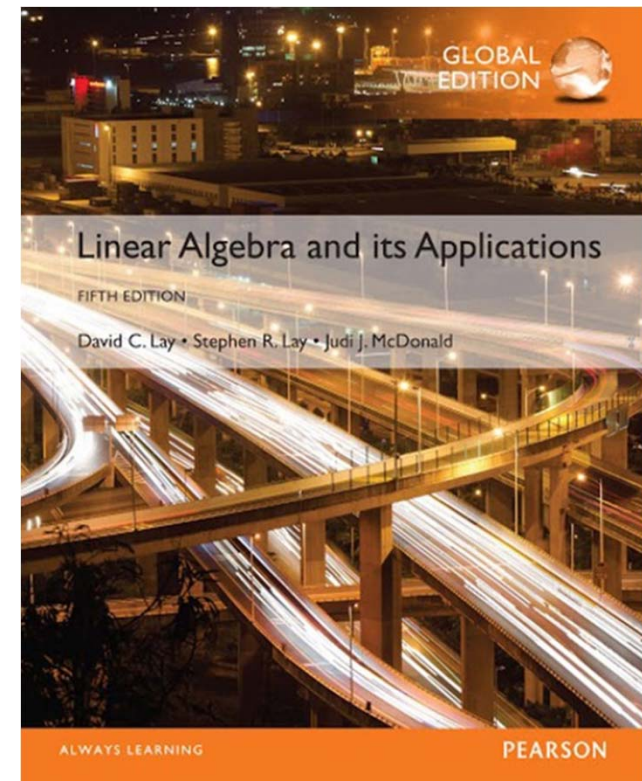


# 3 Determinants

## 3.3

### CRAMER'S RULE, VOLUME, AND LINEAR TRANSFORMATIONS



# CRAMER'S RULE

- **Theorem 7:** Let  $A$  be an invertible  $n \times n$  matrix. For any  $b$  in  $\mathbb{R}^n$ , the unique solution  $x$  of  $Ax=b$  has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \dots, n \quad (1)$$

- **Proof** Denote the columns of  $A$  by  $a_1, \dots, a_n$  and the columns of the  $n \times n$  identity matrix  $I$  by  $e_1, \dots, e_n$ . If  $Ax = b$ , the definition of matrix multiplication shows that

$$\begin{aligned} A \cdot I_i(x) &= A[e_1 \ \dots \ x \ \dots \ e_n] = [Ae_1 \ \dots \ Ax \ \dots \ Ae_n] \\ &= [a_1 \ \dots \ b \ \dots \ a_n] = A_i(b) \end{aligned}$$

# CRAMER'S RULE

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- By the multiplicative property of determinants,

$$(\det A)(\det I_i(x)) = \det A_i(b)$$

- The second determinant on the left is simply  $x_i$ . Hence  $(\det A) \cdot x_i = \det A_i(b)$ . This proves (1) because  $A$  is invertible and  $\det A \neq 0$ .

- **Example 1** Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

# CRAMER'S RULE

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- **Solution** View the system as  $Ax = b$ . Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

- Since  $\det A = 2$ , the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{24 + 30}{2} = 27$$

# A FORMULA FOR $A^{-1}$

■ **Theorem 8:** Let  $A$  be an invertible  $n \times n$  matrix. Then

$$A^{-1} = \frac{1}{\det A} \text{adj} A$$

■ **Example 3** Find the inverse of the matrix  $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$ .

■ **Solution** The nine cofactors are

$$C_{11} = + \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = - \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = - \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = + \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

## A FORMULA FOR $A^{-1}$

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- The adjugate matrix is the *transpose* of the matrix of cofactors. Thus

$$\text{adj}A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

- We could compute  $\det A$  directly, but the following computation provides a check on the calculations above and produces  $\det A$ :

$$(\text{adj}A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = 14I$$

## A FORMULA FOR $A^{-1}$

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- Since  $(\text{adj } A)A = 14I$ , Theorem 8 shows that  $\det A = 14$  and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

# PROOF OF A FORMULA FOR $A^{-1}$

- Let  $\mathbf{e}_j$  be the  $j^{\text{th}}$  column of identity matrix and  $\mathbf{x}$  be the  $j^{\text{th}}$  column of  $A^{-1}$ . we have:  $A\mathbf{x} = \mathbf{e}_j$

- $j^{\text{th}}$  entry of  $\mathbf{x}$  is the  $(i,j)$ -entry of  $A^{-1}$ . By Cramer's rule:

$$\{(i, j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

- A cofactor expansion down column  $i$ :

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

- So,  $\{(i, j)\text{-entry of } A^{-1}\}$  is equal to  $C_{ji}$  divided by  $\det A$ .

- Therefore:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$



# DETERMINANTS AS AREA OR VOLUME

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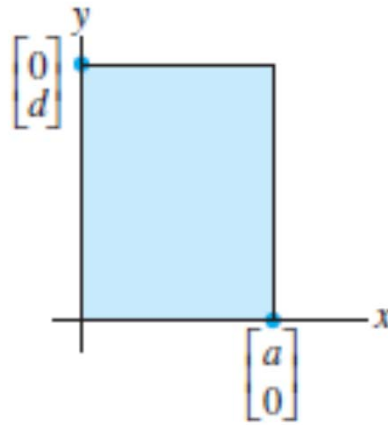
- **Theorem 9:** If  $A$  is a  $2 \times 2$  matrix, the area of the parallelogram determined by the columns of  $A$  is  $|\det A|$ . If  $A$  is a  $3 \times 3$  matrix, the volume of the parallelepiped determined by the columns of  $A$  is  $|\det A|$ .

- **Proof** The theorem is obviously true for any  $2 \times 2$  diagonal matrix:

$$\left| \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad| = \left\{ \begin{array}{l} \text{area of} \\ \text{rectangle} \end{array} \right\}$$

- See Fig. 1 on the next slide.

# DETERMINANTS AS AREA OR VOLUME



**FIGURE 1**

$$\text{Area} = |ad|.$$

- It will suffice to show that any  $2 \times 2$  matrix  $A = [a_1 \ a_2]$  can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor  $|\det A|$ .

# DETERMINANTS AS AREA OR VOLUME

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- It suffices to prove the following simple geometric observation that applies to vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ :
- Let  $a_1$  and  $a_2$  be nonzero vectors. Then for any scalar  $c$ , the area of the parallelogram determined by  $a_1$  and  $a_2$  equals the area of the parallelogram determined by  $a_1$  and  $a_2 + ca_1$ .
- To prove this statement, we may assume that  $a_2$  is not a multiple of  $a_1$ , for otherwise the two parallelograms would be degenerate and have zero area.
- If  $L$  is the line through 0 and  $a_1$ , then  $a_2 + L$  is the line through  $a_2$  parallel to  $L$ , and  $a_2 + ca_1$  is on this line. See Fig. 2 on the next slide.

# DETERMINANTS AS AREA OR VOLUME

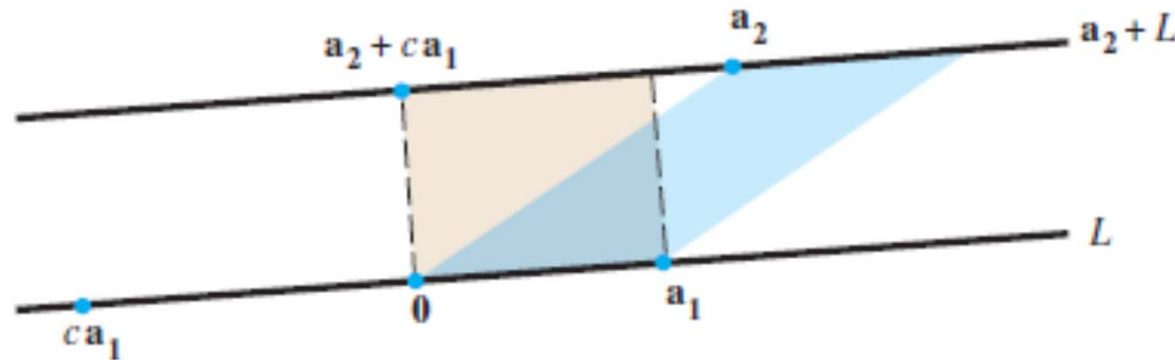
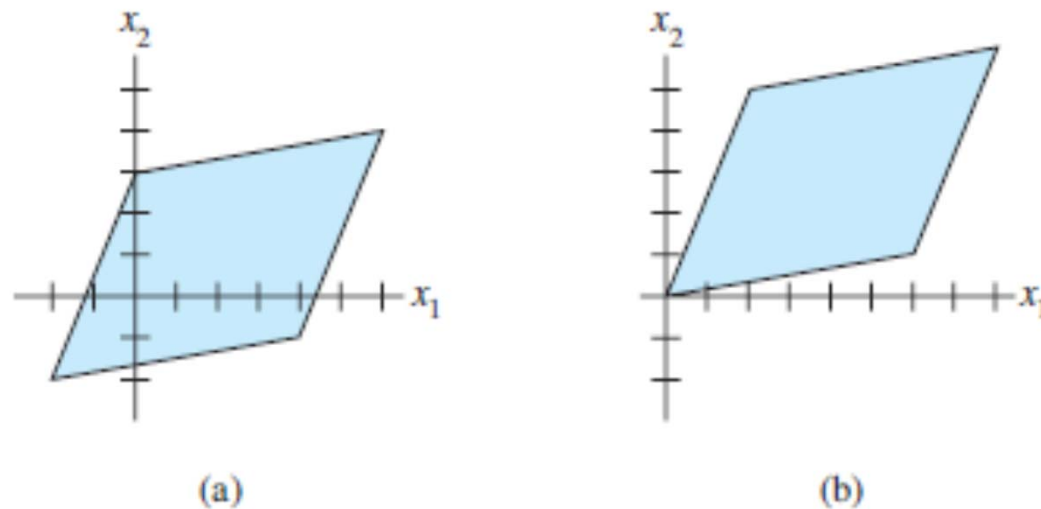


FIGURE 2 Two parallelograms of equal area.

- The points  $a_2$  and  $a_2 + ca_1$  have the same perpendicular distance to  $L$ . Hence the two parallelograms in Fig. 2 have the same area, since they share the base from 0 to  $a_1$ .

# DETERMINANTS AS AREA OR VOLUME

- **Example 4** Calculate the area of the parallelogram determined by the points  $(-2, -2)$ ,  $(0, 3)$ ,  $(4, -1)$ , and  $(6, 4)$ . See Fig. 5(a) below:



**FIGURE 5** Translating a parallelogram does not change its area.

# DETERMINANTS AS AREA OR VOLUME

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- **Solution** First translate the parallelogram to one having the origin as a vertex. For example, subtract the vertex  $(-2, -2)$  from each of the four vertices.
- The new parallelogram has the same area, and its vertices are  $(0, 0)$ ,  $(2, 5)$ ,  $(6, 1)$ , and  $(8, 6)$ . See Fig. 5(b) on the previous slide.
- This parallelogram is determined by the columns of
$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$
- Since  $|\det A| = |-28|$ , the area of the parallelogram is 28.

# LINEAR TRANSFORMATIONS

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- **Theorem 10:** Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation determined by a  $2 \times 2$  matrix  $A$ . If  $S$  is a parallelogram in  $\mathbb{R}^2$ , then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \quad (5)$$

- If  $T$  is determined by a  $3 \times 3$  matrix  $A$ , and if  $S$  is a parallelepiped in  $\mathbb{R}^3$ , then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \quad (6)$$

- **Proof** Consider the  $2 \times 2$  case, with  $A = [a_1 \ a_2]$ . A parallelogram at the origin in  $\mathbb{R}^2$  determined by vectors  $b_1$  and  $b_2$  has the form

$$S = \{s_1 b_1 + s_2 b_2: 0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\}$$

# LINEAR TRANSFORMATIONS

- The image of  $S$  under  $T$  consists of points of the form

$$\begin{aligned} T(s_1b_1 + s_2b_2) &= s_1T(b_1) + s_2T(b_2) \\ &= s_1Ab_1 + s_2Ab_2 \end{aligned}$$

- where  $0 \leq s_1 \leq 1$ ,  $0 \leq s_2 \leq 1$ . It follows that  $T(S)$  is the parallelogram determined by the columns of the matrix  $[Ab_1 \ Ab_2]$ . This matrix can be written as  $AB$ , where  $B = [b_1 \ b_2]$ .
- By Theorem 9 and the product theorem for determinants,

$$\begin{aligned} \{\text{area of } T(S)\} &= |\det AB| = |\det A| \cdot |\det B| \\ &= |\det A| \cdot \{\text{area of } S\} \end{aligned} \quad (7)$$



# LINEAR TRANSFORMATIONS

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- An arbitrary parallelogram has the form  $\mathbf{p} + S$ , where  $\mathbf{p}$  is a vector and  $S$  is a parallelogram at the origin.
- It is easy to see that  $T$  transforms  $\mathbf{p} + S$  into  $T(\mathbf{p}) + T(S)$ . Since translation does not affect the area of a set,

$$\begin{aligned}\{\text{area of } T(\mathbf{p} + S)\} &= \{\text{area of } T(\mathbf{p}) + T(S)\} \\ &= \{\text{area of } T(S)\} && \text{Translation} \\ &= |\det A| \cdot \{\text{area of } S\} && \text{By equation (7)} \\ &= |\det A| \cdot \{\text{area of } \mathbf{p} + S\} && \text{Translation}\end{aligned}$$

- This shows that (5) holds for all parallelograms in  $\mathbb{R}^2$ . The proof of (6) for the  $3 \times 3$  case is analogous.