CPSC 500
Lecture 3: Sept. 11, 2013

Scribe: Paul Harvey

1 Overview

This lecture was divided into two parts with the first part being a very quick review of the Fibonacci sequence shown last class (and also briefly dealt with the shift operator). The second, and larger, part of the class covered mathematical induction in various forms. Mathematical induction is an incredibly powerful tool for computer scientists given how prolific recursive algorithms and structures are in this field.

The first homework assignment was assigned at the end of class.

2 Fibonacci Review

The time required to recursively compute the nth Fibonacci number is proportional to

$$f(n) = f(n-1) + f(n-2)$$

which on a Random Access Machine is modeled by

$$T(n) = T(n-1) + T(n-2) + Overhead(n)$$

This can be further simplified down to

$$A(n) = A(n-1) + A(n-2) + 1$$

and we can use induction on n to show that if $T(1) \geq A(1)$, $T(2) \geq A(2)$, and $Overhead(n) \geq 1$ then $T(n) \geq A(n)$.

Finally, as mentioned in lecture, the Fibonacci function has the following property.

$$f(n) \propto \left(\frac{1+\sqrt{5}}{2}\right)^n$$

2.1 Shift Operator

When discussing the Fibonacci sequence we briefly touched on the shift operator.

Definition (Shift Operator)**.** We define the shift operator σ on a function f as

$$(\sigma f)(n) = f(n+1)$$

which can be generalized to

$$f_t(n) = f(n+t)$$

where t is the amount to shift by - in the above case, 1.

It was pointed out that

$$(\sigma - 1)^{l+1}(\sigma^2 - \sigma - 1)T = 0$$

where T is the model above, given that

$$T(n) - T(n-1) - T(n-2) = poly(n)$$

Just how this operator is used in the proof of the above is a topic left for next week.

3 Mathematical Induction

Definition (Mathematical Induction)**.** If we let $P_1, P_2, P_3, ...$ take their value in $\{true, false\}$. Then,

1. If $P_1 = true$ and, (base case)
2. for each integer $k = 1, 2, 3, ...$ (inductive step)

$$P_k \Rightarrow P_{k+1}$$

then

$$true = P_1 = P_2 = P_3 = ...$$

3.0.1 Example

We will now show a small example of mathematical induction by proving the following theorem.

Theorem 1.

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Proof of Theorem 1. Let $P_n = 1 + 2 + 3 + ... + n = n(n+1)/2$. We then proceed by mathematical induction on n.

P_1 **(base case)**

$$\begin{aligned} 1 &= \frac{1(1+1)}{2} && \text{(by substitution)} \\ 1 &= 1 && \text{(by evaluation)} \end{aligned}$$

$P_k \Rightarrow P_{k+1}$ **(inductive step)**

$$\begin{aligned} 1 + 2 + 3 + ... + k &= \frac{k(k+1)}{2} && \text{(by substitution)} \\ 1 + 2 + 3 + ... + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(add } (k+1)\text{)} \\ &= \frac{(k+1)((k+1)+1)}{2} && \text{(by reduction)} \end{aligned}$$

□

3.1 Alternative Form

If instead we consider the proposition given in the previous section as a requirement for set membership, we arrive at an alternative form of mathematical induction.

Definition (Alternative Form of Mathematical Induction)**.** Let $\mathbb{N} = \{1, 2, 3, ...\}$, and let $A \subseteq \mathbb{N}$ s.t.

1. $1 \in A$, and
2. $\forall k \in \mathbb{N}, k \in A \Rightarrow k + 1 \in A$

Then $A = \mathbb{N}$.

3.1.1 Example

We will now show a small example of this alternative form of mathematical induction by offering an alternative proof of the theorem in the previous example.

Proof of Theorem 1. Let $A = \{n \in \mathbb{N} \mid 1 + 2 + 3 + ... + n = n(n + 1)/2\}$ where $A \subseteq \mathbb{N}$. We then proceed by mathematical induction on n.

$1 \in A$ **(base case)**

$$1 = \frac{1(1+1)}{2} \qquad \text{(by substitution)}$$

$k \in A \Rightarrow k + 1 \in A$ **(inductive step)**

$$\begin{aligned} 1 + 2 + 3 + ... + k &= \frac{k(k+1)}{2} && \text{(by substitution)} \\ 1 + 2 + 3 + ... + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(add } (k+1)) \\ &= \frac{(k+1)((k+1)+1)}{2} && \text{(by reduction)} \end{aligned}$$

Therefore, by induction $A = \mathbb{N}$ and the theorem holds for all $n \in \mathbb{N}$. □

3.2 Strong Form

The final form of mathematical induction presented was that of strong induction.

Definition (Strong Form of Mathematical Induction)**.** Let $P_1, P_2, P_3, ...$ s.t.

1. $P_1 = true$, and
2. $P_1 \wedge P_2 \wedge P_3 \wedge ... \wedge P_k \Rightarrow P_{k+1}$

Then $\forall k \in \mathbb{N}, P_k = true$.

3.2.1 Motivation

Consider the "Joel Sequence",

$$\begin{aligned} J_1 &= 3 \\ J_2 &= 7 \\ J_3 &= 23 \\ J_n &= \frac{J_1 + J_2 + ... + J_{\sqrt{n}}}{\sqrt{n}}, \text{ for } n > 3 \end{aligned}$$

We can make the claim that all "Joel Numbers" are less than 23. Using vanilla mathematical induction would make it tedious to prove this claim, due to the square root term in the subscript. However, strong induction actually makes this easier due to the fact that "Joel Numbers" greater than 3 are in fact an average of a subset of the previous numbers in the sequence.

4 Conclusion

This lecture provided us with a small insight on how computing the Fibonacci sequence could be modeled, with an example using a Random Access Machine. The shift operator was defined, and it was hinted at that it would be used more in the future. Three different forms of mathematical induction were detailed, with accompanying examples for two of them and motivation provided for a third.