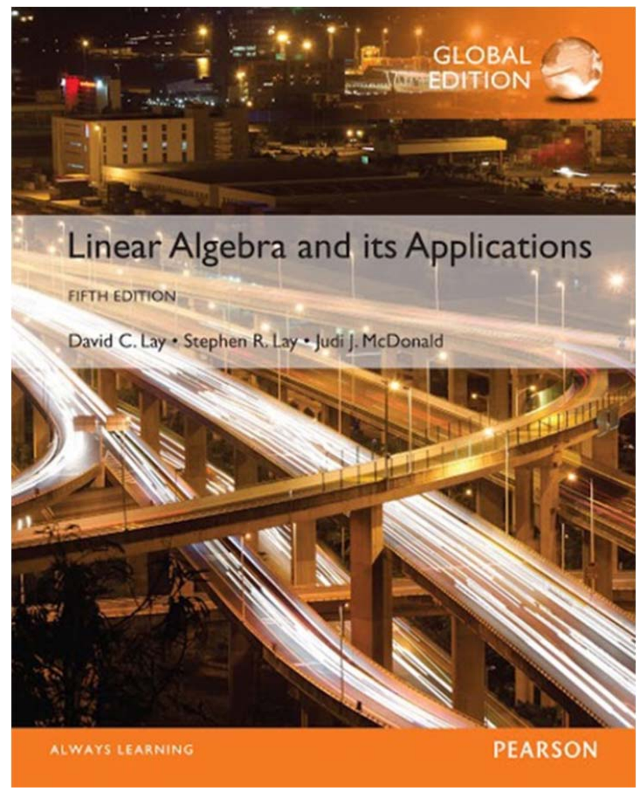
5 Eigenvalues and Eigenvectors

5.8

ITERATIVE ESTIMATES FOR EIGENVALUES



THE POWER METHOD

- The power method applies to an $n \times n$ matrix A with a **strictly dominant eigenvalue** λ_1, which means that λ_1 must be larger in absolute value than all the other eigenvalues.

THE POWER METHOD

- **The power method for estimating a strictly dominant eigenvalue**

1. Select an initial vector x_0, whose largest entry is 1.
2. For $k = 0, 1, \ldots,$
   a) Compute Ax_k.
   b) Let μ_k be an entry in Ax_k whose absolute value is as large as possible.
   c) Compute $x_{k+1} = (1/\mu_k)Ax_k$.
3. For almost all choices of x_0, the sequence $\{\mu_k\}$ approaches the dominant eigenvalue, and the sequence $\{x_k\}$ approaches a corresponding eigenvector.

THE POWER METHOD

- **Example 1** Let $A = \begin{bmatrix} 1.8 & .8 \\ .2 & 1.2 \end{bmatrix}$.
- A has eigenvalues 2 and 1, and the eigenspace for $\lambda_1 =$ 2 is the line through 0 and $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.
- Start from $x = \begin{bmatrix} -.5 \\ 1 \end{bmatrix}$.
- For k = 0, . . . , 8, compute $A^k x$ and construct the line through 0 and $A^k x$. What happens as *k* increases?

THE POWER METHOD

- **Solution** The first three calculations are

$$Ax = \begin{bmatrix} 1.8 & .8 \\ .2 & 1.2 \end{bmatrix} \begin{bmatrix} -.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -.1 \\ 1.1 \end{bmatrix}$$

$$A^2x = \begin{bmatrix} 1.8 & .8 \\ .2 & 1.2 \end{bmatrix} \begin{bmatrix} -.1 \\ 1.1 \end{bmatrix} = \begin{bmatrix} .7 \\ 1.3 \end{bmatrix}$$

$$A^3x = \begin{bmatrix} 1.8 & .8 \\ .2 & 1.2 \end{bmatrix} \begin{bmatrix} .7 \\ 1.3 \end{bmatrix} = \begin{bmatrix} 2.3 \\ 1.7 \end{bmatrix}$$

- Analogous calculations complete Table 1.

TABLE 1 Iterates of a Vector

k	0	1	2	3	4	5	6	7	8
$A^k\mathbf{x}$	$\begin{bmatrix} -.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -.1 \\ 1.1 \end{bmatrix}$	$\begin{bmatrix} .7 \\ 1.3 \end{bmatrix}$	$\begin{bmatrix} 2.3 \\ 1.7 \end{bmatrix}$	$\begin{bmatrix} 5.5 \\ 2.5 \end{bmatrix}$	$\begin{bmatrix} 11.9 \\ 4.1 \end{bmatrix}$	$\begin{bmatrix} 24.7 \\ 7.3 \end{bmatrix}$	$\begin{bmatrix} 50.3 \\ 13.7 \end{bmatrix}$	$\begin{bmatrix} 101.5 \\ 26.5 \end{bmatrix}$

THE POWER METHOD

- The vectors x, Ax, . . . , A^4x are shown in Fig. 1 below:

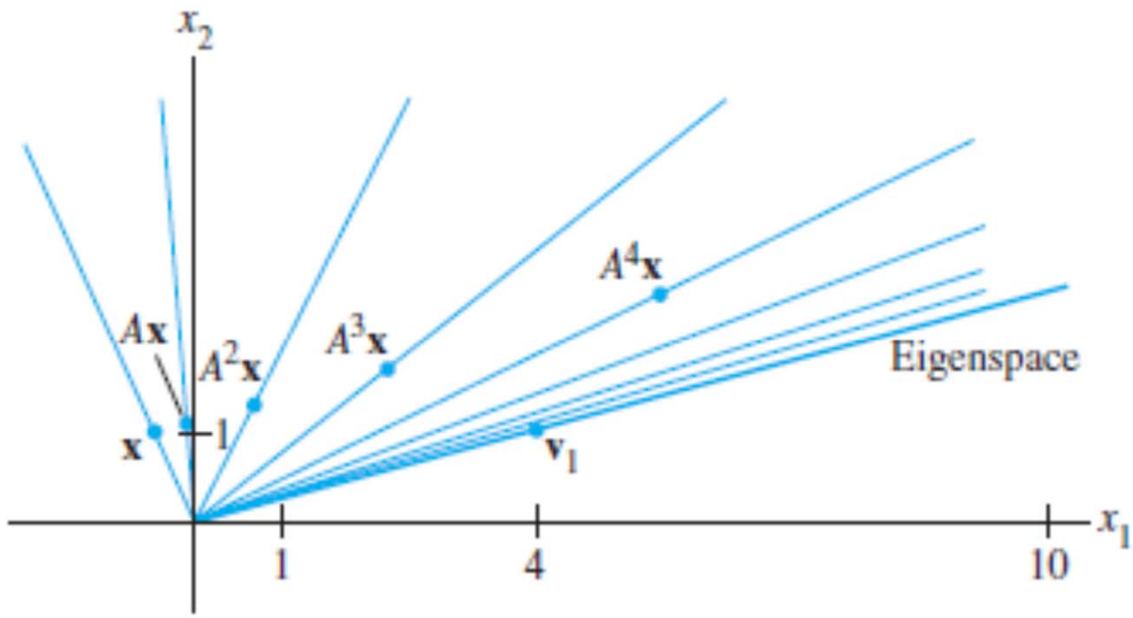


FIGURE 1 Directions determined by $\mathbf{x}$, $A\mathbf{x}$, $A^2\mathbf{x}$, . . . , $A^7\mathbf{x}$.

- The angle between the line (subspace) determined by A^kx and the line (eigenspace) determined by v_1 goes to zero as $k \longmapsto \infty$.

THE INVERSE POWER METHOD

- **The inverse power method for estimating an eigenvalue λ of A**

1. Select an initial estimate α sufficiently close to λ.
2. Select an initial vector x_0 whose largest entry is 1.
3. For $k = 0, 1, \ldots,$
   a) Solve $(A - \alpha I)y_k = x_k$ for y_k.
   b) Let μ_k be an entry in y_k whose absolute value is as large as possible.
   c) Compute $v_k = \alpha + (1/\mu_k)$
   d) Compute $x_{k+1} = (1/\mu_k)y_k$.
4. For almost all choices of x_0, the sequence $\{v_k\}$ approaches the eigenvalue λ of A, and the sequence $\{x_k\}$ approaches a corresponding eigenvector.

THE INVERSE POWER METHOD

- **Example 3** It is not uncommon in some applications to need to know the smallest eigenvalue of a matrix A and to have at hand rough estimates of the eigenvalues.
- Suppose **21**, **3.3**, and **1.9** are estimates for the eigenvalues of the matrix A below.
- Find the smallest eigenvalue, accurate to size decimal places.

$$A = \begin{bmatrix} 10 & -8 & -4 \\ -8 & 13 & 4 \\ -4 & 5 & 4 \end{bmatrix}$$

THE INVERSE POWER METHOD

- **Solution** The smallest eigenvalues seem close together so we use the inverse power method for $A - 1.9I$. Results of a MATLAB calculation are shown in Table 3 below:

TABLE 3 The Inverse Power Method

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .5736 \\ .0646 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .5054 \\ .0045 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .5004 \\ .0003 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .50003 \\ .00002 \\ 1 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 4.45 \\ .50 \\ 7.76 \end{bmatrix}$	$\begin{bmatrix} 5.0131 \\ .0442 \\ 9.9197 \end{bmatrix}$	$\begin{bmatrix} 5.0012 \\ .0031 \\ 9.9949 \end{bmatrix}$	$\begin{bmatrix} 5.0001 \\ .0002 \\ 9.9996 \end{bmatrix}$	$\begin{bmatrix} 5.000006 \\ .000015 \\ 9.999975 \end{bmatrix}$
μ_k	7.76	9.9197	9.9949	9.9996	9.999975
ν_k	2.03	2.0008	2.00005	2.000004	2.0000002

THE INVERSE POWER METHOD

- Here:
  - x_0 was chosen arbitrarily,
  - $y = (A - 1.9I)^{-1}x_k$,
  - μ_k is the largest entry in y_k,
  - $v_k = 1.9 + 1/\mu_k$, and
  - $x_{k+1} = (1/\mu_k)\, y_k$.

- The smallest eigenvalue is 2.
  - The estimated value is 2.0000002.