1. (a) Let f be a flow with size$(f) = 1$. The flow f is composed of a number of simple paths $P_1, P_2, \ldots, P_k$ ($k \geq 1$) from s to t carrying flows of size $a_1, a_2, \ldots, a_k$ respectively, where $\sum_{i=1}^{k} a_i = 1$. (These paths are not necessarily disjoint.) For each edge e, $\sum_{P_i \ni e} a_i = f_e$ so $\sum_e \ell_e f_e = \sum_{i=1}^{k} \ell(P_i) a_i$ where $\ell(P_i)$ is the length of path P_i. Since $\sum_{i=1}^{k} \ell(P_i) a_i$ is an average of st-path lengths, minimizing $\sum_e \ell_e f_e$ is equivalent to finding the shortest path P_i from s to t and assigning $a_i = 1$ (or equivalently $f_e = 1$ for all $e \in P_i$).

   (b) The variables are f_e.

$$
\begin{aligned}
\min \sum_e \ell_e f_e & \\
\sum_w f_{vw} - \sum_u f_{uv} = 0 & \qquad \text{for each vertex } v \neq s, t \\
\sum_w f_{sw} - \sum_u f_{us} = +1 & \\
\sum_w f_{tw} - \sum_u f_{ut} = -1 & \\
f_e \geq 0 & \qquad \text{for each edge } e
\end{aligned}
$$

   (c) The dual has one variable for every constraint. Since each constraint corresponds to a vertex v, call these variables x_v for v a vertex in the graph G. After multiplying each constraint by its corresponding variable and summing up the constraints, we have for every edge (u, v) in G, the coefficient of f_{uv} is $x_u - x_v$, since x_u multiplies all outgoing edges of u and $-x_v$ multiplies all incoming edges to v. To be a lower bound on $\sum_e \ell_e f_e$, we want each such coefficient to be at most ℓ_{uv}. To be the best lower bound on $\sum_e \ell_e f_e$, we want to maximize the right-hand side of the summed constraints, i.e., $x_s - x_t$.

2. Find a maximum matching A in the bipartite graph $G = (V, E)$. For all vertices $v \in V$ that are not adjacent to an edge in A, pick one edge adjacent to v in E (if it exists) and add it to A. The set A is a minimum sized edge cover. Why?

   Let M^* be a maximum matching in G. Let n be the number of vertices in G. First, observe that $|A| = |M^*| + n - 2|M^*| = n - |M^*|$.

   We want to show that the minimum edge cover, A^*, has size at least $n - |M^*|$. Let $S_1, S_2, \ldots, S_k$ be the components of G formed by the edges of A^* where the ith component has vertices $V(S_i)$ and edges $E(S_i)$. A component S_i has no cycle or path of length greater than two (otherwise we could remove an edge from A^* and it would still be an edge cover). Hence, each component is a star in which the number of vertices is one more than the number of edges. Create a matching M by choosing any one edge from each component.

$$
|M| = k = \sum_{i=1}^{k} (|V(S_i)| - |E(S_i)|) = \sum_{i=1}^{k} |V(S_i)| - \sum_{i=1}^{k} |E(S_i)| = n - |A^*|
$$

   since every vertex appears in one component. Thus, $|M^*| \geq |M| = n - |A^*|$ and so $|A^*| \geq n - |M^*|$.

(Thanks to Shailendra Agarwal who thanks `www.cadmo.ethz.ch/education/lectures/FS09/graph_algo/solution_04.pdf`.)

Alternative solution (following the hint):

Let V_1 and V_2 be the partitions of the graph G. Construct a flow network $F = (V \cup \{s,t\}, E_1 \cup E' \cup E_2)$ where

$$\begin{aligned} E' &= \{(u,v) \text{ with capacity } 1 \mid (u,v) \in E, u \in V_1, v \in V_2\} \\ E_1 &= \{(s,u) \text{ with capacity } d(u)-1 \mid u \in V_1\} \\ E_2 &= \{(v,t) \text{ with capacity } d(v)-1 \mid v \in V_2\} \end{aligned}$$

Find an integer-valued maximum flow in this flow network and let A be the edges of E' with flow 0.

3.

	5	10
5	+5	-5
10	-10	+10

This shows how much B (the column player) pays.

$$\begin{aligned} \max z& \\ z &\leq 5x_1 - 10x_2 \\ z &\leq -5x_1 + 10x_2 \\ x_1 + x_2 &= 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The optimal strategy is for A to hide the c-cent coin with probability $d/(c+d)$ and the d-cent coin with probability $c/(c+d)$.