CPSC 500

Lecture 3: Sept. 11, 2013

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1. Overview

In this lecture, we discussed the idea of induction by going through several examples. Sample exam problems were shown as to give a sense of how to write in induction formally. Examples of loose and strong induction were also discussed and compared.

2. Problems that can be solved using induction

2.1 An example with Fibonacci [1]

Sample exam question #0.1 (we slightly modified A for more generality):

Let A: $N \to N$ satisfy $A(n) = A(n-1) + A(n-2) + \widetilde{Overhead(n)}$ for all n>2

Let T: $N \to N$ satisfy $T(n) = T(n-1) + T(n-2) + Overhead(n)$ for all n>2,

Where $Overhead(n)$ is a function of positive integers n for which $Overhead(n) \geq \widetilde{Overhead(n)}$, for all n.

$T(n)$ shares the same form as the time to compute Fibonacci recursively using Random Access Machine. We can use induction to prove that if $T(1) \geq A(1)$ and $T(2) \geq A(2)$, then for all n we have $T(n) \geq A(n)$

2.2 Another example with shift operator [2]

Define:

$$(\sigma f)(n) = f(n+1)$$
$$T(n) - T(n-1) - T(n-2) = poly(n), \deg \leq d$$

Claim:

$$(\sigma - 1)^{d+1}(\sigma^2 - \sigma - 1)T = 0$$

This would be discussed next week.

3. How to write in induction

Two ways to write in induction were introduced.

3.1 Form 1

Let P_1, P_2, P_3,… take their value in {True, False}
(1) if P_1=True, (*this is the base case*), and
(2) for each integer l=1,2,3,…, P_k=>P_{k+1} (*this is the inductive step*)
Then True=P_1=P_2=P_3=…
Sometimes we call P_1, P_2, … as "propositions".

3.2 Form 2

Let N={1,2,3,…}
Let $A \subset N$, s.t.
(1) $1 \in A$
(2) for $k \in \mathbb{N}$, $k \in A$ => $k+1 \in A$
Then A=$\mathbb{N}$

3.3 A concrete example

Now an example was given so as to illustrate the proof step by step.
We might be familiar with the following theorems:

Theorem 1: $1+2+3+...+n = \frac{n(n+1)}{2}$

Theorem 2: $1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$

Theorem 3: $1^3+2^3+3^3+...+n^3 = something(n)$

Now we will show how to prove Theorem 1 in induction. We will introduce two ways to write in induction.

Pf 1:

Let P_n be the proposition that $1+2+3+...+n = \frac{n(n+1)}{2}$

P_1 is true since $1 = \frac{1(1+1)}{2}$

For any $k \in \mathbb{N}$, if P_k is true, then

$$1+2+3+...+k = \frac{k(k+1)}{2}$$

Add $k+1$ to both sides:

$$1+2+3+...+k+k+1 = \frac{k(k+1)}{2}+k+1 = \frac{(k+1)(k+2)}{2}$$

i.e. P_{k+1} is true.
Since the base case is true, and the inductive hypothesis is true,
We have P_n=true for all n.

Hence $1+2+3+...+n=\frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$

Pf 2 (another style):

Let $A=\{n \in \mathbb{N} \mid 1+2+...+n=\frac{n(n+1)}{2}\}$

$A \subset \mathbb{N}$

Claim 1: $1 \in A$ since $1=\frac{1(1+1)}{2}$

Claim 2: If $k \in A$, then $k+1 \in A$

Indeed, $k \in A$ implies that $1+2+3+...+k=\frac{k(k+1)}{2}$

Now adding $k+1$ to both sides:

$1+2+3+...+k+k+1=\frac{k(k+1)}{2}+k+1=\frac{(k+1)(k+2)}{2}$

Hence, $k+1 \in A$

Hence, by induction $A=\mathbb{N}$

Hence, for all $n \in \mathbb{N}$, $1+2+3+...+n=\frac{n(n+1)}{2}$

Tips: if your audience is professionals, some of these steps can be simplified.

4. Strong & loose induction

4.1 Loose induction

We call the forms of induction we introduced in Sec. 2 as loose (or normal) induction.

4.2 Strong induction

Strong induction is written in the following form:

Let P_1, P_2, …, s.t.

The base case is:

(1) P_1=True

Then inductive step may in any of the following case

(2a) P_1&P_2&…&P_k=>P_{k+1}

(2b) $P_1 \wedge P_2 \wedge \ldots \wedge P_k$=>$P_{k+1}$

(2c) …

Basically, compared to the inductive step in loose induction case, P_{k+1} relies on not only P_k, but P_1, P_2, …, P_{k-1} as well.

An example of strong induction is like this:

Say we have a sequence of numbers called "Joel Sequence":

$J_1 = 3,$
$J_2 = 7,$
$J_3 = 23,$

$$J_n = \frac{J_1 + J_2 + ... + J_{\lfloor \sqrt{n} \rfloor}}{\lfloor \sqrt{n} \rfloor}, n \ge 4$$

We can have a theorem like this:

For all n, $J_3 \le 23$

We can show that it is easier to prove this theorem in strong induction since J_n is the average of a subset of $J_1, J_2, ..., J_{n-1}$.

5. Conclusion

This lecture covers the basic concept of induction. We got a glimpse of what kind of problems can be solved by induction, and how to prove this kind of problems in induction. We also learnt the difference between strong induction and loose induction, so that it would be easier for us to adapt our solution to each case.

Homework 1: see the sample exam problems online. Due Sep. 23

Reference

[1] http://www.math.ubc.ca/~jf/courses/CS500/sample_ex.html
[2] http://en.wikipedia.org/wiki/Shift_operator