1. Let $B(X)$ be the best woman that man X is paired with in some stable pairing. We want to show that the proposal algorithm produces the pairing $(X, B(X))$ for all men X. (Note: At this point we don't even know that the $B(X)$ are distinct. For all we know, $B(X)$ may equal $B(Y)$ for $X \neq Y$.) Suppose $G(X) \neq B(X)$ for some man X. Woman $G(X)$ is less preferable to X than woman $B(X)$, thus $B(X)$ rejected X during the proposal algorithm. Suppose man X is the first man rejected by his optimal stable woman during the execution of the proposal algorithm. Let Y be the man that $B(X)$ prefers to X at the time of X's rejection. Man Y has not been rejected by his optimal stable partner, $B(Y)$, so he prefers $B(X)$ to $B(Y)$ or $B(Y) = B(X)$. Now consider a stable pairing P that contains $(X, B(X))$. We know $B(X)$ prefers Y to X, and Y prefers $B(X)$ to his partner in P. (Y prefers $B(X)$ to *any* stable partner not equal to $B(X)$.) Thus any pairing that contains $(X, B(X))$ is not stable, contradicting our definition of $B(X)$ as X's optimal stable partner.

2.

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_x \sum_y (x+y) \Pr[X=x, Y=y] \\
&= \sum_x x \sum_y \Pr[X=x, Y=y] + \sum_y y \sum_x \Pr[X=x, Y=y] \\
&= \sum_x x \Pr[X=x] + \sum_y y \Pr[Y=y] \\
&= \mathbb{E}[X] + \mathbb{E}[Y]
\end{aligned}$$

3.

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_x \sum_y xy \Pr[X=x, Y=y] \\
&= \sum_x \sum_y xy \Pr[X=x] \Pr[Y=y] \\
&= \sum_x x \Pr[X=x] \left(\sum_y y \Pr[Y=y] \right) \\
&= \mathbb{E}[Y] \sum_x x \Pr[X=x] \\
&= \mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}$$

4.

$$
\begin{aligned}
\mathrm{Var}[X+Y] &= \mathbb{E}[((X+Y)-\mathbb{E}[X+Y])^2] \\
&= \mathbb{E}[(X+Y)^2 - 2(X+Y)\mathbb{E}[X+Y] + \mathbb{E}[X+Y]^2] \\
&= \mathbb{E}[(X+Y)^2] - 2\mathbb{E}[X+Y]^2 + \mathbb{E}[X+Y]^2 = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\
&= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - E[Y]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y]
\end{aligned}
$$

5. (a) Let X_i be a random variable that is 1 if the ith number is the maximum of the first i numbers and 0 otherwise. The probability that the ith number is the maximum of the first i numbers in a random ordering of the numbers is $1/i$. So $E[X_i] = 1/i$. The question asks for $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n 1/i = H_n = \ln n + O(1)$.

   (b) (due to K. Parker and G. Shoemaker) Keep track of the maximum M of the first $\lceil n/2 \rceil$ numbers. If, after that, you see a number x greater than M, stop and say "x is the maximum." Let A be the maximum and B be the second largest of the n numbers. You win if B is in the first $\lceil n/2 \rceil$ and A is in the last $\lfloor n/2 \rfloor$ numbers. This occurs with probability $\frac{\lceil n/2 \rceil}{n} \frac{\lfloor n/2 \rfloor}{n-1} > 1/4$.

   A more thorough analysis can be found in Section 5.4.4 of Cormen, Leiserson, Rivest, and Stein.

6. Create n_i copies of hospital i, each with the same preference list for residents. Replace each hospital i in each resident preference list with the n_i copies of that hospital, all of which have the same preference with respect to the other hospitals in the preference list. Run the proposal algorithm with the residents doing the proposing.

7.

A	c	b	a
B	b	c	a
C	b	c	a

a	C	B	A
b	A	B	C
c	B	A	C

Start with (Aa, Bb, Cc) then let run off together Ab, Ac, Bc, Bb, in turn.