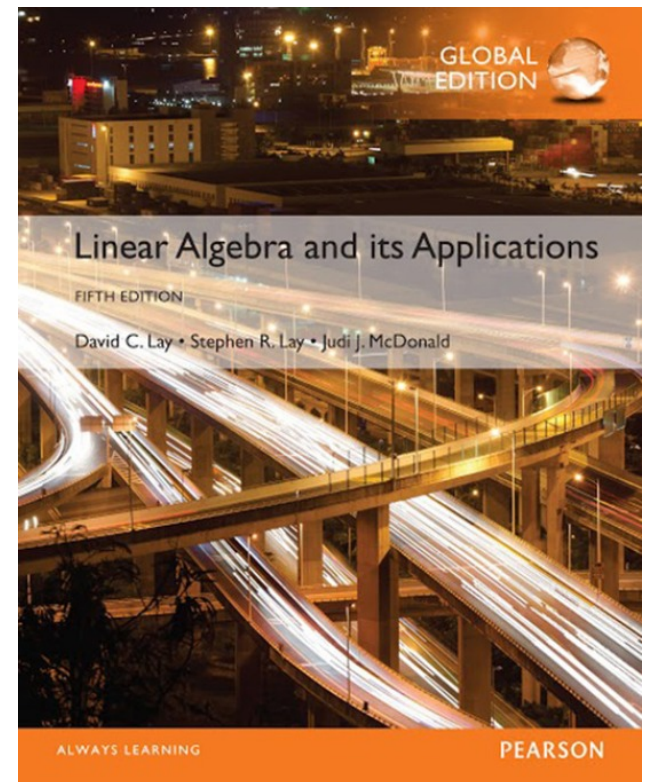


2

Matrix Algebra

2.8

SUBSPACES OF \mathbb{R}^n



SUBSPACES OF \mathbb{R}^n

- **Definition:** A **subspace** of \mathbb{R}^n is any set H in \mathbb{R}^n that has three properties:
 - a) The zero vector is in H .
 - b) For each \mathbf{u} and \mathbf{v} in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .
 - c) For each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

SUBSPACES OF \mathbb{R}^n

- A plane through the origin is the standard way to visualize the subspace in Example 1 on the next slide. See Fig. 1 below:

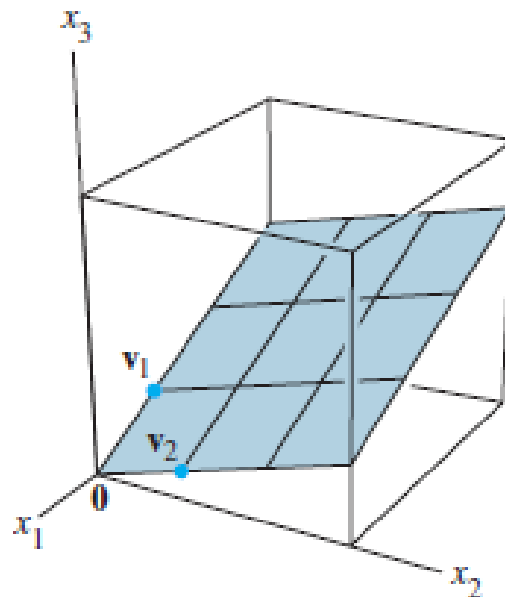


FIGURE 1

$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

SUBSPACES OF \mathbb{R}^n

- **Example 1** If v_1 and v_2 are in \mathbb{R}^n and $H = \text{Span}\{v_1, v_2\}$, then H is a subspace of \mathbb{R}^n . To verify this statement, note that the zero vector is in H (because $0v_1 + 0v_2$ is a linear combination of v_1 and v_2).

- Now take two arbitrary vectors in H , say,

$$u = s_1v_1 + s_2v_2 \quad \text{and} \quad v = t_1v_1 + t_2v_2$$

- Then

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

- which shows that $u + v$ is a linear combination of v_1 and v_2 and hence is in H . Also, for any scalar c , the vector cu is in H , because $cu = c(s_1v_1 + s_2v_2) = cs_1(v_1) + cs_2(v_2)$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **column space** of a matrix A is the set $\text{Col } A$ of all linear combinations of the columns of A .
- If $A = [a_1 \dots a_n]$ with the columns of \mathbb{R}^n , then $\text{Col } A$ is the same as $\text{Span}\{a_1 \dots a_n\}$. Example 4 shows that the column space of an $m \times n$ matrix is a subspace of \mathbb{R}^n .
- **Example 4** Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.
Determine whether b is in the column space of A .

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Solution:** The vector \mathbf{b} is a linear combination of the columns of A if and only if \mathbf{b} can be written as $A\mathbf{x}$ for some \mathbf{x} , that is, if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

- Row reducing the augmented matrix $[A\mathbf{b}]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- We conclude that $A\mathbf{x} = \mathbf{b}$ is consistent and \mathbf{b} is in $\text{Col } A$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **null space** of a matrix A is the set $\text{Nul}A$ of all solutions of the homogenous equation $Ax = 0$.
- **Theorem 12:** The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n . Equivalently, the set of all solutions of a system $Ax = 0$ of m homogenous linear equations in n unknowns is a subspace of \mathbb{R}^n .
- **Proof:** The zero vector is in $\text{Nul}A$ (because $A0 = 0$). To show that $\text{Nul}A$ satisfies that other two properties required for a subspace, take any \mathbf{u} and \mathbf{v} in $\text{Nul}A$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- That is, suppose $A\mathbf{u} = 0$ and $A\mathbf{v} = 0$. Then, by a property of matrix multiplication,

$$A(u + v) = Au + Av = 0 + 0 = 0$$

- Thus $\mathbf{u} + \mathbf{v}$ satisfies $A = 0$, and so $\mathbf{u} + \mathbf{v}$ is in $\text{Nul}A$. Also, for any scalar c , $A(c\mathbf{u}) = c(A\mathbf{u}) = c(0) = 0$, which shows that $c\mathbf{u}$ is in $\text{Nul}A$.

BASIS FOR A SUBSPACE

- **Definition:** A **basis** for a subspace H of \mathbb{R}^n is a linearly independent set in H that spans H .
- **Example 5** The columns of an invertible $n \times n$ matrix form a basis for all of because they are linearly independent and span \mathbb{R}^n , by the Invertible Matrix Theorem.

BASIS FOR A SUBSPACE

- One such matrix is the $n \times n$ identity matrix. Its columns are denoted by e_1, \dots, e_n :

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

- The set $\{e_1, \dots, e_n\}$ is called the **standard basis** for \mathbb{R}^n . See Fig. 3 on the next slide.

BASIS FOR A SUBSPACE

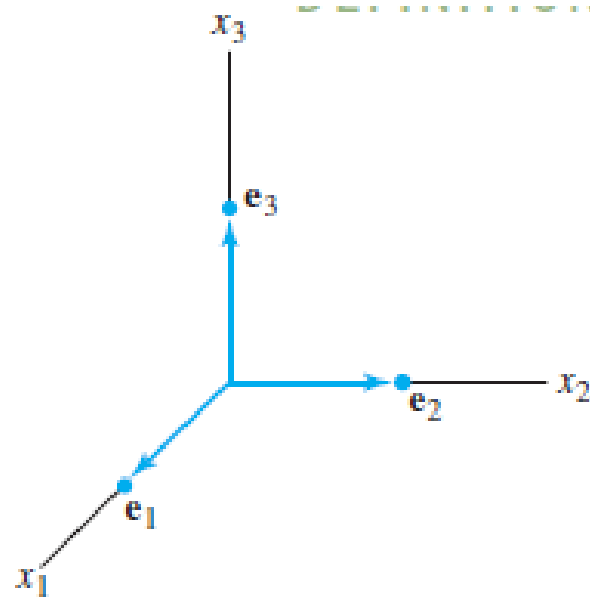


FIGURE 3

The standard basis for \mathbb{R}^3 .

- **Theorem 13:** The pivot columns of a matrix A form a basis for the column space of A .