Properties and Applications of Skew-Symmetric Matrices

Contents

1 Introduction

A skew-symmetric matrix A is a square matrix that satisfies the condition:

$$A^T = -A. \tag{1}$$

These matrices have numerous applications in physics, engineering, and computer graphics.

2 Properties of Skew-Symmetric Matrices

2.1 Determinant

The determinant of an odd-order skew-symmetric matrix is always zero:

$$\det(A) = 0, \quad \text{if } A \text{ is } (2n+1) \times (2n+1). \tag{2}$$

This follows from the property that $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$.

2.2 Eigenvalues

The eigenvalues of a skew-symmetric matrix are either zero or purely imaginary:

$$\lambda = i\mu, \quad \mu \in \mathbb{R}. \tag{3}$$

This follows from the characteristic equation of A and its similarity transformations.

2.3 Trace

Since the diagonal elements of a skew-symmetric matrix are always zero, the trace is also zero:

$$\mathrm{tr}(A) = \sum_i A_{ii} = 0. \tag{4}$$

2.4 Rank

The rank of a skew-symmetric matrix is always an even number. This property follows from the decomposition of A into canonical forms.

3 Operations Involving Skew-Symmetric Matrices

3.1 Addition and Scalar Multiplication

If A and B are skew-symmetric matrices, their sum is also skew-symmetric:

$$(A+B)^T = A^T + B^T = -A - B = -(A+B). \tag{5}$$

Similarly, for any scalar k:

$$(kA)^T = kA^T = k(-A) = -kA. \tag{6}$$

3.2 Matrix Products

If A and B are skew-symmetric, their product is skew-symmetric if and only if they commute:

$$AB = -BA \Rightarrow AB \text{ is skew-symmetric}. \tag{7}$$

4 Applications of Skew-Symmetric Matrices

4.1 Mechanics

Skew-symmetric matrices represent angular momentum and facilitate computations in rotational dynamics.

4.2 Electromagnetism

The electromagnetic field tensor is a skew-symmetric matrix combining electric and magnetic fields.

4.3 Computer Graphics

Used in rotational transformations and cross-product computations for 3D rendering.

4.4 Control Theory

Skew-symmetric matrices describe symmetries in Hamiltonian systems and conservation laws.

4.5 Quantum Mechanics

Appear in the study of spin operators and fundamental transformations.

5 Relation to Symmetric Matrices

Any square matrix B can be decomposed into symmetric and skew-symmetric components:

$$B = \frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T). \tag{8}$$

This allows for easier analysis and classification of matrices.

6 Conclusion

Skew-symmetric matrices play a crucial role in various mathematical and engineering applications. Their properties, including determinant constraints, eigenvalue characteristics, and structural decompositions, make them fundamental tools in theoretical and applied sciences.