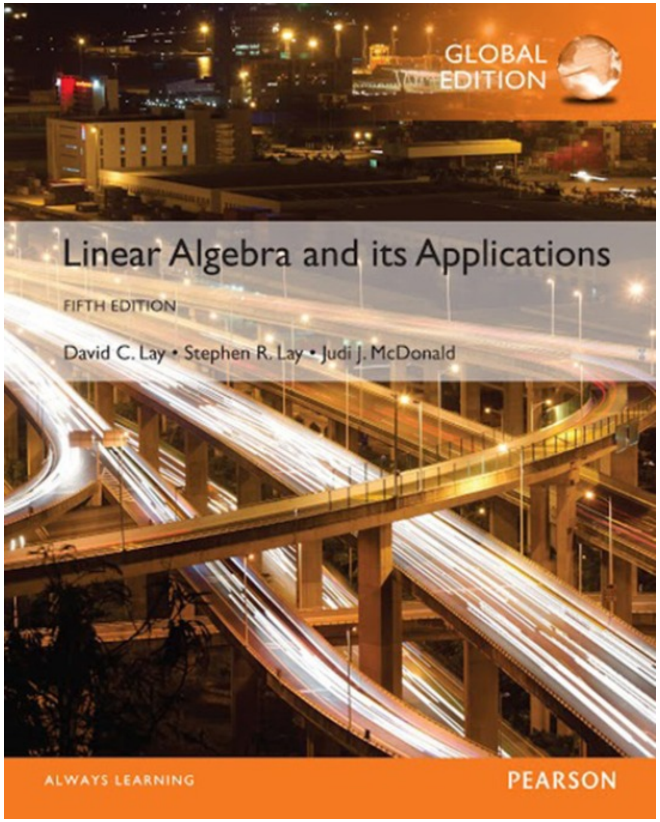
5 Eigenvalues and Eigenvectors

5.2

THE CHARACTERISTIC EQUATION



DETERMINANATS

- Let A be an $n \times n$ matrix, let U be any echelon form obtained from A by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges.
- Then the **determinant** of A, written as det A, is $(-1)^r$ times the product of the diagonal entries $u_{11}, \ldots, u_{nn}$ in U.
- If A is invertible, then $u_{11}, \ldots, u_{nn}$ are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's).

DETERMINANATS

- Otherwise, at least u_{nn} is zero, and the product $u_{11} \ldots u_{nn}$ is zero.

- Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix}, & \text{when A is invertible} \\ 0, & \text{when A is not invertible} \end{cases}$$

DETERMINANATS

- **Example 1:** Compute det A for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

- **Solution:** The following row reduction uses one row interchange:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = U_1$$

DETERMINANATS

- So $\det A$ equals $(-1)^1(1)(-2)(-1) = -2$.
- The following alternative row reduction avoids the row interchange and produces a different echelon form.
- The last step adds $-1/3$ times row 2 to row 3:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} = U_2$$

- This time $\det A$ is $(-1)^0(1)(-6)(1/3) = -2$, the same as before.

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- **Theorem:** Let A be an $n \times n$ matrix. Then A is invertible if and only if:
  - s. The number 0 is *not* an eigenvalue of A.
  - t. The determinant of A is *not* zero.

- **Theorem 3: Properties of Determinants**
- Let A and B be $n \times n$ matrices.
  - *a.* A is invertible if and only if $\det A \neq 0$
  - b. $\det AB = (\det A)(\det B)$.
  - c. $\det A^T = \det A$.

PROPERTIES OF DETERMINANTS

d. If A is triangular, then det A is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

THE CHARACTERISTIC EQUATION

- Theorem 3(a) shows how to determine when a matrix of the form $A-\lambda I$ is *not* invertible.
- The scalar equation $\det(A-\lambda I)=0$ is called the **characteristic equation** of A.
- A scalar λ is an eigenvalue of an $n\times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A-\lambda I)=0$$

PROOF OF CHARACTERISTIC EQUATION

- First, we suppose that λ is a root of the characteristic equation and prove that it is also an eigenvalue of A.
  - We have: $det(A - \lambda I) = 0$.
  - Therefore, $A - \lambda I$ is not invertible and columns of A are not independent.
  - Hence, there exists some $v \neq 0$, such that $(A - \lambda I)v = 0$.
  - This yields $Av = \lambda Iv = \lambda v$, which implies that λ is an eigenvalue of A.
- Walking backwards alongside the above argument provides the proof in the other direction.

THE CHARACTERISTIC EQUATION

- **Example 3:** Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Solution:** Form $A - \lambda I$, and use Theorem 3(d):

THE CHARACTERISTIC EQUATION

$$\det(A-\lambda I)=\det\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$=(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

- The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda)=0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1)=0$$

THE CHARACTERISTIC EQUATION

- Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

- If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A.
- The eigenvalue 5 in Example 3 is said to have *multiplicity* 2 because $(\lambda - 5)$ occurs two times as a factor of the characteristic polynomial.
- In general, the (**algebraic**) **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

SIMILARITY

- If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

- Writing Q for P^{-1}, we have $Q^{-1}BQ = A$.

- So B is also similar to A, and we say simply that A and B **are similar**.

- Changing A into $P^{-1}AP$ is called a **similarity transformation**.

SIMILARITY

- **Theorem 4**: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

- **Proof:** If $B = P^{-1}AP$ then,

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

- Using the multiplicative property (b) in Theorem (3), we compute

$$\begin{aligned} \det(B - \lambda I) &= \det\left[P^{-1}(A - \lambda I)P\right] \\ &= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P) \end{aligned} \qquad (2)$$

SIMILARITY

- Since $\det(P^{-1}) \cdot \det(P) = \det(P^{-1}P) = \det I = 1$, we see from equation (1) that $\det(B - \lambda I) = \det(A - \lambda I)$.

- **Warnings:**
  1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

SIMILARITY

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E). Row operations on a matrix usually change its eigenvalues.