CPSC 500-101 Complexity of Computation

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Lecture 3

1 Introduction

In the first part this lecture, a short review of the analysis of recuisive algorithm for computing Fibonacci number was given. In the second part, the method Mathematical Induction was introduced. It is a method of mathematical proof in recuisive style for proving a given statement for all positive integers, and has the potential to be adapted to more problem with well-founded structures, such as trees. The method has a wide application in computer science, such as in analysis of recuisive algorithm.

2 Review of Fibonacci Computation

Time T(n) for recursively compute Fibonacci f(n)

$$f(n) = f(n-1) + f(n-2) \tag{1}$$

subjects to the following relation

$$T(n) = T(n-1) + T(n-2) + Overhead(n) \tag{2}$$

when we adopt the computing model of Random Access Machine.

We can use induction on n to show that if $T(1) \geq A(1)$, and $T(2) \geq A(2)$, then for all n we have $T(n) \geq A(n)$.

In fact, we have

$$f(n) \propto ((1+\sqrt{5})/2)^n \tag{3}$$

To see this,

Definition 1. *We define shift operation σ over a function as*

$$\sigma \circ f(n) = f(n+1) \tag{4}$$

Since we have

$$T(n) - T(n-1) - T(n-2) = poly(n) \tag{5}$$

we will be able to get

$$(\sigma - 1)^{l+1}(\sigma^2 - \sigma - 1)T = 0 \tag{6}$$

This will be seen next week.

3 Mathematical Induction

Now we show the form of mathematical induction.

Mathematical Induction Form 1. *Let* $P_1, P_2, P_3, \cdots$ *take their values in* $\{True, False\}$ *i.e. each* $P_i \in \{True, False\}$
(1) If $P_1 = True$*, and*
(2) $P_k \to P_{k+1}, k = 1, 2, 3 \cdots$
Then $P_i = True, \forall i \in \mathbb{N}$. [1]

Instructor's Comment (1) is called basis, or base case. (2) is inductive step.

Instructor's Comment The correctness is based on set theory, detailed won't be discussed in class.

Instructor's Comment Sometimes we call P_1, P_2, P_3 proposition.

We can also use mathematical induction in another form.

Mathematical Induction Form 2 (Alternative Form of Mathematical Induction)**.** *Let* $\mathbb{N} = \{1, 2, 3, \cdots \ldots\}$. *Let* $A \subset \mathbb{N}$ *s.t.*
(1) $1 \in A$
(2) $k \in A \Rightarrow k + 1 \in A, \forall k \in \mathbb{N}$
Then $A = \mathbb{N}$.

Finally we show the Strong Induction:

Mathematical Induction Form 3 (Strong Mathematical Induction)**.** *Let* $P_1, P_2, P_3, \cdots$ *s.t.*
(1) If $P_1 = True$*, and*
(2) $P_1 \& P_2 \& P_3, \cdots \& P_k \to P_{k+1}$
Then $P_i = True, \forall i \in \mathbb{N}$.

Instructor's Comment The sign & can take the form $\bigwedge$ or $\wedge$.

We can show an informal example to show the difference between normal induction and Strong Induction. We can define Joel Sequence, $\{J_n\}$ as $J_1 = 3, J_2 = 7, J_3 = 23$, and

$$J_n = \frac{J_1 + J_2 + \cdots + J_{\sqrt{n}}}{\sqrt{n}} \tag{7}$$

for $n \geq 4$, where k is mean biggest integer no bigger than k. We have the claim that for all n, $J_n \leq 23$.

It will be troublesome to use normal induction to prove this claim. To see this, you can try to prove $P_{35} \Rightarrow P_{36}$. However, strong induction will make it easier because J_n is average of some subset of $J_1, J_2, J_3, \cdots J_{n-1}$.

Then we show some formal example of Mathematical Induction.

Theorem 1.

$$1 + 2 + 3 + \cdots + n = n(n + 1)/2, \forall n \in \mathbb{N} \tag{8}$$

[1] In this lecture, we use $\mathbb{N}$ to denote the set of all positive integers.

Theorem 2.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6, \forall n \in \mathbb{N} \tag{9}$$

Theorem 3.

$$1^3 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6, \forall n \in \mathbb{N} \tag{10}$$

Then we proceed to formally prove Theorem 1, the rest left as exercise.

Proof. Let P_n be the proposition that $1 + 2 + 3 + \cdots + n = n(n+1)/2$. P_1 is true since $1 = 1(1+1)/2$. For any $k \in \mathbb{N}$, if P_k is true, then

$$1 + 2 + 3 + \cdots + k = k(k+1)/2 \tag{11}$$

we add k+1 to both sides, then

$$\begin{aligned} & 1 + 2 + 3 + \cdots + k + k + 1 && (12)\\ = \; & k(k+1)/2 + k + 1 && (13)\\ = \; & (k+1)(k+2)/2 && (14) \end{aligned}$$

i.e. P_{k+1} is true.

Since the base case is true, and the inductive hypothesis is true, we have $P_n = True$ for all n. Hence

$$1 + 2 + 3 + \cdots + n = n(n+1)/2 \tag{15}$$

for all $n \in \mathbb{N}$. □

If we use Alternative Form of Mathematical Induction, the proof will be like

Proof. Let $A = \{n \in \mathbb{N} | 1 + 2 + 3 + \cdots + n = n(n+1)/2\}$, apparently $A \subseteq \mathbb{N}$.

Claim 1. *(1)* $1 \in A$ *since* $1 = 1(1+1)/2$

Claim 2. *(2) if* $k \in A$ *, then* $k + 1 \in A$*, indeed.*

$k \in A$ implies that

$$1 + 2 + 3 + \cdots + k = k(k+1)/2 \tag{16}$$

we add k+1 to both sides, then

$$\begin{aligned} & 1 + 2 + 3 + \cdots + k + k + 1 && (17)\\ = \; & k(k+1)/2 + k + 1 && (18)\\ = \; & (k+1)(k+2)/2 && (19) \end{aligned}$$

i.e. $k + 1 \in A$.

Hence, by induction $A = \mathbb{N}$. Hence

$$1 + 2 + 3 + \cdots + n = n(n+1)/2 \tag{20}$$

for all $n \in \mathbb{N}$. □

Instructor's Comment In theory paper, people usually use Mathematical Induction by "for base case ... then we will prove the case of n=k is true implies case of n=k+1 is true ...".

4 Conclusion

In this lecture, we has strengthen our awareness of the techique for calculating the time of computing Fibonacci numbers recursively and were given the way for computing Fibonacci number analytically. Furthermore, various form of the proof techique of Mathematical Induction was introduced, enabling us to do proof recursively. Last but not least, several examples of theorem proving using Mathematical Induction was shown.

References

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[9] Katz, Victor J. (1998). History of Mathematics: An Introduction. Addison-Wesley. ISBN 0-321-01618-1.

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Appendices

There is some important policy about this course mentioned in class.

Course Policy Sample exam problems is on the website.

Course Policy Class will not be held on September 18, during the West Coast National Event of the Truth and Reconciliation Committee.

Course Policy Homework #1 is posted, probably due in a week after.

Course Policy If your grade as computed above falls below the grade required for Theory Breadth, you may raise it to the minimum grade needed to obtain Theory Breadth. Supplementary article presentation will be held in class.

Course Policy Homework should be printed out and should at least be legible.