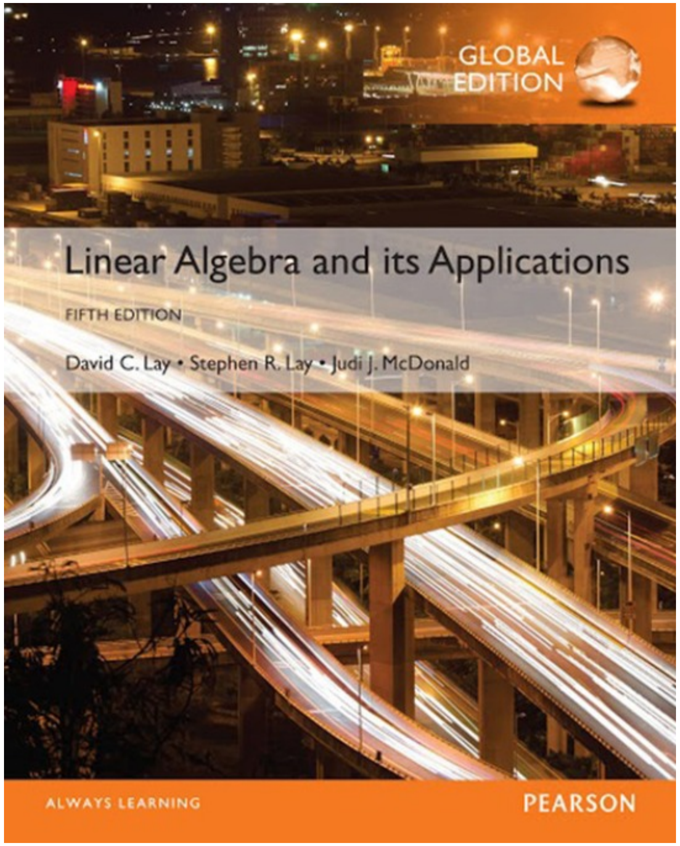
4 Vector Spaces

4.5

THE DIMENSION OF A VECTOR SPACE



DIMENSION OF A VECTOR SPACE

- **Theorem 9:** If a vector space V has a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.
- **Proof:** Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a set in V with more than n vectors.
- The coordinate vectors $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector.

DIMENSION OF A VECTOR SPACE

- So there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1[\mathrm{u}_1]_{\mathrm{B}} + \ldots + c_p[\mathrm{u}_p]_{\mathrm{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ The zero vector in $\mathbb{R}^n$

- Since the coordinate mapping is a linear transformation,

$$\left[c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p\right]_{\mathrm{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

DIMENSION OF A VECTOR SPACE

- The zero vector on the right displays the n weights needed to build the vector $c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p$ from the basis vectors in $\mathcal{B}$.

- That is, $c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p = 0 \cdot \mathrm{b}_1 + \ldots + 0 \cdot \mathrm{b}_n = 0$.

- Since the c_i are not all zero, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly dependent.
- Theorem 9 implies that if a vector space V has a basis $\mathcal{B} = \{\mathrm{b}_1, \ldots, \mathrm{b}_n\}$, then each linearly independent set in V has no more than n vectors.

DIMENSION OF A VECTOR SPACE

- **Theorem 10:** If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

- **Proof:** Let B_1 be a basis of n vectors and B_2 be any other basis (of V).
- Since B_1 is a basis and B_2 is linearly independent, B_2 has no more than n vectors, by Theorem 9.
- Also, since B_2 is a basis and B_1 is linearly independent, B_2 has at least n vectors.
- Thus B_2 consists of exactly n vectors.

DIMENSION OF A VECTOR SPACE

- **Definition:** If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.
- **Example 3:** Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

DIMENSION OF A VECTOR SPACE

- H is the set of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

- Clearly, $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, but $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$.
- By the Spanning Set Theorem, we may discard $\mathbf{v}_3$ and still have a set that spans H.

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- Finally, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent and hence is a basis for H.
- Thus $\dim H = 3$.

- **Theorem 11**: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- **Proof:** If $H = \{0\}$, then certainly $\dim H = 0 \le \dim V$.

- Otherwise, let $S = \{\mathrm{u}_1, \ldots, \mathrm{u}_k\}$ be any linearly independent set in H.

- If S spans H, then S is a basis for H.

- Otherwise, there is some u_{k+1} in H that is not in Span S.

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- But then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ will be linearly independent, because no vector in the set can be a linear combination of vectors that precede it (by Theorem 4).

- So long as the new set does not span H, we can continue this process of expanding S to a larger linearly independent set in H.

- But the number of vectors in a linearly independent expansion of S can never exceed the dimension of V, by Theorem 9.

THE BASIS THEOREM

- So eventually the expansion of S will span H and hence will be a basis for H, and $\dim H \leq \dim V$.

- **Theorem 12:** Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

- **Proof:** By Theorem 11, a linearly independent set S of p elements can be extended to a basis for V.

THE BASIS THEOREM

- But that basis must contain exactly p elements, since $\dim V = p$.
- So S must already be a basis for V.
- Now suppose that S has p elements and spans V.
- Since V is nonzero, the Spanning Set Theorem implies that a subset S' of S is a basis of V.
- Since $\dim V = p$, S' must contain p vectors.
- Hence $S = S'$.

THE DIMENSIONS OF NUL *A* AND COL *A*

- Let A be an $m \times n$ matrix, and suppose the equation $A\mathrm{x} = 0$ has k free variables.

- A spanning set for Nul A will produce exactly k linearly independent vectors—say, $\mathrm{u}_1, \ldots, \mathrm{u}_k$—one for each free variable.

- So $\{\mathrm{u}_1, \ldots, \mathrm{u}_k\}$ is a basis for Nul A, and the number of free variables determines the size of the basis.

DIMENSIONS OF NUL *A* AND COL *A*

- Thus, the dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.

- **Example 5:** Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

DIMENSIONS OF NUL *A* AND COL *A*

- **Solution:** Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three free variable—x_2, x_4 and x_5.
- Hence the dimension of Nul A is 3.
- Also dim Col $A = 2$ because A has two pivot columns.