A Primer in Econometric Theory

Lecture 5: Aymptotics

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Convergence of Random Vectors

The law of large numbers and central limit theorem are pillars of econometrics and statistics

In this lecture, we review both theorems

- first start with the necessary concepts of convergence in probability and distribution

Convergence in Probability

A sequence of random vectors $\{\mathbf{x}_n\}$ is said to **converge in probability** to a random vector $\mathbf{x}$ if,

$$\text{for all } \delta > 0, \quad \mathbb{P}\{\|\mathbf{x}_n - \mathbf{x}\| > \delta\} \to 0 \quad \text{as} \quad n \to \infty \tag{1}$$

In symbols, we write $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$. In the scalar case $\|\mathbf{x}_n - \mathbf{x}\|$ reduces to $|x_n - x|$

Example. If $\mathcal{L}(\mathbf{x}_n) = \mathrm{N}(\mathbf{0}, \sigma_n \mathbf{I})$ and $\sigma_n \to 0$, then $\mathbf{x}_n \xrightarrow{p} \mathbf{0}$ as $n \to \infty$.

The variance is $\sigma_n = 1/n$

With fixed $\delta > 0$, the probability $\mathbb{P}\{|x_n| > \delta\}$ is shown for different values of n. This probability collapses to zero as $n \to \infty$

If we now fix δ at a smaller positive value, $\mathbb{P}\{|x_n| > \delta\}$ can again be made arbitrarily small by increasing n, thus (1) holds

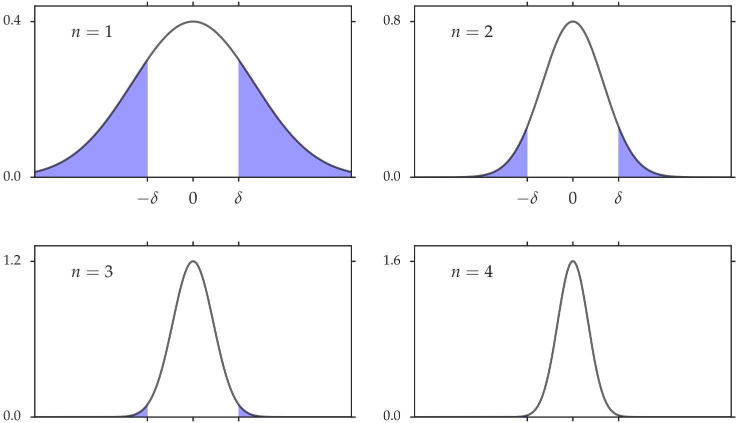

Figure: $\mathbb{P}\{|x_n| > \delta\} \to 0$ when $\mathcal{L}(x_n) = \mathrm{N}(0, 1/n)$

Fact. (6.1.1) The following statements are true:

1. $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \iff \|\mathbf{x}_n - \mathbf{x}\| \xrightarrow{p} 0$
2. $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \implies g(\mathbf{x}_n) \xrightarrow{p} g(\mathbf{x})$ whenever g is continuous at $\mathbf{x}$
3. $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$ and $\mathbf{y}_n \xrightarrow{p} \mathbf{y} \implies \mathbf{x}_n + \mathbf{y}_n \xrightarrow{p} \mathbf{x} + \mathbf{y}$ and $\mathbf{x}_n^\top \mathbf{y}_n \xrightarrow{p} \mathbf{x}^\top \mathbf{y}$
4. $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$ and $\mathbf{a}_n \to \mathbf{a} \implies \mathbf{x}_n + \mathbf{a}_n \xrightarrow{p} \mathbf{x} + \mathbf{a}$ and $\mathbf{x}_n^\top \mathbf{a}_n \xrightarrow{p} \mathbf{x}^\top \mathbf{a}$
5. $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \iff \mathbf{a}^\top \mathbf{x}_n \xrightarrow{p} \mathbf{a}^\top \mathbf{x}$ for any $\mathbf{a} \in \mathbb{R}^K$

Convergence in mean square

The scalar sequence $\{x_n\}$ is said to converge to x **in mean square** if

$$\mathbb{E}\,(x_n - x)^2 \to 0 \quad \text{as} \quad n \to \infty \tag{2}$$

and we write $x_n \stackrel{ms}{\to} x$

Unlike convergence in probability, for convergence in mean square to be defined we require our variables to have finite second moments

Fact. (6.1.2) Let $\{x_n\}$ and x have finite second moments and let α be any constant. The following statements are true:

1. $x_n \stackrel{ms}{\to} x \implies x_n \stackrel{p}{\to} x$.
2. $x_n \stackrel{ms}{\to} \alpha \iff \mathbb{E}\, x_n \to \alpha$ and $\mathrm{var}[x_n] \to 0$.

Part 1. follows from Chebyshev's inequality — $\mathbb{P}\{|x| \geq \delta\} \leq \frac{\mathbb{E}\, x^2}{\delta^2}$

In particular, from monotonicity of $\mathbb{P}$:

$$\mathbb{P}\{|x_n - x| > \delta\} \leq \mathbb{P}\{|x_n - x| \geq \delta\} \leq \frac{\mathbb{E}\,(x_n - x)^2}{\delta^2}$$

Part 2. of the above is implied by:

Fact. (6.1.3) For any $x \in L_2$ and any constant α we have

$$\mathbb{E}\left[(x-\alpha)^2\right] = \mathrm{var}[x] + (\mathbb{E}\left[x\right] - \alpha)^2 \tag{3}$$

Proof is an exercise.

As a prelude to the Law of Large Numbers (LLN), let's consider the effects of averaging over independent random quantities

Let

- x_n be the payoff from holding one dollar of asset n,
- $\mathbb{E}\, x_n = \mu$ and $\mathrm{var}[x_n] = \sigma^2$ for all n, and
- $\mathrm{cov}[x_j, x_k] = 0$ when $j \neq k$.

If we hold just asset 1, then the payoff is x_1, the expected payoff is μ and the variance is σ^2

If we diversify by spreading one dollar evenly over N of these assets, our payoff is

$$\bar{x}_N := \frac{1}{N} \sum_{n=1}^{N} x_n$$

The expected payoff is unchanged at

$$\mathbb{E}\,\bar{x}_N = \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^{N} x_n\right] = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\,x_n = \mu$$

But the variance declines at rate $\frac{1}{N}$ because

$$\mathbb{E}\left[(\bar{x}_N - \mu)^2\right] = \mathbb{E}\left\{\left[\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)\right]^2\right\}$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbb{E}\,(x_i - \mu)(x_j - \mu)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\,(x_i - \mu)^2 = \frac{\sigma^2}{N}$$

The important equality here is the third one, which holds because of the zero covariance between assets

To summarize,

$$\mathbb{E}\,\bar{x}_N = \mu \quad \text{and} \quad \operatorname{var}[\bar{x}_N] = \frac{\sigma^2}{N} \quad \text{for all } N \tag{4}$$

By taking $N \to \infty$ and combining (4) with fact 6.1.2 above we obtain a proof of the **law of large numbers**:

Theorem. (6.1.1) Let $\{x_n\}$ be IID copies of x. If x is integrable, then

$$\frac{1}{N}\sum_{n=1}^{N} x_n \xrightarrow{p} \mathbb{E}\,x \quad \text{as} \quad N \to \infty \tag{5}$$

We assumed finite second moment: see ET page 164 for references on proofs for the LLN without assumption on second moment

We can extend (5) to arbitrary functions of random variables and random vectors:

If $\mathbf{x}$ is any random vector, $\{\mathbf{x}_n\}$ are IID copies and $h \colon \mathbb{R}^N \to \mathbb{R}$ is any $\mathscr{B}$-measurable function such that $h(\mathbf{x})$ is integrable, then

$$\frac{1}{N}\sum_{n=1}^{N} h(\mathbf{x}_n) \xrightarrow{p} \mathbb{E}\, h(\mathbf{x}) \quad \text{as} \quad N \to \infty$$

Proof follows from Theorem (6.1.1) (exercise, or see page 164 of ET)

The law of large numbers applies to probabilities as well as expectations

Fix $B \subset \mathscr{B}(\mathbb{R}^N)$, let $h(\mathbf{s}) = \mathbb{1}_B(\mathbf{s}) = \mathbb{1}\{\mathbf{s} \in B\}$, we have

$$\mathbb{E}\, h(\mathbf{x}) = \mathbb{E}\, \mathbb{1}\{\mathbf{x} \in B\} = \mathbb{P}\{\mathbf{x} \in B\}$$

Combine this equality with the LLN, if $\{\mathbf{x}_n\}$ is IID with distribution P, then

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\{\mathbf{x}_n \in B\} \xrightarrow{p} P(B)$$

The fraction of the sample that falls in B converges to the probability that the distribution assigns to B

To illustrate the law of large numbers, consider flipping a coin until 10 heads have occurred

- probability of heads is 0.4

Let x be the number of tails observed in the process

- random variable is known to have the **negative binomial distribution** with $\mathbb{E} x = 15$

The LLN predicts that if we simulate a large number of observations of x and take the average, we get a value close to 15

Julia code to illustrate LLN:

```
num_reps = 10^6
outcomes = Array(Float64, num_reps)

for i in 1:num_reps
    num_tails = num_heads = 0
    while num_heads < 10
        b = rand()
        num_heads = num_heads + (b < 0.4)
        num_tails = num_tails + (b >= 0.4)
    end
    outcomes[i] = num_tails
end

println(mean(outcomes))
```

What happens when the finite first moment condition in the LLN is not enforced?

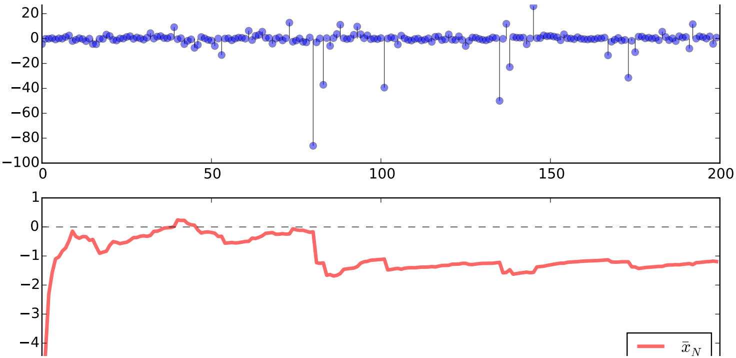

Figure: Samples from the Cauchy distribution and sample mean

Convergence in Distribution

The common notion of convergence of distributions, which is called weak convergence, requires $P_n(B) \to P(B)$ for all "continuity sets" in $\mathbb{R}^K$

Equivalently: $\{P_n\}$ **converges weakly** to P if

$$\int h(\mathbf{s}) P_n(\mathrm{d}\mathbf{s}) \to \int h(\mathbf{s}) P(\mathrm{d}\mathbf{s})$$

$\forall$ continuous bounded $h \colon \mathbb{R}^K \to \mathbb{R}$

and we write $P_n \stackrel{w}{\to} P$

Fact. (6.1.4) Let F_n be the CDF of P_n and let F be the CDF of P. In the univariate case ($K = 1$) we have

$$P_n \xrightarrow{w} P \quad \iff \quad F_n(s) \to F(s)$$

for all s at which F is continuous

Example. It can be shown that the t-distribution with k degrees of freedom converges weakly to the standard normal distribution as $k \to \infty$

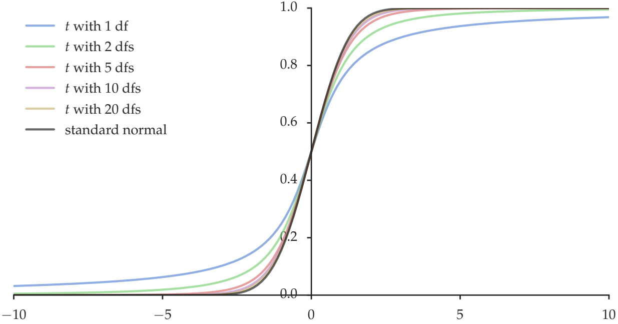

Figure: t-Distribution with k df converges to $\mathrm{N}(0,1)$ as $k \to \infty$

Fact. (6.1.5) Let $\{P_n\}$ and P be absolutely continuous probability measures on $\mathbb{R}^K$, with densities p_n and p

If $p_n(\mathbf{s}) \to p(\mathbf{s})$ for all $\mathbf{s} \in \mathbb{R}^K$, then $P_n \xrightarrow{w} P$

Let $\{\mathbf{x}_n\}$ and $\mathbf{x}$ be random vectors

We say $\mathbf{x}_n \to \mathbf{x}$ **in distribution** if their respective distributions converge weakly

The convergence is symbolized by $\mathbf{x}_n \stackrel{d}{\to} \mathbf{x}$

Thus

$$\mathbf{x}_n \stackrel{d}{\to} \mathbf{x} \iff \mathcal{L}(\mathbf{x}_n) \stackrel{w}{\to} \mathcal{L}(\mathbf{x})$$

Equivalent to:

$$\mathbb{E}\left[h(\mathbf{x}_n)\right] \to \mathbb{E}\left[h(\mathbf{x})\right] \quad \text{for all continuous bounded } h \colon \mathbb{R}^K \to \mathbb{R}$$

(Why?)

Fact. (6.1.6) The following statements are true:

1. If $g \colon \mathbb{R}^K \to \mathbb{R}^J$ is continuous and $\mathbf{x}_n \xrightarrow{d} \mathbf{x}$, then $g(\mathbf{x}_n) \xrightarrow{d} g(\mathbf{x})$.
2. If $\mathbf{a}^\top \mathbf{x}_n \xrightarrow{d} \mathbf{a}^\top \mathbf{x}$ for any $\mathbf{a} \in \mathbb{R}^K$, then $\mathbf{x}_n \xrightarrow{d} \mathbf{x}$.
3. $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \implies \mathbf{x}_n \xrightarrow{d} \mathbf{x}$.
4. If $\mathbf{a}$ is a constant vector and $\mathbf{x}_n \xrightarrow{d} \mathbf{a}$, then $\mathbf{x}_n \xrightarrow{p} \mathbf{a}$

Part 1. called the **continuous mapping theorem**

Part 2. called the Cramér–Wold theorem, or the **Cramér–Wold device**

Fact. (6.1.7) If α is constant, $x_n \xrightarrow{p} \alpha$ and $y_n \xrightarrow{d} y$, then $x_n + y_n \xrightarrow{d} \alpha + y$ and $x_n y_n \xrightarrow{d} \alpha y$

An immediate but useful consequence is that

Fact. (6.1.8) $x_n \xrightarrow{p} 0$ and $y_n \xrightarrow{d} y \implies x_n y_n \xrightarrow{p} 0$

The Central Limit Theorem

The **central limit theorem** is among the most striking and important results in all of mathematics

Theorem. (6.1.2) Let x have finite second moment and let $\{x_n\}$ be IID copies of x. If $\mu := \mathbb{E}\, x$ and $\sigma^2 := \operatorname{var} x$, then

$$\sqrt{N}(\bar{x}_N - \mu) \xrightarrow{d} \mathrm{N}(0, \sigma^2) \quad \text{as} \quad N \to \infty$$

On one hand, $(\bar{x}_N - \mu) \xrightarrow{p} 0$ by the LLN; on the other hand, $\sqrt{N} \to \infty$

If we take the product, these two competing terms just balance

The distribution of the product approaches a zero-mean Gaussian as $N \to \infty$, regardless of the distribution of x

Consider simulating the CLT

Let $Q_N :=$ the distribution of $\sqrt{N}(\bar{x}_N - \mu)$ for $N = 1, \ldots, 5$

Initial distribution $Q = Q_1$ is multi-modal, constructed as a convex combination of three beta distributions

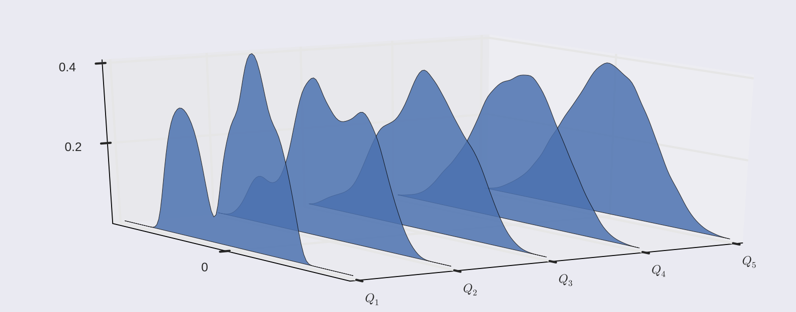

Figure: CLT in action, starting from a beta mixture

Another common statement of the central limit theorem: if all the conditions of the CLT are satisfied, then

$$z_N := \sqrt{N} \left\{ \frac{\bar{x}_N - \mu}{\sigma} \right\} \xrightarrow{d} \mathrm{N}(0,1) \quad \text{as} \quad N \to \infty$$

Python code to illustrate CLT:

```
import numpy as np
import scipy.stats as st

num_reps = 5000
outcomes = np.empty(num_reps)
N, k = 1000, 5      # k = degrees of freedom
chi = st.chi2(k)

for i in range(num_reps):
    xvec = chi.rvs(N)
    outcomes[i] = np.sqrt(N / (2 * k))\
                  *(xvec.mean() - k)
```

The listing generates 5,000 observations of

$$z_N := \sqrt{N} \left\{ \frac{\bar{x}_N - \mu}{\sigma} \right\} \xrightarrow{d} \mathrm{N}(0,1) \quad \text{as} \quad N \to \infty$$

Each x_n is $\chi^2(5)$

- mean of this distribution is 5, and the variance is $2 \times 5 = 10$

The observations of z_N are stored in the vector `outcomes`

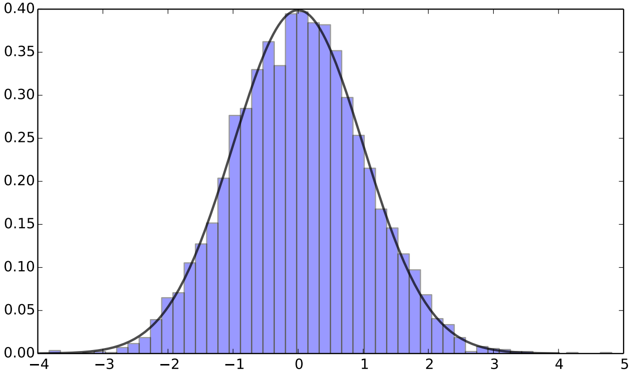

Figure: Observations of z_N in (31) when the underlying distribution is $\chi^2(5)$

Convergence of Random Matrices

Let $\{\mathbf{X}_n\}_{n=1}^{\infty}$ be a sequence of random $N \times K$ matrices. We say that $\mathbf{X}_n$ converges to a random $N \times K$ matrix $\mathbf{X}$ **in probability** and write $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$ if

$$\|\mathbf{X}_n - \mathbf{X}\| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty$$

where $\| \cdot \|$ is the matrix norm defined in §3.3.5

Fact. (6.2.1) Assuming conformability, the following statements are true:

1. If $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$ and $\mathbf{X}_n$ and $\mathbf{X}$ are nonsingular, then $\mathbf{X}_n^{-1} \xrightarrow{p} \mathbf{X}^{-1}$.
2. If $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$ and $\mathbf{Y}_n \xrightarrow{p} \mathbf{Y}$, then

$$\mathbf{X}_n + \mathbf{Y}_n \xrightarrow{p} \mathbf{X} + \mathbf{Y}, \quad \mathbf{X}_n\mathbf{Y}_n \xrightarrow{p} \mathbf{X}\mathbf{Y}, \quad \text{and} \quad \mathbf{Y}_n\mathbf{X}_n \xrightarrow{p} \mathbf{Y}\mathbf{X}$$

3. If $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$ and $\mathbf{A}_n \to \mathbf{A}$, then

$$\mathbf{X}_n + \mathbf{A}_n \xrightarrow{p} \mathbf{X} + \mathbf{A}, \quad \mathbf{X}_n\mathbf{A}_n \xrightarrow{p} \mathbf{X}\mathbf{A}, \quad \text{and} \quad \mathbf{A}_n\mathbf{X}_n \xrightarrow{p} \mathbf{A}\mathbf{X}$$

4. $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$ if and only if $\mathbf{X}_n\mathbf{a} \xrightarrow{p} \mathbf{X}\mathbf{a}$ for any conformable vector $\mathbf{a}$
5. $\mathbf{a}^\intercal\mathbf{X}_n\mathbf{a} \xrightarrow{p} \mathbf{a}^\intercal\mathbf{X}\mathbf{a}$ whenever $\mathbf{a}$ is a conformable constant vector and $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$

In econometrics we often use the vector version of Slutsky's theorem:

Fact. (6.2.2) Let $\mathbf{x}_n$ and $\mathbf{x}$ be random vectors in $\mathbb{R}^K$, let $\mathbf{Y}_n$ be random matrices, and let $\mathbf{C}$ be a constant matrix. Assuming conformability, we have

$$\mathbf{Y}_n \xrightarrow{p} \mathbf{C} \text{ and } \mathbf{x}_n \xrightarrow{d} \mathbf{x} \quad \implies \quad \mathbf{Y}_n \mathbf{x}_n \xrightarrow{d} \mathbf{C}\mathbf{x}$$

$$\text{and} \quad \mathbf{Y}_n + \mathbf{x}_n \xrightarrow{d} \mathbf{C} + \mathbf{x}$$

The scalar LLN and CLT extend to the vector case:

Theorem. (6.2.1) Let $\mathbf{x}$ be a random vector in $\mathbb{R}^K$ and let $\{\mathbf{x}_n\}$ be IID copies of $\mathbf{x}$. If $\boldsymbol{\mu} := \mathbb{E}\,\mathbf{x}$ is finite, then

$$\bar{\mathbf{x}}_N := \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}_n \xrightarrow{p} \boldsymbol{\mu} \quad \text{as} \quad N \to \infty \tag{6}$$

If, in addition, $\mathbb{E}\,\|\mathbf{x}\|^2 < \infty$, then

$$\sqrt{N}\,(\bar{\mathbf{x}}_N - \boldsymbol{\mu}) \xrightarrow{d} \mathrm{N}(\mathbf{0}, \Sigma) \quad \text{where} \quad \Sigma := \operatorname{var} \mathbf{x} \tag{7}$$

Here $\frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n$ should be understood in terms of vector addition and scalar multiplication

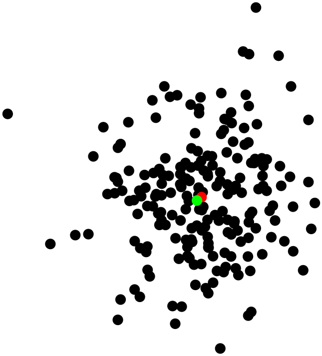

Figure: LLN, vector case

Vector LLN in theorem 6.2.1 follows from the scalar LLN

- let $\mathbf{x}_n$ be $\{\mathbf{x}_n\}$ be IID copies of $\mathbf{x}$
- let $\mathbf{a}$ be any constant vector in $\mathbb{R}^K$
- define $y_n := \mathbf{a}^\mathsf{T} \mathbf{x}_n$
- define $y := \mathbf{a}^\mathsf{T} \mathbf{x}$

The sequence $\{y_n\}$ is IID (see fact 5.1.10 on page 137) with the same distribution as y

By the scalar LLN

$$\frac{1}{N} \sum_{n=1}^{N} y_n \xrightarrow{p} \mathbb{E}\, y = \mathbb{E}\left[\mathbf{a}^\mathsf{T} \mathbf{x}\right] = \mathbf{a}^\mathsf{T} \mathbb{E}\left[\mathbf{x}\right] = \mathbf{a}^\mathsf{T} \boldsymbol{\mu}$$

At the same time:

$$\frac{1}{N}\sum_{n=1}^{N} y_n = \frac{1}{N}\sum_{n=1}^{N} \mathbf{a}^{\mathsf{T}}\mathbf{x}_n = \mathbf{a}^{\mathsf{T}}\left[\frac{1}{N}\sum_{n=1}^{N}\mathbf{x}_n\right] = \mathbf{a}^{\mathsf{T}}\bar{\mathbf{x}}_N$$

Thus

$$\mathbf{a}^{\mathsf{T}}\bar{\mathbf{x}}_N \stackrel{p}{\to} \mathbf{a}^{\mathsf{T}}\boldsymbol{\mu} \quad \text{for any} \quad \mathbf{a} \in \mathbb{R}^K$$

The claim $\bar{\mathbf{x}}_N \stackrel{p}{\to} \boldsymbol{\mu}$ now follows (recall fact 6.1.1 above)

Fact. (6.2.3) Let $\mathbf{X}$ be a random matrix and let $\{\mathbf{X}_n\}$ be IID copies of $\mathbf{X}$. If $\mathbb{E}\,\|\mathbf{X}\| < \infty$, then

$$\frac{1}{N}\sum_{n=1}^{N}\mathbf{X}_n \xrightarrow{p} \mathbb{E}\,\mathbf{X} \quad \text{as} \quad N \to \infty \tag{8}$$

Proof. Since $\mathbf{X}_n\mathbf{a}$ is a vector with expectation $\mathbb{E}\,[\mathbf{X}]\mathbf{a}$, the following

$$\frac{1}{N}\sum_{n=1}^{N}\mathbf{X}_n\mathbf{a} \xrightarrow{p} \mathbb{E}\,[\mathbf{X}]\mathbf{a}$$

for any conformable vector $\mathbf{a}$, is immediate from the vector LLN (theorem 6.2.1)

The proof for Fact 6.2.3 is then complete by recalling the following from fact 6.1.1

$$\mathbf{x}_n \xrightarrow{p} \mathbf{x} \quad \iff \quad \mathbf{a}^\top \mathbf{x}_n \xrightarrow{p} \mathbf{a}^\top \mathbf{x} \quad \text{for any} \quad \mathbf{a} \in \mathbb{R}^K$$

The Delta Method

We showed the asymptotic normality result in the central limit theorem is preserved under linear transformations (fact 6.2.2)

The result also holds for functions that are locally almost linear — for differentiable functions

Theorem. (6.2.2) Let $g\colon \mathbb{R}^K \to \mathbb{R}$, let $\boldsymbol{\theta}$ be a point in the domain of g, and let $\{\mathbf{t}_n\}$ be a sequence of random vectors in $\mathbb{R}^K$. If

1. $\sqrt{n}(\mathbf{t}_n - \boldsymbol{\theta}) \stackrel{d}{\to} \mathrm{N}(0, \boldsymbol{\Sigma})$ for some positive definite $\boldsymbol{\Sigma}$ and
2. $\nabla g(\boldsymbol{\theta})$ exists, is continuous, and each element is nonzero

then

$$\sqrt{n}\{g(\mathbf{t}_n) - g(\boldsymbol{\theta})\} \stackrel{d}{\to} \mathrm{N}(0, \nabla g(\boldsymbol{\theta})^\top \boldsymbol{\Sigma} \nabla g(\boldsymbol{\theta})) \quad \text{as} \quad n \to \infty \tag{9}$$

The term $\nabla g(\boldsymbol{\theta})$ is the **gradient vector** of g at $\boldsymbol{\theta}$:

$$\nabla g(\boldsymbol{\theta}) := \begin{pmatrix} g_1'(\boldsymbol{\theta}) \\ \vdots \\ g_K'(\boldsymbol{\theta}) \end{pmatrix} \quad \text{where} \quad g_k'(\boldsymbol{\theta}) := \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_k}$$

In the scalar case, (9) translates to

$$\sqrt{n}\{g(t_n) - g(\theta)\} \xrightarrow{d} \mathrm{N}(0, g'(\theta)^2\sigma^2) \quad \text{as} \quad n \to \infty$$