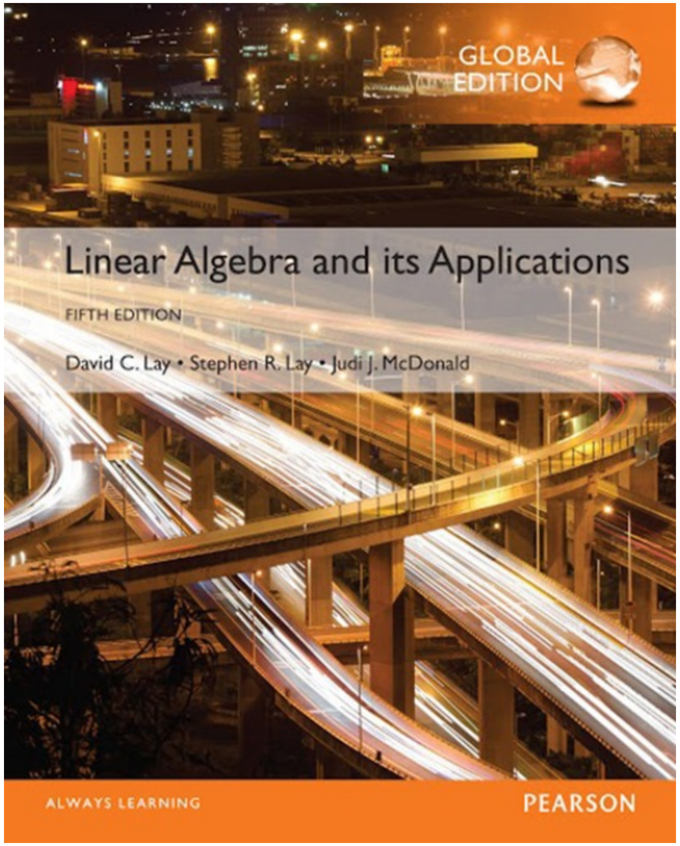
4 Vector Spaces

4.1

VECTOR SPACES AND SUBSPACES



VECTOR SPACES AND SUBSPACES

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.

  1. The sum of **u** and **v**, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
  2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
  3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
  4. There is a zero vector $\mathbf{0}$ in V such that

     $$\mathbf{u}+\mathbf{0}=\mathbf{u}.$$

5. For each **u** in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of **u** by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

- Using these axioms, we can show that the zero vector in Axiom 4 is unique, and the vector $-\mathbf{u}$, called the **negative** of **u**, in Axiom 5 is unique for each **u** in V.

VECTOR SPACES AND SUBSPACES

- For each **u** in V and scalar c,

$$0\mathbf{u} = \mathbf{0} \qquad (1)$$

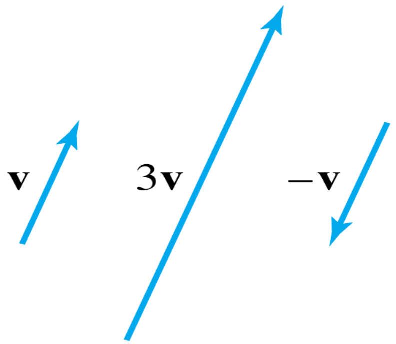
$$c\mathbf{0} = \mathbf{0} \qquad (2)$$

$$-\mathbf{u} = (-1)\mathbf{u} \qquad (3)$$

- **Example 2:** Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule, and for each **v** in V, define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of **v**, pointing in the same direction as **v** if $c \geq 0$ and otherwise pointing in the opposite direction.

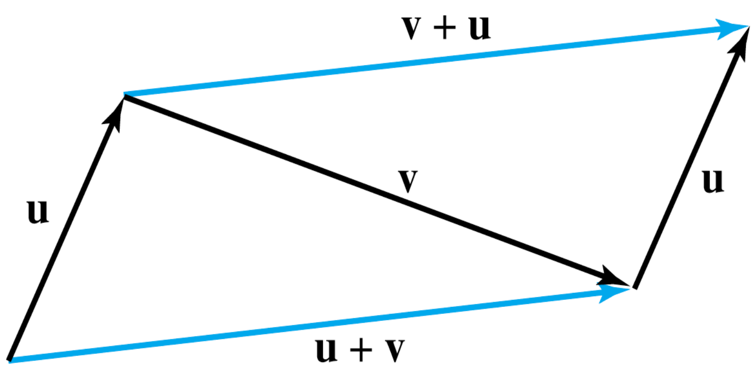
VECTOR SPACES AND SUBSPACES

- See the following figure below. Show that V is a vector space.

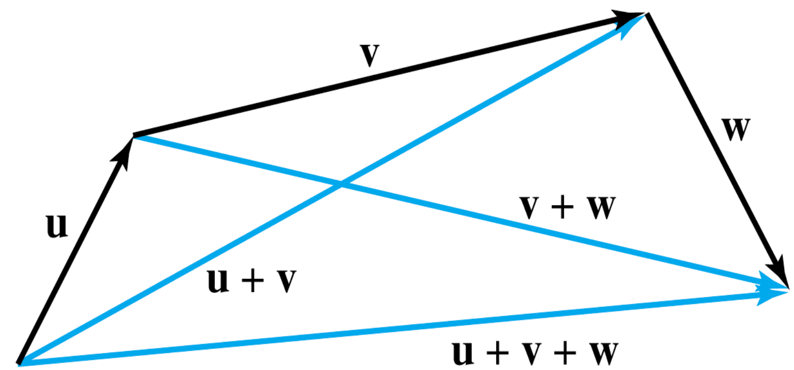


- **Solution:** The definition of V is geometric, using concepts of length and direction.
- No $x\ y\ z$-coordinate system is involved.
- An arrow of zero length is a single point and represents the zero vector.
- The negative of $\mathbf{v}$ is $(-1)\mathbf{v}$.
- So Axioms 1, 4, 5, 6, and 10 are evident. See the figures on the next slide.

SUBSPACES



$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$



$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$

- **Definition:** A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The zero vector of V is in H.
  b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

SUBSPACES

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- Properties (a), (b), and (c) guarantee that a subspace H of V is itself a *vector space*, under the vector space operations already defined in V.
- Every subspace is a vector space.
- Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

A SUBSPACE SPANNED BY A SET

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

- As the term **linear combination** refers to any sum of scalar multiples of vectors, and Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

A SUBSPACE SPANNED BY A SET

- **Example 10:** Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.
- **Solution:** The zero vector is in H, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
- To show that H is closed under vector addition, take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

- By Axioms 2, 3, and 8 for the vector space V,

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2\end{aligned}$$

A SUBSPACE SPANNED BY A SET

- So $\mathbf{u}+\mathbf{w}$ is in H.

- Furthermore, if c is any scalar, then by Axioms 7 and 9,
$$c\mathbf{u}=c(s_1\mathbf{v}_1+s_2\mathbf{v}_2)=(cs_1)\mathbf{v}_1+(cs_2)\mathbf{v}_2$$
which shows that $c\mathbf{u}$ is in H and H is closed under scalar multiplication.

- Thus H is a subspace of V.

A SUBSPACE SPANNED BY A SET

- **Theorem 1:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

- We call Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

- Give any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in H such that

$$H = \mathrm{Span}\{\mathrm{v}_1,\ldots\mathrm{v}_p\}.$$