Subject : Date

$\iint f_{XY}(x,y)\,dx\,dy = 1$ مسئله ۱. الف) می‌دانیم می‌بایست

$$\Rightarrow \int_0^2\int_0^1 xy(c-x-y)\,dx\,dy = \int_0^2\int_0^1 (cxy - x^2y - xy^2)\,dx\,dy$$

$$= \int_0^2 \left(\frac{cyx^2}{2} - \frac{yx^3}{3} - \frac{y^2x^2}{2}\Big|_0^1\right) dy$$

$$= \int_0^2 \left(\frac{cy}{2} - \frac{y}{3} - \frac{y^2}{2}\right) dy = \frac{cy^2}{4} - \frac{y^2}{6} - \frac{y^3}{6}\Big|_0^2$$

$$= c - \frac{2}{3} - \frac{4}{3} = 1 \Rightarrow \boxed{c = 3}$$

ب) $f_X(x) = \int_0^2 f_{XY}(x,y)\,dy = \int_0^2 xy(3-x-y)\,dy$

$$= \int_0^2 (3xy - x^2y - xy^2)\,dy = \frac{3xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3}\Big|_0^2$$

$$\Rightarrow f_X(x) = 6x - 2x^2 - \frac{8}{3}x = -2x^2 - \frac{10}{3}x$$

$$f_X(x) = \begin{cases} -2x^2 - \frac{10}{3}x & 0<x<1 \\ 0 & o.w \end{cases}$$

$E[\underbrace{E[X|Y]}_{\text{تابعی از } Y}]$ مسئله ۲. الف) در حالت پیوسته :

LOTUS $\int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, f_Y(y)\,dx\,dy$

conditional $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x \int_{\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$

marginal $\int_{-\infty}^{\infty} x\, f_X(x)\,dx = E[X]$

Subject :

Date

$E[E[X|Y]]$ در حالت گسسته: (همانند حالت پیوسته)

$$= \sum_y E[X|Y=y]\,P(Y=y) = \sum_y \sum_x x\,P(X=x \mid Y=y)\,P(Y=y)$$

$$\overset{\text{conditional}}{=} \sum_y \sum_x x\,P(X=x, Y=y) = \sum_x x \sum_y P(X=x, Y=y)$$

$$\overset{\text{marginal}}{=} \sum_x x\,P(X=x) = E[X]$$

ب)

1) $E[Var(X|Y)] = E[E[X^2|Y]] - E[(E[X|Y])^2]$

$$\xrightarrow{\text{طبق الف}} * \quad E[Var(X|Y)] = E[X^2] - E[(E[X|Y])^2]$$

2) $Var(E[X|Y]) = E[(E[X|Y])^2] - E[E[X|Y]]^2$

$$\xrightarrow{\text{طبق الف}} ** \quad Var(E[X|Y]) = E[(E[X|Y])^2] - E[X]^2$$

$$\xrightarrow{* + **} \underbrace{E[X^2] - E[X]^2}_{Var(X)} = E[Var(X|Y)] + Var(E[X|Y])$$

پ)

می‌دانیم $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$\rightarrow Cov(aX+b, cY+d) = E\left[(aX+b-\mu_{aX+b})(cY+d-\mu_{cY+d})\right]$$

$$= E\left[(aX+b - a\mu_X - b)(cY+d - c\mu_Y - d)\right]$$

$$= E[a(X-\mu_X)\,c\,(Y-\mu_Y)] = ac\,E[(X-\mu_X)(Y-\mu_Y)]$$

$$= ac\,Cov(X,Y)$$

Subject :

مسئله ۳) الف)

$$\underbrace{\mathcal{L}(\theta \mid x_1, x_2, \dots, x_N)}_{\text{Likelihood}} = p(x_1 \mid \theta) \dots p(x_N \mid \theta)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{x_1-\theta}{\sigma}\right)^2/2} \times \dots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{x_N-\theta}{\sigma}\right)^2/2}$$

$$\xrightarrow[\text{likelihood}]{\log} \ln \mathcal{L}(\theta \mid X) = \frac{-1}{\sigma\sqrt{2\pi}} \left(\frac{\left(\frac{x_1-\theta}{\sigma}\right)^2}{2} + \dots + \frac{\left(\frac{x_N-\theta}{\sigma}\right)^2}{2} \right)$$

$$\rightarrow \frac{d \ln \mathcal{L}(\theta \mid X)}{d\theta} = 0 \rightarrow \frac{\theta - x_1}{\sigma^2} + \dots + \frac{\theta - x_N}{\sigma^2} = 0$$

$$\Rightarrow \frac{N\theta}{\sigma^2} = \frac{x_1 + x_2 + \dots + x_N}{\sigma^2} \Rightarrow \hat{\theta}_{MLE} = \frac{x_1 + x_2 + \dots + x_N}{N}$$

ب)

$$\hat{\theta}_{MAP}(x) = \underset{\theta}{\operatorname{argmax}}\, f(\theta \mid X) = \underset{\theta}{\operatorname{argmax}}\, \underbrace{f(x \mid \theta)}_{\text{likelihood}} \underbrace{f(\theta)}_{\text{prior}}$$

$$\xrightarrow{\ln} \frac{-1}{\sigma\sqrt{2\pi}} \left(\frac{\left(\frac{x_1-\theta}{\sigma}\right)^2}{2} + \dots + \frac{\left(\frac{x_N-\theta}{\sigma}\right)^2}{2} + \underbrace{\left(\frac{\theta - v}{\beta}\right)^2/2}_{\text{prior}} \right)$$

$$\rightarrow \frac{d}{d\theta} MAP = 0 \rightarrow \frac{\theta - x_1}{\sigma^2} + \dots + \frac{\theta - x_n}{\sigma^2} + \frac{\theta - v}{\beta^2} = 0$$

$$\Rightarrow \frac{N\theta}{\sigma^2} + \frac{\theta}{\beta^2} = \frac{x_1 + \dots + x_N}{\sigma^2} + \frac{v}{\beta^2}$$

$$\Rightarrow \theta\left(\frac{N\beta^2 + \sigma^2}{\sigma^2\beta^2}\right) = \frac{x_1 + \dots + x_N}{\sigma^2} + \frac{v}{\beta^2}$$

$$\Rightarrow \hat{\theta}_{MAP} = \frac{\beta^2}{N\beta^2 + \sigma^2}(x_1 + \dots + x_N) + \frac{\sigma^2}{N\beta^2 + \sigma^2} v$$

$$\Rightarrow = \frac{\beta^2 N}{N\beta^2 + \sigma^2}\left(\frac{x_1 + \dots + x_N}{N}\right) + \frac{\sigma^2}{N\beta^2 + \sigma^2} v$$

Subject :

پ)

$$\lim_{N \to \infty} \hat{\theta}_{MAP} = \underbrace{\frac{N\beta^2}{N\beta^2}}_{\to 1} \left(\frac{x_1 + \cdots + x_N}{N} \right) + \underbrace{\frac{1}{N}}_{\to 0}$$

$$= \frac{x_1 + \cdots + x_N}{N}$$

$$\Rightarrow n \to \infty : \hat{\theta}_{MAP} = \hat{\theta}_{MLE}$$

مسئله ۴. الف)

$X =$ متغیر تصادفی پذیرفته شدن

$X \sim Bern(p)$, p: احتمال پذیرفته شدن : $P(X=1)$

حال با توجه به data داریم:

$$p = \frac{۷}{۱۲}$$

→ آنتروپی: $H(X) = -\sum_x P(X=x) \log_۲ P(X=x)$

$$= -\left(P(X=0) \log_۲ P(X=0) + P(X=1) \log_۲ P(X=1) \right)$$

$$= -\left(\frac{5}{12} \log_2 \frac{5}{12} + \frac{7}{12} \log_2 \frac{7}{12} \right)$$

$$= 0{,}9798$$

ب) متغیر تصادفی مقاله $= P$ ، متغیر تصادفی توصیه‌نامه $= R$ ، متغیر تصادفی پذیرفته شدن $= X$

$$\Rightarrow IG(X, R) = H(X) - H(X \mid R)$$

$$H(X \mid R) = -\sum p(x,r) \log \frac{p(x,r)}{p(r)}$$

$$\Rightarrow H(X \mid R) = -\left[\frac{1}{6} \log \frac{1/6}{1/3} \times 2 + \frac{1}{6} \log \frac{1/6}{1/3} \times 2 + \frac{1}{12} \log \frac{1/12}{1/3} + \frac{1}{4} \log \frac{1/4}{1/3} \right] = 0{,}9371$$

Subject : Date

$\rightarrow IG(x, R) = H(x) - H(x \mid R) = 0{,}9798 - 0{,}9371 = 0{,}0427$

$IG(x, P) = H(x) - H(x \mid P)$

$\rightarrow H(x \mid P) = -\left[\frac{5}{12} \log \frac{5/12}{5/12} + \frac{2}{12} \log \frac{2/12}{7/12} + 0\right.$

$\left. + \frac{5}{12} \log \frac{5/12}{7/12} \right] = 0{,}503$

$\Rightarrow IG(x, Z) = 0{,}9798 - 0{,}503 = 0{,}4768$

$a_{n\times 1}$, $x_{n\times 1}$, $A_{n\times n}$

Subject : Date

الف) $\frac{d}{dx}\underbrace{a^T x}_{\text{scalar}} = \left[\frac{d}{dx_1}a^T x \quad \frac{d}{dx_2}a^T x \;\dots\; \frac{d}{dx_n}a^T x\right]$ (سئوال ۵.۱

$x^T a = a^T x = a_1x_1 + a_2x_2 + \dots + a_nx_n \rightarrow \frac{d a^T x}{dx_j} = a_j \;,\; \frac{d x^T a}{dx_j} = a_j$

$\Rightarrow \frac{d}{dx}x^T a = \frac{d}{dx}a^T x = [a_1 \quad a_2 \;\dots\; a_n] = a^T$

ب) $\frac{d}{dx}\underbrace{x^T x}_{\text{scalar}} = \left[\frac{d}{dx_1}x^T x \;\dots\; \frac{d}{dx_n}x^T x\right]$

$x^T x = x_1^2 + \dots + x_n^2 \Rightarrow \frac{d}{dx_j}x^T x = 2x_j$

$\Rightarrow \frac{d}{dx}x^T x = [2x_1 \quad 2x_2 \;\dots\; 2x_n] = 2x^T$

ج) $\frac{d}{dx}\underbrace{(x^T a)^2}_{\text{scalar}} = \left[\frac{d}{dx_1}(x^T a)^2 \;\dots\; \frac{d}{dx_n}(x^T a)^2\right]$

$(x^T a)^2 = (a_1x_1 + \dots + a_nx_n)^2 \rightarrow \frac{d(x^T a)^2}{dx_j} = 2a_j(a_1x_1 + \dots + a_nx_n)$

$\Rightarrow \frac{d}{dx}\underbrace{(x^T a)^2}_{\text{scalar}} = [2(a_1x_1 + \dots + a_nx_n)a_1 \quad 2(a_1x_1 + \dots + a_nx_n)a_2 \;\dots\; 2(a_1x_1 + \dots + a_nx_n)a_n]$

$= 2\underbrace{(a_1x_1 + \dots + a_nx_n)}_{x^T a}\underbrace{[a_1 \quad a_2 \;\dots\; a_n]}_{a^T}$

$= 2x^T a a^T$

د) $\frac{d}{dx}\underbrace{Ax}_{\text{vector}} = \begin{bmatrix} \frac{d}{dx_1}(Ax)_1 & \dots & \frac{d}{dx_n}(Ax)_1 \\ \vdots & \ddots & \vdots \\ \frac{d}{dx_1}(Ax)_n & \dots & \frac{d}{x_n}(Ax)_n \end{bmatrix}$

$(Ax)_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \Rightarrow \frac{d(Ax)_i}{dx_j} = a_{ij}$

MICRO

Subject :

Date

$$\Rightarrow \frac{d}{dx_j}(Ax)_i = a_{ij} \Rightarrow \frac{d}{dx}Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = A$$

ث) $\frac{d}{dx}\underbrace{x^TA}_{\text{vector}} \longrightarrow x^TA = [(x^TA)_1, (x^TA)_2 \cdots (x^TA)_n]$

$$\Rightarrow \frac{d}{dx}x^TA = \begin{bmatrix} \frac{d}{dx_1}(x^TA)_1 & \frac{d}{dx_1}(x^TA)_1 & \cdots & \frac{d}{dx_1}(x^TA)_1 \\ \vdots & \ddots & & \vdots \\ \frac{d}{x_1}(x^TA)_n & \frac{d}{x_2}(x^TA)_n & \cdots & \frac{d}{x_n}(x^TA)_n \end{bmatrix}$$

$$(x^TA)_i = x_1 a_{1i} + x_2 a_{2i} + \cdots + x_n a_{ni}$$

$$\Rightarrow \frac{d}{x_j}(x^TA)_i = a_{ji} \Rightarrow \frac{d}{dx}x^TA = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & & & \\ \vdots & & \ddots & \\ a_{1n} & & \cdots & a_{nn} \end{bmatrix} = A^T$$

ج) $\frac{d}{dx}x^TAx : \quad f(x) = x^TAx$

$$\rightarrow f(x+h) = (x+h)^TA(x+h) = x^TAx + x^TAh + h^TAx + h^TAh$$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{x^TAh + h^TAx + h^TAh}{h}$$

$$h^TAx : \text{scalar} \longrightarrow h^TAx = (h^TAx)^T = x^TA^Th$$

$$\Rightarrow \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x^TA + x^TA^T)h + h^TAh}{h} \overset{h\to 0}{=\!=} x^TA + x^TA^T$$

$$= x^T(A+A^T)$$

Subject :

(مسئله‌ی ۵.۲

$$\alpha = \sum_{j=1}^{n} x_j y_j \quad , \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \\ \vdots \\ z_n \end{bmatrix}$$

$$\frac{\partial \alpha}{\partial z_k} = \sum_{j=1}^{n} \left(x_j \frac{\partial y_j}{\partial z_k} + y_j \frac{\partial x_j}{\partial z_k} \right)$$

$$\Longrightarrow k = 1, 2, \ldots, n \Rightarrow \frac{\partial \alpha}{\partial z} = \frac{\partial \alpha}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial \alpha}{\partial x} \frac{\partial x}{\partial z}$$

$$= x^T \frac{\partial y}{\partial z} + y^T \frac{\partial x}{\partial z}$$

(مسئله‌ی ۵.۳

$$HH^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} = A$$

$$Ax = \lambda x \longrightarrow (A - \lambda I)x = 0 \longrightarrow \det(A - \lambda I) = 0$$

$$\rightarrow \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6$$

$$= (\lambda - 6)(\lambda - 1) = 0 \longrightarrow \lambda_1 = 6 \;,\; \lambda_2 = 1$$

$$\Rightarrow \lambda_1 = 6 \longrightarrow \underbrace{N}_{Null}(A - 6I) = N\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \Longrightarrow \underbrace{x_1}_{\text{بردار ویژه}} = \overbrace{\begin{bmatrix} ۲ \\ ۱ \end{bmatrix}}^{\text{و هر ضریبی از آن}}$$

$$= \lambda_2 = 1 \longrightarrow N(A - I) = N\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \Longrightarrow \underbrace{x_2}_{\text{بردار ویژه}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

و هر ضریبی از آن

Subject :

Date ....

مسئله‌ی ۵.۴) می‌دانیم برای ماتریس A ، $\det(A)=\det(A^T)$ ⊛

حال برای بدست آوردن مقادیر ویژه یک ماتریس می‌بایست معادله $\det(A-\lambda I)=0$

را حل کنیم. با توجه به ⊛ ، می‌توانیم $\det((A-\lambda I)^T)=0$ را حل کنیم

حال داریم:

$$\det((A-\lambda I)^T)=\det(A^T-\lambda \cancel{I^T})$$

از این معادله، نتیجه می‌شود که λ مقدار ویژه A^T نیز محسوب می‌شود $\Leftarrow \det(A^T-\lambda I)=0$.

مسئله‌ی ۶.

$$\frac{\partial Y}{\partial X}=\frac{\partial}{\partial x}AX+\frac{\partial Z}{\partial X}$$

$$\frac{\partial AX}{\partial X}=A \longrightarrow$$ طبق مسئله‌ی ۵، ت

$$Z=\begin{bmatrix}(x_1x_2+1)^2\\(x_2x_3+1)^2\\(x_3x_1+1)^2\end{bmatrix}=\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}$$

$$\Rightarrow \frac{\partial Z}{\partial X}=\begin{bmatrix}\frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3}\\ \vdots & \ddots & \vdots\\ \frac{\partial z_3}{\partial x_1} & \cdots & \frac{\partial z_3}{\partial x_3}\end{bmatrix}$$

$$=\begin{bmatrix}2(x_1x_2+1)x_2 & 2(x_1x_2+1)x_1 & 0\\ 0 & 2(x_2x_3+1)x_3 & 2(x_2x_3+1)x_2\\ 2(x_3x_1+1)x_3 & 0 & 2(x_3x_1+1)x_1\end{bmatrix}$$

$$\Rightarrow \frac{\partial Y}{\partial X}=\begin{bmatrix}1 & 2 & -1\\ 0 & 3 & 2\\ 1 & 4 & -2\end{bmatrix}+\begin{bmatrix}24 & 8 & 0\\ 0 & 4 & -12\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}25 & 10 & -1\\ 0 & 7 & -10\\ 1 & 4 & -2\end{bmatrix}$$