1 Question 1

1. After the transformation there's no guarantee that these points can still form a valid triangle. Given the description, with the multiplicative errors $(1+\epsilon)$ and $(1-\epsilon)$, these points can become co-linear which effectively is a triangle with area 0, hence the ratio does not have any bounds.

2. Here we can provide a guarantee that the new transformed points are going to make a valid triangle, and then we can bound the ratio of the areas. For calculating the area, we're going to be using Heron's formula.

   As long as the triangle inequality is satisfied we're good! A right angled isosceles has lengths of $l, l, \sqrt{2}l$. We know $l + l > \sqrt{2}l$. If we want to violate this condition by a multiplicative error we have to decrease $2l$ and increase $\sqrt{2}l$. Even if we replace $2l$ with $2(1-\epsilon)l$ and $\sqrt{2}l$ with $(1+\epsilon)\sqrt{2}l$, we will have:

$$(1-\epsilon)2l > (1+\epsilon)\sqrt{2}l \Rightarrow (1-\epsilon)2 > (1+\epsilon)\sqrt{2} \tag{1}$$

$$\Rightarrow \frac{2}{1+\epsilon} - 1 > \frac{\sqrt{2}}{2} \Rightarrow \frac{2}{1+\epsilon} > \frac{\sqrt{2}+2}{2} \tag{2}$$

$$\Rightarrow \frac{4}{\sqrt{2}+2} > 1+\epsilon \Rightarrow 0.1715 > \epsilon \tag{3}$$

$$\text{similarly for other sides we'd have: } 0.4142 > \epsilon \tag{4}$$

   Therefore it's guaranteed to have a triangle if $\epsilon < 0.1715$.

$$s = \frac{a+b+c}{2} \quad A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5}$$

$$A_f \leq \sqrt{(1+\epsilon)s(1+\epsilon)(s-a)(1+\epsilon)(s-b)(1+\epsilon)(s-c)} \tag{6}$$

$$\Rightarrow A_f \leq (1+\epsilon)^2 A \tag{7}$$

$$\text{and similarly} \quad (1-\epsilon)^2 A \leq A_f \tag{8}$$

   Therefore the area is guaranteed to be preserved with multiplicative error of $(1+\epsilon)^2$.

3. For each triangle $ABC \in \binom{V}{3}$ (3 noncolinear points), we add a point P to the affine 2d-plane defined by ABC such that it makes a right angled isosceles triangle PAB (See Fig. 1) to guarantee the bound on errors on that plane and for the projection we'd use JL with $\binom{V}{3} + V$ points which will project into a space $\mathbb{R}^T$ which in practice will have a dimension of $\frac{4\log(V^3+V)}{\epsilon^2/2-\epsilon^3/3}$.

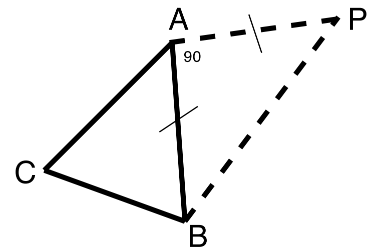

Figure 1: Let's add a point P to each plane defined by the three points in V to create a right angled isosceles triangle.

2 Question 2

1. **Additive product error.**

$$\|u+v\|^2 - \|u-v\|^2 = 4u^Tv \tag{9}$$

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \tag{10}$$

$$\frac{1}{2}((10)\text{ - }(9))\text{: } \|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2u^Tv = 2 - 2u^Tv \Rightarrow u^Tv = 1/2(2 - \|u-v\|^2) \tag{11}$$

$$|f(u)^Tf(v) - u^Tv| = |1/2(\|f(u)\|^2 + \|f(v)\|^2 - \|f(u)-f(v)\|^2) - 1/2(2 - \|u-v\|^2)| \tag{12}$$

$$\leq |1/2((1+\epsilon)^2\|u\|^2 + (1+\epsilon)^2\|v\|^2 - (1-\epsilon)^2\|u-v\|^2) - 1/2(2 - \|u-v\|^2)| \tag{13}$$

$$\|u\|^2 = \|v\|^2 = 1, \|u-v\| \leq \|u\| + \|v\| \leq 2 \Rightarrow 0 \leq \|u-v\|^2 \leq 4 \tag{14}$$

$$|f(u)^Tf(v) - u^Tv| \leq |(1+\epsilon)^2 - 1| = \epsilon^2 + 2\epsilon \leq 3\epsilon \tag{15}$$

2. **New margin**

$$\min \frac{|f(w)^T f(x_i)|}{\|f(w)\|\|f(v)\|} \geq (1-\alpha)\gamma \quad \forall_{x_i}, \quad \alpha, \epsilon \in (0,1) \quad \epsilon = K\alpha\gamma \quad \gamma > 0 \tag{16}$$

$$-3\epsilon \leq f(u)^T f(v) - u^T v \leq 3\epsilon \Rightarrow u^T v - 3\epsilon \leq f(u)^T f(v) \tag{17}$$

$$u^T v \geq \gamma \tag{18}$$

$$\|f(w)\| \leq (1+\epsilon)\|w\|, \|f(v)\| \leq (1+\epsilon)\|v\| \tag{19}$$

$$(19), (17), (18) \Rightarrow \frac{\gamma - 3\epsilon}{(1+\epsilon)^2} \geq \frac{\gamma - 3\epsilon}{1+3\epsilon} \geq (1-\alpha)\gamma \tag{20}$$

$$\gamma - 3K\alpha\gamma \geq (1+3K\alpha\gamma)(1-\alpha)\gamma \tag{21}$$

$$1 - 3K\alpha \geq (1+3K\alpha\gamma)(1-\alpha) \Rightarrow -3K\alpha \geq -\alpha + 3K\alpha\gamma - 3K\alpha^2\gamma \tag{22}$$

$$\Rightarrow -3K\alpha \geq -\alpha + 3K\alpha\gamma - 3K\alpha\gamma \tag{23}$$

$$\Rightarrow -3K \geq -1 + 3K\gamma - 3K\gamma \tag{24}$$

$$\Rightarrow -3K \geq -1 \Rightarrow K \geq \frac{1}{3} \tag{25}$$

So if $K \geq 1/3$ everything will work. Given that we have guaranteed the minimum margin, the consistency of $f(w)$ in the the new space follows.

3 Question 3

The idea is to normalize By_1 and approximate with ϵ-net again. Recursively do this and solve the resultant geometric series. Assuming ϵ' is the ϵ for JL and ϵ is the ϵ for the ϵ-net:

$$\|By_0\| \leq \|Bp_0\| + \|By_1\| \tag{26}$$

$$\bar{y_1} = \frac{y_1}{\|y_1\|} \Rightarrow \|y_1\|\bar{y_1} = y_1 \tag{27}$$

$$\|By_1\| = \|y_1\|\|B\bar{y_1}\| = \epsilon\|B\bar{y_1}\| \tag{28}$$

$$(1), (3) \Rightarrow \|By_0\| \leq (1+\epsilon') + (\epsilon)(\|B\bar{y_1}\|) \tag{29}$$

$$\Rightarrow \|By_0\| \leq (1+\epsilon') + (\epsilon)((1+\epsilon') + (\epsilon)(\|B\bar{y_2}\|)) \tag{30}$$

$$\Rightarrow \|By_0\| \leq (1+\epsilon') + (\epsilon)((1+\epsilon') + (\epsilon)((1+\epsilon') + (\epsilon)(\ldots))) \tag{31}$$

$$\Rightarrow \|By\| \leq (1+\epsilon')(\frac{1}{1-\epsilon}) \tag{32}$$

$$\Rightarrow (1+\epsilon')(\frac{1}{1-\epsilon}) \leq 1+\alpha \Rightarrow \epsilon' \leq \frac{2\alpha - \alpha^2}{3} \Rightarrow \epsilon' \leq \frac{\alpha}{3}, \epsilon = \frac{\alpha}{3} \tag{33}$$

Similarly we can show $\|By\| \geq 1 - \alpha$. In the original lemma we had $m \geq \Omega(K \log(\frac{1}{\epsilon})/\alpha^2)$. Replacing with the new parameters we'd get $m = \mathrm{O}(K \log(\frac{1}{\alpha})/\alpha^2)$.