CO 250 Winter 2021: Assignment 1 problems

Due: Wednesday January 20 at 11:59pm EST

Preamble.

- You will receive one Crowdmark link for each problem. Submit your solutions on Crowdmark.
- You should justify all of your solutions, unless you are explicitly asked not to.
- You are graded on both your accuracy and your presentation. A correct solution that is poorly presented may not receive full marks. The markers will not spend a lot of time figuring out something you write that is not clear.
- When discussing assignment problems on Piazza, use private posts if you think that you might spoil an answer or major thinking point of a problem.
- Read the assignment policies on LEARN for what is allowed and not allowed in working on the assignment.
- Reproduction, sharing or online posting of his document is strictly forbidden.

Assignment problems.

A1-1 This course requires the use of results from linear algebra, so it is a good idea to take some time to review some of the topics from it. In particular, you should be familiar with matrix multiplication, solutions of linear systems, linear independence, basis, rank, and the geometry of hyperplanes.

Here are some review questions from linear algebra.

Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcrcr} -4x_1 & + & 3x_2 & + & 2x_3 & - & 2x_4 & + & 2x_5 & = & -17 \\ 10x_1 & - & 2x_2 & + & 6x_3 & + & 3x_4 & - & 5x_5 & = & 40 \\ 3x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 12 \end{array}$$

Note: You can use computational tools to do the calculations. In your solutions, state which calculation you are performing, and then state the result of the calculation.

(a) Write down $A \in \mathbb{R}^{3\times 5}, x \in \mathbb{R}^5, b \in \mathbb{R}^3$ so that this system of linear equations is equivalent to $Ax = b$.

(b) Prove that the rank of A is 3.

(c) Define what the columnspace of a matrix is in general. Then determine what is the columnspace of A in this question, and determine a basis for it.

(d) Find one possible solution x to this system.

(e) Prove that we can replace b by any vector in $\mathbb{R}^3$, and this system has a solution. Given any $b \in \mathbb{R}^3$, determine a quick way to find a solution x using one multiplication.

A1-2 The Hamilton Steel Company received an order to produce 1000kg of a type of steel with very restrictive composition requirements. The carbon content must be between 3.00% and 3.50%, the manganese content must be between 1.35% and 1.65%, and the silicon content must be between 2.70% and 3.00%.

The company can produce the steel by blending several existing materials available on the market. The following table summarizes the composition of each material and its cost are.

Material	Carbon %	Manganese %	Silicon %	Cost per kg
A	4.0	0.9	2.5	0.65
B	0.0	60.0	18.0	0.22
C	15.0	0.0	30.0	0.17
D	0.1	0.3	0.0	0.05

Note that when you blend the materials, the chemical content is the weighted average of the respective content in these materials. For example, if we blend 3kg of material A with 1kg of material C, we would have a carbon content of $(3 \cdot 4.0 + 1 \cdot 15.0)/4 = 6.75\%$.

The goal is to obtain enough materials to blend and produce the steel satisfying the composition requirements, while minimizing the total cost.

(a) Formulate this as a linear program. Clearly explain your variables, objection function and constraints. Do not solve the LP. Write down the full LP at the end.

(b) Suppose we now have the added condition that the differences between the amounts we buy for the 4 materials must be at most 100kg. Explain how to modify your LP from part (a) so that we get a new LP where this condition is satisfied.

A1-3 Martin's Bakery is famous for its freshly baked apple pies (not peis). There are two steps to making an apple pie: (1) process fresh apples and pastry to form raw pies; and (2) bake the raw pies. We now consider the workflow of making pies for a 7-day period $\{1, 2, 3, 4, 5, 6, 7\}$.

For each day i, a supply of s_i kilograms of fresh apples come into the bakery. Each kilogram is enough to fill one pie. There is an unlimited supply of pastry. We can produce up to α_i raw pies on day i (step 1), and we can bake up to β_i raw pies on day i (step 2). Based on historical data, we expect to sell up to d_i baked pies. You may assume that $s_i, d_i, \alpha_i, \beta_i$ are constants that are given to you.

Now freshness is very important in Martin's Bakery. If we can take the apples from day i, go through both steps to produce baked apple pies and sell them on the same day, then this is the highest quality pies that we can make, and these pies can sell for \$15 each. But we do have the option of storing unused apples, raw pies or baked pies. There is no cost to storing them, but the freshness decreases, and the end product is less valuable. So baked pies produced using apples from k days earlier will only sell for $\$15 - 2k$. (That is, we lose \$2 of value for each day the apples do not turn into baked pies and be sold.) For example, we can take 10kg of apples from day 2, turn them into raw pies on day 3, bake them on day 5, then sell them on day 6. This will produce 10 baked pies that sell for \$7 each (as they are 4 days late).

We want to maximize the total revenue for selling the baked apple pies over 7 days. We may make the (unreasonable) assumption that we can make/sell fractional pies.

Formulate this as a linear program. Clearly explain your variables, objection function and constraints. Do not solve the LP. Write down the full LP at the end.