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1 Introduction

The stable marriage problem is described, an algorithmic solution is provided, solution is contextualized, and computational complexity is explored. The stable marriage problem was initially motivated by a story of hospitals attempting to find student doctors. However, it is also a story of student doctors trying to attend their choice institutions. Individual ambition resulted in an uncomfortably competitive environment where everyone suffered more. The hospitals began working together to solve this problem. A solution was sought out where no student doctor-hospital pair would be motivated to abandon previous commitments. For example, if a talented student doctor ended up at a less prestigious hospital, he and a more prestigious hospital could be motivated to abandon their previous partners for each other.

A solution is the Proposal Algorithm (Gale and Shapely (1962)) where the student doctors and hospitals are replaced by men and women seeking marriage partners, where each individual has a list of preferred candidates ordinally listed. The algorithm provides pairings which are stable; that is, if every pair were to marry, no two individuals from two different pairs would be tempted to leave their current partners and elope. The solution has potentially undesirable qualities, but it remains powerful and impressive.

2 Description of the problem

For a given set of N men, M, and N women, W, each with an ordinal list describing who the individual would prefer to pair with. The problem requires men to choose only and all women, and women to choose only and all men. An example of an input for this problem would be the following, where the leftmost column of each matrix is the index of the man or woman and the entries to the right of the individual's semicolon ordinate choice partners where, say, leftmost decides most preference.

$$M = \begin{pmatrix} A & ; & b & c & a & d \\ B & ; & d & c & a & b \\ C & ; & b & c & a & d \\ D & ; & d & b & a & c \end{pmatrix} \quad and \;\; W = \begin{pmatrix} a & ; & C & B & A & D \\ b & ; & D & A & B & C \\ c & ; & C & A & B & D \\ d & ; & A & D & C & B \end{pmatrix}$$

Notice that in this example, if Ad and Bb where married pairs A and b would be motivated to leave their marriages for each other, thus any pairing including Ad, Bb would be called unstable.

3 Notation

Hereon the term $x >_y z$ will be defined as x is ordinated above z according to the preferences of y. That is, individual y would choose x over z.

4 The proposal algorithm

The algorithm is as follows.

INPUT: $M\ men, W\ women,\ both\ sets\ of\ size\ N$
OUTPUT: $P\ pairs$
$P = \emptyset$
while $\exists\ unpaired\ man\ X$ {
  Let y be the most preferred woman on the list of X.
  if $Zy \in P$ and $X >_y Z$ then {
    remove Zy from P (Z and y are now unpaired)
    remove y from Z's list
  }
  if y is unpaired then {
    add Xy to P
  } else {
    remove y from X's list
  }
}
return P

5 Remarks on the proposal algorithm

1. Once paired, a woman is always paired. Also, her sequence of partners is strictly improving.

2. The algorithm terminates when, for the first time, every woman has recieved a proposal.

3. Thus, (by 2) no man ever runs out of women to propose to.

4. The algorithm requires between N and $N^2 - (N-1)$ cycles of the while loop. Next class we will use randomization to show the expected number of cycles is $O(N \log N)$.

   Define instability as follows: a pairing P is unstable if there exists $Xx, Yy \in P$ such that $X >_y Y$ and $y >_X x$

5. The proposal algorithm produces no unstable marriages.

   Proof:

   If a man X is paired $Xx \in P$ but there is a woman he prefers y, $y >_X x$, then he has already proposed to her and has been rejected by her or she left him for another man Y such that $Y >_y X$, because every man proposes to his most preferred partners before moving down hist list of preferences.

   If a woman, y, is paired $Yy \in P$ then she has rejected any man, X, who would rather be with her for another man, Z, such that $Y >_y Z$ and $Z \geq_y X$.

   So if a man prefers a partner above his own, then the preferred partner does not to prefer to be with him. Also, if a woman prefers to be with a man above her partner, then he has not proposed to her so he does not prefer to be with her, because all men ask their preferred partners first.

6. This algorithm greatly favours the preferences of M. For a long time, hospitals played this role until it was recently decided that the preferences of an individual should outweigh those of the institution.The following two theorems from (Gusfield and Irving (1998)) capture this notion well.

Th 1.2.2: All possible executions of the Gale-Shapely algorithm (with men as proposers) yield the same stable matching, and in this stable matching, each man has the best partner that he can have in any stable matching.

Th 1.2.3: In the man-optimal stable matching, each woman has the worst partner that she can have in any stable matching.

7. The men have no incentive to lie about their preferences.

8. The stable marriage problem is similar to the roommate problem where pairs are made from a single set of individuals with preferences. A stable solution does not always exist for the roommate problem.

9. The proposal algorithm was designed to ensure stable marriages, not to ensure global optimality of satisfaction.

6 Discussion

The proposal algorithm provides stable solutions for the stable marriage problem in polynomial time. This result is non-trivial and valuable in the example of student doctors pairing with hospitals. The proposal algorithms must be used with the caveats of bias toward the preferences of set M, and no guarentee of the optimality of any universal utility function of everyone's preferences (i.e. not everyone may be happy, but all relationships may be stable). However, these caveats were also not sought motivations; only stability was sought. Therefore the proposal algorithm provides exactly what it promises efficiently.

References

Gale, D. and L. S. Shapely (1962), "College admissions and the stability of marriage." *The American Mathematical Monthly*, 69, 9–15.

Gusfield, Dan and Robert W. Irving (1998), *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, Cambridge, Massachusetts.