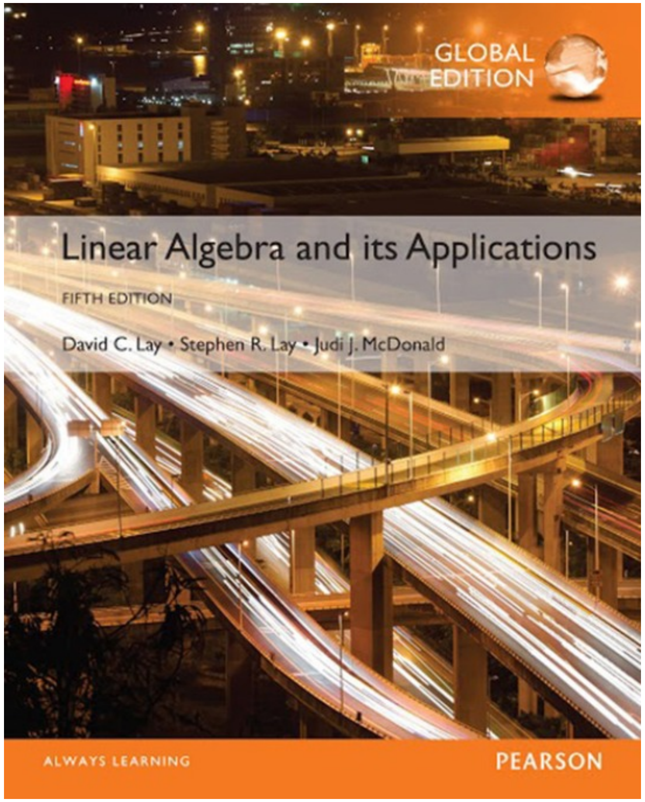
4 Vector Spaces

4.4

COORDINATE SYSTEMS



THE UNIQUE REPRESENTATION THEOREM

- **Theorem 7:** Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space V. Then for each **x** in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \qquad (1)$$

- **Proof:** Since $\mathcal{B}$ spans V, there exist scalars such that (1) holds.
- Suppose **x** also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

for scalars $d_1, \ldots, d_n$.

THE UNIQUE REPRESENTATION THEOREM

- Then, subtracting, we have
  $$0 = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \ldots + (c_n - d_n)\mathbf{b}_n \qquad (2)$$

- Since $\mathcal{B}$ is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \le j \le n$.

- **Definition:** Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and **x** is in V. **The coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinate of x**) are the weights c_1, …, c_n such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$

THE UNIQUE REPRESENTATION THEOREM

- If $c_1, \ldots, c_n$ are the **B**-coordinates of **x**, then the vector in $\mathbb{R}^n$

$$[\mathrm{x}]_{\mathrm{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

  is the **coordinate vector of x (relative to B)**, or the **B-coordinatevector ofx**.

- The mapping $\mathrm{x} \mapsto [\mathrm{x}]_{\mathrm{B}}$ is the **coordinate mapping (determined by B)**.

COORDINATES IN $\mathbb{R}^n$

- When a basis $\mathcal{B}$ for $\mathbb{R}^n$ is fixed, the $\mathcal{B}$-coordinate vector of a specified $\mathbf{x}$ is easily found, as in the example below.
- **Example 1:** Let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{b_1, b_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.
- **Solution:** The $\mathcal{B}$-coordinate c_1, c_2 of $\mathbf{x}$ satisfy

$$\underbrace{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b_1} + \underbrace{c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b_2} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\mathbf{x}}$$

COORDINATES IN $\mathbb{R}^n$

or

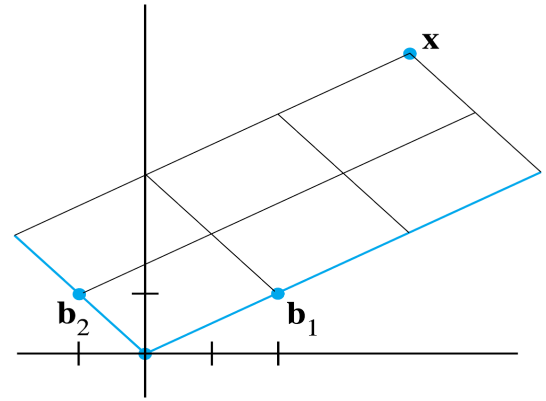
$$\begin{array}{cc} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ \mathbf{b}_1 \quad \mathbf{b}_2 & \qquad\qquad \mathbf{x} \end{array} \qquad (3)$$

- This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left.
- In any case, the solution is $c_1 = 3, c_2 = 2$.
- Thus $\mathrm{x} = 3\mathrm{b}_1 + 2\mathrm{b}_2$ and

$$[\mathrm{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

COORDINATES IN $\mathbb{R}^n$

- See the following figure.



The $\mathcal{B}$-coordinate vector of **x** is (3, 2).

- The matrix in (3) changes the B-coordinates of a vector **x** into the standard coordinates for **x**.
- An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $\mathrm{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.
- Let $P_{\mathrm{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$

COORDINATES IN $\mathbb{R}^n$

- Then the vector equation

$$\mathrm{x} = c_1\mathrm{b}_1 + c_2\mathrm{b}_2 + \ldots + c_n\mathrm{b}_n$$

is equivalent to

(4) $$\boxed{\mathrm{x} = P_{\mathcal{B}}[\mathrm{x}]_{\mathcal{B}}}$$

- $P_{\mathcal{B}}$ is called the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

- Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

- Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem).

COORDINATES IN $\mathbb{R}^n$

- Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts **x** into its $\mathcal{B}$-coordinate vector:

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

- The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping.

- Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by the Invertible Matrix Theorem.

THE COORDINATE MAPPING

- **Theorem 8:** Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

- **Proof:** Take two typical vectors in V, say,

$$\mathbf{u} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

$$\mathbf{w} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

- Then, using vector operations,

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{b}_1 + \ldots + (c_n + d_n)\mathbf{b}_n$$

THE COORDINATE MAPPING

- It follows that

$$[\mathbf{u}+\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{w}]_{\mathcal{B}}$$

- So the coordinate mapping preserves addition.

- If r is any scalar, then

$$r\mathbf{u} = r(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = (rc_1)\mathbf{b}_1 + \ldots + (rc_n)\mathbf{b}_n$$

THE COORDINATE MAPPING

- So

$$[r\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathbf{u}]_{\mathcal{B}}$$

- Thus the coordinate mapping also preserves scalar multiplication and hence is a linear transformation.
- The linearity of the coordinate mapping extends to linear combinations.
- If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, cp$ are scalars, then

$$\left[c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p\right]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \ldots + c_p\left[\mathbf{u}_p\right]_{\mathcal{B}} \quad (5)$$

THE COORDINATE MAPPING

- In words, (5) says that the B-coordinate vector of a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ is the *same* linear combination of their coordinate vectors.
- The coordinate mapping in Theorem 8 is an important example of an *isomorphism* from V onto $\mathbb{R}^n$.
- In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W.
- The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces.

THE COORDINATE MAPPING

- *Every vector space calculation in V is accurately reproduced in W, and vice versa.*
- In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$.
- **Example 7:** Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

  and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

THE COORDINATE MAPPING

- **Solution:** If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1\begin{bmatrix}3\\6\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}3\\12\\7\end{bmatrix}$$

- The scalars c_1 and c_2, if they exist, are the B-coordinates of $\mathbf{x}$.

THE COORDINATE MAPPING

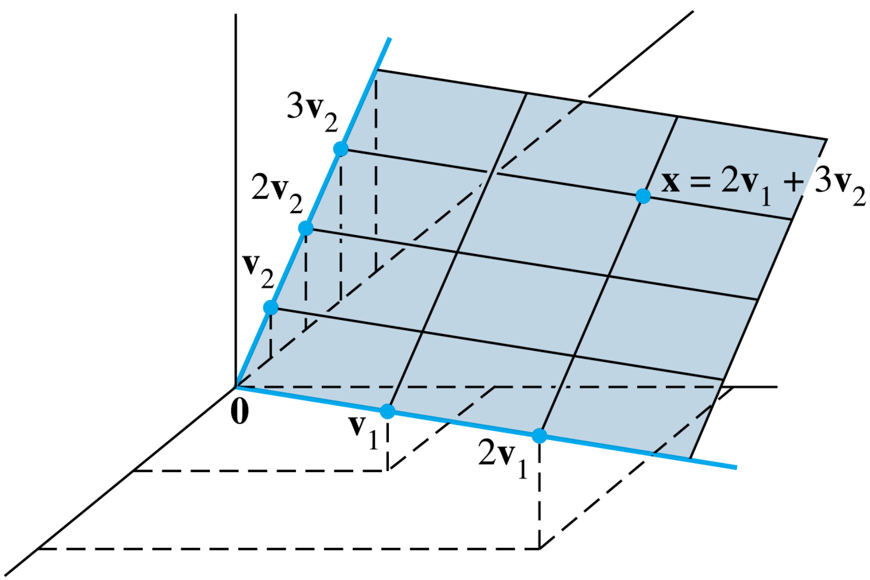
- Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

THE COORDINATE MAPPING

- The coordinate system on H determined by $\mathcal{B}$ is shown in the following figure.



A coordinate system on a plane H in $\mathbb{R}^3$.