CPSC 500 Fundamentals of Algorithm Design and Analysis (V1)

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1. Overview

This lecture firstly reviewed the duality problem in linear programming following the content of last week. The introductory maximization problem with the two types of chocolate is recalled to introduce the duality theorem and duality general form. The two players' zero-sum games problem is analyzed using the example of "Rock-Paper-Scissors" game. The motivation of which is to find the strategy for each player. Then another example of strategy achievement with a different payoff matrix is given to explain the duality problem further. In general, every linear maximization problem could be converted to a dual minimization problem, and they relate to each other to help solve the original problem, thereby taking advantage of some computational efficiencies.

2. Review on Duality in Linear Program

Example: Recall the example of LP with two types of chocolate given in previous lecture:

$$\textbf{\textit{max}}\ \ x_1 + 6\,x_2$$

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1\,,\,x_2 \geq 0.$$

Motivation: Investigate the issue by describing what we expect of three multipliers, called y_1, y_2, y_3.

Multiplier	**Inequality**
y_1	$x_1 \quad\quad \leq 200$
y_2	$\quad\quad x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

After the multiplication and addition steps, we get the ***bound***:

$$(y_1 + y_2)\,x_1 + (y_2 + y_3)\,x_2 \ \leq 200\,y_1 + 300\,y_2 + 400\,y_3$$

Therefore, finding the set of multipliers that gives ***the best upper bound*** on the original LP is tantamount to solving the new LP:

$$\textbf{\textit{min}}\ \ 200\,y_1 + 300\,y_2 + 400\,y_3$$

$$y_1 + y_2 \geq 1$$

$$y_2 + y_3 \geq 6$$

$$y_1\,,\,y_2\,,y_3 \geq 0.$$

By design, any feasible value of this *Dual* LP is an upper bound on the original *Primal* LP. So if somehow find a pair of *Primal* and *Dual* feasible values that are equal, then they must both be optimal. By analyzing this problem, we could carry out the duality theorems below.

- **Theorem 1:** *If a LP has a bounded optimum, then so does its dual, and the two optimum valves coincide.*
- **Theorem 2:** *Dual of dual is primal.*

General Form of Duality:

Primal: **maximize** $\sum_{j=1}^{n} c_j x_j$

$$\text{Such that } \sum_{j=1}^{n} a_{ij} x_j \le b_i \text{ for } i \in I$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i \notin I, 1 \le i \le m$$

$$x_k \ge 0 \text{ for } k \in N$$

Dual: **minimize** $\sum_{j=1}^{m} b_j y_j$

$$\text{Such that } \sum_{k=1}^{m} a_{kj} y_k \ge c_j \text{ for } j \in N$$

$$\sum_{k=1}^{m} a_{kj} y_k = c_j \text{ for } j \notin N, 1 \le j \le n$$

$$y_k \ge 0 \text{ for } k \in I$$

Comment: Equality in *Primal* can convert to the unconstraint variables in *Dual*, and equality in *Dual* can convert to the unconstraint variables in *Primal*.

To better explain the advantage of *Duality* problem in finding the optimum solution, a typical game example in life is then introduced.

3. Zero-sum Games Problem

Concept: The schoolyard rock-paper-scissors game is a typical example of conflict situations in life by matrix games, illustrated in Figure 1, which is specified by the payoff matrix. Assume that there are two players, called *Row* and *Column*, and they each pick a move from {r, p, s} (representing for Rock, Paper, Scissor). Hence we know that one player's gain is the other's loss. Further assume that the *Column* player should pay some amount to *Row* player if lost, and the payoff matrix ***G*** is defined as below:

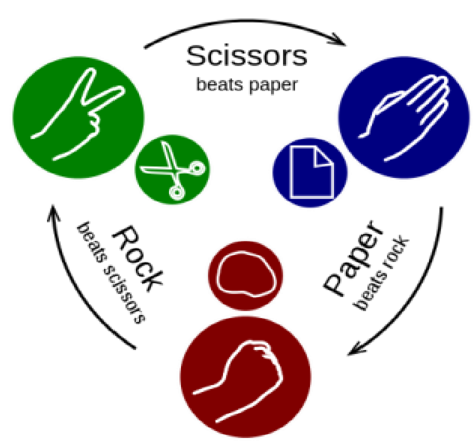

Figure 1. Illustration of rock-paper-scissors game (Credit from Wiki)

G		Column		
		r	p	s
Row	r	0	-1	1
	p	1	0	-1
	s	-1	1	0

Discussion: Assume the two players play this game repeatedly. Then if *Row* always plays ***r*** (Rock), then Column can win every time since he will definitely make the move ***p*** to win. And this is a pure strategy to play this game. Another side, a mixed strategy is a probability distribution on moves:

$$(x_1, x_2, x_3) \text{ with } \sum x_i = 1 \text{ for } Row\text{, and } (y_1, y_2, y_3) \text{ with } \sum y_i = 1 \text{ for } Column.$$

Therefore the expected (average) payoff is

$$\sum_{i,j} G_{i,j} \text{ Pr } [Row \text{ plays } i \text{ and } Col \text{ plays } j] = \sum_{i,j} G_{i,j}\, x_i y_j.$$

Obviously, *Row* wants to maximize this while *Col* wants to minimize it. Assume that each player plays the one move randomly. Then if Row plays (1/3, 1/3, 1/3), we have the expected payoff **0**, indicating that *Col* can't hope for better (smaller) result. If *Col* plays (1/3, 1/3, 1/3), we have the expected payoff **0**, indicating that Row can't hope for better (larger) result. From above discussion, it's obvious that among these two situations, one is lower bounding, the other is upper bounding. Hence the optimum case would be the strategy that **each player makes the move randomly**.

The average payoff might well be due to the high level of symmetry in rock-paper-scissors. In general games, this does not always happen. To yield this concept to a more general case, we following discuss a non-symmetrical matrix example.

4. Non-symmetrical Payoff Matrix Game Example

Derived from the same parameter definition from above problem, given the payoff matrix G:

G		Column	
		1	2
Row	1	3	-1
	2	-2	1

Consider the situation that *Row* plays (1/2, 1/2) then:

If *Col* plays**1**, *Col* has to pay 1/2; If *Col* plays **2**, *Col* has to pay 0. Therefore *Col*'s best response is (0, 1), which indicating a pure strategy.

In general case, if *Row* plays (x_1, x_2), then *Col* chooses the best pure strategy and pays:

$$\boldsymbol{min}\ \{3x_1 - 2x_2, -x_1 + x_2\} \quad (*)$$

Then *Row* should picks x_1, x_2 to maximize the (*) formula.

Here comes the general form of the LP: for fixed x_1, x_2, which indicates the probability, we have

$$\boldsymbol{max}\quad z \quad \text{(from the view of } Row\text{)}$$

$$z \le 3x_1 - 2x_2$$

$$z \le -x_1 + x_2$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \ge 0$$

$$\text{given } z = \underbrace{\boldsymbol{min}\ \{3x_1 - 2x_2, -x_1 + x_2\}}_{\text{Payoff from } Col\text{'s best response to } x}$$

Instead, if *Col* plays (y_1, y_2), then *Row* chooses the best pure strategy and pays

$$\boldsymbol{max}\{3y_1 - y_2, -2y_1 + y_2\} \quad (**)$$

Then *Col* should picks y_1, y_2 to minimize the (**) formula.

Here comes the general form of the LP: for fixed y_1, y_2, which indicates the probability, we have

$$\boldsymbol{min}\quad w \quad \text{(from the view of } Col\text{)}$$

$$w \ge 3y_1 - y_2$$

$$w \ge -2y_1 + y_2$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \ge 0$$

$$\text{given } w = \underbrace{\boldsymbol{max}\ \{3y_1 - y_2, -2y_1 + y_2\}}_{\text{outcome of } Row\text{'s best response to } y}$$

Actually, the above two LPs are dual to each other, meaning that the optimal solution should be the case of $z = w$. Let V be the solution of this LP or its *Dual*, we have:

- *Row* can choose a play (x_1, x_2) that yields V, no matter what *Col* does,
- *Col* can choose a play (y_1, y_2) that yields V, no matter what *Row* does.

In above example, $V = 1/7$ and will be realized hen *Row* plays optimum mixed strategy (3/7, 4/7) and *Col* plays optimum mixed strategy (2/7, 5/7).

Corollary: To arbitrary games the existence of mixed strategies that are optimal for both players can achieve the same value, which is a fundamental result of game theory called ***min-max theorem***, illustrated as follows:

$$\max_x \min_y \sum_{i,j} G_{i,j} x_i y_j = \min_y \max_x \sum_{i,j} G_{i,j} x_i y_j$$

5. Summary

This lecture mainly presented the *Duality* in Linear Programming (LP), using the examples of maximizing the chocolate's profit problem, rock-paper-scissors game strategies, and non-symmetrical payoff matrix strategies, respectively. The *Duality* in LP behaves useful and efficient in constituting a proof of optimality. And this is much similar to the max-flow min-cut theorem. The LP topic of this course ended in this lecture, with covering Chocolate profit problem, Simplex algorithm, Network flow problem, Bipartite matching and Duality in the past 4 lectures. Next topic will be NP-complete problems.

References

[1] http://en.wikipedia.org/wiki/Linear_programming#Duality

[2] Lecture notes online: http://www.princeton.edu/~rvdb/542/lectures/lec5.pdf

[3] Book chapters online: http://web.mit.edu/15.053/www/AMP-Chapter-04.pdf