1. Dominating Set is in NP since a set of vertices D that forms a dominating set of size k in $G = (V, E)$ can be verified in polynomial time by checking that D has size k and for every vertex $v \in V$, that either $v \in D$ or there is an edge $(u, v) \in E$ with $u \in D$.

   One can reduce SAT to Dominating Set in polynomial time as follows: Given a formula Φ in conjunctive normal form with n variables $x_1, x_2, \ldots, x_n$ and m clauses, form a graph $G = (V, E)$ with vertices

$$V = \{x_i, \overline{x_i}, a_i | i = 1, 2, \ldots, n\} \cup \{c_j | j = 1, 2, \ldots, m\}$$

   and edges

$$\begin{aligned} E = \{&(x_i, \overline{x_i}), (\overline{x_i}, a_i), (a_i, x_i) | i = 1, 2, \ldots, n\} \\ \cup \{&(x_i, c_j) | \exists i, j \text{ such that } x_i \text{ occurs in } c_j\} \\ \cup \{&(\overline{x_i}, c_j) | \exists i, j \text{ such that } \overline{x_i} \text{ occurs in } c_j\} \end{aligned}$$

   Claim The graph G has a dominating set of size n if and only if Φ is satisfiable.

   Proof. If Φ is satisfiable then let D be the set of vertices corresponding to the true literals of a truth assignment (i.e., if x_i is *true* then $x_i \in D$ otherwise $\overline{x_i} \in D$ for all i.) The set D is a dominating set of size n. Why?

   If D is a dominating set of size n then exactly one of $a_i, x_i, \overline{x_i}$ must be in D (because a_i is only adjacent to x_i and $\overline{x_i}$) and thus no c_j may be in D. If $x_i \in D$ then set x_i to *true* otherwise set x_i to *false*. The result is a satisfying truth assignment of Φ. Why? □

2(b) To find a maximum independent set in G, first find its size by iteratively querying the black-box with larger and larger bounds. Let k be this maximum size.

For every vertex v in G
    Let G' be G with v and all adjacent edges removed.
    If G' has an independent set of size k then
        Set $G = G'$
Return the vertices in G.

2(d) Given a bipartite graph $G = (V, E)$ with V partitioned into L and R (so that for all $(u, v) \in E$, $u \in L$ and $v \in R$), construct the flow network $F = (V', E')$ where $V' = \{s, t\} \cup V$, $E' = E \cup \{(s, u) | u \in L\} \cup \{(v, t) | v \in R\}$, and the capacities $c(u, v) = \infty$ for $(u, v) \in E$, $c(s, u) = 1$ for $u \in L$, and $c(v, t) = 1$ for $v \in R$.

Claim Any cut (S, T) in F with capacity $C < \infty$ defines an independent set $(S \cap L) \cup (T \cap R)$ in G of size $|V| - C$.

Proof. Since (S, T) has finite capacity, there is no edge $(u, v) \in E$ such that $u \in S$ and $v \in T$. So there is no edge in G between a vertex in $S \cap L$ and a vertex in $T \cap R$. Since $S \cap L$ is a subset of L, there is no edge in G between two vertices in $S \cap L$. Similarly for $T \cap R$. Thus $(S \cap L) \cup (T \cap R)$ is an independent set.

The capacity of (S, T) is the number of edges (s, u) with $u \in T$ plus the number of edges (v, t) with $v \in S$. Thus $C = |L \setminus S| + |R \setminus T|$ which is equal to $|V| - |(S \cap L) \cup (T \cap R)|$. So the independent set $(S \cap L) \cup (T \cap R)$ has size $|V| - C$. $\square$

Claim Any independent set I in G corresponds to cut (S, T) with $S = \{s\} \cup (I \cap L) \cup (R \setminus I)$, $T = \{t\} \cup (I \cap R) \cup (L \setminus I)$), and capacity $|V| - |I|$.

Proof. First (S, T) is a cut since $s \in S$, $t \in T$, $S \cap T = \emptyset$, and $S \cup T = V'$. Since I is an independent set in G, there is no edge from $I \cap L$ to $I \cap R$ in F. So the only edges from S to T are from s to $u \in L \setminus I$ or from $v \in R \setminus I$ to t. The number of such edges is $|V| - |I|$. $\square$

These two claims imply that we can find a maximum sized independent set in G by finding a minimum capacity cut (S, T) in F.

3. The problem CliqueAndIS is in NP since given a clique of size k and an independent set of size k, we can verify in polynomial time that the clique and independent set exist in the input graph G and are of size k.

   To show that it is NP-hard, we reduce Clique to it. Given a graph G and value k, we construct a graph G' which is G with k additional isolated vertices. If $\langle G', k \rangle$ is in CliqueAndIS then G contains a clique of size k because any clique in G' cannot use any of the isolated vertices. On the other hand, if $\langle G', k \rangle$ is not in CliqueAndIS then G' must not contain a k-clique because it certainly contains a size k independet set (the isolated vertices).

4. Let $C_i = (V_i, E_i)$ be the cycle containing i vertices created by the closest-point heuristic. The heuristic starts by creating C_1 and then adds a vertex to create C_2, etc. We show that the length of the cycle C_i (denoted $\|C_i\|$) is at most twice the length of the minimum length tour OPT_i of the vertices in V_i for all i, by induction on i.

   The claim is certainly true for $i \leq 3$ since $C_i = \mathrm{OPT}_i$ for $i \leq 3$. Suppose it is true for C_{i-1}. We want to show it is true for C_i. Let u be the vertex added to C_{i-1} to make C_i. Let v be the closest vertex in C_{i-1} to u. Let w be the vertex that follows v in C_{i-1}. (So edges uv and uw are part of the cycle C_i.) Since u is the closest vertex to V_{i-1} that is not in V_{i-1} (and it is closest to $v \in V_{i-1}$), $\|\mathrm{OPT}_i\| \geq \|\mathrm{OPT}_{i-1}\| + \|uv\|$. By the triangle inequality, $\|uv\| + \|vw\| \geq \|uw\|$, which implies $\|uv\| \geq \|uw\| - \|vw\|$. So

   $$\|C_i\| = \|C_{i-1}\| + \|uv\| + \|uw\| - \|vw\| \leq \|C_{i-1}\| + 2\|uv\| \leq 2\|\mathrm{OPT}_{i-1}\| + 2\|uv\| \leq 2\|\mathrm{OPT}_i\|.$$