تمرین اول الگوریتم

1. Answer to the five mentioned questions for the following problems:
    a. Bubble Sort
    b. Sequential Search
    c. Binary Search

2. Let $p(n) = \sum_{i=0}^{d} a_i n^i$, where $a_d > 0$, be a degree-d polynomial in n, and let k be a constant. Use the definitions of the asymptotic notations to prove the following properties:
    a. If $k \geq d$, then $p(n) = O(n^k)$.
    b. If $k \leq d$, then $p(n) = \Omega(n^k)$.
    c. If $k = d$, then $p(n) = \Theta(n^k)$.

3. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures:
    a. $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.
    b. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$.
    c. $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.
    d. Either $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$ holds.

4. Solve the following recurrence equations:
    a. $T(n) = T(n/2) + 1$.
    b. $T(n) = 4T(n/2) + n^3$.
    c. $T(n) = 2T(n/4) + \sqrt{n}$.
    d. $T(n) = 3T(n/2) + n\log(n)$.
    e. $T(n) = 2T(n/2) + n/\log(n)$.
    f. $T(n) = 4T(n/2) + n^3$.
    g. $T(n) = T(\sqrt{n}) + 1$.
    h. $T(n) = T(n-1) + 1/n$

5. Suppose that $S_n = \{1, 2, 3, \cdots, n\}$. An involution over the set S_n is a permutation $\pi: S_n \to S_n$ of order at most 2 (i.e. $1 \leq \forall i \leq n,\ \pi^2(i) = i$). Derive a recurrence equation to count the number of involution for a set of size n and then try to solve it using Generating Functions.

6. Try to obtain a closed form for the following recurrence equation:

$$p_n = \begin{cases} 1 & \text{if } n \leq 2 \\ 2 & \text{if } n = 3 \\ p_{n-1} + (n-1)p_{n-2} - p_{n-3} + p_{n-4} & \text{if } n \geq 4. \end{cases}$$

7. Try to solve the Fast Multiplication by dividing each number into three parts and analyze it.

8. Show the details of Matrix Multiplication in which each matrix is divided into nine blocks (each of size $n/3 \times n/3$).

9. Draw a comparison tree for five elements and then show that at most six comparisons are enough to find the median of five elements

10. Try to solve the select problem where each group contains j elements, instead of 5. The analyze you algorithm.