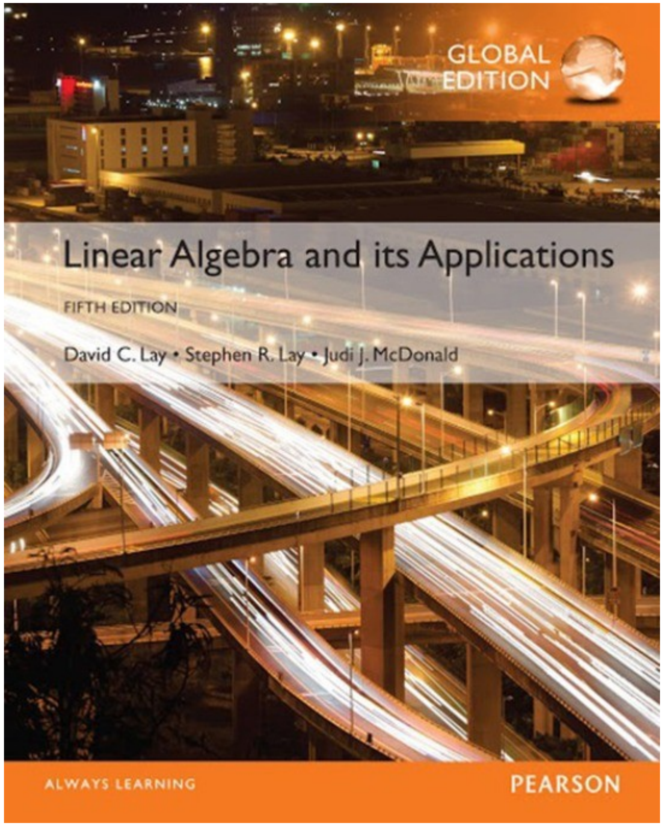
9 OPTIMIZATION

LINEAR PROGRAMMING



LINEAR PROGRAMMING

- A linear programming problem consists of:
  - (objective function) a linear function f from $\mathbb{R}^n$ into $\mathbb{R}$
  - (constraints) A system of linear inequalities in variables $x_1, \ldots, x_n$.
- The goal is to find a solution $\mathbf{x}$ that maximizes $f(\mathbf{x})$.

$$
\begin{array}{ll}
\text{Maximize} & 2x_1 + 3x_2 \\
\text{subject to} & 3x_1 + 2x_2 \leq 1200 \\
 & \;\;x_1 + 2x_2 \leq 800 \\
 & \;\;x_1 + \;\;x_2 \leq 450 \\
\text{and } x_1 \geq 0, x_2 \geq 0. &
\end{array}
$$

CANONICAL LINEAR PROGRAMMING

- Given: $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ in $\mathbb{R}^m$ $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$ an $m \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$
- Find:

an n-tuple $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ to maximize

$$f(x_1, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

- Subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}$$

and

$$x_j \ge 0 \quad \text{for } j = 1, \ldots, n$$

CANONICAL LINEAR PROGRAMMING

- This may be restated in the vector-matrix form:

$$\text{Maximize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$$
$$\text{subject to the constraints } A\mathbf{x} \leq \mathbf{b}$$
$$\text{and } \mathbf{x} \geq \mathbf{0}$$

- Feasible solution: a vector **x** that satisfies all the constraints.
- Feasible solutions ($\mathcal{F}$): the set of all feasible solutions.

- A vector $\overline{\mathbf{x}}$ in $\mathcal{F}$ is an optimal solution if

$$f(\overline{\mathbf{x}}) = \max_{\mathbf{x}\in\mathcal{F}} f(\mathbf{x})$$

CANONICAL LINEAR PROGRAMMING

- A canonical form is not very restrictive
  - To minimize a function, replace it with maximizing the negation of the function
  - A constraint of the form $a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$
  - can be replaced by $-a_{i1}x_1 - \cdots - a_{in}x_n \leq -b_i$
  - An equality constraint $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$
  - can be replaced with two inequalities:

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$$
$$-a_{i1}x_1 - \cdots - a_{in}x_n \leq -b_i$$

CANONICAL LINEAR PROGRAMMING

- If the constraints are inconsistent, the problem becomes infeasible.
- If the objective function takes on arbitrarily larges values, the problems becomes unbounded.
- Infeasible:

$$\begin{array}{ll} \text{Maximize} & 5x \\ \text{subject to} & x \leq 3 \\ & -x \leq -4 \\ & x \geq 0 \end{array}$$

- Unbounded:

$$\begin{array}{ll} \text{Maximize} & 5x \\ \text{subject to} & -x \leq 3 \\ & x \geq 0 \end{array}$$

CANONICAL LINEAR PROGRAMMING

- **Theorem.** If:
  - the feasible set is nonempty, and
  - the objective function is bounded on the feasible set,
- Then:
  - the canonical linear programming problem has at least one optimal solution.
  - At least one of the optimal solutions is an extreme point in the feasible solution.
- To find an optimal solution, we can evaluate objective function at each of the extreme points of the feasible set and select the point that gives the largest value.

CANONICAL LINEAR PROGRAMMING

- **Example**:

$$\text{Maximize } f(x_1, x_2) = 2x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 \qquad\quad &\le 30 \\ x_2 &\le 20 \\ x_1 + 2x_2 &\le 54 \end{aligned}$$

$$\text{and } x_1 \ge 0, x_2 \ge 0.$$

- **Solution**: The figure in the next slide shows the 5 extreme points (corresponding to the 5 vertices of the feasible set)
- They are found by solving appropriate pairs of linear equations
  - For example, $(14,20)$ is found by solving $x_1 + 2x_2 = 54$ and $x_2 = 20$.

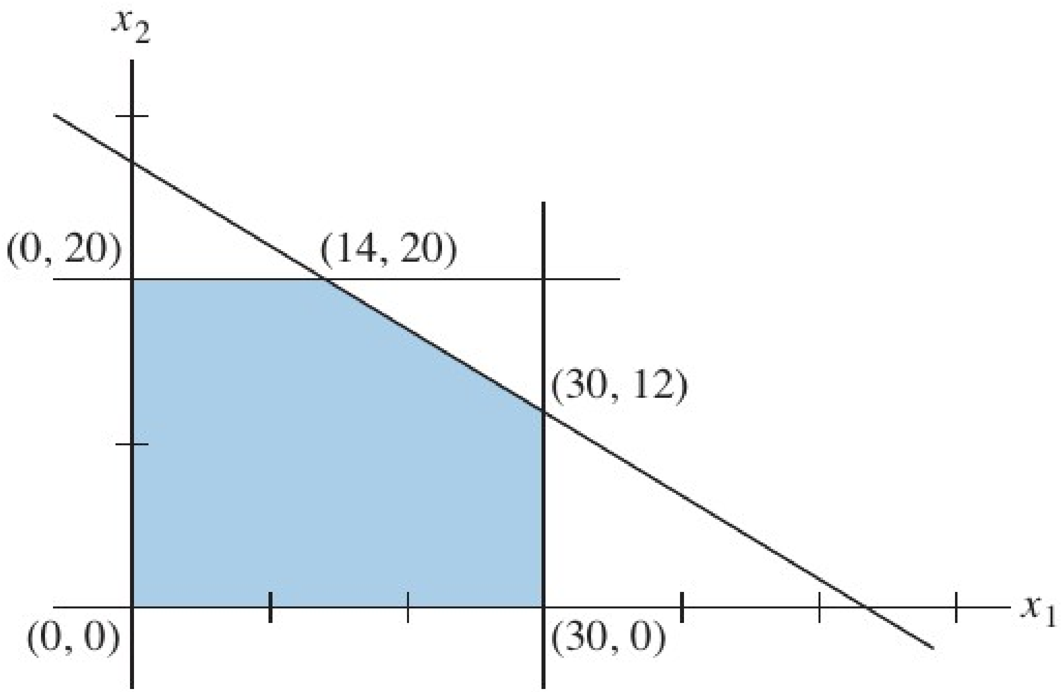
CANONICAL LINEAR PROGRAMMING

- **Solution:**

$$\text{Maximize } f(x_1, x_2) = 2x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 \quad\quad\quad &\le 30 \\ x_2 &\le 20 \\ x_1 + 2x_2 &\le 54 \end{aligned}$$

and $x_1 \ge 0, x_2 \ge 0.$



(x_1, x_2)	$2x_1 + 3x_2$
(0, 0)	0
(30, 0)	60
(30, 12)	96 ←
(14, 20)	88
(0, 20)	60

SIMPLEX ALGORITHM

1) Select an extreme point **x** of the feasible set.
2) Consider all edges of the feasible set that join to **x**. If the objective function does not increase by moving along any of these edges, **x** is an optimal solution.
3) If f is increasing by moving along one or more of the edges, follow the path that gives the largest increase and move to the extreme point at the opposite end.
4) Repeat steps 2 and 3.

SIMPLEX ALGORITHM

- In the following, we assume vector **b** is non-negative and the problem is in the canonical form (we have *max* and ≤).
- **Slack variable**: a non-negative variable added to the smaller side of an inequality to convert it into equality.
  - Example: $5x_1+7x_2 \leq 80 \;\rightarrow\; 5x_1+7x_2+x_3 = 80$ and $x_3 \geq 0$
- If A is $m\times n$, the addition of m slack variables in $A\mathbf{x}\leq\mathbf{b}$ gives a linear system with m equations and $n+m$ variable.
- A solution to this equation is called **basic feasible solution**, if at most m of the variables are positive (and all variables are non-negative).
- **Basic variables**: occur only in one equation and has a coefficient of 1.

SIMPLEX ALGORITHM

- **Example**: Find a basic solution for the following:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\le 60 \\ 3x_1 + x_2 + 5x_3 &\le 46 \\ x_1 + 2x_2 + x_3 &\le 50 \end{aligned}$$

- **Solution**:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + x_4 \qquad\qquad &= 60 \\ 3x_1 + x_2 + 5x_3 \qquad + x_5 \qquad &= 46 \\ x_1 + 2x_2 + x_3 \qquad\qquad + x_6 &= 50 \end{aligned}$$

- The following simple is the basic feasible solution (it corresponds to the extreme point (0,0,0)):

$$x_1 = x_2 = x_3 = 0, \quad x_4 = 60, \quad x_5 = 46, \quad \text{and} \quad x_6 = 50$$

- Basic variables (x_4,x_5,x_6) are said to be "in" the solution.
- Variables (x_1,x_2,x_3) are said to be "out" the solution.

SIMPLEX ALGORITHM

- In simplex, the role a variable plays, changes!
- Consider the system (of constraints):

$$
\begin{array}{c}
a_{11}x_1 + \cdots + a_{1k}x_k + \cdots + a_{1n}x_n = b_1 \\
\vdots \\
a_{i1}x_1 + \cdots + a_{ik}x_k + \cdots + a_{in}x_n = b_i \\
\vdots \\
a_{m1}x_1 + \cdots + a_{mk}x_k + \cdots + a_{mn}x_n = b_m
\end{array}
$$

SIMPLEX ALGORITHM

- Suppose we bring x_k “in” the solution, using equation p to pivot on the entry $a_{pk}\, x_k$.
- The basic solution of the resulting system is feasible iff the following conditions are satisfied:
  - Coefficient a_{pk} of x_k must be positive (when the p^{th} equation is divided by a_{pk}, the new b_p must be positive).
  - Ratio b_p/a_{pk} must be the smallest among all ratios b_i/a_{ik} for which $a_{ik}>0$ (in this case, the new terms b_i will be positive).

SIMPLEX ALGORITHM

- **Example**:

$$\text{Maximize } 25x_1 + 33x_2 + 18x_3$$

$$\begin{aligned} \text{subject to} \qquad 2x_1 + 3x_2 + 4x_3 &\le 60 \\ 3x_1 + x_2 + 5x_3 &\le 46 \\ x_1 + 2x_2 + x_3 &\le 50 \end{aligned}$$

$$\text{and } x_j \ge 0 \text{ for } j = 1, \ldots, 3.$$

- **Solution**:
- First, add *slack variables*.
- Then, change the objective function $25x_1 + 33x_2 + 18x_3$ into an *equation* by introducing a new variable $M = 25x_1 + 33x_2 + 18x_3$.
- Now, the goal is to maximize M where

$$-25x_1 - 33x_2 - 18x_3 + M = 0$$

SIMPLEX ALGORITHM

- Among all solutions of the following, find a solution for which $x_j \geq 0$ (j = 1,…,6) and M is as large as possible.

$$\begin{aligned}
2x_1 + 3x_2 + 4x_3 + x_4 \qquad\qquad &= 60 \\
3x_1 + x_2 + 5x_3 \quad + x_5 \qquad &= 46 \\
x_1 + 2x_2 + x_3 \qquad + x_6 \quad &= 50 \\
-25x_1 - 33x_2 - 18x_3 \qquad\quad + M &= 0
\end{aligned}$$

- First, form **initial simplex tableau**:

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array} \\
\left[\begin{array}{ccccccc|c}
2 & 3 & 4 & 1 & 0 & 0 & 0 & 60 \\
3 & 1 & 5 & 0 & 1 & 0 & 0 & 46 \\
1 & 2 & 1 & 0 & 0 & 1 & 0 & 50 \\
\hline
-25 & -33 & -18 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

- Basic solution is $x_1 = x_2 = x_3 = 0$, $x_4 = 60$, $x_5 = 46$, $x_6 = 50$, $M = 0$

SIMPLEX ALGORITHM

- M will rise when any of x_1, x_2 or x_3 rises ($M=25x_1+33x_2+18x_3$).
- Coefficient of x_2 is the largest of the three coefficients
  - Bringing x_2 into the solution will cause the greatest increase in M.
- To bring x_2 into the solution:
  - Compute the ratios b_i/a_{i2}, for each row i except the last one.
    - 60/3, 46/1, 50/2
    - 60/3 is the smallest, so row 1 is selected

$$
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\
\end{array}
$$
$$
\left[\begin{array}{ccccccc|c}
2 & 3 & 4 & 1 & 0 & 0 & 0 & 60 \\
3 & 1 & 5 & 0 & 1 & 0 & 0 & 46 \\
1 & 2 & 1 & 0 & 0 & 1 & 0 & 50 \\
\hline
-25 & -33 & -18 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

SIMPLEX ALGORITHM

- The result is:

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} \frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 20 \\ \frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 0 & 26 \\ -\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 0 & 10 \\ \hline -3 & 0 & 26 & 11 & 0 & 0 & 1 & 660 \end{array}\right]$$

- The basic feasible solution is: $x_1 = x_3 = x_4 = 0$, $x_2 = 20$, $x_5 = 26$, $x_6 = 10$, $M = 660$.
- $M = 660 + 3x_1 - 26x_3 - 11x_4$ and all the variables are non-negative
  - M rises only if x_1 increases.
  - Coefficients of x_3 and x_4 are negative, so their rise will decrease M.
  - So x_1 comes into the solution.

SIMPLEX ALGORITHM

- Compute the ratios b_i/a_{i1} for rows 1 and 2: 30 and 78/7.
  - The second one is smaller, so row 2 is selected:

$$
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
\frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 20 \\
\frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 0 & 26 \\
-\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 0 & 10 \\
\hline
-3 & 0 & 26 & 11 & 0 & 0 & 1 & 660
\end{array}\right]
$$

- The result is:

$$
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & \frac{88}{7} \\
1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{78}{7} \\
0 & 0 & -\frac{8}{7} & -\frac{5}{7} & \frac{1}{7} & 1 & 0 & \frac{96}{7} \\
\hline
0 & 0 & \frac{215}{7} & \frac{74}{7} & \frac{9}{7} & 0 & 1 & \frac{4854}{7}
\end{array}\right]
$$

SIMPLEX ALGORITHM

- The basic feasible solution is: $x_3=x_4=x_5=0$, $x_1=78/7$, $x_2=88/7$, $x_6=96/7$, $M=4854/7$
- $M = 4854/7 - 215/7\, x_3 - 74/7\, x_4 - 9/7\, x_5$
- All the coefficients are negative, so M can not be larger than 4854/7 (because x_3 and x_4 and x_5 are non-negative)!
- So the solution is optimal
  - The maximum value is 4854/7
  - It occurs when $x_1=78/7$ and $x_2=88/7$ and $x_3=0$.
    - The value of x_6 is not part of the solution, because it is *slack* variable!

SIMPLEX ALGORITHM FOR CANONICAL LINEAR PROGRAMMING

1) Change inequality constraints into equality by adding *slack variables*. Let M be a variable equal to objective function. Below the constraints equations, write:

$$- (\text{objective function}) + M = 0$$

2) Set up *initial simplex tableau*. The slack variables (and M) provide the initial basic feasible solution.
3) Check the last row of the tableau for optimality. If all the entries to the left of vertical line are non-negative, the solution is optimal. If some are negative, choose variable x_k for which the entry in the last row is as negative as possible.
4) Bring x_k into the solution. Do this by pivoting on the positive entry a_{pk} for which the non-negative ratio b_i/a_{ik} is the smallest.

SIMPLEX ALGORITHM FOR CANONICAL LINEAR PROGRAMMING

5) Repeat Steps 3-4, until all the entries in the last row are non-negative.

MINIMIZATION PROBLEMS

- **Example**: Minimize $x_1 + 2x_2$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2 &\geq 14 \\ x_1 - x_2 &\leq 2 \end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0.$

- **Solution**: the minimum of *f* over a set occurs at the same point as the maximum of -*f* over the same set.
- Moreover in the canonical form, the constraints must use ≤ .

$$\begin{aligned} \text{Maximize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -14 \\ & x_1 - x_2 \leq 2 \end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0.$

- Let $M = -x_1 - 2x_2$ and add slack variables:

MINIMIZATION PROBLEMS

$$\begin{aligned} -x_1 - x_2 + x_3 \qquad\qquad &= -14 \\ x_1 - x_2 \qquad + x_4 \qquad &= 2 \\ x_1 + 2x_2 \qquad\qquad + M &= 0 \end{aligned}$$

- The **initial simplex tableau** will be:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 0 & -14 \\ 1 & -1 & 0 & 1 & 0 & 2 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

- Corresponding basic solution: $x_1 = x_2 = 0$, $x_3 = -14$, $x_4 = 2$, $M=0$.
- Since x_3 is negative, this solution is not feasible.
  - Each term in the augmented column above horizontal line must be non-negative.

MINIMIZATION PROBLEMS

- For this, we find another negative entry in the same row
  - If there is no such entry, the problem has no feasible solution.
  - This negative entry corresponds to variable we bring into solution.
  - In our example, either x_1 or x_2 can be brought into.
- We bring x_2:
  - We select the entry a_{i2} (in column 2) for which the ratio b_i/a_{i2} is the smallest non-negative number.
  - In our example, only -14/-1 is non-negative, so the first row is selected:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 14 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ \hline -1 & 0 & 2 & 0 & 1 & -28 \end{array}\right]$$

MINIMIZATION PROBLEMS

- Now all entries in the augmented column (except the bottom entry) are non-negative, simplex can start.
- In the end, we will have the following tableau:

$$
\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & M & \\
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 6 \\
1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 8 \\
\hline
0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & -20
\end{array}\right]
$$

- Thus, maximum feasible value of $-x_1 - 2x_2$ is -20, when $x_1 = 8$ and $x_2 = 6$.
  - So, the minimum value of $x_1 + 2x_2$ is 20.

DUALITY

- Associated with each canonical maximization problem, there is a related minimization problem, called the *dual* problem:

Primal Problem P		**Dual Problem P^***	
Maximize	$f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$	Minimize	$g(\mathbf{y}) = \mathbf{b}^T\mathbf{y}$
subject to	$A\mathbf{x} \le \mathbf{b}$	subject to	$A^T\mathbf{y} \ge \mathbf{c}$
	$\mathbf{x} \ge \mathbf{0}$		$\mathbf{y} \ge \mathbf{0}$

Example: find the dual of the following primal problem:

$$\begin{array}{ll} \text{Maximize} & 5x_1 + 7x_2 \\ \text{subject to} & 2x_1 + 3x_2 \le 25 \\ & 7x_1 + 4x_2 \le 16 \\ & x_1 + 9x_2 \le 21 \\ \multicolumn{2}{l}{\text{and } x_1 \ge 0,\ x_2 \ge 0.} \end{array}$$

Solution:

$$\begin{array}{ll} \text{Minimize} & 25y_1 + 16y_2 + 21y_3 \\ \text{subject to} & 2y_1 + 7y_2 + y_3 \ge 5 \\ & 3y_1 + 4y_2 + 9y_3 \ge 7 \\ \multicolumn{2}{l}{\text{and } y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0.} \end{array}$$

DUALITY

- The dual of the dual problem is the original problem.
- **The Duality Theorem**: Let P be a linear programming problem with feasible set F and let P^* be the dual problem with feasible set F^*.
  - If F and F^* are both nonempty, P and P^* both have optimal solutions (say $\mathbf{x}$ and $\mathbf{y}$, respectively), and $f(\mathbf{x}) = g(\mathbf{y})$.
  - If one of the problems P or P^* has an optimal solution ($\mathbf{x}$ or $\mathbf{y}$, respectively), so does the other and $f(\mathbf{x}) = g(\mathbf{y})$.
  - If either P or P^* is solved by the simplex method, the solution of its dual is displayed in the bottom row of the final tableau in the columns associated with the *slack* variables.

DUALITY

- **Example**: Consider the following primal and dual problems:
  - The prime problem P:

$$
\begin{array}{ll}
\text{Maximize} & f(x_1, x_2, x_3) = 25x_1 + 33x_2 + 18x_3 \\
\text{subject to} & 2x_1 + 3x_2 + 4x_3 \le 60 \\
 & 3x_1 + x_2 + 5x_3 \le 46 \\
 & x_1 + 2x_2 + x_3 \le 50 \\
\multicolumn{2}{l}{\text{and } x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.}
\end{array}
$$

  - The dual problem P^*:

$$
\begin{array}{ll}
\text{Minimize} & g(y_1, y_2, y_3) = 60y_1 + 46y_2 + 50y_3 \\
\text{subject to} & 2y_1 + 3y_2 + y_3 \ge 25 \\
 & 3y_1 + y_2 + 2y_3 \ge 33 \\
 & 4y_1 + 5y_2 + y_3 \ge 18 \\
\multicolumn{2}{l}{\text{and } y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0.}
\end{array}
$$

DUALITY

- The final tableau for the primal problem is:

$$\begin{array}{c}
\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\
\left[\begin{array}{ccccccc|c}
0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & \frac{88}{7} \\
1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{78}{7} \\
0 & 0 & -\frac{8}{7} & -\frac{5}{7} & \frac{1}{7} & 1 & 0 & \frac{96}{7} \\
\hline
0 & 0 & \frac{215}{7} & \frac{74}{7} & \frac{9}{7} & 0 & 1 & \frac{4854}{7}
\end{array}\right]
\end{array}$$

- The *slack variables* are x_4 and x_5 and x_6.
- They give the optimal solution to the dual problem P^*.
  - Thus: $y_1 = 74/7$ and $y_2 = 9/7$ and $y_3 = 0$.
  - $g(74/7, 9/7, 0) = 60(74/7) + 46(9/7) + 50\,(0) = 4854/7$.