Exercise 1:

Find and analyze the first bifurcation points of the logistic map analytically.

Logestic map: $x_{n+1} = rx_n(1 - x_n)$

Fixed − point analysis: $x^* = rx^*(1 - x^*) \rightarrow \begin{cases} x^* = 0 \\ x^* = 1 - \dfrac{1}{r} \end{cases}$

Stability analysis: $f(x_n) = rx_n(1 - x_n)$

$x_n = x^* + \epsilon_n$

$x_{n+1} = x^* + \epsilon_{n+1}$

$\rightarrow f(x_n) = f(x^* + \epsilon_n) \rightarrow$ Taylor series expansion

$\rightarrow x^* + \epsilon_{n+1} = f(x^*) + \epsilon_n f'(x^*) + \cdots$ (neglect higher order terms)

$\rightarrow \ \epsilon_{n+1} = \epsilon_n f'(x^*)$

As $|\epsilon| << 1, x^*$ is stable if $|f'(x^*)| < 1$

$f'(x) = r - 2rx \rightarrow f'(0) = r$

So, x^* is stable if $r < 1$

At $r = 1$ (first bifurcation point), $x^* = 1 - \dfrac{1}{r}$

$f'(x^*) = r - 2r\left(1 - \dfrac{1}{r}\right) = 2 - r, |f'(x^*)| < 1$

$\rightarrow |f'(x^*)| = |2 - r| \ \rightarrow$ The model is stable for $1 < r < 3$

Exercise 2:

Part 1: Calculate the fixed-points of the Lorenz Map.

Lorenz map: $\begin{cases} \dot{x} = \rho(y - x) \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz \end{cases}$

Fixed − point analysis: if we consider $\mathrm{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathrm{F} = \begin{pmatrix} \rho(y - x) \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz \end{pmatrix} \rightarrow \dot{\mathrm{X}} = \mathrm{F}$

$$\text{So, the fixed} - \text{points are given by } F = 0 \rightarrow \begin{cases} \rho(y-x) = 0 \\ -xz + rx - y = 0 \\ xy - bz = 0 \end{cases}$$

$$\rightarrow \begin{cases} x = y \\ x = \sqrt{b(r-1)} \\ z = \dfrac{x^2}{b} \end{cases} \rightarrow \text{The first fixed} - \text{point is: } (x^*, y^*, z^*) = (0,0,0)$$

$$\rightarrow \text{The second, and third fixed} - \text{points are: } (x^*, y^*, z^*) = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, (r-1)\right)$$

$$and \ \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, (r-1)\right)$$

Part 2: Are the fixed-points stable?

Stability analysis with Lyapunov function:

We consider a concentric ellipsoids about origin as the potential function:

We need to show $\dot{V} \leq 0$ for all $(x, y, z) \neq 0$.

$$V(x(t), y(t), z(t)) = \frac{1}{\rho}x^2 + y^2 + z^2 \rightarrow \frac{dV}{dt} = \frac{x\dot{x}}{\rho} + y\dot{y} + z\dot{z}$$

$$\rightarrow \dot{V} = (yx - x^2) + (ryx - y^2 - xzy) + (-bz^2 + xzy) = (r+1)xy - x^2 - y^2 - bz^2$$

$$\rightarrow \dot{V} = -\left(x - \frac{r+1}{2}y\right)^2 - \left(1 - \left(\frac{r+1}{2}\right)^2\right)y^2 - bz^2$$

$\rightarrow$ So, for $r < 1, \dot{V} < 0$ for all $(x, y, z) \neq 0$ and zero only at the origin $\rightarrow$

Trajectories move to smaller V, penetrating smaller and smaller ellipsoids as t $\rightarrow\infty$
($\dot{V}$ is negative definite)

$$\text{For } t \rightarrow \infty, V(x(t), y(t), z(t)) \rightarrow 0 \text{ and } (x(t), y(t), z(t)) \rightarrow 0$$

So, for $r < 1$ origin is stable.

As the system is symmetric in (x, y), for example, if (x(t), y(t), z(t)) is a solution, then (−x(t),−y(t), z(t)) is a solution as well. Because of the symmetry, it suffices to only consider just one of the other fixed-points. The characteristic equation is:

$$c_{A(\pm q)}(\lambda) = \lambda^3 + (b + \rho + 1)\lambda^2 + \beta(\rho + r)\lambda + 2b\rho(r-1)$$

Solving the eigenvalues:

$$\lambda_1 = -(b + \rho + 1)$$

$$\lambda_{\pm} = i\sqrt{\frac{2\rho(\rho+1)}{\rho - b - 1}}$$

The value for $r_{bifurcation}$ can be determined considering purely imaginary roots. Let λ = iμ , plugging this back into the characteristic equation:

$$c_{A(\pm q)}(\lambda) = i\mu^3 - (b+\rho+1)\mu^2 + i\mu\beta(\rho+r) + 2b\rho(r-1) = 0$$

Taking real and imaginary parts, reveals that:

$$\mu^2 = \frac{2b\rho(r-1)}{(b+\rho+1)}$$

$$\mu^3 = \mu\beta(\rho+r)$$

As $\mu^3 \neq 0$, equating the two expressions and solving for r, $r_{bifurcation}$ will be:

$$r_{bifurcation} = \frac{\rho(\rho+b+3)}{\rho-b-1}$$

If $(\rho > b+1)$, then the linearization for $q_{\pm}$ are stable for $1 < r < r_{bifurcation}$.

For $r > r_{bifurcation}$, there is no stability.

Exercise 3:

Part 1: Calculate the fixed-points of the SIR model.

SIR model: $\dot{s}(t) = -\alpha s(t)i(t) + \gamma r(t)$

$$\dot{i}(t) = \alpha s(t)i(t) - \beta i(t)$$

$$\dot{r}(t) = \beta i(t) - \gamma r(t)$$

$N(t) = s(t) + i(t) + r(t)$, where N is the whole population at any given time.

Also, s, i, and r are separately ≤ N.

The population of a system must always be positive.

$\rightarrow N > 0,\ s(t) > 0, i(t) > 0, r(t) \geq 0$; thus, the lower bound for s, i, and r is 0.

We can re − write equations to see N in the formula: $\dot{s}(t) = \Lambda - \beta s\frac{1}{N} - \mu s$

$$\dot{i}(t) = \beta s\frac{1}{N} - (\mu+\gamma)i$$

$$\dot{r}(t) = \gamma i - \mu r$$

$\dot{N}(t) = \Lambda - \mu N$, The answer of this differential equation is: $N(t) = \frac{\Lambda}{\mu} + \frac{c}{e^{\mu t}}$

$$\lim_{t\to\infty}\left(\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}\right)=\frac{\Lambda}{\mu}\rightarrow \quad \text{So, } N\approx\frac{\Lambda}{\mu} \text{for large values of t.}$$

This implies that $s(t)\le\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}, i(t)\le\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}},\ r(t)\le\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}$ for every time point.

Therefore, the upper bound for s, i, r is $\frac{\Lambda}{\mu}$.

So, we have found a limit system for the model.

Part 2: Are the fixed-points stable?

Stability analysis with Lyapunov function:

First, by solving the equation of SIR model for s, i, and r, we will have:

$$s=\frac{\Lambda^2}{\mu(\beta+\Lambda)}$$

$$i=\frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}$$

$$r=\frac{\beta\Lambda}{\gamma(\beta+\Lambda)(\mu+\gamma)}$$

Also, we are able to write the $\dot{r}\ (t)$ equation of the SIR model based of N, i, and s:

$$r(t)=-s(t)-i(t)+N(t)$$

$$\dot{r}\ (t)=i(\mu+\gamma)-\mu(N-s)$$

Only two variables are listed in the SIR model with three equations. Therefore, $\dot{r}\ (t)$ may be disregarde.

We consider a circle about origin as the potential function:

$$V(x(t),y(t))=x^2+y^2$$

$$x=s-\frac{\Lambda^2}{\mu(\beta+\Lambda)}$$

$$y=i-\frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}$$

$$V=\left(s-\frac{\Lambda^2}{\mu(\beta+\Lambda)}\right)^2+\left(i-\frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}\right)^2$$

$$\dot{V}=2\left(s-\frac{\Lambda^2}{\mu(\beta+\Lambda)}\right)\left(\Lambda-\beta s\frac{1}{N}-\mu s\right)+2\left(i-\frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}\right)(\beta s\frac{1}{N}-(\mu+\gamma)i)$$

$$\dot{V}=-2\left[\frac{\mu(\beta+\Lambda)}{\Lambda^2}\left(s-\frac{\Lambda^2}{\mu(\beta+\Lambda)}\right)^2+\frac{(\beta+\Lambda)+(\mu+\gamma)}{\beta\Lambda}\left(i-\frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}\right)^2\right]$$

The terms inside the brackets is only ever non-negative numbers since all of our terms are positive. So, the overall sign of $\dot{V}$ is determined by the factor of -2 outside the brackets. Thus, $\dot{V}$ is negative definite and system is globally stable.