Computational Complexity

Assignment 1

Due date: 30 Azar

Problem 1. Let c_1, c_2, c_3 are three positive integers such that $c_3 > 2$. prove or disprove that

$$c_1.n^{2^n}.2^{c_2 n.c_3^n} \in 2^{2^{O(n)}}.$$

Problem 2. Let A, B be two languages in P and let the strings $x \in A$ and $y \notin A$ are given. Prove that there is a polytime reduction from A to B.

Problem 3. Consider the languages

- $A_1 = \{ < M > \ | \ L(M) \text{ is empty } \}$
- $A_2 = \{ < M, x > \ | \ M \text{ accepts the string } x \ \}$
- $B_1 = \{ \ (< M >, < N >) \mid L(N)$ is the compliment of $L(M)$, i.e. $L(N) = \Sigma^* \setminus L(M) \ \}$
- $B_2 = \{ \ (< M >, x) \mid M$ accepts the strings x and x^R (the reverse of string x)$\}$

where M, N are Turing machine and $< M >$ is the representation of a machine M and $L(M)$ is the language accepted by M. For each $i \in \{1, 2\}$, we know that A_i is undecidable, use this fact and prove that B_i is undecidable.

Problem 4. Define a TM M to be ***oblivious*** if its head movement does not depend on the input but only on the input length. That is, M is oblivious if for every input $x \in \{0,1\}^*$ and $i \in \mathbb{N}$, the location of each of M's heads at the ith step of execution on input x is only a function of $|x|$ and i. Show that for every $T : \mathbb{N} \to \mathbb{N}$, if $L \in \mathbf{DTIME}(T(n))$ then there is an oblivious TM that decides L in time $O(T(n)^2)$.

Problem 5. In the *Two Hamilton Cycle* problem, we have a graph G and a Hamilton cycle C in G and asked whether G contains another Hamilton cycle. What is the complexity of this problem (polynomial solvable or NP-hard)?

Problem 6. In the *2023-SAT* problem, the input is a boolean formula ϕ and the query is if ϕ has at least 2023 different satisfying assignments. Prove that *2023-SAT* is NP-complete. (Hint: Make a reduction from SAT problem.)

Problem 7. Let X be a set and $P = \{P_1, \cdots, P_n\}$ be a collection of subsets of X such that $\cup_{i=1}^n P_i = X$. A *nice sub-collection* for P is a subset $\{P'_1, \cdots, P'_k\}$ of P such that $\cup_{i=1}^k P'_i = X$. In *nice sub-collection* problem, the input is a collection P and integer k and the question is if P has a nice sub-collection of size at most k. Prove that this problem is NP-complete.

Problem 8. Prove that PRIMES $\in NP$ without using the fact that PRIMES $\in P$.
Hint: A number n is prime iff for every prime factor q of $n-1$, there exists a number $a \in \{2, \ldots, n-1\}$ satisfying $a^{n-1} = 1 \pmod{n}$ but $a^{(n-1)/q} \neq 1 \pmod{n}$. Use this fact and induction to guess certificates for prime factors of $n-1$.

Problem 9. A language is called unary if every string in it is of the form 1^i (the string of i ones) for some $i > 0$. Show that if there exists an NP-complete unary language then $P = NP$.

Good luck!