# Computational Complexity

Assignment 1

Due date: 30 Azar

**Problem 1.** Let $c_1, c_2, c_3$ are three positive integers such that $c_3 > 2$. Show or disprove that

$$c_1.n^{2^n}.2^{c_2 n.c_3^n} \in 2^{2^{O(n)}}.$$

**Problem 2.** Let $A, B$ be two languages in $P$ and let the strings $x \in A$ and $y \notin A$ are given. Prove that there is a polytime reduction from $A$ to $B$.

**Problem 3.** Consider the languages

- $A_1 = \{ < M > \ | \ L(M) \text{ is empty } \}$
- $A_2 = \{ < M, x > \ | \ M \text{ accepts the string } x \ \}$
- $B_1 = \{ \ (< M >, < N >) \mid L(N) \text{ is the complete of } L(M) \ \}$
- $B_2 = \{ \ (< M >, x) \mid M \text{ accepts the strings } x \text{ and } x^R \text{ (the revers of string } x \text{ )}\}$

where $M, N$ are Turing machine and $< M >$ is the representation of a machine $M$. For each $i \in \{1, 2\}$, we know that $A_i$ is undecidable, use this fact and prove that $B_i$ is undecidable.

**Problem 4.** Define a TM $M$ to be ***oblivious*** if its head movement does not depend on the input but only on the input length. That is, $M$ is oblivious if for every input $x \in \{0, 1\}^*$ and $i \in \mathbb{N}$, the location of each of $M$'s heads at the $i$th step of execution on input $x$ is only a function of $|x|$ and $i$. Show that for every $T : \mathbb{N} \to \mathbb{N}$, if $L \in \mathbf{DTIME}(T(n))$ then there is an oblivious TM that decides $L$ in time $O(T(n)^2)$.

**Problem 5.** In the *Two Hamilton Cycle* problem, we have a graph $G$ and a Hamilton cycle $C$ in $G$ and asked whether $G$ contains another Hamilton cycle. What is the complexity of this problem (polynomial solvable or NP-hard)?

**Problem 6.** In the *2023-SAT* problem, the input is a boolean formula $\phi$ and the query is if $\phi$ has at least 2023 different satisfying assignments. Prove that *2023-SAT* is NP-complete. (Hint: Make a reduction from SAT problem.)

**Problem 7.** Consider a partition $P = \{P_1, \cdots, P_n\}$ of set $\cup_{i=1}^{n} P_i$. A *nice sub-partition* of size k for $P$ is a subset $\{P'_1, \cdots, P'_k\}$ of $P$ such that $\cup_{i=1}^{k} P'_i = \cup_{i=1}^{n} P_i$. In *nice sub-partition* problem, the input is a partition $P$ and integer $k$ and the question is $P$ has a nice sub-partition of size at least $k$. Prove that this problem is NP-complete.

**Problem 8.** Prove that PRIMES$\in NP$ without using the fact that PRIMES$\in P$.
Hint: A number $n$ is prime iff for every prime factor $q$ of $n-1$, there exists a number $a \in \{2, \ldots, n-1\}$ satisfying $a^{n-1} = 1 \pmod n$ but $a^{(n-1)/q} \neq 1 \pmod n$. Use this fact and induction to guess certificates for prime factors of $n - 1$.

**Problem 9.** A language is called unary if every string in it is of the form $1^i$ (the string of $i$ ones) for some $i > 0$. Show that if there exists an $NP$-complete unary language then $P = NP$.

**Good luck!**