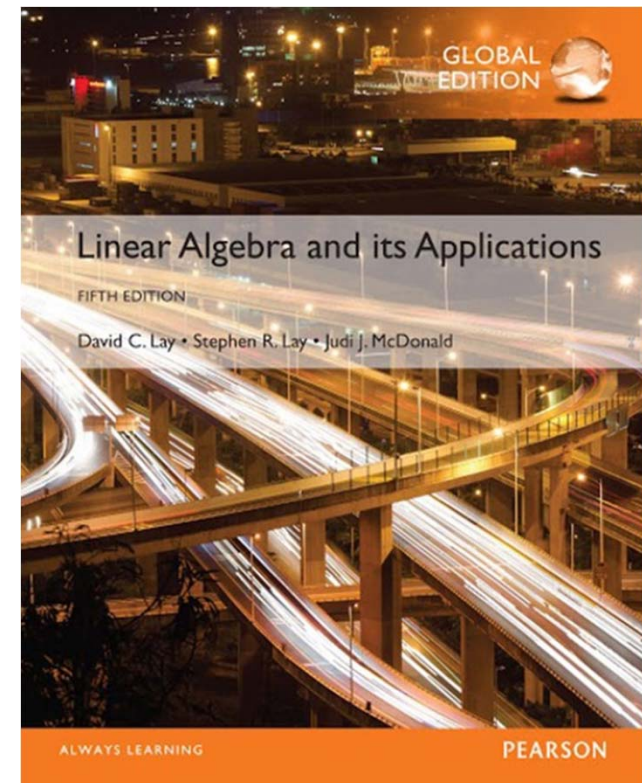


# 2

## Matrix Algebra

### 2.2

### THE INVERSE OF A MATRIX



# MATRIX OPERATIONS

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- An  $n \times n$  matrix  $A$  is said to be **invertible** if there is an  $n \times n$  matrix  $C$  such that

$$CA = I \quad \text{and} \quad AC = I$$

where  $I = I_n$ , the  $n \times n$  identity matrix.

- In this case,  $C$  is an **inverse** of  $A$ .
- In fact,  $C$  is uniquely determined by  $A$ , because if  $B$  were another inverse of  $A$ , then

$$B = BI = B(AC) = (BA)C = IC = C.$$

- This unique inverse is denoted by  $A^{-1}$ , so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

# MATRIX OPERATIONS

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- **Theorem 4:** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then

$A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If  $ad - bc = 0$ , then  $A$  is not invertible.

- The quantity  $ad - bc$  is called the **determinant** of  $A$ , and we write  $\det A = ad - bc$
- This theorem says that a  $2 \times 2$  matrix  $A$  is invertible if and only if  $\det A \neq 0$

# MATRIX OPERATIONS

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- **Theorem 5:** If  $A$  is an invertible  $n \times n$  matrix, then for each  $\mathbf{b}$  in  $\mathbb{R}^n$ , the equation  $A\mathbf{x} = \mathbf{b}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .
- **Proof:** Take any  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- A solution exists because if  $A^{-1}\mathbf{b}$  is substituted for  $\mathbf{x}$ , then  $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$ .
- So  $A^{-1}\mathbf{b}$  is a solution.
- To prove that the solution is unique, show that if  $\mathbf{u}$  is any solution, then  $\mathbf{u}$  must be  $A^{-1}\mathbf{b}$ .
- If  $A\mathbf{u} = \mathbf{b}$ , we can multiply both sides by  $A^{-1}$  and obtain  $A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}$ ,  $I\mathbf{u} = A^{-1}\mathbf{b}$ , and  $\mathbf{u} = A^{-1}\mathbf{b}$ .

# MATRIX OPERATIONS

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- **Theorem 6:**

- a. If  $A$  is an invertible matrix, then  $A^{-1}$  is invertible and
$$(A^{-1})^{-1} = A$$

- a. If  $A$  and  $B$  are  $n \times n$  invertible matrices, then so is  $AB$ , and the inverse of  $AB$  is the product of the inverses of  $A$  and  $B$  in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

- a. If  $A$  is an invertible matrix, then so is  $A^T$ , and the inverse of  $A^T$  is the transpose of  $A^{-1}$ . That is,

$$(A^T)^{-1} = (A^{-1})^T$$

# MATRIX OPERATIONS

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- **Proof:** To verify statement (a), find a matrix  $C$  such that

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I$$

- These equations are satisfied with  $A$  in place of  $C$ . Hence  $A^{-1}$  is invertible, and  $A$  is its inverse.
- Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

- A similar calculation shows that  $(B^{-1}A^{-1})(AB) = I$ .
- For statement (c), use Theorem 3(d), read from right to left,  $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ .
- Similarly,  $A^T (A^{-1})^T = I^T = I$ .

# ELEMENTARY MATRICES

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- Hence  $A^T$  is invertible, and its inverse is  $(A^{-1})^T$ .
- The generalization of Theorem 6(b) is as follows:  
The product of  $n \times n$  invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.
- An invertible matrix  $A$  is row equivalent to an identity matrix, and we can find  $A^{-1}$  by *watching the row reduction of  $A$  to  $I$* .
- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

# ELEMENTARY MATRICES

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■ **Example 5:** Let  $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ ,  $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute  $E_1A$ ,  $E_2A$ , and  $E_3A$ , and describe how these products can be obtained by elementary row operations on  $A$ .



# ELEMENTARY MATRICES

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- **Solution:** Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g - 4a & h - 4b & i - 4c \end{bmatrix}, E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

- Addition of  $-4$  times row 1 of  $A$  to row 3 produces  $E_1A$ .

# ELEMENTARY MATRICES

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- An interchange of rows 1 and 2 of  $A$  produces  $E_2A$ , and multiplication of row 3 of  $A$  by 5 produces  $E_3A$ .
- Left-multiplication (that is, multiplication on the left) by  $E_1$  in Example 1 has the same effect on any  $3 \times n$  matrix.
- Since  $E_1 \cdot I = E_1$ , we see that  $E_1$  *itself* is produced by this same row operation on the identity.

# ELEMENTARY MATRICES

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- Example 5 illustrates the following general fact about elementary matrices.
- If an elementary row operation is performed on an  $m \times n$  matrix  $A$ , the resulting matrix can be written as  $EA$ , where the  $m \times m$  matrix  $E$  is created by performing the same row operation on  $I_m$ .
- Each elementary matrix  $E$  is invertible. The inverse of  $E$  is the elementary matrix of the same type that transforms  $E$  back into  $I$ .

# ELEMENTARY MATRICES

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- **Theorem 7:** An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .
- **Proof:** Suppose that  $A$  is invertible.
- Then, since the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  (Theorem 5),  $A$  has a pivot position in every row.
- Because  $A$  is square, the  $n$  pivot positions must be on the diagonal, which implies that the reduced echelon form of  $A$  is  $I_n$ . That is,  $A$  is row equivalent to  $I_n$ .

# ELEMENTARY MATRICES

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- Now suppose, conversely, that .
- Then, since each step of the row reduction of  $A$  corresponds to left-multiplication by an elementary matrix, there exist elementary matrices  $E_1, \dots, E_p$  such that
- That is,

$$(1) \quad E_p \dots E_1 A = I_n$$

- Since the product  $E_p \dots E_1$  of invertible matrices is invertible, (1) leads to

$$(E_p \dots E_1)^{-1} (E_p \dots E_1) A = (E_p \dots E_1)^{-1} I_n$$

$$A = (E_p \dots E_1)^{-1}$$

## ALGORITHM FOR FINDING $A^{-1}$

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- Thus  $A$  is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = \left[ (E_p \dots E_1)^{-1} \right]^{-1} = E_p \dots E_1.$$

- Then  $A^{-1} = E_p \dots E_1 \cdot I_n$ , which says that  $A^{-1}$  results from applying  $E_1, \dots, E_p$  successively to  $I_n$ .
- This is the same sequence in (1) that reduced  $A$  to  $I_n$ .
- Row reduce the augmented matrix  $[A \ I]$ . If  $A$  is row equivalent to  $I$ , then  $[A \ I]$  is row equivalent to  $\begin{bmatrix} I & A^{-1} \end{bmatrix}$ . Otherwise,  $A$  does not have an inverse.

## ALGORITHM FOR FINDING $A^{-1}$

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- **Example 2:** Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}, \text{ if it exists.}$$

- **Solution:**

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

## ALGORITHM FOR FINDING $A^{-1}$

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$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$



## ALGORITHM FOR FINDING $A^{-1}$

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- Theorem 7 shows, since  $A \sim I$ , that  $A$  is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## ANOTHER VIEW OF MATRIX INVERSION

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- It is not necessary to check that  $A^{-1}A = I$  since  $A$  is invertible.