Draft

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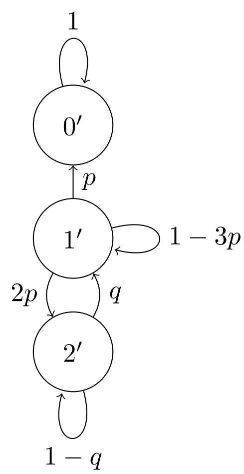
0.1 Part 1 (2nd approach)

0.1.1 Model

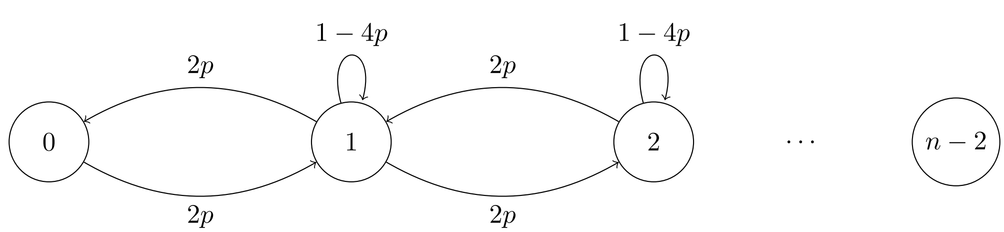
We model the problem with three systems 0, 1 and 2 corresponding to the following set of states of $\{n\}$, $\{0, n-2\}$ and $\{1, 2, 3, \ldots, n-3\}$ of previous model respectively. We denote the upper bound of stay of our random walker on expectation in system i by T_i.



We compute an upper bound for expected equilibrium time starting from any arbitrary system (T).

$$
\begin{aligned}
T &= p \times (T_2 + 2) \\
&\quad + p \times (2T_2 + 4) \times 2p \\
&\quad + p \times (3T_2 + 6) \times (2p)^2 \\
&\vdots \\
&= \frac{1}{2p} \sum_{i=1}^{\infty} pi\,(T_2 + 2)\,(2p)^i \\
&= \frac{p\,(T_2 + 2)}{(1 - 2p)^2}
\end{aligned} \tag{1}
$$

To calculate the value of T_2 we can use the settings that we had in the first approach.



We consider states $1, 2, \ldots, n-3$ corresponding to the configuration with $c = i \in \{0, 1, \ldots, n\}$. Let t_c denote the expected reaching time to state 0 or $n-2$, starting from state c. The changes in c are governed by the following non-homogeneous linear recurrence relation

$$t_c = 2pt_{c-1} + 2pt_{c+1} + (1-4p)t_c + 1 = \frac{1}{2}t_{c-1} + \frac{1}{2}t_{c+1} + \frac{1}{4p}, \quad c = 1, \ldots, n-3, \tag{2}$$

and following initial conditions

$$t_0 = 0 \qquad t_{n-2} = 0 \tag{3}$$

We solve the recurrence same way as the first approach. thus, the final closed form is:

$$t_c = \frac{c(n-c-2)}{4p} \tag{4}$$

For every $c \in \{0, 1, 2, 3, \ldots, n-2\}$ we have:

$$t_c \leq t_{\frac{n-2}{2}}$$

So

$$T_2 = t_{\frac{n-2}{2}} \tag{5}$$

By (1) and (5) we can infer that the expected equilibrium time would be at most

$$\frac{p\left(\frac{(n-2)^2}{16p} + 2\right)}{(1-2p)^2} \tag{6}$$

0.2 Extending 2nd approach to m systems

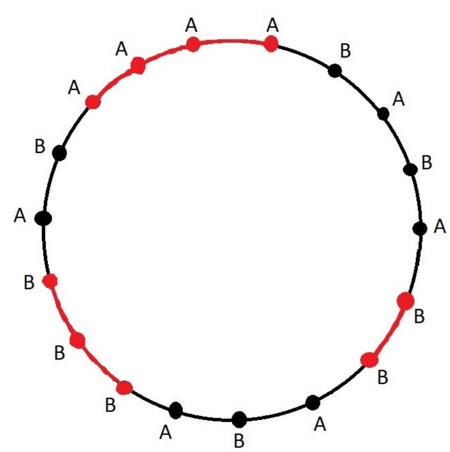
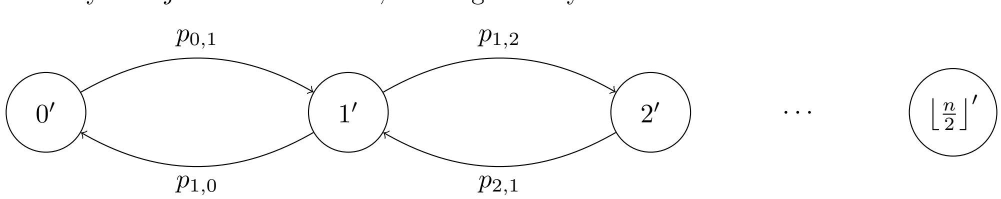


Figure 1: configuration with 3 red arcs of length one, two and three

For each set of configurations with same number of arcs, we consider a unique system that includes all these configurations. we denote the upper bound of stay of our random walker in system i by T_i. also $p_{i,j}$ denotes the probability of reaching from system i to system j in one step. and finally, $t_{i,j}$ denotes the reaching time to system j for the first time, starting from system i.



We may have following recurrence relation for ts:

$$t_{c,0} = t_{c,c-1} + t_{c-1,c-2} + \ldots + t_{1,0} \tag{7}$$

$$t_{c,c-1} = p_{c,c+1} t_{c+1,c-1} + (1 - p_{c,c+1} - p_{c,c-1})\, t_{c,c-1} + p_{c,c-1} + 1 \tag{8}$$

0.3 New approach for main problem (asynchronous version) (probably similar to the phase transition paper)

0.3.1 Model

For any configuration c with n agents there exists a string of 0s and 1s with length n, each 0s and 1s corresponding to black arc (equilibrium) and red arc (not equilibrium) respectively. the activation of any agent between arcs i and $i+1$ leads to one of the following modifications in the string:

00	11
01	10
10	01
11	11

We denote the number of red arcs in configuration c by $r(c)$. also we denote configuration in next step (current configuration is c) by $\delta(c)$. We denote the $r(\delta(c)) - r(c)$ by Δr and compute its expected value as follows:

$$\begin{aligned}\mathbb{E}[\Delta r] &= \mathbb{E}[\Delta r|\text{activation of agent between two red arcs}] \times \mathbb{P}\{\text{activation of agent between two red arcs}\} \\ &\quad + \mathbb{E}[\Delta r|\text{activation of agent not between two red arcs}] \times \mathbb{P}\{\text{activation of agent not between two red arcs}\} \\ &= -2 \times \mathbb{P}\{\text{activation of agent between two red arcs}\}\end{aligned} \tag{9}$$

If we have such a lower bound for $\mathbb{P}\{\text{activation of agent between two red arcs}\}$ in any configuration, we can calculate an upper bound for the expected time of reaching to equilibrium through dividing number of red arcs by $\mathbb{E}[\Delta r]$. (failed. because it can be zero :( )