Assignment 1: Sample solutions and comments

In the following, the n-th Fibonacci number, $f(n)$, is given by:
$f(1) = f(2) = 1$, and $f(n) = f(n-1) + f(n-2)$, for all $n > 2$.

- [Problem 0.1]

  Claim 0.1. *Let $A : \mathbb{N} \to \mathbb{N}$ satisfy $A(n) = A(n-1) + A(n-2) + 1$ for all $n > 2$. Let $T : \mathbb{N} \to \mathbb{N}$ satisfy $T(n) = T(n-1) + T(n-2) + Overhead(n)$ for all $n > 2$, where $Overhead(n)$ is a function of positive integers n for which $Overhead(n) \geq 1$ for all n. If $T(1) \geq A(1)$ and $T(2) \geq A(2)$, then $T(n) \geq A(n)$ for $n \geq 1$.*

  Proof. The base cases for $n = 1$ and $n = 2$ are true by the statement of the claim.

  Let $n > 2$ and suppose that $T(n') \geq A(n')$ for all n' with $1 \leq n' < n$ (this is the induction hypothesis). Then,

  $$\begin{aligned} T(n) &= T(n-1) + T(n-2) + \text{Overhead}(n) \text{ (by definition)} \\ &\geq T(n-1) + T(n-2) + 1 \text{ (since Overhead}(n) \geq 1 \text{ for all } n) \\ &\geq A(n-1) + A(n-2) + 1 \\ &\qquad \text{(by the induction hypothesis, since } 1 \leq n-2 \leq n-1 < n) \\ &= A(n) \text{ (by definition of } A(n)). \end{aligned}$$

  We can now apply the principle of strong mathematical induction (see mathematical induction form 3 of Keqian Li's notes from lecture 3, September 11). Let P_i be the predicate that $T(i) \geq A(i)$. By hypothesis, P_1 and P_2 both hold. The reasoning above shows that $P_{n-2} \& P_{n-1} \to P_n$ for all $n > 2$. The principle implies that therefore P_n holds for all $n \geq 1$, and thus $T(n) \geq A(n)$ for all $n \geq 1$.

  What does this tell us about a naive method for computing Fibonacci numbers? Consider a method for computing $f(n)$, the nth Fibonacci number, by recursively computing $f(n-1)$ and $f(n-2)$ and then summing the two. Let $T(n)$ be the time (number of operations) needed by the algorithm to compute $f(n)$. Then $T(n)$ is at least the time to compute $f(n-1)$ plus the time to compute $f(n-2)$, plus the time to compute their sum. Thus, $T(n) \geq T(n-1) + T(n-2) + 1$. We know that $f(n) \propto ((1 + \sqrt{5})/2)^n \geq (3.23/2)^n \geq (1.6)^n$ (see lecture notes 3). Thus $f(n)$ grows exponentially with n and by the above claim, so does the running time, $T(n)$ of the naive method for computing Fibonacci numbers. ☐

- [Problem 0.3]

  Claim 0.3. *For any $n > 1$, $f(n)f(n) - f(n-1)f(n+1)$ is -1 if n is even, and $+1$ if n is odd.*

  Proof. The base case is when $n = 2$. Then $f(2)f(2) - f(1)f(3) = 1 - 2 = -1$.

  Now let $n \geq 3$ and suppose that $f(n-1)f(n-1) - f(n-1)f(n)$ is -1 if $n-1$ is even, and $+1$ if $n-1$ is odd (this is the induction hypothesis).

  Then, applying the fact that $f(n) = f(n-1) + f(n-2)$, we have

$$\begin{aligned} f(n)f(n) &= f(n)(f(n-1) + f(n-2)) \\ &= f(n)f(n-1) + f(n)f(n-2). \end{aligned} \tag{1}$$

  Similarly, since $f(n+1) = f(n) + f(n-1)$,

$$\begin{aligned} f(n-1)f(n+1) &= f(n-1)(f(n) + f(n-1)) \\ &= f(n)f(n-1) + f(n-1)f(n-1). \end{aligned} \tag{2}$$

  Subtracting each side of (2) from (1), we get

$$f(n)f(n) - f(n-1)f(n+1) = f(n)f(n-2) - f(n-1)f(n-1). \tag{3}$$

  By the principle of mathematical induction (form 1 from the notes of lecture 3), since $2 \leq n-1 < n$, we have that $f(n-1)f(n-1) - f(n)f(n-2)$ is -1 if $n-1$ is even and $+1$ if $n-1$ is odd. Substituting these values into the right side of (3), we have that $f(n)f(n) - f(n-1)f(n+1) = -1$ if n is even and $+1$ if n is odd, proving the claim. □

- [Problem 0.5]

  Claim 0.5. *$f(n) \leq 2^{n-1}$, for all integers $n \geq 1$.*

  Proof. Let P_n denote the proposition: $(f(n) \leq 2^{n-1})\&(f(n+1) \leq 2^n)$. We will prove that P_n holds for all $n \in \mathbb{N}$, which clearly establishes the claim. (Note: we could also use strong induction and avoid the slightly clumsy form of the proposition).

  [basis] P_1 follows immediately from the fact that $f(1) = f(2) = 1$.

  [induction step] Let k be an arbitrary element of $\mathbb{N}$, and suppose that P_k is true. Then both (a) $f(k) \leq 2^{k-1}$ and (b) $f(k+1) \leq 2^k$ must hold. Hence, (c) $f(k+2) = f(k+1) + f(k) \leq 2^k + 2^{k-1} \leq 2^{k+1}$. But P_{k+1} follows immediately from (b) and (c).

  Thus by the Principle of Mathematical Induction (Form 1), it follows that P_n holds for all $n \in \mathbb{N}$ □

- [Problem 0.7]

Claim 0.7. *$f(n) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$, for all $n \in \mathbb{N}$.*

Proof. Let P_n denote the proposition: $f(n) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$. We will prove that P_n holds for all $n \in \mathbb{N}$, using the strong form of mathematical induction.

[basis] To establish the basis it suffices to note that $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^1 - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^1 = 1$ and $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^2 - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^2 = 1$.

[induction step] Let $k > 2$ be an arbitrary element of $\mathbb{N}$, and suppose that P_i is true, for all $i \in \mathbb{N}$ satisfying $i \leq k$. Then, in particular, both

(a) $f(k-1) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k-1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k-1}$ and

(b) $f(k) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^k - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^k$. Hence,

$$\begin{aligned}
f(k+1) &= f(k) + f(k-1) \\
&= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^k - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^k + \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k-1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k-1} \\
&\qquad \text{by (a) and (b) above} \\
&= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k-1}(1 + \frac{1+\sqrt{5}}{2}) - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k-1}(1 + \frac{1-\sqrt{5}}{2}) \\
&= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k-1}(\frac{1+\sqrt{5}}{2})^2 - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k-1}(\frac{1-\sqrt{5}}{2})^2 \\
&= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k+1}.
\end{aligned}$$

Thus by the Principle of Mathematical Induction (Strong Form), it follows that P_n holds for all $n \in \mathbb{N}$ □