Integer Programs

5.1 Week 10: Solving Integer Programs

5.1.1 Definition: Convex Hull

Let C be a subset of $\mathbb{R}^n$
The convex hull of C is the smallest convex set that contains C

5.1.2 Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational
Then, the convex hull of all integer poitns in P is a polyhedron

5.1.3 Theorem

- IP is infeasible if and only if LP is infeasible
- IP is unbounded if and only if LP is unbounded
- an optimal solution to IP is an optimal solution to LP
- an extreme optimal solution to LP is an optimal solution to IP

5.1.4 Cutting Plane Scheme

$$\max \{c^T x : Ax \leq b, x \text{integer}\} \quad (IP)$$

- Let (P) denote $\max\{c^T x : Ax \leq b\}$
- If (P) is infeasible, then STOP. (IP) is also infeasible
- Let $\overline{x}$ be the optimal solution to (P)
- If $\overline{x}$ is integral, then STOP. $\overline{x}$ is also optimal for (IP)
- Find a cutting plane $a^T x \leq \beta$ for $\overline{x}$
- Add a constraint $a^T x \leq \beta$ to the system $Ax \leq b$
- Repeat

5.1.5 Definition: Floor

Let $a \in \mathbb{R}$, then $\lfloor a \rfloor$ denotes the largest integer $\leq a$

6 Module 6: Nonlinear Optimization

6.1 Week 11: Nonlinear Optimization

6.1.1 Definition: Nonlinear Program

A nonlinear program (NLP) is a problem of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad (i = 1, \cdots, k) \end{aligned}$$

6.1.2 Definition: Local Optimum

Consider

$$min\{f(x) : x \in S\} \qquad (P)$$

$s \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \text{ where } ||x' - x|| \leq \delta \text{ and we have } f(x) \leq f(x')$$

6.1.3 Proposition

Consider

$$min\{c^T x : x \in S\} \qquad (p)$$

If S is convex and x is a local optimum, then x is optimal

6.1.4 Proposition

If $g_1, \cdots, g_k$ are all convex, then the feasible region of (P) is convex

6.1.5 Definition

Function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $a, b \in \mathbb{R}^n$

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b) \text{ for all } 0 \leq \lambda \leq 1$$

6.1.6 Proposition

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function $\beta \in \mathbb{R}$
It follows that $S = \{x \in \mathbb{R}^n : g(x) \leq \beta\}$ is a convex set

6.1.7 Proposition

(P):

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad (i = 1, \cdots, k) \end{aligned}$$

If all functions g_i are convex, then the feasible region of (P) is convex

6.1.8 Definition: Epigraph

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The epigraph of f is then given by

$$epi(f) = \{ \begin{bmatrix} y \\ x \end{bmatrix} : y \geq f(x), x \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$$

6.1.9 Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. It follows that

1. f is convex $\Rightarrow$ $epi(f)$ is convex
2. $epi(f)$ is convex $\Rightarrow$ f is convex

6.1.10 Defintion: Subgradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\overline{x} \in \mathbb{R}^n$
Then, $s \in \mathbb{R}^n$ is a subgradient of f at $\overline{x}$ if

$$h(x) := f(\overline{x}) + s^T(x - \overline{x}) \leq f(x) \text{ for all } x \in \mathbb{R}^n$$

6.1.11 Definition: Supporting

Let $C \in \mathbb{R}^n$ be a convex set and let $\overline{x} \in C$
The halfspace $F = \{x : s^T x \leq \beta\}$ is supporting C at $\overline{x}$ if

1. $C \subseteq F$ and
2. $s^T\overline{x} = \beta$. That is, $\overline{x}$ is on the boundary of F

6.1.12 Proposition

Let $g : \mathbb{R}^n \to \mathbb{R}$ be convex and let $\overline{x}$ where $g(\overline{x}) = 0$
Let s be a subgradient of g at $\overline{x}$
Let $C = \{x : g(x) \leq 0\}$
Let $F = \{x : h(x) := g(\overline{x}) + s^T(x - \overline{x}) \leq 0\}$
Then, F is a supporting halfspace of C at $\overline{x}$

6.1.13 Proposition

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad (i = 1, \cdots, k) \end{aligned}$$

$g_1, \cdots, g_k$ all convex
$\overline{x}$ is a feasible solution
$\forall i \in I$, $g_i(\overline{x}) = 0$
$\forall i \in I$, s^i subgradient for g_i at $\overline{x}$
If $-c \in \text{cone}\{s^i : i \in I\}$ then $\overline{x}$ is optimal

6.1.14 Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\overline{x} \in \mathbb{R}^n$
If the gradient $\nabla f(\overline{x})$ of f exists at $\overline{x}$, then it is a subgradient

6.1.15 Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be function and let $\overline{x} \in \mathbb{R}^n$
If the partial derivative $\dfrac{\partial f(x)}{\partial x_j}$ exists for f at $\overline{x}$ for all $j = 1, \cdots, n$, then the gradient $\nabla f(\overline{x})$ is obtained by evaluating for $\overline{x}$

$$\begin{bmatrix} \dfrac{\partial f(x)}{\partial x_1} & \cdots & \dfrac{\partial f(x)}{\partial x_n} \end{bmatrix}^T$$

6.1.16 Definition: Slater Point

A feasible solution to $\overline{x}$ is a slater point of

$$\begin{aligned} \min \;\; & c^T x \\ \text{s.t.} \;\; & g_i(x) \leq 0 \quad (i = 1, \cdots, k) \end{aligned}$$

if $g_i(\overline{x}) < 0$ for all $i = 1, \cdots, k$

6.1.17 The Karush-Kuhn-Tucker (KKT) Theorem

Consider the following NLP:

$$\begin{aligned} \min \;\; & c^T x \\ \text{s.t.} \;\; & g_i(x) \leq 0 \quad (i = 1, \cdots, k) \end{aligned}$$

Suppose that

1. $g_1, \cdots, g_k$ are all convex
2. there exists a slater point
3. $\overline{x}$ is a feasible solution
4. I is the set of indices i for which $g_i(\overline{x}) = 0$
5. for all $i \in I$ there exists a gradient $\nabla g_i(\overline{x})$ of g_i at $\overline{x}$

Then $\overline{x}$ is optimal $\iff$ $-c \in cone\{\nabla g_i(\overline{x}) : i \in I\}$