Math 138 Notes

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Contents

1 Week 1

1.1 Lecture 1 - Riemann Sums and Area

1.1.1 Areas under a curve

Approximate the area under the curve by a bunch of rectangles
Example: $f(x) = x^2$ from 0 to 1

- 1 rectangle: $1 \times 1 = 1$
- 2 rectangles (same width): $\frac{1}{2} + \frac{1}{2} \times \frac{1}{4} = \frac{5}{8}$
- 3 rectangles: $\frac{1}{3} \times \frac{1}{9} + \frac{1}{3} \times \frac{4}{9} + \frac{1}{3} \times 1 = \frac{14}{27}$
- n rectangles: base $\times$ height
  - base: $\frac{1}{n}$
  - height: $(\frac{i}{n})^2$
  - $A_i = \frac{1}{n} \cdot (\frac{i}{n})^2$
  - approximate area under $f(x)$: $R_n = A_1 + A_2 + \cdots + A_n = \displaystyle\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{i^2}{n^3}$

Area under $f(x) = x^2$ from 0 to 1: $R_n = \displaystyle\sum_{i=1}^{n} f(c_i)\Delta t_i$

- Δt_i is the width of the i^{th} rectangle, $\Delta t_i = \frac{1}{n}$
- c_i is the right end point of the i^{th} rectangle, making $f(c_i)$ the height of the rectangle, $c_i = \frac{i}{n}$
- $f(c_i)\Delta t_i$ is the area of the i^{th} rectangle

1.1.2 Riemann sums

Let f be a bounded function on $[a, b]$

1. divide the interval $[a, b]$ into n segments
   - $a = t_0 < t_1 < t_2 < \cdots < t_n = b$
   - set of points $\{t_i\}$ is called a **partition** P
   - $\Delta t_i = t_i - t_{i-1}$, width of the subinterval $[t_{i-1}, t_i]$
   - width of the largest subinterval is called the **norm of the partition**, $||P|| = \max_i\{\Delta t_i\}$
2. determine the height of the rectangles
   - for each $i = 1, 2, \cdots, n$ choose a point $c_i \in [t_{i-1}, t_i]$
   - height of the i^{th} rectangle if $f(c_i)$, can be negative

The **Riemann sum** is defined as $S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$

The value of the sum depends on

- partition (how many subintervals, the choice of t_i)
- choice of c_i within each subinterval

Terminology:

- if all the widths of the n-subintervals are the same, call it a **regular n-partition** of $[a, b]$
  - width: $\Delta t_i = \Delta t = \frac{b-a}{n}$
  - $t_i = a + i\Delta t$
- choose $c_i = t_i$ for all i, call it a **right hand Riemann sum**, denoted R_n
- choose $c_i = t_{i-1}$ for all i, call it a **left hand Riemann sum**, denoted L_n

1.1.3 The Definite Integral

Let f be a bounded function on $[a, b]$

- f is called **integrable on [a,b]** if there is a unique $I \in \mathbb{R}$ such that if for any sequence of partitions P_n with $\lim_{n\to\infty} ||P_n||$, the sequence of Riemann sums S_n corresponding to P_n to I, $\lim_{n\to\infty} S_n = I$
- I is called the **integral of f from a to b**, $I = \int_a^b f(t)dt$
- if a sequence of partitions P_n for which S_n does not converge, then it shows the function is not integrable
- if two sequence of partitions P_n and Q_n where respective Riemann sums converge to different values, the function is not integrable

Theorem:
Let f be continuous (or be bounded and have finitely many discontinuities) on $[a, b]$. Then f is integrable on $[a, b]$ and $\int_a^b f(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta t_i$ where the right hand side is the limit of the Riemann sum with any regular n-partition
In particular, $\int_a^b f(t)dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$
Theorem: Integrability Theorem for Continuous Functions
Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$
Definition: Regular n-partition
When for the interval $[a, b]$ the partition has n subintervals of equal length. That is, $\Delta t = \frac{b-a}{n}$ and $t_i = t_0 + i\Delta t$
Definition: Regular Right Hand Riemann Sum
We take $c_i = t_i$ for all i:

$$S_n = \sum_{i=1}^{n} f(t_i)\Delta t_i = \sum_{i=1}^{n} f(c_i)\Delta t = \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

We can also define LHRS similarly

$$S_n = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i = \sum_{i=1}^{n} f(t_{i-1})\Delta t = \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

Put the ideas together

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

1.2 Lecture 2

Note

- The integrability theorem also holds for functions that are bounded and have finitely many discontinuous
- Since a definite integral is the limit of a sequence, many limit laws will hold

1.2.1 Theorem: Properties of Integrals

If f is integrable on $[a, b]$, then

1. For $c \in \mathbb{R}$, $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
2. $\displaystyle\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. If $m \leq f(x) \leq M$ for $x \in [a, b]$, then $\displaystyle m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$
4. If $0 \leq f(x)$ for $x \in [a, b]$, then $\displaystyle 0 \leq \int_a^b f(x)dx$
5. If $f(x) \leq g(x)$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$
6. $|f|$ is integrable on $[a, b]$ and $\displaystyle|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

1.2.2 Definition: Additional Properties of Integrals

1. If $f(a)$ is defined, then $\int_a^a f(x)dx = 0$
2. If f is integrable on $[a,b]$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. Theorem: If f is integrable on an interval I containing a, b, and c, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

* c does NOT need to be between a & b

1.2.3 Geometric Interpretation of the Integral

We note that $\int_a^b f(x)dx$ returns the signed area between f and the x-axis.
That is, if $f(x) \leq 0$, then $\int_a^b f(x)dx \leq 0$
That is, in general, $\int_a^b f(x)dx$ is the unsigned area under f that lies above the x-axis <u>minus</u> your unsigned area above the graph & under the x-axis

1.3 Lecture 3 - Average Value of a Function

1.3.1 Definition: Average Value of Function on $[a,b]$

If f is continuous on $[a,b]$, the average value of f on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$

A Geometric Interpretation

If f is continuous on $[a,b]$, EVT says there exists $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for $x \in [a,b]$ and $f(c_1) = m$, $f(c_2) = M$ for some $c_1, c_2 \in [a,b]$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$$

$$f(c_1) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(c_2)$$

IVT says there exists c between c_1 and c_2 such that $f(c) = \frac{1}{b-a} \displaystyle\int_a^b f(x)dx$

1.3.2 Theorem: Average Value Theorem / MVT for Integrals

Assume f is continuous on $[a, b]$. There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

Note: the theorem also holds even if $b < a$

$$\begin{aligned} f(c) &= \frac{1}{a-b} \int_b^a f(x)dx \\ &= \frac{1}{a-b}(-\int_a^b f(x)dx) \\ &= \frac{1}{b-a} \int_a^b f(x)dx \end{aligned}$$

2 Week 2

2.1 Lecture 4

2.1.1 Integral Function

Let f be continuous on $[a, b]$
Define $G(x) = \displaystyle\int_a^x f(t)dt$ for $x \in [a, b]$

- $G(x)$ is the function that returns the signed area under f from a to x

2.1.2 Theorem: Fundamental Theorem of Calculus (Part I)

If f is continuous on an open interval I containing $x = a$ and if

$$G(x) = \int_a^x f(t)dt$$

Then $G(x)$ is differentiable for all $x \in I$ and

$$G'(x) = f(x)$$

That is

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

2.1.3 Definition: Anti-Derivative

Given a function f, an anti-derivative of f is a function F such that $F'(x) = f(x)$
*Note: Anti-derivative are <u>not</u> unique
The collection of all anti-derivatives of $f(x)$ is denoted by $\int f(x)dx = F(x) + c$ where $c \in \mathbb{R}$ and F is any anti-derivative. This is called an indefinite integral (no bounds)
*Note: by the Anti-Derivative Theorem, any two anti-derivatives of f differe by a constant

A bunch of anti-derivatives

- $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c$, if $n \neq -1$
- $\displaystyle\int \frac{1}{x}dx = \ln|x| + c$
- $\displaystyle\int e^x = e^x + c$
- $\displaystyle\int \sin(x)dx = -\cos(x) + c, \int \cos(x)dx = \sin(x) + c$
- $\displaystyle\int \sec^2(x)dx = \tan(x) + c$
- $\displaystyle\int \frac{1}{1+x^2}dx = \arctan(x) + c$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + c, \int \frac{-1}{\sqrt{1-x^2}}dx = \arccos(x) + c$
- $\displaystyle\int \sec(x)\tan(x)dx = \sec(x) + c$

- $\int a^x dx = \frac{a^x}{\ln(a)} + c$, for $a > 0, a \neq 1$

2.2 Lecture 5

2.2.1 Theorem: Fundamental Theorem of Calculus II (FTC2)

If f is continuous on $[a, b]$ and F is any anti-derivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

2.2.2 Corollary: Extended Version of FTC

If f is continuous and g & h and differentiable, then:

$$\frac{d}{dx}[\int_{g(x)}^{f(x)} f(t)dt] = f(h(x))h'(x) - f(g(x))g'(x)$$

Also known as Leibniz Formula

2.2.3 U-Substitution

This technique is in essen a reverse chain-rule

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Good time for u-substitution: when the function and its anti-derivative are presented

2.3 Lecture 6

2.3.1 Theorem: Change of Variable

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

3 Week 3

3.1 Lecture 7

3.1.1 Trignometry Substitution

There are 3 situations where this technique is useful

If you see	Trig substitution	Range for θ
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$

Notes

- The range of θ is key so that that $\sin / \tan / \sec$ are invertible
- You don't have to state the restriction when you saw a trig substitution
- Remember:

$$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$$
$$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$$
$$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$$

3.2 Lecture 8

3.2.1 Integration By Part (IBP)

Integration by part is related to product rule
Product Rule:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

Integrate both sides:

$$\int \frac{d}{dx}[u(x)v(x)]dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

By FTC:

$$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

Rearrange:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$
$$\int udv = uv - \int vdu$$

(Integration By Parts Formula)

3.2.2 IBP Rule of Thumb

Let u be whatever shows up first in the acronym

L I A T E
L: log
I: inverse trig
A: algebric (polynomials, etc.)
T: trig
E: exponent

Note

For definite integral:

- find indefinite integral first, and sub in the bounds after
- do the bounds as you go along

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$
$$[uv]_a^b = u(b)v(b) - u(a)v(a)$$

3.3 Letcture 9

3.3.1 Integration By Partial Functions

This is used when integrating rational functions

$$\int \frac{p(x)}{q(x)}dx$$

when p & q are polynomials
Specifically when $deg(p) < deg(p)$, if $deg(p) \geq deg(q)$, we have to do long division (or synthetic devision) first
We need to decompose $\frac{p}{q}$ into partial fractions
The first step is to ensure that q is in fully factored form
Once we are in a form when $deg(p) < deg(q)$ and q is fully factored, we decompose on the factors we see in q

Name	Appearance	Split
Dintinct Linear	$(5x+1)(x-3)\cdots$	$\frac{A}{5x+1}+\frac{B}{x-3}+\cdots$
Repeated Linear	$(5x+1)^7$	$\frac{A}{5x+1}+\frac{B}{(5x+1)^2}+\cdots+\frac{G}{(5x+1)^7}$
Dintinct, Irreducible Linear	$(5x^2+3x+1)(x^2+3)\cdots$	$\frac{Ax+B}{5x^2+3x+1}+\frac{Cx+D}{x^2+3}+\cdots$
Repeated, Irreducible Linear	$(5x^2+3x+1)^7$	$\frac{Ax+B}{5x^2+3x+1}+\frac{Cx+D}{(5x^2+3x+1)^2}+\cdots+\frac{Mx+N}{(5x^2+3x+1)^7}$

Next, we'd set $\frac{p}{q}$ equal to our decomposition and solve for coefficients. Then integrate.

4 Week 4

More Integrals

1. Continuous functions over infinite intervals
2. Functions with infinite discontinuities

4.1 Lecture 10 - Continuous functions over infinite intervals

4.1.1 Type I Improper Integrals: Infinite Intervals

Integrals of form:

$$\int_{-\infty}^{a} f(x)dx, \int_{a}^{\infty} f(x)dx, \int_{-\infty}^{\infty} f(x)dx$$

Key Idea:

Approach problem with limits

4.1.2 Type I Solution:

- $\displaystyle\int_{-\infty}^{a} f(x)dx = \lim_{b\to-\infty}\int_{b}^{a} f(x)dx$
- $\displaystyle\int_{a}^{\infty} f(x)dx = \lim_{b\to\infty}\int_{a}^{b} f(x)dx$
- $\displaystyle\int_{-\infty}^{-\infty} f(x)dx = \lim_{b_1\to-\infty}\int_{b_1}^{0} f(x)dx + \lim_{b_2\to\infty}\int_{0}^{b_2} f(x)dx$

  * Do not try to say $\displaystyle\int_{-\infty}^{-\infty} f(x)dx = \lim_{b\to\infty}\int_{-b}^{b} f(x)dx$

We say an improper integral converges if all of the limits exist (and are finite). The integral <u>diverges</u> if <u>any</u> limit fails to exist (include: $\pm\infty$)

4.1.3 Theorem: P-Test for Type I Integrals

$\displaystyle\int_{1}^{\infty}\frac{1}{x^p}dx$ converges if and only if $p > 1$
If $p > 1$, $\displaystyle\int_{1}^{\infty}\frac{1}{x^p}dx = \frac{1}{p-1}$

4.1.4 Properties of Type I Integrals

Suppose $\int_a^\infty f(x)dx$ & $\int_a^\infty g(x)dx$ converges

1. $\int_a^\infty cf(x)dx$ converges for any $c \in \mathbb{R}$, and $\int_a^\infty cf(x)dx = c\int_a^\infty f(x)dx$
2. $\int_a^\infty [f(x)+g(x)]dx$ converges and $\int_a^\infty [f(x)+g(x)]dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$
3. If $f(x) \leq g(x)$ for all $x \geq a$, then $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$
4. If $a < c < \infty$, then $\int_c^\infty f(x)dx$ converges and $\int_a^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx$

In general, finding the value an integral converges to (if in fact it does) can become quite difficult. We do however have a way of comparing a more complex improper integral to a simpler one

4.1.5 Theorem: Comparison Theorem (For Type I)

Supppose f & g are continuous functions where $0 \leq g(x) \leq f(x)$ for $x \geq a$

1. If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges too
2. If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges too

4.2 Lecture 11

4.2.1 Definition: Absolute Convergence

Let f be integrable on $[a,b]$ for all $b \geq a$
We say that $\int_a^\infty f(x)dx$ converges absolutely if $\int_a^\infty |f(x)|dx$ converges

4.2.2 Theorem: Absolute Converges Theorems (ACT)

Let f be integrable on $[a, b]$ for all $b \geq a$
Then $|f|$ is integrable on $[a, b]$ for all $b \geq a$, and if $\displaystyle\int_a^\infty |f(x)|dx$ converges, then so does $\displaystyle\int_a^\infty f(x)dx$
In particular, if $|f(x)| \leq g(x)$ for $x \geq a$, and if $\displaystyle\int_a^\infty g(x)dx$ converges, so does $\displaystyle\int_a^\infty f(x)dx$

4.2.3 Type II: Improper Integrals

Consider $\displaystyle\int_a^b f(x)dx$:

- If f has an infinite discontinuities at $x = a$, then $\displaystyle\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$
- If f has an infinite discontinuities at $x = a$, then $\displaystyle\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$
- If f is not continuous at c, $a < c < b$, then $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ and use two previous points to replace problem bounds with limits

Again, if all limits exist, we say the integral converges
If any limit fials to exist, we say the integral diverges

4.3 Lecture 12

4.3.1 Theorem: P-Test for Type II

$\displaystyle\int_0^1 \frac{1}{x^p}dx$ converges if and only if $p < 1$, $\displaystyle\int_0^1 \frac{1}{x^p}dx = \frac{1}{1-p}$

4.3.2 Area between two curves

Suppose we want to calculate the are between two curves, $f\&g$
This is the area under $f(x)$ minus the area $g(x)$
Using all of the same idea as before, we divide the area into infinitely many thin rectangles, and integrate
So for $f \geq g$, the area bounded by $f\&g$ from $x = a$ to $x = b$ is

$$\int_a^b f(x) - g(x)dx \text{ top - bottom}$$

Actual Formula

Area between $f\&g$ from $x = a$ to $x = b$ is

$$\int_a^b |f(x) - g(x)|dx$$

So, to deal with this we just split the integral where top/bottom switch
We can use the same idea to compute areas where x is a function of y, "right - left"

5 Week 5

5.1 Lecture 13

5.1.1 Disk Method

$$V = \int_a^b \pi[R(x)]^2 dx$$

$R(x)$ is the radius at each x-value

5.1.2 Washers Method

$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)dx$$

$R(x)$ is the radius of the outer circle, $r(x)$ is the radius of the inner circle

5.2 Lecture 14

5.2.1 Method: Cylindrical Shells

Imagine a 3D shape created by rotation (around y-axis)
We can build the shape from the inside out by building concentric cylindrical shells of thickness $\Delta x(dx)$

$$V_{shell} = 2\pi r\Delta x = \int_a^b 2\pi rh dx$$

5.3 Lecture 15

	functions of x	functions of y
rotate around a vertical line	shells	washers / disks
rotate around a horizontal line	washers / disks	shells

- If you go down one path, and meet restriction (region split or nasty integral) then reframe the problem
  - change the function to be in other variables

5.3.1 Differentiable Equations (DEs)

Definition: Ordinary Differential Equations (ODE)

An equation containing derivatives of a dependent variable (ie. function) $y = f(x)$ is called an ordinary differential equation
(To contrast: partial DEs (PDE) are multivariable functions)

Definition: Order

The order of a DE is the highest derivative present

Definition: Linear

An ODE is called linear if it only contains linear functions of y (y, y', y'', etc.)

Definition: General Solution

The general solution of an ODE is the collection of all possible functions that solve the ODE, including arbitrary constants

Definition: Particular Solution

A solution in which all arbitrary constants have been determined

Definition: Initial Condition (IC)

Extra information about y and/or its derivatives at specific points that allow us to find a particular solution

Definition: Initial Value Problem (IVP)

An DE with IC(s)

Definition: Constant Solution / Equilibrium Solution

A solution to a DE which is a constant function

Definition: Direction Field

For ODEs of form

$$\frac{dy}{dx} = f(x, y)$$

we can create a visualization called a direction field
The direction field shows the family of curves (general solution) satisfying the DE. An IC would be point, and te particular solution follows the field through the point

6 Week 6

6.1 Lecture 16

6.1.1 Definition: Separable DE

A separable ODE is a first order DE that can be written as

$$\frac{dy}{dx} = g(y)h(x)$$

that is, we can factor the RHS into a product of a function of y and a function of x

Solving Technique

$$\begin{aligned}
\frac{dy}{dx} &= g(y)h(x) \\
\frac{1}{g(y)}\frac{dy}{dx} &= h(x) \\
\int \frac{1}{g(y)}\frac{dy}{dx}dx &= \int h(x)dx \\
\int \frac{1}{g(f(x))}f'(x)dx &= \int h(x)dx \\
\int \frac{1}{g(y)}dy &= \int h(x)dx
\end{aligned}$$

6.2 Lecture 17

6.2.1 Definition: First Order Linear DE (FOLDE)

A DE of form:

$$A(x)y' + B(x)y = C(x) \text{ where } A(x) \neq 0$$

that can be written as:

$$y' + P(x)y = Q(x)$$

is a FOLDE
Solve such DE using an "integrating factor"
our FOLDE algorithm:

1. Write in form $\dfrac{dy}{dx} + P(x)y = Q(x)$
2. Find $\mu(x) = e^{\int P(x)dx}$
3. Multiply both sides by $\mu(x)$, collapse the LHS by product rule
4. Integrate both sides (+C !) and solve for y

6.2.2 Formula for FOLDE solution

$$y = \frac{1}{\mu(x)}[\int \mu(x)Q(x)dx], \text{ where } \mu(x) = e^{\int P(x)dx}$$

6.2.3 Newton's Law for Cooling

$$\frac{dT}{dt} = -k(T - T_{room}) \quad k > 0$$

- $\frac{dT}{dt}$ rate of change in temp
- k constant
- T temperature function
- T_{room} srround room temperature

$$T(t) = Ce^{-kt} + T_{room} \quad \text{For } c, k \in \mathbb{R}$$

6.3 Lecture 18

6.3.1 Population Growth

1. Expoential / Natural Growth

$$\frac{dP}{dt} = kP$$

2. Logistic Model

$$\frac{dP}{dt} = kP(1 - \frac{P}{M})$$

   - $k > 0$
   - M carrying capacity

7 Week 7

7.1 Lecture 19

7.1.1 Definition: Infinite Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

7.1.2 Definition: Sequence of Partial Sum

If $\sum_{n=1}^{\infty} a_n$ is a series, we define its sequence of partial sums $\{S_n\}$ as:

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

7.1.3 Defintion: Convergence / Divergence of Series

A series $\sum_{n=1}^{\infty} a_n$ ($\sum a_n$) converges to $S \in \mathbb{R}$ if $\lim_{n\to\infty} S_n = S$. Here S is called the sum of the series.
If $\{S_n\}$ diverges, we say the series diverges

7.1.4 Geometric Series

A geometric series is of form $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots$ for some $r \in \mathbb{R}$
$\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$ if $|r| < 1$ and diverges otherwise

A common other version for Geometric Series

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r} \quad \text{if } |r| < 1$$

7.2 Lecture 20

7.2.1 Theorem: Arithmetic for Series

Suppose $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ and $k \in \mathbb{R}$

1. $\sum_{n=1}^{\infty} ka_n = kA$

2. $\displaystyle\sum_{n=1}^{\infty}(a_n \pm b_n) = A \pm B$

3. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=j}^{\infty} a_n$ also converges for each $j \geq 1$

4. If $\displaystyle\sum_{n=j}^{\infty} a_n$ converges for some j, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges

   Convergence only depends on the tail

   Add or Subtract finite number terms does not affect convergence

7.3 Lecture 21 - Series Tests

7.3.1 Theorem

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$

7.3.2 Theorem: Divergence Test

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ (or DNE), then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges

Note:

- always the first test to try
- only tell about divergence, not convergence
  - result: $\displaystyle\sum_{n=1}^{\infty} a_n$

    $\neq 0$, series diverges

    $= 0$, no information

8 Week 8

8.1 Lecture 22

8.1.1 Integral Test

Let $\sum a_n$ be a positive series. Suppose that $a_n = f(n)$ where f is a continuous, positive, decreasing function for all $x \geq N$. Then, $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ either both converge or both diverge

Note

- Must always state (if a simple function) or show (complicated function) the continuous, positive, decreasing native
- These qualities only need to take place beyond cutoff N
- The integral test does not say the series converges to the integral value

8.1.2 P-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

8.1.3 Bounds on the Remainder (Error) in the Integral Test

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx \text{ where } R_n = S - S_n$$

We can rewrite:

$$\int_{n+1}^{\infty} f(x)dx \leq S - S_n \leq \int_n^{\infty} f(x)dx$$

$$S_n + \int_{n+1}^{\infty} f(x)dx \leq S \leq S_n + \int_n^{\infty} f(x)dx$$

8.2 Lecture 23

8.2.1 Theorem: (Direct) Comparison Test

Assume $0 \leq a_n \leq b_n$ for $n \in \mathbb{N}$ (eventually)

1. If $\sum b_n$ converges, then $\sum a_n$ converges too
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges too

8.2.2 Theorem: Limit Comparison Test (LCT)

If $a_n \geq 0$ and $b_n \geq 0$ for $n \in \mathbb{N}$ (eventually) and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $L \neq 0$, $L < \infty$, then either both a_n and b_n converge or both diverge

Note

Use LCT for:

- $\sum \frac{\text{powers of n}}{\text{powers of n}} \to$ dominating terms
- "almost" geometric series

8.2.3 Theorem: LCT (full version)

If $a_n \geq 0$ and $b_n > 0$ for $n \in \mathbb{N}$ (eventually) and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, then:

1. If $0 < L < \infty$ then $\sum a_n$ converges iff $\sum b_n$ converges
2. If $L = 0$ and $\sum b_n$ converges then $\sum a_n$ converges (if $\sum a_n$ diverges then $\sum b_n$ diverges)
3. If $L = \infty$ and $\sum a_n$ converges then $\sum b_n$ converges (if $\sum a_n$ diverges then $\sum b_n$ diverges)

8.3 Lecture 24

8.3.1 Alternating Series Test

Assume that

1. $a_n > 0$ for all n
2. $a_{n+1} \leq a_n$ for all n
3. $\lim_{n\to\infty} a_n = 0$

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges

8.3.2 Theorem: Error/Remainder Bound for Alternating Test

$$|R_n| = |S - S_n| \leq a_{n+1},\ a_{n+1} \text{ is the next unused term}$$

9 Week 9

9.1 Lecture 25

9.1.1 Definition: Absolutely Convergent

A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges

9.1.2 Definition: Conditional Convergence

A series is conditionally convergent if it is convergent but not absolutely convergent

9.1.3 Theorem: Absolute Convergence Theorem (ACT)

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges

9.2 Lecture 26

9.2.1 Theorem: Ratio Test

Let $\sum a_n$ be any series and suppose $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$ where $L \in \mathbb{R}$ or $L = \infty$

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges
3. If $L = 1$, inconclusive (no info)

Note

Ratio test is good to use when

- factorial
- 'almost' geo series

9.2.2 Theorem

For $x \in \mathbb{R}$, $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$

9.2.3 Theorem: Root Test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be any series and assume $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, $L \in \mathbb{R}$ or $L = \infty$

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $L = 1$, inconclusive (no info)

9.3 Lecture 27 - Practices

No notes

10 Week 10

10.1 Lecture 28

10.1.1 Definition: Power Series

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \to$ center 0
$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \to$ center a
Where $a_i \in \mathbb{R} \forall i$
The domain of a power series is the collection of all $a \in \mathbb{R}$ for which the power series converges
Domain is never empty, series always converges to a_0 at centre

Convention for $\sum_{n=0}^{\infty} a_n (x-a)^n$

1. When $n = 0$, the term is a_0 for all x, including $x = a$
2. If the first feew term are zero ($a_0 = a_1 = \cdots = a_k = 0$), then $\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=k+1}^{\infty} a_n (x-a)^n$ (throw out term where $a_n = 0$)

10.1.2 Theorem

For a given power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, there are 3 possibilities:

1. The series converges only when $x = a$
2. The series converges $\forall x \in \mathbb{R}$
3. There exists $R \in \mathbb{R}$ such that the series converges absolutely for $|x - a| < R$, diverges if $|x - a| > R$ and may converge or diverge when $|x - a| = R$

10.2 Lecture 29

10.2.1 Geometric Series

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, $I = (-1, 1), R = 1$

10.2.2 Theorem: Abel's Theorem

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I, then f is continuous on I

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ & $g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ with radii of convergence R_f & R_g and invervals of convergence I_f & I_g

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-a)^n$
   If $R_f \neq R_g$ then the radius of convergence is $R = \{R_f, R_g\}$ and the interval of convergence is $I = I_f \cap I_g$
   If $R_f = R_g$, then $R \geq R_f$

2. $(x-a)^k f(x) = \sum_{n=0}^{\infty} a_n(x-a)^{n+k}$
   Where the radius is R_f & the interval is I_f (no change)

3. If $c \in \mathbb{R}$, $c \neq 0$, and $a = 0$ (so $\sum_{n=0}^{\infty} a_n x^n$)
   Then $f(cx^k) = \sum_{n=0}^{\infty} a_n c^n x^{nk}$ —— Where the radius comes from solving $|cx^k| < R_f$
   If $R_f = \infty$ then $R = \infty$
   The interval is $I = \{x \in \mathbb{R} | cx^k \in I_f\}$

10.2.3 Theorem

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $R > 0$, then $f(x)$ is differentiable (thus continuous, integrable) on $(a-R, a+r)$ and

1. $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ (change starting index since if $n = 0$, term is zero)

2. $\int f(x) = \sum_{n=0}^{\infty} (\frac{a_n(x-a)^{n+1}}{n+1}) + C$

Note

While the radius of convergence will not change via differentiate/integrate, the interval of convergence may change

10.3 Lecture 30

10.3.1 Proposition

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$

11 Week 11

11.1 Lecture 31

11.1.1 Definition: n^{th} degree Taylor Polynomials

If f is n-times differentiable at $x = a$, the n^{th} degree Taylor polynomials for f centered $x = a$ is

$$T_{n,a}(X) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

11.1.2 Definition: Remainder

If f is n-times differentiable at $x = a$, we define the n^{th} degree Taylor remainder function centered at $x = a$ to be

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

The error in using $T_{n,a}(x)$ to approximate $f(x)$ is given by Error$= |R_{n,a}(x)|$

11.1.3 Theorem: Taylor's Theorem

Assume f is $(n+1)$ times differentiable on an interval I containing $x = a$. Let $x \in I$. Then, there exists a point c between $x\&a$ such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

11.1.4 Corollary: Taylor's Inequality

$|R_{n,a}(x)| \leq \dfrac{M|x-a|^{n+1}}{(n+1)!}$ where $|f^{n+1}(c)| \leq M$ for all c between $x\&a$

11.1.5 Theorem

If $f(x)$ has a power series representation about $x = a$, $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ for $|x-a| < R$, $R > 0$ then $a_n = \dfrac{f^n(a)}{n!}$

That is $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$ (The Taylor series for f centered at $x = a$)

Special case: $a = 0$, $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}x^n$ is called the Maclaurin series for f

11.1.6 Theorem: Convergence Theorem for Taylor Series

Assume f has derivatives of all orders on an interval I containing $x = a$
Assume also there exists $M \in \mathbb{R}$ such that $|f^k(x)| \leq M$ for all $k \in \mathbb{N}$ and

$x \in I$
Then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \text{ for } x \in I$$

11.2 Lecture 32

11.2.1 Corollary

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$

11.2.2 Corollary

Both $\sin(x)$ & $\cos(x)$ are equal to their Taylor series for all $x \in \mathbb{R}$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ for all } x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ for all } x \in \mathbb{R}$$

11.3 Lecture 33

11.3.1 Theorem: Generalized Binomial Theorem

Let $k \in \mathbb{R}$. Then for all $x \in (-1, 1)$:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \text{ where } \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

$\binom{k}{0} = 1$