Causality

Granger Causality vs Transfer Entropy

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- If X_t causes Y_t, b_{2k} should not be equal to 0.
- Hypothesis testing:

$$H_0 : b_{2k} = 0 \quad \text{for} \quad k \in \{1, 2, 3, \ldots, L\}$$

$$G = \frac{\frac{R_2 - R_1}{L}}{\frac{R_1}{N-2L}} \sim F_{L,N-2L}$$

1. Granger Causality
2. Transfer Entropy
3. Conclusions

Transfer Entropy

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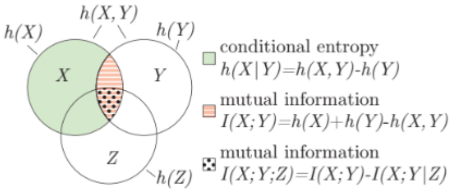

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- Transfer Entropy:

$$T^t_{X\to Y} \equiv I(Y_t;X_{t-1}|Y_{t-1}) \geq 0$$

$$Y_t = f(X_{t-1}) + f(Y_{t-1}) + \varepsilon_t$$

General (k, ℓ) $-$ *HistoryDefinitionofTransferEntropy*

- General Transfer Entropy:

$$T_{X\to Y}^{(K,\ell)} \equiv I(Y_t; X_{t-1}^{\ell} | Y_{t-1}^{k})$$

$$X_{t-1}^{\ell} = (X_{t-1}, X_{t-2}, \ldots, X_{t-\ell})$$

$$Y_{t-1}^{k} = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-k})$$

- Multivariate Transfer Entropy:

$$T_{X\to Y|Z}^{(K,\ell,m)} \equiv I(Y_t; X_{t-1}^{\ell} | Y_{t-1}^{k}, Z_{t-1}^{m})$$

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2. Select variables from CY:
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   - Find the candidate with maximum information contribution, c^*, and perform a significance test; if significant, add c^* to Z and remove it from CY. We use the maximum statistic to test for significance.
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4. Prune Z: Test and remove redundant variables in Z using the minimum statistic.

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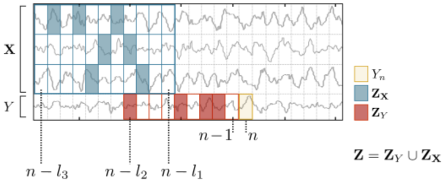
5. Perform statistical tests on the final set Z:
   - Perform an omnibus test to test the collective information transfer from all the relevant (informative) source variables to the target: $I(Y_n; Z_X | Z_Y)$.
   - If the omnibus test is significant, perform a sequential maximum statistic on each selected variable $z \in Z$ to obtain each variable's final information contribution and p-value.



Conclusion

- Granger causality is particularly well-suited for analyzing linear time series data.
- Transfer entropy, being a non-parametric method, is versatile and effective for both linear and non-linear time series analysis.

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Thanks!