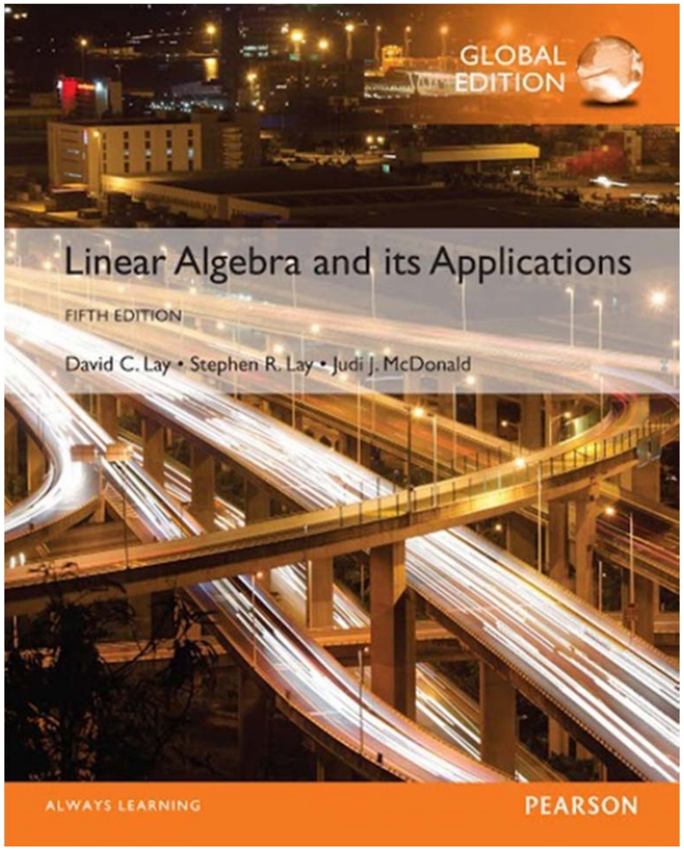
1 Linear Equations in Linear Algebra

1.10

LINEAR MODELS IN BUSINESS, SCIENCE, AND ENGINEERING



DIFFERENCE EQUATIONS

- In many fields such as ecology, economics, and engineering, a need arises to model mathematically a dynamic system that changes over time.
- Several features of the system are each measured at discrete time intervals, producing a sequence of vectors $x_0, x_1, x_2,$ …..The entries in x_k provide information about the *state* of the system at the time of the *k*th measurement.
- If there is a matrix *A* such that $x_1 = Ax_0, x_2 = Ax_1,$ and, in general

$$x_{k+1} = Ax_k \quad \text{for } k = 0, 1, 2, \ldots$$

- then (5) is called a **linear difference equation** (or **recurrence relation**)

DIFFERENCE EQUATIONS

- Given such an equation, one can compute x_1, x_2, and so on, provided x_0 is known.
- The discussion below illustrates how a difference equation might arise.
- A subject of interest to demographers is the movement of populations or groups of people from one region to another. The simple model here considers the changes in the population of a certain city and its surrounding suburbs over a period of years.

DIFFERENCE EQUATIONS

- Fix an initial year—say, 2000—and denote the populations of the city and suburbs that year by r_0 and s_0, respectively. Let x_0 be the population vector

$$\mathbf{x}_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} \begin{array}{l} \text{City population, 2000} \\ \text{Suburban population, 2000} \end{array}$$

- For 2001 and subsequent years, denote the populations of the city and suburbs by the vectors

$$\mathbf{x}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} r_3 \\ s_3 \end{bmatrix}, \ldots$$

- Our goal is to describe mathematically how these vectors might be related.

DIFFERENCE EQUATIONS

- Suppose demographic studies show that each year about 5% of the city's population moves to the suburbs (and 95% remains in the city), while 3% of the suburban population moves to the city (and 97% remains in the suburbs). See Fig. 2 below:

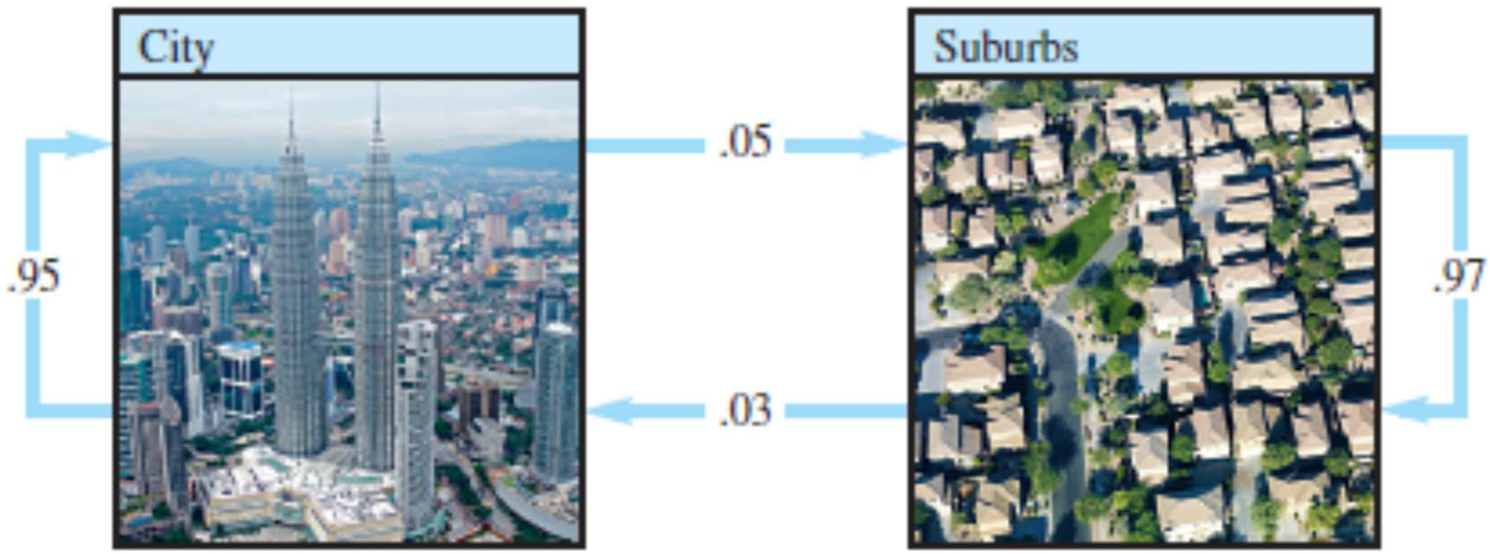


FIGURE 2 Annual percentage migration between city and suburbs.

DIFFERENCE EQUATIONS

- After 1 year, the original r_0 persons in the city are now distributed between city and suburbs as

$$\begin{bmatrix} .95r_0 \\ .05r_0 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} \begin{array}{l} \text{Remain in city} \\ \text{Move to suburbs} \end{array} \qquad (6)$$

- The so persons in the suburbs in 2000 are distributed 1 year later as

$$s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} \begin{array}{l} \text{Move to city} \\ \text{Remain in suburbs} \end{array} \qquad (7)$$

DIFFERENCE EQUATIONS

- The vectors in (6) and (7) account for all of the populations in 2001. Thus

$$\begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} + s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

- That is,

$$\mathbf{x}_1 = M\mathbf{x}_0 \qquad (8)$$

- Where M is the migration matrix determined by the following table:

$$\begin{array}{ccl} \multicolumn{2}{c}{\text{From:}} & \\ \text{City} & \text{Suburbs} & \text{To:} \\ \multicolumn{2}{c}{\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}} & \begin{array}{l} \text{City} \\ \text{Suburbs} \end{array} \end{array}$$

DIFFERENCE EQUATIONS

- Equation (8) describes how the population changes from 2000 to 2001. If the migration percentages remain constant, then the change from 2001 to 2002 is given by

$$\mathbf{x}_2 = M\mathbf{x}_1$$

- And similarly for 2002 to 2003 and subsequent years. In general,

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots \tag{9}$$

- The sequence of vectors $\{x_0, x_1, x_2, \ldots\}$ describes the population of the city/suburban region over a period of years.