In The Name of God

Model Predictive Control

Fall Semester of 2024

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Exercise 2

Optimal Control and Linear MPC

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APAC Advanced Process Automation & Control

Analytical Questions

Question 1

For the system in Equation 1, design an optimal controller using discrete dynamic programming. Provide the optimal cost values and the optimal control signal for each step. Only two iterations are required, with all calculations to be completed manually, including printed dynamic programming tables for each possible stage.
The system dynamics are given by:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where the state matrix A and input matrix B are as follows:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2}$$

The cost function is defined as:

$$J = \|x(N)\|^2 + 2\sum_{i=0}^{N-1} \|u(i)\|^2 \tag{3}$$

The input and state constraints are:

$$|x(k)| \leq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad |u(k)| \leq 0.5 \tag{4}$$

The input and state quantization levels are set to 0.5.

Hint: To learn more, refer to Chapter 3 of *Optimal Control Theory: An Introduction* by Donald E. Kirk.

Simulation Questions

Question 2

Model: The small signal model of a **temperature and humidity control chamber** is depicted in Figure 1 and can be described by the following continuous-time transfer function model:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

where:

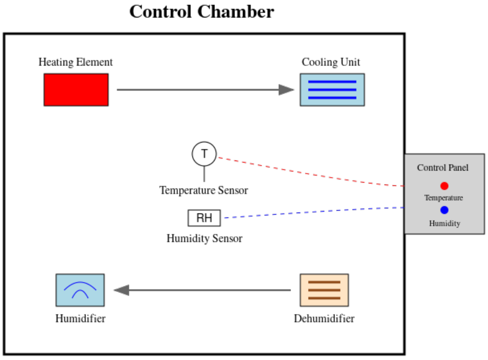

Figure 1: Temperature and humidity control chamber

- The manipulated variables $u_1(s)$ and $u_2(s)$ represent **cooling/heating power** (negative for cooling, positive for heating) and **humidifier/dehumidifier power** (negative for dehumidifying, positive for humidifying), respectively.
- The controlled variables $y_1(s)$ and $y_2(s)$ are the **temperature deviation from room temperature** (20°C) and the **relative humidity deviation from 50%**, respectively.

The four transfer functions are given as:

$$g_{11}(s) = \frac{1.5}{1+0.8s}, \quad g_{12}(s) = \frac{4}{1+0.4s}$$
$$g_{21}(s) = \frac{2}{1+0.6s}, \quad g_{22}(s) = \frac{3}{1+0.5s}$$

Constraints:
The constraints are imposed on both the amplitudes and the rates of change of the manipulated variables u_1 and u_2, as well as on the measured outputs y_1 and y_2. Assuming zero initial conditions, the constraints are specified as follows:
- Cooling/Heating Power (u_1):

$$-50\,W \leq u_1 \leq 50\,W$$
$$-5\,\frac{W}{min} \leq \dot{u}_1 \leq 5\,\frac{W}{min}$$

- Humidifier/Dehumidifier Power (u_2):

$$-20\,W \leq u_2 \leq 20\,W$$
$$-3\,\frac{W}{min} \leq \dot{u}_2 \leq 3\,\frac{W}{min}$$

- Temperature Deviation (y_1):

$$-15^{\circ}C \leq y_1 \leq 25^{\circ}C$$

(This corresponds to an actual temperature range of 5°C to 45°C.)
- Humidity Deviation (y_2):

$$-30\% \leq y_2 \leq 40\%$$

(This corresponds to an actual relative humidity range of 20% to 90%.)

Weights:
To account for the relative importance of output tracking and control effort, the following weighting matrices R and Q are used in the cost function:
- Control effort weighting matrix (R):

$$R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

This imposes a lower penalty on temperature control (u_1) compared to humidity control (u_2).
- Output weighting matrix (Q):

$$Q = \begin{bmatrix} 15 & 0 \\ 0 & 25 \end{bmatrix}$$

This gives higher priority to tracking the humidity deviation (y_2).

Part 1

Design a linear MPC for the given system to track the following reference signals:

- Temperature Deviation (y_1): A step change from -10°C to 15°C at $t = 10$ minutes (corresponding to an actual temperature change from 10°C to 35°C).
- Humidity Deviation (y_2): A step change from -20% to $+30\%$ at $t = 5$ minutes (corresponding to an actual RH change from 30% to 80%).

The MPC controller should achieve good tracking performance with minimum overshoot, fast response, and minimum tracking error.

Part 2

Effect of control parameters: Analyze the effect of changes in each of the five control parameters on the output.

Question 3

Stabilizing a two-wheeled inverted pendulum, also known as a balancing robot (as shown in Figure 2), has been extensively studied due to its inherently unstable and multivariable nature, compounded by highly non-linear dynamics.

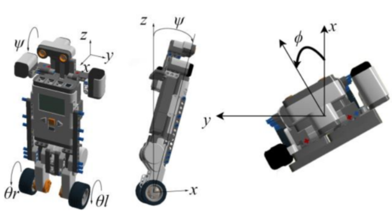

Figure 2: Balancing robot

The mathematical model was derived using the Lagrangian approach and subsequently linearized around the equilibrium point, assuming that the pitch angle approaches zero. The continuous-time space-state representation of the model with appropriate parameters is:

$$
\begin{bmatrix} \dot{\theta} \\ \dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -698.30 & -416.80 & 416.80 \\ 0 & 139.96 & 53.41 & -53.41 \end{bmatrix} \begin{bmatrix} \theta \\ \psi \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 405.11 & 405.11 \\ -51.91 & -51.91 \end{bmatrix} \begin{bmatrix} u_l \\ u_r \end{bmatrix} \tag{5}
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \psi \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{6}
$$

The system receives as inputs the voltages of each spare tire (u_l and u_r) and its outputs are the body pitch angle ψ [rad], the average angular position of the wheels θ [rad] and their velocities $\dot{\psi}$ [rad/s], $\dot{\theta}$ [rad/s].

Part 1

Select a suitable sampling time and derive a discrete-time state-space model using MATLAB. Display the system's poles before and after discretization. Additionally, verify the observability and controllability of the discretized system.

Part 2

Set up the discrete-time state-space model in Simulink. Assume the initial condition is $x_0 = \begin{bmatrix} 0 & 0.1 & 0 & 0 \end{bmatrix}^T$ and plot the system's zero-input response.

Part 3

Add the disturbance, indicated by Figure 3, to the second output of the system. Design a Linear MPC and an discrete-time LQR controller (with suboptimal gains) to stabilize the system and regulate all states to zero in the presence of the disturbance. Adjust control parameters to ensure that the settling time $t_s < 4$ seconds. Plot the state trajectories and control inputs for both controllers and compare the results.

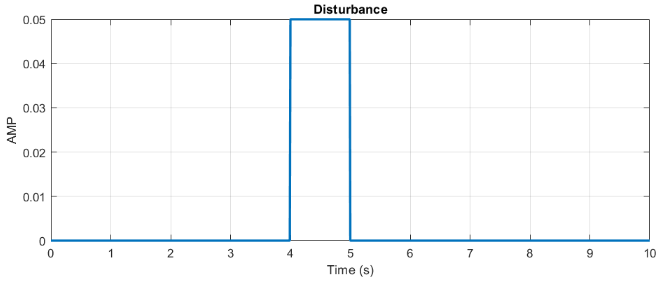

Figure 3: Additive disturbance

The following sections should be simulated exclusively with Linear MPC:

Part 4

Simulate the system under the given constraints, once using hard constraints and once using soft constraints, and compare the results.

$$-0.3 \leq \theta \leq 1.5 \tag{7}$$

Part 5

Simulate the system under the given constraints, once using hard constraints and once using soft constraints, and compare the results with previous part.

$$-0.3 \leq \theta \leq 1.6 \tag{8}$$

Hint: Set the Simulink configuration to use *discrete* solver with *Fixed-step* type.

Notes on Submitting Assignments

- Submit the answers to analytical questions, along with simulation reports and explanations of written code, in a single PDF file. Place the software simulation files (MATLAB and Simulink) in a folder named Simulation.
- Writing reports in LaTeX format will contribute 10% of the bonus points. To receive the bonus, you must place the LaTeX executable files in a folder named LaTeX, alongside the other files.
- Place all the requested files in a single ZIP file, named in the following format and upload it [**HW#-StudentID-StudentName**] (replace # with the assignment number).
- For each day of delay in submission, 10% of the assignment grade will be deducted.
- If you have any questions or ambiguities, please ask them in the Telegram group, the link to which is available on the VC.