Optional background reading from Randomized Algorithms, Motwani, Raghavan (MR).

The Stable Marriage Problem	pages 53–59
The Markov and Chebyshev Inequalities	pages 45–47

1. $\star$ Let $G(X)$ be the woman that is paired with man X by the Gale-Shapley proposal algorithm. Prove that there is *no* stable pairing P such that X prefers $P(X)$ (his partner in pairing P) to $G(X)$.

2. Prove that the expectation of a sum of random variables is the sum of the expectation of these random variables.

3. Prove that the expectation of a product of independent random variables is the product of the expectation of these random variables.

4. Use the previous two results to prove that the variance of a sum of pairwise independent random variables is the sum of the variances of these random variables.

5. Suppose I have a sequence of n different numbers. I reorder the sequence randomly (each ordering equally likely) and give you the numbers one at a time in this random order. You are interested in determining the maximum number in the sequence.

   (a) An obvious approach is to keep the maximum of the numbers given so far, updating this maximum as needed. What is the expected number of times you update the maximum (as a function of n)?

   (b) Suppose after you see each number, I ask "Is this the maximum of all n numbers?" If you say "no" then I give you the next number. If you say "yes" then you win if the number truly is the maximum number in the sequence, otherwise you lose.

   Describe an algorithm for you to follow that results in a probability of winning that is at least $1/4$. Note that your algorithm knows n from the start.

The remaining questions are optional.

6. Suppose we want to solve the original motivating problem of stably matching hospitals and residents. Let h be the number of hospitals and r the number of residents. The ith hospital ($i = 1, 2, \ldots, h$) has n_i open positions for residents. Suppose that $r \geq \sum_{i=1}^{h} n_i$. Assume that every resident includes every hospital in their preference list and every hospital includes every resident in their preference list. How would you solve this problem? Why is your solution correct? Suppose that $r < \sum_{i=1}^{h} n_i$. Does your solution still work? If not, how would you fix it?

7. We gave an iterative process that starts with an arbitrary pairing and repeatedly modifies it by placating a dissatisfied couple (i.e. letting them run off together and pairing up their former partners). Give an example of preference lists for the three men and women, an initial (unstable) pairing, and a sequence of dissatisfied couples that shows that this process may never terminate.

8. (from Motwani & Raghavan Exercise 3.9) The use of the variance of a random variable in bounding its deviation from its expectation is called *the second moment method.* In an analogous fashion, we can speak of the *kth moment method*: let k be even, and suppose we have a random variable X for which the kth moment, $\mu_k = \mathbf{E}[(X - \mathbf{E}[X])^k]$, exists. Show that
$$\mathbf{Pr}[|X - \mathbf{E}[X]| > t \sqrt[k]{\mu_k}] \leq \frac{1}{t^k}.$$

9. Given $a > 0$, describe a random variable Z taking non-negative values so that
$$\mathbf{Pr}[Z \geq a] = \mathbf{E}[Z]/a.$$

   In other words, I'm asking for an example of when Markov's inequality is tight.