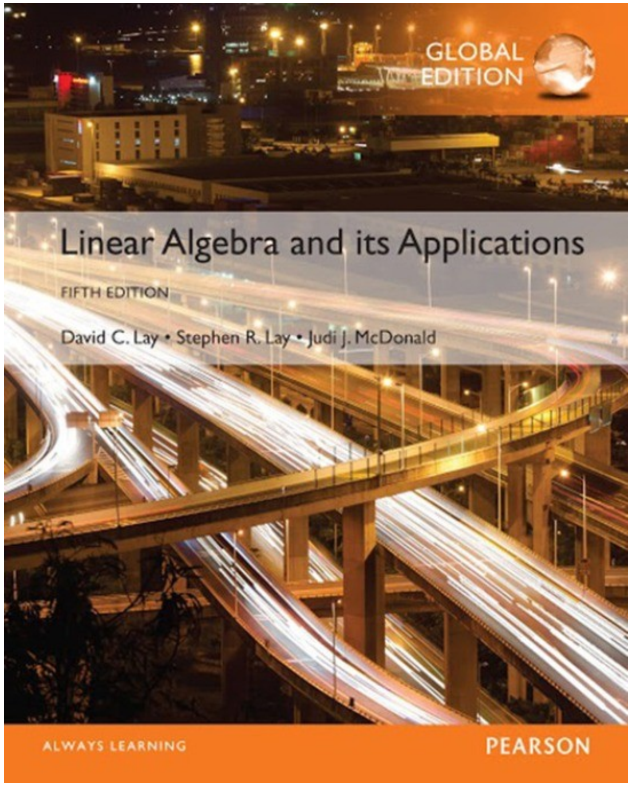
3 Determinants

3.1

INTRODUCTION TO DETERMINANTS



INTRODUCTION TO DETERMINANTS

- **Definition**: For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} det A_{12} + \cdots + (-1)^{1+n} a^{1n} det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det A_{1j}$$

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- **Example 1** Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute $\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$:

$$detA = 1 \cdot det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0 - 2) - 5(0 - 0) + 0(-4 - 0) = -2$$

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- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets.
  - Thus the calculation in Example 1 can be written as

$$detA = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

  - To state the next theorem, it is convenient to write the definition of det A in a slightly different form. Given $A = [a_{ij}]$, the **(i, j)-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} detA_{ij} \qquad (4)$$

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- Then

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}}$$

- This formula is called a **cofactor expansion across the first row** of A.

- **Theorem 1**: The determinant of an $n \times n$ matrix A can be computed by a cofactor across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}}$$

- The cofactor expansion down the j th column is

$$\boxed{\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}}$$

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- **Example 2** Use a cofactor expansion across the third row to compute det A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute

$$detA = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$
$$= (-1)^{3+1}a_{31}detA_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33}$$
$$= 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$
$$= 0 + 2(-1) + 0 = -2$$

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- **Theorem 2:** If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ * & a_{22} & 0 & 0 \\ * & * & a_{33} & 0 \\ * & * & * \;\cdots & a_{nn} \end{bmatrix}$$

Lower triangular matrix