CPSC 500 Fundamentals of Algorithm Design and Analysis (Updated V1)

Lecturer: Prof. Will Evans (Cover for Prof. Joel Friedman this time) Lecture Date: Sept.23rd, 2013

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1. Overview

In this lecture given by Prof. Will Evans, something not described before is introduced. By starting with a story sharing about the "Residence-Medical students" phenomenon derived from 1940s (Period of WWII), a mathematical problem about matching Hospital-Residence is introduced. Then in the following part, the problem of stable marriage is inducted and a simple example of "men marrying women" is given to explain the problem. In the third part, the proposal algorithm is introduced to help provide a solution to the given problem. It is the method which has been proved and utilized for long time to have applications in a variety of real-world situations, like solving these kinds of Residence-Medical students' problems. The proposal algorithm will always produce a stable solution, but not the necessarily optimal from all individuals' points of view.

2. Stable Marriage Problem

Definition: The stable marriage problem (SMP) is the problem of finding a stable matching between two sets of elements given a set of preferences for each element. A matching is a mapping from the elements of one set to the elements of the other set.

Example: Given n men and n women, where each person has ranked all members of the opposite gender with a unique number between 1 and n in order of preference, then try to marry the men and women together.

Goals: Minimize the number of dissatisfied pairs, said there are no two people of opposite gender who would both rather have each other than their current partners.

Specific Case given in the lecture: Assume $n = 4$, preferences shown as below:

Men (Capital letter)	***Women (Lowercase letter)***
A | b c d a	a | C B A D
B | d c a b	b | D A B C
C | b c a d	c | C A B D
D | d b a c	d | A D C B

The basic and simple algorithm to deal this problem is to do as following: While there exists a dissatisfied pair, like Xx ($X \in \{A, B, C, D\}$, $x \in \{a, b, c, d\}$), that is the currently matching is (Xx, Yy), then we change the marriage to be (Xy, Yx).

This algorithm is nice and simple, but it doesn't converge necessarily. The partners swap makes one pair happier but may decrease the happiness of the other. In fact, the process may loop and produce a pairing that occurred before. Hence, we need to consider another stable algorithm.

3. Proposal Algorithm

3.1 Theory and Algorithm

Concept: Let the men propose to the women and let the women accept or reject these proposals based on their preferences.

Definition: Involve a number of "rounds" (or "iterations"). In each subsequent round, an unpaired man proposes to his most preferred woman (who hasn't rejected him). If she is unpaired or prefers the new proposal to her current partner, she accepts, otherwise she rejects.

Algorithm:

```
function  StableMatching {
        Initialize P as the pairing set, that P=∅ (start)
        while ∃ unpaired man X
        (who proposes to first woman y in his preference list)
        {
            if   y is paired (with Y), but prefers X
                 remove Yy from P
                 remove y from Y's preference list
                 (results to the fact that Y and y are unpaired)
            if   y is unpaired
                 add Xy to P
            else
                 remove y from X's preference list
        }
}
```

3.2 Results and Discussion

- **Lemma:** From the time a woman receives first proposal, she remains paired (and her sequence of partners is strictly improving from her view).
- **Corollary 1:** The process terminates when for first time, every woman has received a proposal.
- **Corollary 2:** As a consequence of last statement, no man runs out of woman to propose to.
- **Corollary 3:** The process terminates after at most n^2 proposals, said $(n^2 - (n - 1)) \rightarrow \boldsymbol{O}(n^2)$
- **Theorem:** The final pairing is stable.

Q: It there any possibility that they are not stable at last?

A: No. To explain this, we suppose that *Xy* (of *Xx* and *Yy*) is dissatisfied. We can get the fact that *X* prefers *y* to *x*, but *X* is rejected by *y*, which means *y* should be paired with *Z* (another guy) she prefers to *X*. But *y*'s final partner *Y* is at least as preferable to her as *Z*. Then we carry out the conclusion that *y* prefers *Y* to *X*, which is a conflict to the assumption.

Q: The given problem make the men propose first, how about we let the women get to propose first, dose the matching result come out different stable pairing?

A: Yes, it would be different. If the men do the proposals, then the resulting stable pairing is man-optimal, which means no man can be paired (in a stable pairing) with a better woman than in this stable pairing. However, there might be a better stable pairing from the women's point of view. Here comes out a conclusion that the proposal algorithm for solving the stable marriage problem **always produces a stable solution, but not the necessarily optimal from all individuals' points of view.**

P.S. Originally, the hospitals did the proposing in the Residence matching problem, but that has changed recently. Now the interns do the proposing.

Average case analysis: Assume that men's preference lists are chosen independently and uniformly at random. Women's preference lists can be arbitrary but must be fixed in advance.

What is the average number of proposals made by the proposal algorithm? Clock Solitaire is a simpler example of the type of analysis we need to do.

3.3 Card Game "Clock Solitaire" (Cont'd in next lecture)

Motivation: To help explain the computational complexity of the average number of proposals needed in solving the stable marriage problem, Clock Solitaire is just a more active and vivid case example, to be continued in next lecture.

Definition: Clock patience, also known as clock solitaire is a solitaire card game with the cards laid out to represent the face of a clock. 52 cards used here, divided into 13 piles with 4 cards each randomly. Then put the 12 piles with the circle like a clock representing the 12 hour, the rest pile in the center of this clock, with all cards faced down, as illustrated in Figure 1.

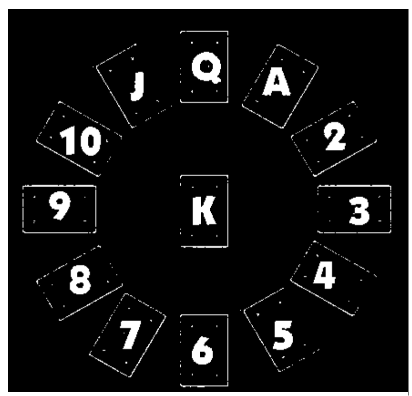

Figure 1

Rules: Start with revealing a card in the central pile, it will point to the corresponding pile, then reveal one card from the pointed pile. If a King is revealed, it is placed face up under the central pile. Play continues in this fashion and the game is won if all the cards are revealed. The game is lost if all four Kings are revealed and face-down cards are still present.

Results and discussions: To be continued in next lecture…

e.g. Q: What is the probability of wining?

At each draw any unrevealed card is equally likely to appear. Thus, the process of playing this game is exactly equivalent to repeatedly drawing a card uniformly at random from a deck of 52 cards. Then a winning game corresponds to the situation where the first 51 cards drawn contain exactly 3 Kings.

4. Summary

This lecture mainly presents a widely-use algorithm to settle the stable marriage problem, with discussing some interesting examples. The proposal algorithm is proved efficient to solve these matching problems. The traditional form of the algorithm is optimal for the initiator of the proposals and the stable, suitor-optimal solution may or may not be optimal for the reviewer of the proposals. It is meaningful for students to have a clear understanding about dealing with matching problem in the future. The game example "Clock Solitaire" proposed makes sense to help explain the average number of average analysis, and perform an example of the principle of deferred decisions, hope it could be discussed more in the next lecture.

References

[1] http://en.wikipedia.org/wiki/Hospital_resident#Matching_algorithm

[2] http://en.wikipedia.org/wiki/Stable_marriage_problem

[3] http://en.wikipedia.org/wiki/Clock_patience

[4] D Gale, L Shapley, College admissions and the stability of marriage, Amer. Math., 69 (1962), pp.9–15

Appendices

The National Resident Matching Program (NRMP) (or the Match) is a United States-based private non-profit non-governmental organization created in 1952 to help match medical school students with residency programs. The NRMP is sponsored by the American Board of Medical Specialties (ABMS), the American Medical Association (AMA), the Association of American Medical Colleges (AAMC), the American Hospital Association (AHA), and the Council of Medical Specialty Societies (CMSS).

Stable marriage problem (SMP) in mathematics, economics, and computer science, the SMP is the problem of finding a stable matching between two sets of elements given a set of preferences for each element. A matching is stable when there does not exist any alternative pairing (A, B) in which both A and B are individually better off than they would be with the element to which they are currently matched.

Similar problems:

- **The weighted matching problem** seeks to find a matching in a weighted bipartite graph that has maximum weight. Maximum weighted matching does not have to be stable, but in some applications a maximum weighted matching is better than a stable one.
- **The stable roommates' problem** is similar to the stable marriage problem, but differs in that all participants belong to a single pool (instead of being divided into equal numbers of "men" and "women").

P.S. It's interesting that some researcher argue that the college admissions problem is not equivalent to the marriage problem, please refer to the link if interested.

http://www.sciencedirect.com/science/article/pii/0022053185901061