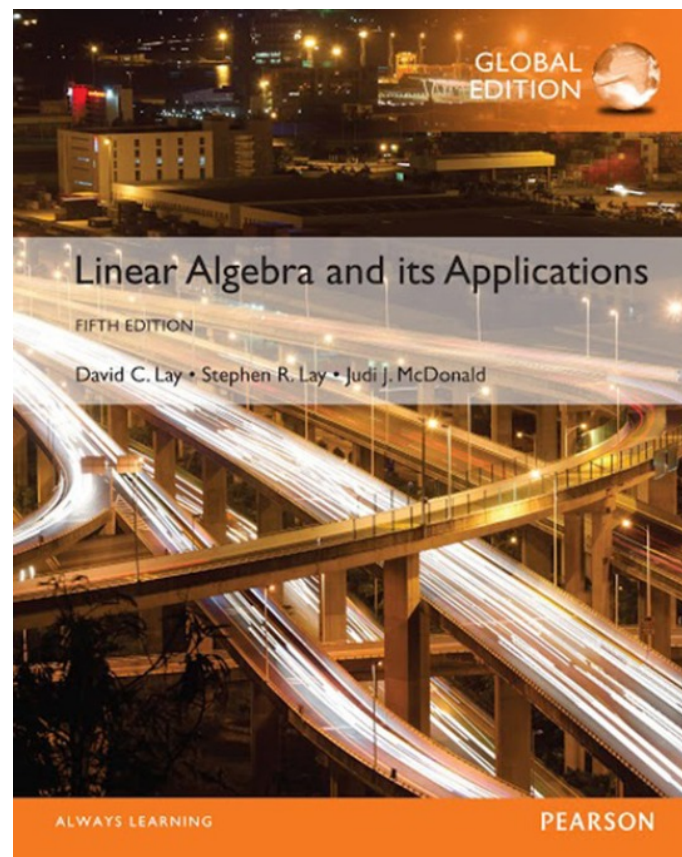


4 Vector Spaces

4.6

RANK



THE ROW SPACE

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in \mathbb{R}^n .
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by $\text{Row } A$.
- Each row has n entries, so $\text{Row } A$ is a subspace of \mathbb{R}^n .
- Since the rows of A are identified with the columns of A^T , we could also write $\text{Col } A^T$ in place of $\text{Row } A$.

THE ROW SPACE

- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B .
- **Proof:** If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A .
- It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A .

THE ROW SPACE

- Thus the row space of B is contained in the row space of A .
- Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B .
- So the two row spaces are the same.

THE ROW SPACE

- If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it. (Apply Theorem 4 to the nonzero rows of B in reverse order, with the first row last).
- Thus the nonzero rows of B form a basis of the (common) row space of B and A .

THE ROW SPACE

- **Example 2:** Find bases for the row space, the column space, and the null space of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution:** To find bases for the row space and the column space, row reduce A to an echelon form:

THE ROW SPACE

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B).
- Thus
Basis for Row A : $\{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

THE ROW SPACE

- For the column space, observe from B that the pivots are in columns 1, 2, and 4.

- Hence columns 1, 2, and 4 of A (not B) form a basis for Col A :

$$\text{Basis for Col } A: \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

- Notice that any echelon form of A provides (in its nonzero rows) a basis for Row A and also identifies the pivot columns of A for Col A .

THE ROW SPACE

- However, for $\text{Nul } A$, we need the *reduced echelon form*.
- Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THE ROW SPACE

- The equation $A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$, that is,

$$x_1 + x_3 + x_5 = 0$$

$$x_2 - 2x_3 + 3x_5 = 0$$

$$x_4 - 5x_5 = 0$$

- So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables.

THE ROW SPACE

- The calculations show that

$$\text{Basis for Nul } A : \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

- Observe that, unlike the basis for Col A , the bases for Row A and Nul A have no simple connection with the entries in A itself.

THE RANK THEOREM

- **Definition:** The **rank** of A is the dimension of the column space of A .
- Since Row A is the same as Col A^T , the dimension of the row space of A is the rank of A^T .
- The dimension of the null space is sometimes called the **nullity** of A .
- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A , also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

THE RANK THEOREM

- **Proof:** By Theorem 6, rank A is the number of pivot columns in A .
- Equivalently, rank A is the number of pivot positions in an echelon form B of A .
- Since B has a nonzero row for each pivot, and since these rows form a basis for the row space of A , the rank of A is also the dimension of the row space.
- The dimension of $\text{Nul } A$ equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
- Expressed another way, the dimension of $\text{Nul } A$ is the number of columns of A that are *not* pivot columns.

THE RANK THEOREM

- (It is the number of these columns, not the columns themselves, that is related to $\text{Nul } A$).

- Obviously,

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{nonpivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{number of} \\ \text{columns} \end{array} \right\}$$

- This proves the theorem.

THE RANK THEOREM

- **Example 3:**

- a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A ?
- b. Could a 6×9 matrix have a two-dimensional null space?

- **Solution:**

- a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence $\text{rank } A = 7$.
- b. No. If a 6×9 matrix, call it B , has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem.

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- But the columns of B are vectors in \mathbb{R}^6 , and so the dimension of $\text{Col } B$ cannot exceed 6; that is, $\text{rank } B$ cannot exceed 6.

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.
 - m. The columns of A form a basis of \mathbb{R}^n .
 - n. $\text{Col } A = \mathbb{R}^n$
 - o. $\text{Dim Col } A = n$
 - p. $\text{rank } A = n$

RANK AND THE INVERTIBLE MATRIX THEOREM

q. $\text{Nul } A = \{0\}$

r. $\text{Dim Nul } A = 0$

- **Proof:** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning.
- The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:
 $(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$

RANK AND THE INVERTIBLE MATRIX THEOREM

- Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n , implies (n), because $\text{Col } A$ is precisely the set of all \mathbf{b} such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.
- The implications $(n) \Rightarrow (o) \Rightarrow (p)$ follow from the definitions of dimension and rank.
- If the rank of A is n , the number of columns of A , then $\dim \text{Nul } A = 0$, by the Rank Theorem, and so $\text{Nul } A = \{\mathbf{0}\}$.

RANK AND THE INVERTIBLE MATRIX THEOREM

- Thus $(p) \Rightarrow (r) \Rightarrow (q)$.
- Also, (q) implies that the equation $Ax = 0$ has only the trivial solution, which is statement (d) .
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.