Math 136 Notes

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1 Chapter 1 Vectors in Euclidean Space ($\mathbb{R}^n$)

1.1 Introduction

Linear algebra is the algebraic theory behind geometric ideas such as

- lines
- planes
- perpendicular/parallel
- rotation/reflection/translation
- megnitude + direction

  $\cdots$ and their generalization

Definition

Let $n \in \mathbb{N}$

$$\mathbb{R}^n = \{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R}\}$$

We call $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ a vector in $\mathbb{R}^n$ and we call x_i the i^{th} component of the vector

Example: $\begin{bmatrix} 1 \\ \pi \\ \frac{3}{7} \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}$

1.2 Geometric Interpretation

Case 1 let $\vec{v} \in \mathbb{R}^2$, say $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Case 2 let $\vec{v} \in \mathbb{R}^3$, say $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

1.3 Operations

To do linear algebra we need the following operations:

Definition: let $\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \vec{v} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$, then

1. addition: $\vec{u} + \vec{v} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$

2. scalar multiplication: for $c \in \mathbb{R}$, $c\vec{u} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$

Example: In $\mathbb{R}^2$, $2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

More on Scalar Multiplication:

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$

2. $(cd)\vec{v} = c(d\vec{v})$

3. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

4. $0\vec{v} = \vec{0}$

5. If $c\vec{v} = \vec{0}$, then $c = 0$ or $\vec{v} = \vec{0}$

Standard Basis
In $\mathbb{R}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $\varepsilon = \{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$ is called the standard basis of $\mathbb{R}^n$

Note that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_n\vec{e_n}$

1.4 Vector in $\mathbb{C}^n$

Definition:

$$\mathbb{C}^n = \{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1 \cdots x_n \in \mathbb{C}\}$$

1.5 Dot Product

Let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$

We define the dot product: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

1.6 Properties of Dot Product

If $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w}$ are vectors in $\mathbb{R}^n$

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4. $\vec{v} \cdot \vec{v} \geq 0$, with $\vec{v} \cdot \vec{v} = 0$ iff $\vec{v} = \vec{0}$

1.7 Length

Length of vector $\vec{v} \in \mathbb{R}^n$ is $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$. It is also called **norm** or **magnitude**. If $c \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$, then $||c\vec{v}|| = |c|||\vec{v}||$

1.8 Unit Vector

$\vec{v} \in \mathbb{R}^n$ is unit vector if $||\vec{v}|| = 1$

1.9 Normalization

When $\vec{v} \in \mathbb{R}^n$ is a non-zero vector, we produce a unit vector

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

in the direction of $\vec{v}$ by scale $\vec{v}$. This process is called normalization

1.10 Angle

Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^n$. The angle θ, in radians $(0 \leq \theta \leq \pi)$, between $\vec{u}$ and $\vec{v}$ is such that

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$$

1.11 Orthogonality

We say that the two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal / perpendicular if $\vec{u} \cdot \vec{v} = 0$

1.12 Projection and Perpendicular

1.12.1 Projection

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$. The projection of $\vec{v}$ onto $\vec{w}$ is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v} \cdot \vec{w})}{||\vec{w}||^2}\vec{w} = \frac{(\vec{v} \cdot \vec{w})}{\vec{w} \cdot \vec{w}}\vec{w}$$

1.12.2 Perpendicular

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$. The perpendicular of $\vec{v}$ onto $\vec{w}$ is defined by

$$\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$$

Note:
We have

$$\text{proj}_{\vec{w}}\vec{v} = (\vec{v} \cdot \hat{w})\hat{w}$$

length of $\vec{w}$ is irrelevant
Also

$$\begin{aligned}\text{proj}_{\vec{w}}\vec{v} &= \frac{||\vec{v}||||\vec{w}||\cos\theta}{||\vec{w}||^2}\vec{w} \\ &= ||\vec{v}||\cos\theta\hat{w}\end{aligned}$$

$||\vec{v}||\cos\theta$ is the scalar component of $\vec{v}$ over $\vec{w}$

1.13 Fields

$\mathbb{R}$ and $\mathbb{C}$

- can always add, subtract and multiply and stay in the set
- can always divide by a non-zero element and stay in the set
- "everything works" (commutative, associative, distributive, inverses exist etc.)

1.14 Cross Product

Definition:
Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$
The cross product of $\vec{u}$ and $\vec{v}$ is the vecttor in $\mathbb{R}^3$ given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Properties:
Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ and let $\vec{z} = \vec{u} \times \vec{v}$. Then

1. $\vec{z} \cdot \vec{u} = 0$ and $\vec{z} \cdot \vec{v} = 0$
2. $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$
3. If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$

More Properties:
Notes:

- only defined for $\mathbb{R}^3$
- $||\vec{u} \times \vec{v}||$ gives the area of a parallelogram

Linearity of the Cross Product:
If $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, then

1. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

2 Chapter 2 - Span, Lines and Planes

2.1 Linear Combinations and Spans

2.1.1 Linear Combination

Let $c_1, c_2, \cdots, c_k \in \mathbb{F}$ and let $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ be vectors in $\mathbb{F}^n$. We refer any vector of the form $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$

2.1.2 Span

Let span of $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ to be the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$. That is, Span

$$\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\} = \{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} : c_1, c_2, \cdots, c_k \in \mathbb{F}\}$$

We refer to $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ as a spanning set for Span$\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$. We also say that Span$\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is spanned by $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$

2.2 Geometry

- span$\{\vec{0}\} = \vec{0}$
- For $\vec{v} \neq \vec{0}$, span$\{\vec{0}\}$ is a line through origin with direction $\vec{v}$
- For $\vec{v}, \vec{w} \neq \vec{0}$, if $\vec{v}$ and $\vec{w}$ are not parallel, then span$\{\vec{v}, \vec{w}\}$ is a plane

2.3 Parametric Equations of a Line in $\mathbb{R}^2$

Let p, q be fixed real numbers and $q \neq 0$. The parametric equations of a line in $\mathbb{R}^2$ through the point (x_1, y_1) with slope $\frac{p}{q}$ are

$$\begin{cases} x & = x_1 + qt \\ y & = y_1 + pt \end{cases}, t \in \mathbb{R}$$

2.4 Vectors and Lines

2.4.1 Vector Equation of a Line in $\mathbb{R}^2$

Let $\begin{bmatrix} q \\ p \end{bmatrix}$ be a non-zero vector in $\mathbb{R}^2$. The expression

$$\vec{l} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\begin{bmatrix} q \\ p \end{bmatrix}, t \in \mathbb{R}$$

is a vector equation of $\mathcal{L}$ in $\mathbb{R}^2$ through $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ with direction $\begin{bmatrix} q \\ p \end{bmatrix}$

2.4.2 Line in $\mathbb{R}^2$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ with $\vec{v} \neq \vec{0}$. We refer to the set of vectors

$$\mathcal{L} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$$

as a line $\mathcal{L}$ in $\mathbb{R}^2$ through $\vec{u}$ with direction $\vec{v}$

2.5 Equations in $\mathbb{R}^n$

2.5.1 Vector Equation of a Line in $\mathbb{R}^n$

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The expression

$$\vec{l} = \vec{u} + t\vec{v}, t \in \mathbb{R}$$

is a vector equation of line $\mathcal{L}$ in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$

2.5.2 Parametric Equations of a Line in $\mathbb{R}^n$

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. From above, consider

$$\vec{l} = \vec{u} + t\vec{v}, t \in \mathbb{R}$$

The parametric equations of the line $\mathcal{L}$ in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$ are

$$\begin{cases} l_1 = & u_1 + tv_1 \\ l_2 = & u_2 + tv_2 \\ & \vdots \\ l_n = & u_n + tv_n \end{cases}, t \in \mathbb{R}$$

2.6 Lines in $\mathbb{R}^n$

2.6.1 Line in $\mathbb{R}^n$

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. We refer to the set of vectors

$$\mathcal{L} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$$

as a line $\mathcal{L}$ in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$

Notes

- If l_1 and l_2 are lines with direction $\vec{v_1}$ and $\vec{v_2}$, respectively, we say that they have the same direction if $c\vec{v_1} = \vec{v_2}$ for some non-zero $c \in \mathbb{R}$
- Lines through the origin can be described as the span of a vector

2.7 Planes Through the Origin in $\mathbb{R}^n$

2.7.1 Plane in $\mathbb{R}^n$ Through the Origin

Let $\vec{v}, \vec{w}$ be non-zero vectors in $\mathbb{R}^n$ with $\vec{v} \neq c\vec{w}$ for any $c \in \mathbb{R}$. Then

$$\mathcal{P} = \text{Span}\{\vec{v}, \vec{w}\} = \{s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$

is a plane in $\mathbb{R}^n$ through the origin with direction vectors $\vec{v}$ and $\vec{w}$

2.7.2 Vector Equation of a Plane in $\mathbb{R}^n$ Through the Origin

Let $\vec{v}, \vec{w}$ be non-zero vectors in $\mathbb{R}^n$ with $\vec{v} \neq c\vec{w}$ for any $c \in \mathbb{R}$. The expression

$$\vec{p} = s\vec{v} + t\vec{w}$$

is a vector equation of the plane in $\mathbb{R}^n$ through the origin with direction vectors $\vec{v}$ and $\vec{w}$

2.8 Arbitrary Planes in $\mathbb{R}^n$

2.8.1 Planes in $\mathbb{R}^n$

Let $\vec{u} \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be non-zero vectors in $\mathbb{R}^n$ with $\vec{v} \neq c\vec{w}$, for any $c \in \mathbb{R}$. Then

$$\mathcal{P} = \{\vec{u} + s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$

is a plane in $\mathbb{R}^n$ through $\vec{u}$ with direction vectors $\vec{v}$ and $\vec{w}$. We say that $\vec{v}$ and $\vec{w}$ are parallel to $\mathcal{P}$

2.8.2 Vector Equation of a Plane

Let $\vec{u} \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be non-zero vectors in $\mathbb{R}^n$ with $\vec{v} \neq c\vec{w}$, for any $c \in \mathbb{R}$. Then

$$\vec{p} = \vec{u} + s\vec{v} + t\vec{w}, s, t \in \mathbb{R}$$

is a vector equation of the plane in $\mathbb{R}^n$ through $\vec{u}$ with direction vectors $\vec{v}$ and $\vec{w}$

2.9 Planes in $\mathbb{R}^3$

2.9.1 Normal Vector

Let $\vec{v}$ and $\vec{w}$ be non-zero vectors in $\mathbb{R}^3$ with $\vec{v} \neq c\vec{w}$, for any $c \in \mathbb{R}$. The vector $\vec{n} = \vec{v} \times \vec{w}$ is referred to as a normal vector to the plane with direction vectors $\vec{v}$ and $\vec{w}$

2.9.2 Normal Form, Scalar Equation of a Plane in $\mathbb{R}^3$

Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ with direction vectors $\vec{v}$ and $\vec{w}$ and a normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$. Let $\vec{u} \in \mathcal{P}$ and $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{P}$ where $\vec{p} \neq \vec{u}$. A normal form of $\mathcal{P}$ is given by

$$\vec{n} \cdot (\vec{p} - \vec{u}) = 0$$

Expanding this, we arrive at a scalar equation (general form) of $\mathcal{P}$, $ax + by + cz = d$, where $d = \vec{n} \cdot \vec{u}$

3 Chapter 3 - Systems of Linear Equations

3.1 Linear Equations

3.1.1 Linear Equation, Coefficient, Constant Term

A linear equation in n variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n, b \in \mathbb{F}$
The scalars $a_1, a_2, \cdots, a_n$ are the coefficients of $x_1, x_2, \cdots, x_n$, and b is the constant term

3.2 Systems of Linear Equations

A system of linear equations is a collection of m linear equations in n variables, $x_1, \cdots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Use the convention that a_{ij} is the coefficient of x_j in the i^{th} equation

3.3 Solution to the System

We say that the scalars $y_1, y_2, \cdots, y_n$ in $\mathbb{F}$ solve a system of lienar equations if, when we set $x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n$ in the system, then each of the equations is satisfied:

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &= b_1 \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &= b_2 \\ &\vdots \\ a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n &= b_m \end{aligned}$$

We also say that the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is a solution to the system

3.4 Important Facts

3.4.1 Solution Set

The set of all solutions to a system of linear equations is called the solution set to the system

3.4.2 Theorem: The solution Set to a System of Linear Equations

The solution set to a system of lienar equations is exactly one of the following:

(a) empty (no solutions)

(b) contains exactly one element (unique solution)

(c) contains an infinite number of elements (has one or more parameters)

3.5 System Terminology

3.5.1 Inconsistent and Consistent Systems

If the solution set to a system of linear equations is empty, we say that the system is inconsistent.
If the solution set has a unique solution or infinitely many solutions, we say that the system is consistent

3.5.2 Equivalent Systems

We say that two linear systems are equivalent whenever they have the same solution set

3.6 Elementary Operations

Consider a system of m linear equations in n variables. The equations are ordered and labelled from e_1 to e_m. The following three operations are known as elementary operations.

- Equation swap (interchange equations e_i and e_j):

$$e_i \iff e_j$$

- Equation scale (replace equation e_i by m times e_i, $m \neq 0$)

$$e_i \to me_i, m \in \mathbb{R} \setminus 0$$

- Equation addition (replace e_j by e_j plus a multiple of e_i)

$$e_j \to ce_i + e_j$$

$$i \neq j, c \in \mathbb{F}$$

3.7 More Important Facts

3.7.1 Theorem - Elementary Operations

If a single elementary operation of any type is performed on a system of linear equations, then the system produced will be equivalent to the original system

3.7.2 Trivial Equation

We refer to the equation $0 = 0$ as the trivial equation. Any other equation is known as a non-trivial equation

3.8 Matrix

An $m \times n$ matrix, A, is a rectangular array of scalars with m rows and n columns. The scalar in the i^{th} row and j^{th} column is the $(i, j)^{th}$ entry and is denoted a_{ij} or A_{ij}. That is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

3.9 Coefficient Matrix

For a given system of linear equations,

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

the coefficients matrix, A, of the system is the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The entry a_{ij} is the coefficient of the variable x_j in the i^{th} equation

3.10 Augmented Matrix

For a given system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the augmented matrix, $[A|\vec{b}]$, of the system is

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

where $\vec{b}$ is a column whose entries are the constant terms on the right-hand side of the equations

3.11 Manipulating Rows

3.11.1 Elementary Row Operation (ERO)

Elementary Row Operations (EROs) are the operations performed on the coefficient and/or augmented matrix which correspond to the elementary operations performed on the system of equations

Row swap	$e_i \leftrightarrow e_j$	$R_i \leftrightarrow R_j$
Row scale	$e_i \to ce_i, c \neq 0$	$R_i \to cR_i, c \neq 0$
Row addition	$e_i \to ce_j + e_i, i \neq j$	$R_i \to cR_j + R_i, i \neq j$

3.11.2 Zero Row

We refer to a row that has all zero entries as a zero row

3.11.3 Row Equivalent

If a matrix B is obtained from a matrix A by a infinite number of EROs, then we say that B is row equivalent to A

3.12 REF

3.12.1 Leading Entry, Leading One

The leftmost non-zero entry in any non-zero row of a matrix is called the leading entry of that row. If the leading entry is a 1, that it is called a leading one.

3.12.2 Row Echelon Form

We say that a matrix is row echelon form (REF) whenever both of the following two conditions are satisfied:

1. All zero rows occur as the final rows in the matrix
2. The leading entry in any non-zero row appears in a column to the right of the columns containing the leading entried of any of the rows above it

We say that the matrix R is a row echelon form of matrix A to mean that R is in row echelon form and that R can be obtained from A by performing a finite number of $EROs$ to A

3.13 RREF

3.13.1 Pivot, Pivot Position, Pivot Column, Pivot Row

If a matrix is in REF, then the leading entries are referred to as pivots and their positions in the matrix are called pivot positions. Any column that contains a pivot position is called a pivot column. Any row that contains a pivot position is called a pivot row.

3.13.2 Reduced Row Echelon Form

We say that a matrix is in reduced row echelon form (RREF) whenever all of the following three conditions are satisfied:

1. It is in REF
2. All its pivots are leading ones
3. The only non-zero entry in a pivot column is the pivot itself

Important Result

3.13.3 Theorem - Unique RREF

Let A be a matrix with REFs R_1 and R_2. Then R_1 and R_2 will have the same set of pivot positions. Moreover, there is a unique matrix R such that R is the RREF of A

3.13.4 RREF(A)

We say that the matrix R is the reduced row echelon form of matrix A, and we write $R = RREF(A)$, if R is in reduced row echelon form and if R can be obtained from A by performing a finite number of EROs to A

3.14 Obtaining an REF

Algorithm

1. Conside the leftmost non-zero column of the matrix. Use EROs to obtain a leading entry in the top position of this column. This entr is now a pivot and this row is now a pivot row
2. Use EROs to change all other entries below the pivot in this pivot column to 0
3. Consider the submatrix formed by covering up the current pivot row and all previous pivot rows. If there are no more rows or if the only remaining rows are zero rows, we are finished. Otherwise, repeat steps 1 and 2 on the submatrix. Continus in this manner, convering up the current pivot row to obtain a matrix with one less row until remain or we obtain a submatrix with only zero rows

3.15 Obtaining the RREF from an REF

Algorithm

Start with a matrix in REF

1. Select the rightmost pivot column. If the pivot is not already 1, use EROs to change it to 1

2. Use EROs to change all entries above the pivot in this pivot column to 0

3. Consider the submatrix formed by covering up the current pivot row and all other rows below it. If there are no more rows, then we are finished. Otherwise, repeat steps 1 and 2 on the submatrix until no rows remain

3.16 Variables in RREF to a Solution

3.16.1 Basic Variable, Free Variable

Consider a system of linear equations. Let R be an REF of the coefficient matrix of this system. If the i^{th} column of this matrix contians a pivot, then we call x_i a **basic variable**. Otherwise, we call x_i a **free variable**

3.17 Characterizing Consistent Systems

3.17.1 Notation

We use $M_{m\times n}(\mathbb{F})$ to denote the set of all $m \times n$ matrices with entries from $\mathbb{F}$, or more specifically, sometimes we replace $\mathbb{F}$ with $\mathbb{R}$ or $\mathbb{C}$. If $m = n$ it is common to abbreviate $M_{m\times n}(\mathbb{F})$ as $M_n(\mathbb{F})$

3.17.2 Rank

Let $A \in M_{m\times n}(\mathbb{F})$ such that RREF(A) has exactly r pivots. Then we say that the rank of A is r, and we write $rank(A) = r$

3.17.3 Rank Bounds

If $A \in M_{m\times n}(\mathbb{F})$, then $rank(A) \leq min(m, n)$

3.17.4 Consistent System Test

Let A be the coefficient matrix of a system of linear equations and let $[A|\vec{b}]$ be the augmented matrix of the system. The system is consistent if and only if $rank(A) = rank([A|\vec{b}])$

3.18 System Rank Theorem

Let $A \in M_{m\times n}(\mathbb{F})$ with $rank(A) = r$

(a) Let $\vec{b} \in \mathbb{F}^m$. If the system of linear equations with augmented matrix $[A|\vec{b}]$ is consistent, then the solution set to this system will contain $n-r$ parameters

(b) The system with augmented matrix $[A|\vec{b}]$ is consistent for every $\vec{b} \in \mathbb{F}^m$ if and only if $r = m$

3.19 Definitions

3.19.1 Nullity

Let $A \in M_{m\times n}(\mathbb{F})$ with $\text{rank}(A) = r$. We define the nullity of A, written nullity(A), to be the integer $n - r$

3.19.2 Homogeneous and Non-homogeneous Syystems

We say that a system of linear equations is homogeneous if all the constant terms on the right-hand side of the equations are zero. Otherwise we say that the system is non-homogeneous

3.19.3 Trivial Solution

For a homogeneous system with variables $x_1, x_2, \cdots, x_n$, the trivial solution is the solution $x_1 = x_2 = \cdots = x_n = 0$

3.19.4 Nullspace

The solution set of a homogeneous system of linear equations with coefficient matrix A, written Null(A), is called the nullspace of A.

3.20 Matrix-Vector Multiplication

3.20.1 Row Vector

A row vector is a matrix with exactly one row. For a matrix $A \in M_{m\times n}(\mathbb{F})$, we will denote the i^{th} row of A by $r\vec{o}w_i(A)$. That is,

$$r\vec{o}w_i(A) = [a_{i1}\ a_{i2}\ \cdots\ a_{in}]$$

3.20.2 Matrix-Vector Multiplication in Terms of the Individual Entries

Let $A \in M_{m\times n}(\mathbb{F})$ and $\vec{x} \in \mathbb{F}^n$. We define the product $A\vec{x}$ as follows:

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

3.20.3 Linearity of Matrix-Vector Multiplication

Let $A \in M_{m\times n}(\mathbb{F})$. Let $\vec{x}, \vec{y} \in \mathbb{F}^n$ and $c \in \mathbb{F}$. Then

a $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

b $A(c\vec{x}) = cA\vec{x}$

3.21 Another Way of Representing Linear Systems of Equations

Example

The system of equations

$$\begin{aligned} 5x_1 + 4x_2 - 7x_3 + 2x_4 &= -1 \\ -6x_1 - 2x_2 + 3x_3 &= 2 \\ +3x_2 + 5x_3 + 5x_4 &= 7 \end{aligned}$$

can be represented as $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 5 & 4 & -7 & 2 \\ -6 & -2 & 3 & 0 \\ 0 & 3 & 5 & 5 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$$

3.22 Solve Homogeneous System

3.22.1 Particular Solution

Let $A\vec{x} = \vec{b}$ be a consistent system of linear equations. We refer to a solution of this system, $\vec{x_p}$ as a particular solution to this system

3.22.2 Associated Homogeneous System

Let $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$, be a non-homogeneous system of linear equations. The associated homogeneous system is the system $A\vec{x} = \vec{0}$

3.22.3 Solutions to $A\vec{x} = \vec{0} and A\vec{x} = \vec{b}$

Let $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$, be a consistent non-homogeneous system of linear equations with solution set T. Let $A\vec{x} = \vec{0}$ be the associated homogeneous system with solution set S. If $\vec{y} \in T$, then $T = \{\vec{w} + \vec{y} : \vec{w} \in S\}$

4 Matrices

4.1 Column Space

Let $A \in M_{m\times n}$, column space of A is $Col(A) = \text{span}\{\vec{a_1}, \vec{a_2}, \cdots, \vec{a_n}\}$
Let $A \in M_{m\times n}(\mathbb{F})$ and $\vec{b} \in \mathbb{F}^m$. The system of linear equations $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b} \in Col(A)$

4.2 Transpose

Let $A \in M_{m\times n}$. Define A^T by $(A^T)_{ij} = (A)_{ji}$, $A^T \in M_{n\times m}$

4.3 Row Space

Let $A \in M_{m\times n}(\mathbb{F})$. We define $Row(A)$ to the span of the transposed rows of A. $Row(A) = \text{span}\{(r\vec{o}w_1(A))^T, \cdots, (r\vec{o}w_n(A))^T\}$

4.4 Column Extraction Lemma

Let $A = [\vec{a_1}, \vec{a_2}, \cdots, \vec{a_n}] \in M_{m\times n}(\mathbb{F})$. Then $A\vec{e_i} = \vec{a_i}$ for all $i = 1, \cdots, n$

4.5 Matrix Equality

Let $A, B \in M_{m\times n}(\mathbb{F})$. Then $A = B$ if and only if $a_{ij} = b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$

4.6 Equaliity of Matrices

Let $A, B \in M_{m\times n}(\mathbb{F})$. Then $A = B$ if and only if $A\vec{x} = B\vec{x}$ for all $\vec{x} \in \mathbb{F}^n$

4.7 Matrix Multiplication

Let $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{m\times p}(\mathbb{F})$. We define $AB = C$ to be the matrix $C \in M_{n\times p}(\mathbb{F})$, constructed as:

$$C = AB = A[\vec{b_1}, \vec{b_2}, \cdots, \vec{b_p}] = [A\vec{b_1}, A\vec{b_2}, \cdots, A\vec{b_p}]$$

That is, j^{th} column C, $\vec{c_j}$, is obtained by multiplying the matrix A by the j^{th} column of the matrix B:

$$\vec{c_j} = A\vec{b_j} \text{ for } j = 1, \cdots, p$$

4.8 Matrix Addition

Let $A, B \in M_{m\times n}(\mathbb{F})$. We define $A + B = C$ to be the matrix $C \in M_{m\times n}(\mathbb{F})$ whose $(i, j)^{th}$ entry is

$$c_{ij} = a_{ij} + b_{ij} \text{ for all } i = 1, \cdots, m \text{ and } j = 1, \cdots, n$$

Matrix Addition

If $A, B, C \in M_{m\times n}(\mathbb{F})$

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C) = A + B + C$

Proposition of Matrix Multiplication

If $A, B \in M_{m\times n}(\mathbb{F})$, $C, D \in M_{n\times p}(\mathbb{F})$ and $E \in M_{p\times q}(\mathbb{F})$

1. $(A + B)C = AC + BC$
2. $A(C + D) = AC + AD$
3. $(AC)E = A(CE) = ACE$

4.9 Additive Inverse

If $A \in M_{m\times n}(\mathbb{F})$. We define the additive inverse of A to be the matrix $-A$ whose ij^{th} entry is $-a_{ij}$ for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$

4.10 Zero Matrix

The $m \times n$ zero matrix, all of those entries are 0, is denoted $0_{m\times n} \in M_{m\times n}(\mathbb{F})$

Properties of the Additive Inverse and the Zero Matrix

If $A \in M_{m\times n}(\mathbb{F})$, $0 \in M_{m\times n}(\mathbb{F})$

1. $0 + A = A + 0 = A$
2. $A + (-A) = (-A) + A = 0$

4.11 Multiplication of a matrix by a scalar

Let $A \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$. We define the matrix $cA \in M_{m\times n}(\mathbb{F})$ whose $(i, j)^{th}$ entry is $(cA)_{ij} = c(A_{ij}) = ca_{ij}$ for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$

Properties of Multiplication of a Matrix by a Scalar

If $A, B \in M_{m\times n}(\mathbb{F})$, $C \in M_{n\times k}(\mathbb{F})$, and $r, s \in \mathbb{F}$

1. $s(A + B) = sA + sB$
2. $(r + s)A = rA + sA$
3. $r(sA) = (rs)A$
4. $s(AC) = (sA)C = A(sC)$

4.12 Properties of the Matrix Transpose

If $A, B \in M_{m\times n}(\mathbb{F})$, $C \in M_{n\times k}(\mathbb{F})$, and $s \in \mathbb{F}$

1. $(A+B)^T = A^T + B^T$
2. $(sA)^T = s(A^T)$
3. $(AC)^T = C^T A^T$
4. $(A^T)^T = A$

4.13 Elementary Matrices

4.13.1 Definition

A matrix that can be obtained by performing a single ERO on the identity matrix is called an elementary matrix

4.13.2 Proposition

Let $A \in M_{m\times n}(\mathbb{F})$ and suppose that a single ERO is performed on it to produce matrix B. Suppose, also, that we perform the same EOR on the matrix I_m to produce the elementary matrix E. Then

$$B = EA$$

4.13.3 Corollary

Let $A \in M_{m\times n}(\mathbb{F})$ and suppose that a finite number of EROs, numbered 1 through k, are performed on A to produce a matrix B. Let E_i denote the elementary matrix corresponding to the i^{th} ERO ($1 \leq i \leq k$) applied to I_m. Then

$$B = E_k \cdots E_2 E_1 A$$

4.14 Invertibility

4.14.1 Invertible Matrix

We say that an $n \times n$ matrix A is invertible if there exist $n \times n$ matrices B and C such that $AB = CA = I_n$

4.14.2 Proposition: Equality of Left and Right Inverses

Let $A \in M_{m\times n}(\mathbb{F})$. If there exists matrices B and C in $M_{n\times n}(\mathbb{F})$ such that $AB = CA = I_n$, then $B = C$

4.14.3 Theorem: Left Invertible Iff Right Invertible

For $A \in M_{m\times n}(\mathbb{F})$, there exists an $n \times n$ matrix B such that $AB = I_n$ iff there exists an $n \times n$ matrix C such that $CA = I_n$

4.14.4 Inverse of a Matrix

If an $n\times n$ matrix A is invertible, we refer to the matrix B such that $AB = I_n$ as the inverse of A. We denote the inverse of A by A^{-1}. The inverse of A satisfies

$$AA^{-1} = A^{-1}A = I_n$$

4.14.5 Theorem: Invertibility Criteria - First Version

Let $A \in M_{n\times n}(\mathbb{F})$. The following three conditions are equivalent:

(a) A is invertible

(b) rank(A) = n

(c) RREF(A) = I_n

4.15 Algorithm for Checking Invertibility and Finding the Inverse

The following algorithm allows you to determine whether an $n \times n$ matrix A is invertible, and it is, the algorithm will provide the inverse of A

1. Construct a super-augmented matrix $[A|I_n]$
2. Find the RREF, $[R|B]$, of $[A|I_n]$
3. If $R \neq I_n$, conclude that A is not invertible. If $R = I_n$, conclude that A is invertible, and that $A^{-1} = B$

4.15.1 Proposition: Inverse of a 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if $ad - bc \neq 0$. Furthurmore, if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

5 Chapter 5 - Linear Transformation

5.1 From Matrices to Functions

5.1.1 Function Determined by a Matrix

Let $A \in M_{m \times n}(\mathbb{F})$. The function determined by the matrix A is the function

$$T_A : \mathbb{F}^n \Rightarrow \mathbb{F}^m$$

defined by

$$T_A(\vec{x}) = A\vec{x}$$

5.1.2 Function Determined by a Matrix is Linear

Let $A \in M_{m \times n}(\mathbb{F})$ and let T_A be the function determined by the matrix A. Then T_A is linear; that is, for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$, the following two properties hold:

- $T_A(\vec{x} + \vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$
- $T_A(c\vec{x}) = cT_A(\vec{x})$

5.2 Linear Transformation - Definition

5.2.1 Linear Transformation

We say that the function T: $\mathbb{F}^n \Rightarrow \mathbb{F}^m$ is a linear transformation if, for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$, the following two properties hold:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (linearity over addition)
2. $T(c\vec{x}) = cT(\vec{x})$ (linearity over scalar multiplication)

- We will call T_A the linear transformation determined by A
- We refer to $\mathbb{F}^n$ here as the domain of T and $\mathbb{F}^m$ as the codomain of T, as we would for any function
- Sometimes we use the term linear mapping instead

5.2.2 Alternative Characterization of a Linear Transformation

Let T: $\mathbb{F}^n \Rightarrow \mathbb{F}^m$ be a function. Then T is a linear transformation if and only if for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$

$$T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y})$$

5.2.3 Zero Maps to Zero

Let T: $\mathbb{F}^n \Rightarrow \mathbb{F}^m$ be a linear transformation. Then

$$T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$$

5.3 Range

5.3.1 Definition

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the range of T, enoted Range(T), to be the set of all outputs of T. That is,

$$\text{Range(T)} = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}$$

The range of T is a subset of $\mathbb{F}^m$

5.3.2 Range of a Linear Transformation

Let $A \in M_{m\times n}(\mathbb{F})$, and let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be the lienar transformation determined by A. Then

$$\text{Range}(T_A) = \text{Col}(A)$$

5.4 Onto

5.4.1 Definition

We say that the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is onto (or surjective) if Range(T)$= \mathbb{F}^m$

5.4.2 Onto Criteria

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent:

(a) T_A is onto

(b) Col(A)$= \mathbb{F}^m$

(c) rank(A)$= m$

5.5 Kernel

5.5.1 Definition

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the kernel of T, denoted Ker(T), to be the set of inputs of T whose output is zero. That is

$$\text{Ker(T)} = \{\vec{x} \in \mathbb{F}^n : T(\vec{x}) = \vec{0}_{\mathbb{F}^m}\}$$

The kernel of T is a subset of $\mathbb{F}^n$

5.5.2 Kernel of a Linear Transformation

Let $A \in M_{m\times n}(\mathbb{F})$ and let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by A. Then

$$\text{Ker}(T_A) = \text{Null(A)}$$

5.6 One-to-One

5.6.1 Definition

We say that the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is one-to-one (or injective) if, whenever $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$

5.6.2 One-to-One Test

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then

$$\text{T is one-to-one if and only if Ker(T)} = \{\vec{0}_{\mathbb{F}^n}\}$$

5.6.3 One-to-One Criteria

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent

1. T_A is one-to-one
2. Null(A) $= \{\vec{0}_{\mathbb{F}^n}\}$
3. nullity(A) $= 0$
4. rank(A) $=$ n

5.7 Invertibility Criteria (More)

Let $A \in M_{m\times n}(\mathbb{F})$ be a square matrix and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent

1. A is invertible
2. T_A is one-to-one
3. T_A is onto
4. Null(A) $= \{\vec{0}\}$. That is, the only solution to the homogeneous system $A\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$
5. Col(A) $= \mathbb{F}^n$. That is, for every $\vec{b} \in \mathbb{F}^n$, the system $A\vec{x} = \vec{b}$ is consistent
6. nullity(A) $= 0$
7. rank(A) $=$ n
8. RREF(A) $= I_n$

5.8 Standard Matrix

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the standard matrix of T, denoted by $[T]_\epsilon$, to be $m \times n$ matrix whose columns are the images under T of the vectors in the standard basis of $\mathbb{F}^n$

$$
\begin{aligned}
[T]_\epsilon &= [T(\vec{e_1})\ T(\vec{e_2})\ \cdots\ T(\vec{e_n})] \\
&= [T(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix})\ T(\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix})\ \cdots\ T(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix})]
\end{aligned}
$$

Remarkable Result

5.8.1 Every Linear Transformation is Determined by a matrix

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation and let $[T]_\epsilon$ be the standard matrix of T. Then for all $\vec{x} \in \mathbb{F}^n$

$$T(\vec{x}) = [T]_\epsilon \vec{x}$$

That is, $T = T_{[T]_\epsilon}$ is the linear transformation determined by the matrix $[T]_\epsilon$

5.8.2 Proposition 5.5.3 (Corollary)

Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Then there is a real number $m \in \mathbb{R}$ such that $T(x) = mx$ for all $x \in \mathbb{R}$

5.9 Properties of Standard Matrices

Let $A \in M_{m\times n}(\mathbb{F})$, let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by A, and let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation

1. $T_{[T]_\epsilon} = T$
2. $[T_A]_\epsilon = A$
3. T is onto if and only if rank($[T]_\epsilon$) = m
4. T is one-to-one if and only if rank($[T]_\epsilon$) = n

5.10 Composition of Linear Transformations

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. We define the function $T_2 \circ T_1 : \mathbb{F}^n \to \mathbb{F}^p$ by

$$(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$$

The function $T_2 \circ T_1$ is called the composite function of T_2 and T_1

5.11 Result

5.11.1 Composition of Linear Transformation is Linear

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. Then $T_2 \circ T_1$ is a linear transformation

5.11.2 The Standard Matrix of a Composition of Linear Transformations

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. Then the standard matrix of $T_2 \circ T_1$ is equal to the product of standard matrices of T_2 and T_1. That is

$$[T_2 \circ T_1]_\epsilon = [T_2]_\epsilon [T_1]_\epsilon$$

5.12 Special Cases

5.12.1 Identitty Transformation

The linear transformation $id_n : \mathbb{F}^n \to \mathbb{F}^n$ such that $id_n(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{F}^n$ is called the identity transformation

5.12.2 T^P

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ and let $p \geq 0$ be an integer. We recursively define the p^{th} power of T, denoted T^P as follows:

$$T^0 = id_n \text{ and for } p > 0, T^P = T \circ T^{P-1}$$

5.12.3 Corollary

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation and let $p > 1$ be an integer. Then the standard matrix of T^P is the p^{th} power of the standard matrix of T. That is

$$[T^P]_\epsilon = ([T]_\epsilon)^p$$

6 Chapter 6 - Determinant

6.1 Definition of Determinant

6.1.1 Determinant of a 1×1 and 2×2 Matrix

Let $A = [a_{11}] \in M_{1\times 1}(\mathbb{F})$
The determinant of A, denoted by det(A), is

$$det(A) = a_{11}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}(\mathbb{F})$
The determinant of A is

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

6.1.2 $(i,j)^{th}$ Submatrix, $(i,j)^{th}$ Minor

Let $A \in M_{n\times n}(\mathbb{F})$
The $(i,j)^{th}$ submatrix of A, denoted by $M_{ij}(A)$ is the $(n-1)\times(n-1)$ matrix obtained from A by removing the i^{th} row and the j^{th} column from A
The determinant of $M_{ij}(A)$ is known as the $(i,j)^{th}$ minor of A

6.1.3 Determinant of an $n \times n$ Matrix

Let $A \in M_{n\times n}(\mathbb{F})$ for $n \geq 2$
The determinant function, $det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ by

$$det(A) = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} det(M_{1j}(A))$$

6.1.4 i^{th} Row Expansion of the Determinant

Let $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ and let $i \in \{1, \cdots, n\}$. Then

$$det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} det(M_{ij}(A))$$

6.1.5 j^{th} Column Expansion of the Determinant

Let $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ and let $j \in \{1, \cdots, n\}$. Then

$$det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} det(M_{ij}(A))$$

6.1.6 Easy Determinants for a Square Matrix $A \in M_{n\times n}(\mathbb{F})$

- If A has a row consisting only of zeros, then det(A) = 0
- If A has a column consisting only of zeros, then det(A) = 0
- If $A = \begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is upper triangular (also works for lower triangular), $det(A) = a_{11}a_{22}\cdots a_{nn}$
- $det(I_n) = 1$
- Let $A \in M_{n\times n}(\mathbb{F})$. $det(A) = det(A^T)$

6.2 Elementary Row Operations and the Determinant

Theorem

Let $A \in M_{n\times n}(\mathbb{F})$

1. (Row sawp) If B is obtained from A by interchanging two rows, then det(B) = -det(A)
2. (Row scale) If B is obtained from A by multiplying a row by $m \neq 0$, then det(B) = m det(A)

3. (Row addition) If B is obtained from A by adding a non-zero multiple of one row to another row, then det(B) = det(A)

   remains true if instances of "row" replaced with "column"

4. If A has two identical rows (or two identical columns), then det(A) = 0

5. Determinants of Elementary Matrices
   If E is the elementary matrix of the indicated type, then

   (a) (Row swap) det(E) = -1
   (b) (Row scale by $m \neq 0$) det(E) = m
   (c) (Row addition) det(E) = 1

6. Determinant after one ERO
   Suppose we perform a single ERO on A to produce the matrix B where the corresponding elementary matrix is E. Then det(B) = det(E)det(A)

7. Determinant after k EROs
   Suppose we perform a sequence of k EROs on the matrix A to obtain the matrix B. Suppose that the elementary matrix corresponding too the ith ERO is E_i, so
   $$B = E_k \cdots E_2 E_1 A$$
   Then
   $$det(B) = det(E_k \cdots E_2 E_1 A) = det(E_k) \cdots det(E_2) det(E_1) det(A)$$

6.3 The Determinant and Invertibility

1. Invertible iff the Determinant is Non-Zero
   Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible if and only if $det(A) \neq 0$

2. Determinant of a Product
   Let $A, B \in M_{n\times n}(\mathbb{F})$. Then $det(AB) = det(A)det(B)$

3. Corollary
   Let $A, B \in M_{n\times n}(\mathbb{F})$. Then $det(AB) = det(BA)$

4. Determinant of Inverse
   Let $A \in M_{n\times n}(\mathbb{F})$ be invertible. Then $det(A^{-1}) = \dfrac{1}{det(A)}$

6.4 An Expression for A^{-1}

6.4.1 Cofactor

Let $A \in M_{n\times n}(\mathbb{F})$. The $(i,j)^{th}$ cofactor of A, denoted by $C_{ij}(A)$ is defined by

$$C_{ij}(A) = (-1)^{i+j}det(M_{ij}(A))$$

6.4.2 Adjugate of a Matrix

Let $A \in M_{n\times n}(\mathbb{F})$. The adjugate of A, denoted by $adj(A)$, is the $n \times n$ matrix whose $(i,j)^{th}$ entry is

$$(adj(A))_{ij} = C_{ij}(A)$$

The conjugate of A is the transpose of the matrix of cofactors of A

6.4.3 Theorem

Let $A \in M_{n\times n}(\mathbb{F})$. Then

$$\text{A } adj(A) = adj(A)A = det(A)I_n$$

6.4.4 Inverse by Adjugate

Let $A \in M_{n\times n}(\mathbb{F})$. If $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(a)}adj(A)$$

6.4.5 Cramer's Rule

Let $A \in M_{n\times n}(\mathbb{F})$ and consider the equation $A\vec{x} = \vec{b}$, where $\vec{b} \in \mathbb{F}^n$ and $det(A) \neq 0$
If B_j is constructed from A by replacing the j^{th} column of A by the column vector $\vec{b}$, then the solution $\vec{x}$ to the equation

$$A\vec{x} = \vec{b}$$

is given by

$$x_j = \frac{det(B_j)}{det(A)}, \quad \text{for all } j = 1, \cdots, n$$

6.4.6 Determinant and Geometry

Area of Parallelogram

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$
The are of the parallelogram with sides $\vec{v}$ and $\vec{w}$ is

$$|det(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix})|$$

7 Chapter 7 - Eigenvalues and Diagonalization

7.1 Eigenvector, Eigenvalue

7.1.1 Eigenvector, Eigenvalue and Eigenpair

Let $A \in M_{n\times n}(\mathbb{F})$. A non-zero vector $\vec{x}$ is an eigenvector of A over $\mathbb{F}$ if there exists a scalar $\lambda \in \mathbb{F}$ such that

$$A\vec{x} = \lambda\vec{x}$$

The scalar λ is called an eigenvalue of A over $\mathbb{F}$, and the pair $(\lambda, \vec{x})$ is an eigenpair of A over $\mathbb{F}$

7.1.2 Eigenvalue Equation or Eigenvalue Problem

Let $A \in M_{n\times n}(\mathbb{F})$. We refer ro the equation

$$A\vec{x} = \lambda\vec{x} \quad \text{or} \quad (A - \lambda I)\vec{x} = \vec{0}$$

as the eigenvalue equation for the matrix A over $\mathbb{F}$. It is also sometimes referred to as the eigenvalue problem

7.1.3 Characteristic Polynomial and Characteristic Equation

Let $A \in M_{n\times n}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. The characteristic polynomial of A, denoted by $C_A(\lambda)$, is

$$C_A(\lambda) = det(A - \lambda I)$$

The characteristic equation of A is

$$C_A(\lambda) = 0$$

7.2 Trace

7.2.1 Definition

Let $A \in M_{n\times n}(\mathbb{F})$. We define the trace of A by

$$tr(A) = \sum_{i=1}^{\infty} a_{ii}$$

That is, the trace of a square matrix is the sum of its diagonal entries

7.3 Properties of the Characteristic Polynomial

7.3.1 Features of the Characteristic Polynomial

Let $A \in M_{n\times n}(\mathbb{F})$ have characteristic polynomial $C_A(\lambda) = det(A - \lambda I)$. Then $C_A(\lambda)$ is a degree n polynomial in λ of the form

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

where

1. $c_n = (-1)^n$
2. $c_{n-1} = (-1)^{(n-1)}tr(A)$
3. $c_0 = det(A)$

7.4 Connecting to Eigenvalues

7.4.1 Characteristic Polynomial and Eigenvalues over $\mathbb{C}$

Let $A \in M_{n\times n}(\mathbb{F})$ have characteristic polynomial

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

and n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (possibly repeated) in $\mathbb{C}$. Then

1. $c_{n-1} = (-1)^{(n-1)} \displaystyle\sum_{i=1}^{n} \lambda_i$
2. $c_0 = \prod_{i=1}^{n} \lambda_i$

7.4.2 Eigenvalues and Trace / Determinant

Let $A \in M_{n\times n}(\mathbb{F})$ have n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (possibly repeated) in $\mathbb{C}$. Then

1. $\displaystyle\sum_{i=1}^{n} \lambda_i = tr(A)$

2. $\displaystyle\prod_{i=1}^{n} \lambda_i = det(A)$

7.5 Eigenspaces

7.5.1 Linear Combinations of Eigenvectors

Let $c, d \in \mathbb{F}$ and suppose that $(\lambda_1, \vec{x})$ and $(\lambda_1, \vec{y})$ are eigenpairs of a matrix A over $\mathbb{F}$ with the same eigenvalue λ_1. If $c\vec{x} + d\vec{y} \neq \vec{0}$, then $(\lambda_1, c\vec{x} + d\vec{y})$ is also an eigenpair for A with eigenvalue λ_1

7.5.2 Eigenspace

Let $A \in M_{n\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. The eigenspace of A associated with λ, denoted by $E_\lambda(A)$, is the solution set to the system $(A - \lambda I)\vec{x} = \vec{0}$ over $\mathbb{F}$. That is

$$E_\lambda(A) = Null(A - \lambda I)$$

If the choice of A is clear, we abbreviate this as E_λ

7.6 Motivating Fact and Definition

7.6.1 Fact

If $D = diag(d_1, \cdots, d_n) \in M_{n\times n}(\mathbb{F})$, then $D^k = diag(d_1^k, \cdots, d_n^k)$ for all $k \in \mathbb{N}$

7.6.2 Similar

Let $A, B \in M_{n\times n}(\mathbb{F})$. We say that A is similar to B over $\mathbb{F}$ if there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $P^{-1}AP = D$ We say that the matrix P diagonalizes A

7.7 Proposition

7.7.1 n Distinctt Eigenvalues $\Rightarrow$ Diagonalizable

If $A \in M_{n\times n}(\mathbb{F})$ has n distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ in $\mathbb{F}$, then A is diagonalizable over $\mathbb{F}$

More specifically, if we let $(\lambda_1, \vec{v_1}), (\lambda_2, \vec{v_2}), \cdots, (\lambda_n, \vec{v_n})$ be eigenpairs of A over $\mathbb{F}$, and if we let $P = [\vec{v_1}\vec{v_2}\cdots\vec{v_n}]$ be the matrix whose columns are eigenvectors corresponding to the distinct eigenvalues, then

1. P is invertible
2. $P^{-1}AP = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$

8 Chapter 8 - Subspaces and Bases

8.1 Subspace

8.1.1 Definition

A subset V of $\mathbb{F}^n$ is called a subspace of $\mathbb{F}^n$ if the following properties are all satisfied

1. $\vec{0} \in V$
2. For all $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$
3. For all $\vec{x} \in V$ and $c \in \mathbb{F}$, $c\vec{x} \in V$

8.1.2 Proposition: Examples of Subspaces

1. $\{\vec{0}\}$ and $\mathbb{F}^n$ are subspaces of $\mathbb{F}^n$
2. If $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is a subset of $\mathbb{F}^n$, then $Span\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is a subspace of $\mathbb{F}^n$
3. If $A \in M_{m\times n}(\mathbb{F})$, then the solution set to the homogeneous system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{F}^n$
4. If $A \in M_{m\times n}(\mathbb{F})$, then $Col(A)$ is a subspace of $\mathbb{F}^m$

5. If $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation, then the range of T, $Range(T)$, is a subspace of $\mathbb{F}^m$

6. If $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation, then the kernel of T, $Ker(T)$, is a subspace of $\mathbb{F}^n$

7. If $A \in M_{n\times n}(\mathbb{F})$ and if $\lambda \in \mathbb{F}$, then the eigenspace E_λ is a subspace of $\mathbb{F}^n$

8.1.3 Subspace Test

Let V be a subset of $\mathbb{F}^n$. Then V is a subspace of $\mathbb{F}^n$ if and only if

1. V is non-empty

2. for all $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{F}$, $c\vec{x} + \vec{y} \in V$

8.2 Linear Dependence

The vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k} \in \mathbb{F}^n$ are linearly dependent if there exists scalars $c_1, c_2, \cdots, c_k \in \mathbb{F}$, not all zero, such that $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}$
If $U = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$, then we say thhat the seet U is a linearly dependent set (or simply that U is linearly dependent) to mean that the vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly dependent

8.3 Linear Independence

The vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k} \in \mathbb{F}^n$ are linearly independent if there do not exist scalars $c_1, c_2, \cdots, c_k \in \mathbb{F}$, not all zero, such that $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}$
Equivalently we say that $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k} \in \mathbb{F}^n$ are linearly independent if the only solution to the equation

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}$$

is the trivial solution $c_1 = c_2 = \cdots = c_k = 0$
If $U = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$, then we say thhat the seet U is a linearly independent set (or simply that U is linearly independent) to mean that the vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly independent

8.4 Basis

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a finite set of vectors contained in V. We say that $\mathcal{B}$ is a basis for V if

1. $\mathcal{B}$ is linearly independent
2. $V = Span(\mathcal{B})$

8.5 Linear Dependence Proposition

8.5.1 Proposition 8.3.1

1. The vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly dependent if and only if one of the vectors can be written as a linear combination of some of the other vectors
2. The vecotrs $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly independent if and only if

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0} \quad (c_i \in \mathbb{F}) \text{ implies } c_1 = \cdots = c_k = 0$$

8.5.2 Proposition 8.3.2

Let $S \subseteq \mathbb{F}^n$

1. If $\vec{0} \in S$, then S is linearly dependent
2. If $S = \{\vec{x}\}$ contains only one vector, then S is linearly dependent if and only if $\vec{x} = \vec{0}$
3. If $S = \{\vec{x}, \vec{y}\}$ contains only two vectors, then S is linearly dependent if and only if one of the vectors is a multiple of the other

8.6 Pivots and Linear Independence

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$
Let A $= [\vec{v_1} \quad \vec{v_2} \quad \cdots \quad \vec{v_k}]$ be the $n \times k$ matrix whose columns are the vectors in S
Suppose $rank(A) = r$ and A has pivots in columns $q_1, q_2, \cdots, q_r$
Let $U = \{\vec{v_{q1}}, \vec{v_{q2}}, \cdots, \vec{v_{qr}}\}$, the set of columns of A that correspond to the pivot columns labelled above
Then

1. S is linearly independent if and only if $r = k$

2. U is linearly independent

3. If $\vec{v}$ is in S but not in U then the set $\{\vec{v_{q1}}, \vec{v_{q2}}, \cdots, \vec{v_{qr}}, \vec{v}\}$ is linearly dependent

4. $Span(U) = Span(S)$

8.6.1 Bound on Number of Linearly Independent Vectors

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$
If $n < k$, then S is linearly dependent

8.7 Subspaces and Spanning Sets

8.7.1 Every Subspace Has a Spanning Set

Let V be a subspace of $\mathbb{F}^n$. Then there exist vectors $\vec{v_1}, \cdots, \vec{v_k} \in V$ such that

$$V = Span\{\vec{v_1}, \cdots, \vec{v_k}\}$$

8.7.2 Every Subspace Has a Basis

Let V be a subspace of $\mathbb{F}^n$. Then V has a basis

8.7.3 Span of Subset

Let V be a subspace of $\mathbb{F}^n$ and let $S = \{\vec{v_1}, \cdots, \vec{v_k}\} \subseteq V$. Then $Span(S) \subseteq V$

8.7.4 Spans $\mathbb{F}^n$ iff rank is n

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$
Let A $= [\vec{v_1} \quad \vec{v_2} \quad \cdots \quad \vec{v_k}]$ be the matrix whose columns are the vectors in S. Then

$$Span(S) = \mathbb{F}^n \text{ if and only if } rank(A) = n$$

8.7.5 Size of Basis for $\mathbb{F}^n$

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. If S is a basis for $\mathbb{F}^n$, then $k = n$

8.7.6 n Vectors in $\mathbb{F}^n$ Span iff Independent

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a set of n vectors in $\mathbb{F}^n$. Then S is linearly independent if and only if $Span(S) = \mathbb{F}^n$

8.7.7 Bass From a Spanning Set of Linearly Independent Set

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a subset of $\mathbb{F}^n$

a) If $Span(S) = \mathbb{F}^n$, then there exists a subset $\mathcal{B}$ of S which is a basis for $\mathbb{F}^n$

b) If $Span(S) \neq \mathbb{F}^n$ and S is linearly independent, then therre exist vectors $\vec{v}_{k+1}, \cdots, \vec{v}_n$ in $\mathbb{F}^n$ such that $\mathcal{B} = \{\vec{v_1}, \cdots, \vec{v_k}, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$

8.8 Bases for $Col(A)$ and $Null(A)$

8.8.1 Basis for $Col(A)$

Let $A = [\vec{a_1} \quad \cdots \quad \vec{a_n}] \in M_{m\times n}(\mathbb{F})$ and suppose that $RREF(A)$ has pivots in columns $q_1, \cdots, q_r$, where $r = rank(A)$. Then $\{\vec{a_{q1}}, \cdots, \vec{a_{qr}}\}$ is a basis for $Col(A)$

8.8.2 Basis for $Null(A)$

Let $A \in M_{m\times n}(\mathbb{F})$ and consider the homogeneous linear system $A\vec{x} = \vec{0}$. Suppose that, after applying the Gauss-Jordan Algorithm, we obtain k free parameters so that the solution set to this system is given by

$$Null(A) = \{t_1\vec{x_1} + \cdots + t_k\vec{x_k} : t_1, \cdots, t_k \in \mathbb{F}\}$$

Here, $k = nullity(A) = n - rank(A)$ and the parameters t_i and the vectors $\vec{x_i}$ for $1 \leq i \leq k$ are obtained using the method outlined earlier. Then $\{\vec{x_1}, \cdots, \vec{x_k}\}$ is a basis for $Null(A)$

8.9 Dimension

8.9.1 Definition

The number of elements in a basis for a subspace V of $\mathbb{F}^n$ is called the dimension of V. We denote this number by $dim(V)$

8.9.2 Well-deinfed Dimension

Let V be a subspace of $\mathbb{F}^n$. If $\mathcal{B} = \{\vec{v_1}, \cdots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \cdots, \vec{w_l}\}$ are bases for V, then $k = l$

8.10 Dimension Propositions

8.10.1 Bound on Dimension of Subspace

Let V be a subspace of $\mathbb{F}^n$. Then $dim(V) \leq n$

8.10.2 Rank and Nullity as Dimensions

Let $A \in M_{m\times n}(\mathbb{F})$. Then

1. $rank(A) = dim(Col(A))$
2. $nullity(A) = dim(Null(A))$

8.10.3 Rank-Nullity Theorem

Let $A \in M_{m\times n}(\mathbb{F})$. Then

$$\begin{aligned} n &= rank(A) + nullity(A) \\ &= dim(Col(A)) + dim(Null(A)) \end{aligned}$$

8.11 Basis

8.11.1 Unique Representation Theorem

Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a basis for $\mathbb{F}^n$. Then, for every vector $\vec{v} \in \mathbb{F}^n$, there exists unique scalars $c_1, c_2, \cdots, c_n \in \mathbb{F}$ such that

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$$

8.11.2 Coordinates and Components

Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a basis for $\mathbb{F}^n$. Let the vector $\vec{v} \in \mathbb{F}^n$ have representation

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \sum_{i=1}^{n} c_i\vec{v_i} \quad (c_i \in \mathbb{F})$$

We call the scalars $c_1, c_2, \cdots, c_n$ the coordinates (or components) of $\vec{v}$ with respect to $\mathcal{B}$, or the $\mathcal{B}$-coordinates of $\vec{v}$

8.11.3 Coordinate Vector

Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. Let $\vec{v} \in \mathbb{F}^n$ have coordinates $c_1, c_2, \cdots, c_n$ with respect to $\mathcal{B}$, where the ordering of the scalars c_i matches the ordering in $\mathcal{B}$, that is

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{v_i}$$

Then the coordinates vector of $\vec{v}$ with respect to $\mathcal{B}$ (or the $\mathcal{B}$-coordinates vector of $\vec{v}$) is the column vector in $\mathbb{F}^n$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

8.12 Linearity

8.12.1 Linearity of Taking Coordinates

Let $\mathcal{B} = \{\vec{v_1}, \cdots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. Then the function $[\quad]_{\mathcal{B}} : \mathbb{F}^n \to \mathbb{F}^n$ defined by sending $\vec{x}$ to $[\vec{x}]_{\mathcal{B}}$ is linear:

1. For all $\vec{v}$, $\vec{u} \in V$, $[\vec{v} + \vec{u}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} + [\vec{u}]_{\mathcal{B}}$
2. For all $\vec{v} \in V$ and $c \in \mathbb{F}$, $[c\vec{v}]_{\mathcal{B}} = c[\vec{v}]_{\mathcal{B}}$

8.13 Change of Basis

8.13.1 Change-of-Basis Matrix, Change-of-Coordinate Matrix

Let $\mathcal{B} = \{\vec{v_1}, \cdots, \vec{v_n}\}$ and $\mathcal{C} = \{\vec{w_1}, \cdots, \vec{w_n}\}$ be ordered bases for $\mathbb{F}^n$
The *change-of-basis* (or *change-of-coordinates*) matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is the $n \times n$ matrix

$${}_{\mathcal{C}}[I]_{\mathcal{B}} = [[\vec{v_1}]_{\mathcal{C}}, \cdots, [\vec{v_n}]_{\mathcal{C}}]$$

whose columns are the $\mathcal{C}$-coordinates of the vectors $\vec{v_i}$ in $\mathcal{B}$

8.13.2 Changing a Basis

Let $\mathcal{B} = \{\vec{v_1}, \cdots, \vec{v_n}\}$ and $\mathcal{C} = \{\vec{w_1}, \cdots, \vec{w_n}\}$ be ordered bases for $\mathbb{F}^n$. Then $[\vec{x}]_\mathcal{C} =_\mathcal{C} [I]_\mathcal{B}[\vec{x}]_\mathcal{B}$ and $[\vec{x}]_\mathcal{B} =_\mathcal{B} [I]_\mathcal{C}[\vec{x}]_\mathcal{C}$ for all $\vec{x} \in \mathbb{F}^n$

8.13.3 Corollary

Let $\vec{x} = [\vec{x}]_\epsilon$ be a vector in $\mathbb{F}^n$, where ϵ is the standard basis for $\mathbb{F}^n$. If $\mathcal{C}$ is any ordered basis for $\mathbb{F}^n$ then $[\vec{x}]_\mathcal{C} =_\mathcal{C} [I]_\epsilon[\vec{x}]_\epsilon$

8.13.4 Inverse of Change-of-Basis Matrix

Let $\mathcal{B}$ and $\mathcal{C}$ be two ordered bases of $\mathbb{F}^n$. Then

$$_\mathcal{B}[I]_{\mathcal{C}\mathcal{C}}[I]_\mathcal{B} = I_n \quad \text{and} \quad _\mathcal{C}[I]_{\mathcal{B}\mathcal{B}}[I]_\mathcal{C} = I_n$$

In other words, $_\mathcal{B}[I]_\mathcal{C} = (_\mathcal{C}[I]_\mathcal{B})^{-1}$ and $_\mathcal{C}[I]_\mathcal{B} = (_\mathcal{B}[I]_\mathcal{C})^{-1}$

9 Chapter 9 - Diagonalization

9.1 Linear Operators

A linear Transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ where n = m is called a linear operator

$$T(\vec{x}) = [T]_\epsilon\vec{x} = [T(\vec{e_1}) \cdots T(\vec{e_n})]\vec{x}$$

9.2 Matrix Representations

9.2.1 $\mathcal{B} - Matrix of T$

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a lienar operator and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. We define the $\mathcal{B} - matrix$ of T to be the matrix $[T]_\mathcal{B}$ constructed as follows:

$$[T]_\epsilon\vec{x} = [[T(\vec{v_1})]_\mathcal{B} \quad [T(\vec{v_2})]_\mathcal{B} \cdots [T(\vec{v_n})]_\mathcal{B}]$$

9.2.2 Proposition 9.1.3

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. If $\vec{v} \in \mathbb{F}^n$, then

$$[T(\vec{v})]_\mathcal{B} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B}$$

9.3 Definitions Addressing "Repeated" Eigenvalues

9.3.1 Algebraic Multiplicity

Let λ_i be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. The algebraic multiplicity of λ_i denoted by a_{λ_i}, is the largest positive integer such that $(\lambda - \lambda_i)^{a_{\lambda_i}}$ divides the characteristic polynomial $C_A(\lambda)$
In other word, a_{λ_i} gives the number of times that $(\lambda - \lambda_i)$ terms occur in the fully factorized form of $C_A(\lambda)$

9.3.2 Geometric Multiplicity

Let λ_i be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. The geometric multiplicity of λ_i, denoted by g_{λ_i}, is the dimensioin of the eigenspace E_{λ_i}. That is $g_{\lambda_i} = dim(E_{\lambda_i})$

9.3.3 Geometric and Algebraic Multiplicities

Let λ_i be an eigenvalue of the matrix $A \in M_{n\times n}(\mathbb{F})$. Then

$$1 \leq g_{\lambda_i} \leq a_{\lambda_i}$$

9.4 Merging all the Eigenspaces

9.4.1 Proposition 9.4.9

Let $A \in M_{n\times n}(\mathbb{F})$ with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. If their corresponding eigenspaces, $E_{\lambda_1}, E_{\lambda_2}, \cdots, E_{\lambda_k}$ have bases $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_k$, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is linearly independent

9.4.2 Theorem 9.4.10

Let $A \in M_{n\times n}(\mathbb{F})$. Suppose that the complete factorization of the characteristic polynomial of A into irreducible factors over $\mathbb{F}$ is given by

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_{\lambda_1}} \cdots (\lambda - \lambda_k)^{a_{\lambda_k}} h(\lambda)$$

where $\lambda_1, \cdots, \lambda_k$ are all of the distinct eigenvalues of A in $\mathbb{F}$ with corresponding algebraic multiplicities $a_{\lambda_1} \cdots a_{\lambda_k}$ and $h(\lambda)$ is a polynomial in λ that is irreducible over $\mathbb{F}$
Then A is diagonalizable over $\mathbb{F}$ if and only if $h(\lambda)$ is a constant polynomial and $a_{\lambda_1} = g_{\lambda_i}$, for each $i = 1, \cdots, k$