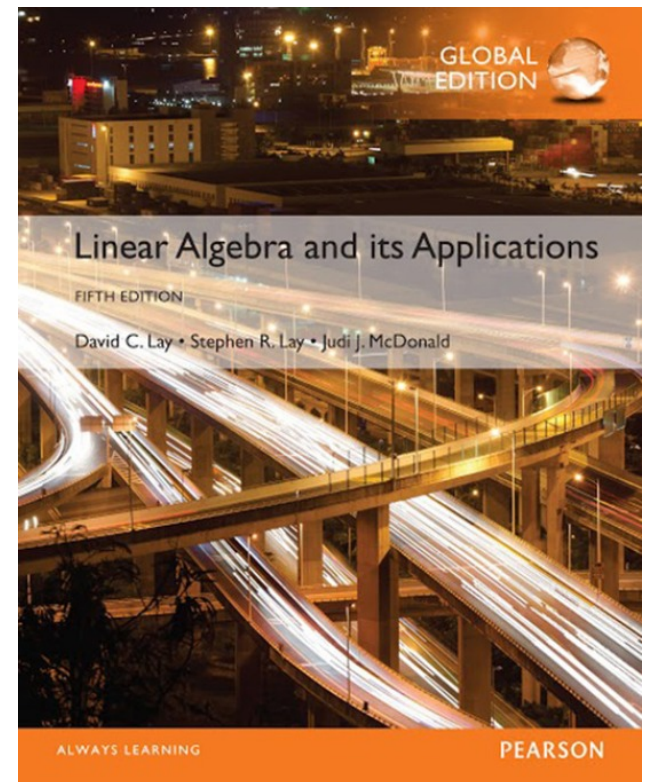


4

Determinants

4.7

CHANGE OF BASIS



CHANGE OF BASIS

- **Example 1** Consider two bases $\beta = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ for a vector space V , such that

$$b_1 = 4c_1 + c_2 \quad \text{and} \quad b_2 = -6c_1 + c_2 \quad (1)$$

- Suppose

$$x = 3b_1 + b_2 \quad (2)$$

- That is, suppose $[x]_\beta = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_{\mathcal{C}}$.

CHANGE OF BASIS

- **Solution** Apply the coordinate mapping determined by C to \mathbf{x} in (2). Since the coordinate mapping is a linear transformation,

$$\begin{aligned}[x]_C &= [3b_1 + b_2]_C \\ &= [3b_1]_C + [b_2]_C\end{aligned}$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[x]_C = [b_1]_C [b_2]_C \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad (3)$$

CHANGE OF BASIS

- This formula gives $[x]_C$, once we know the columns of the matrix. From (1),

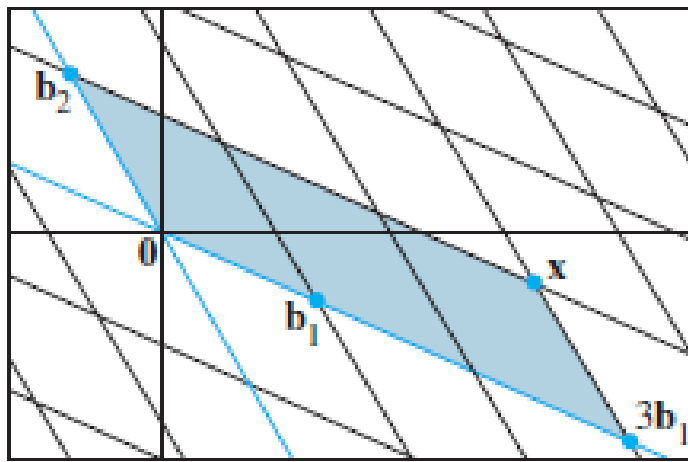
$$[b_1]_C = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } [b_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

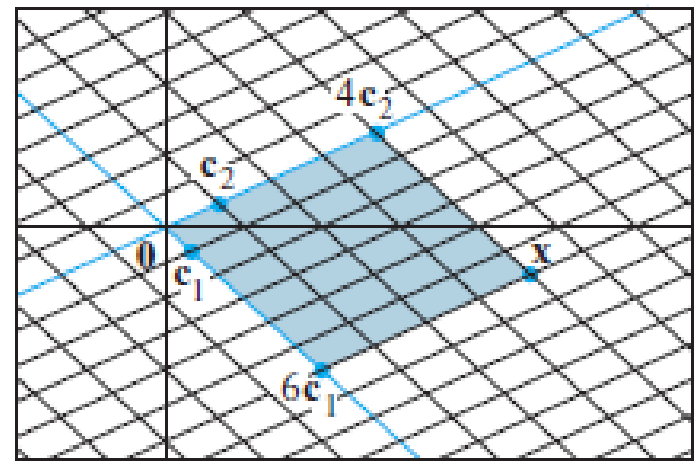
$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

- The C -coordinates of \mathbf{x} match those of the \mathbf{x} in Fig. 1, as seen on the next slide.

CHANGE OF BASIS



(a)



(b)

FIGURE 1 Two coordinate systems for the same vector space.

CHANGE OF BASIS

- **Theorem 15:** Let $\beta = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_2\}$ for a vector space V . Then there is a unique $n \times n$ matrix ${}^P c \leftarrow \beta$ such that

$$[x]_C = {}^P c \leftarrow \beta [x]_\beta \quad (4)$$

- The columns of ${}^P c \leftarrow \beta$ are the C-coordinate vectors of the vectors in the basis β . That is,

$${}^P c \leftarrow \beta = [[b_1]_C [b_2]_C \quad \dots \quad [b_n]_C] \quad (5)$$

CHANGE OF BASIS

- The matrix ${}^P_C \leftarrow \beta$ in Theorem 15 is called the **change-of-coordinates matrix from β to \mathcal{C}** . Multiplication by ${}^P_C \leftarrow \beta$ converts β -coordinates into \mathcal{C} -coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

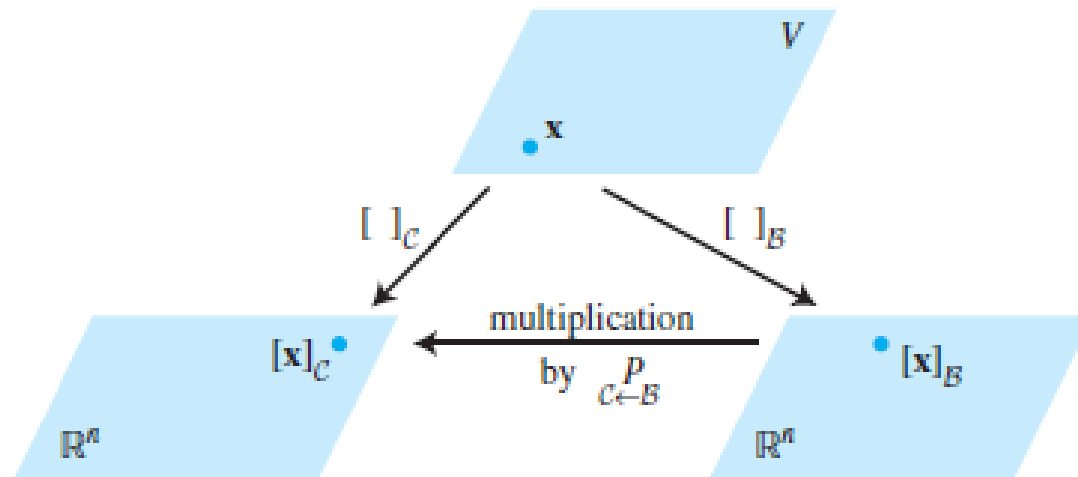


FIGURE 2 Two coordinate systems for V .

CHANGE OF BASIS

- The columns of $c \stackrel{P}{\leftarrow} \beta$ are linearly independent because they are the coordinate vectors of the linearly independent set β .
- Since $c \stackrel{P}{\leftarrow} \beta$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $(c \stackrel{P}{\leftarrow} \beta)^{-1}$ yields

$$(c \stackrel{P}{\leftarrow} \beta)^{-1} [x]_c = [x]_\beta$$

- Thus $(c \stackrel{P}{\leftarrow} \beta)^{-1}$ is the matrix that converts C-coordinates into β -coordinates. That is,

$$(c \stackrel{P}{\leftarrow} \beta)^{-1} = \beta \stackrel{P}{\leftarrow} c \quad (6)$$

CHANGE OF BASIS IN \mathbb{R}^n

- If $\beta = \{b_1, \dots, b_n\}$ and \mathcal{E} is the standard basis $\{e_1, \dots, e_n\}$ in \mathbb{R}^n , then $[b_1]_{\mathcal{E}} = b_1$, and likewise for the other vectors in β . In this case, $\mathcal{E} \xleftarrow{P} \beta$ is the same as the change-of-coordinates matrix P_β introduced in Section 4.4, namely,

$$P_\beta = [b_1 \ b_2 \ \dots \ b_n]$$

- To change coordinates between two nonstandard bases in \mathbb{R}^n , we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

CHANGE OF BASIS IN \mathbb{R}^n

- **Example 2** Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for \mathbb{R}^n given by $\beta = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$. Find the change-of-coordinates matrix from β to C .
- **Solution** The matrix $\beta \xleftarrow{P} C$ involves the C -coordinate vectors of b_1 and b_2 . Let $[b_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[b_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[c_1 \ c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1 \quad \text{and} \quad [c_1 \ c_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_2$$

CHANGE OF BASIS IN \mathbb{R}^n

- To solve both systems simultaneously, augment the coefficient matrix with b_1 and b_2 , and row reduce:

$$[c_1 \ c_2 : b_1 \ b_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array} \right] \quad (7)$$

- Thus

$$[b_1]_c = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{ and } [b_2]_c = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$${}^P_{c \leftarrow \beta} = \begin{bmatrix} [b_1]_c & [b_2]_c \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$