Exercise 4:

Study the stability of the other fixed points and discuss their interpretations in biological systems.

Competitive population growth model:

$\dot{x} = x(t)[\epsilon_1 - \alpha_1 x(t) - \beta_1 y(t)]$

$\dot{y} = y(t)[\epsilon_2 - \alpha_2 y(t) - \beta_2 x(t)]$

Linearization:

$$\begin{cases} x(t) = x_0 + \delta x(t) \\ y(t) = y_0 + \delta y(t) \end{cases}$$

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} \approx A = \begin{bmatrix} \epsilon_1 - 2\alpha_1 x_0 - \beta_1 y_0 & -\beta_1 x_0 \\ -\beta_2 y_0 & \epsilon_2 - 2\alpha_2 y_0 - \beta_2 x_0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix},$$

$$A = \begin{bmatrix} \epsilon_1 - 2\alpha_1 x_0 - \beta_1 y_0 & -\beta_1 x_0 \\ -\beta_2 y_0 & \epsilon_2 - 2\alpha_2 y_0 - \beta_2 x_0 \end{bmatrix}$$

Fixed-points ($\dot{x} = \dot{y} = 0$):

$$x^* = y^* = 0 \quad , \quad x^* = 0, y^* = \frac{\epsilon_2}{\alpha_2}$$

$$y^* = 0, x^* = \frac{\epsilon_1}{\alpha_1} \quad , \quad \alpha_1 x^* + \beta_1 y^* = \epsilon_1, \;\; \alpha_2 x^* + \beta_2 y^* = \epsilon_2$$

Stability analysis at $x^* = 0, y^* = 0$:

$$A = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \rightarrow \varphi(t) = \begin{bmatrix} e^{\epsilon_1 t} u(t) & 0 \\ 0 & e^{\epsilon_2 t} u(t) \end{bmatrix}$$

$e^{\epsilon_1 t} u(t)$ and $e^{\epsilon_2 t} u(t)$ are exponentially growing functions of time for $\epsilon_1, \epsilon_2 > 0$. Therefore, $x^* = y^* = 0$ is an unstable fixed point.

Interpretations in biological systems:

Which shows that at $x^* = y^* = 0$ point, the population of both species tends to increase exponentially. At this point, a sufficient amount of resources is available and the population of both species is limited.

Stability analysis at $\boldsymbol{x^* = 0, y^* = \frac{\epsilon_2}{\alpha_2}}$:

$$A = \begin{bmatrix} \epsilon_1 - \dfrac{\beta_1}{\alpha_2}\epsilon_2 & 0 \\ -\dfrac{\beta_2}{\alpha_2}\epsilon_2 & -\epsilon_2 \end{bmatrix} \to \varphi(t) = \begin{bmatrix} e^{(\epsilon_1 - \frac{\beta_1}{\alpha_2}\epsilon_2)t}u(t) & 0 \\ e^{-\frac{\beta_2}{\alpha_2}\epsilon_2 t}u(t) & e^{-\epsilon_2 t}u(t) \end{bmatrix}$$

If $\beta_1\epsilon_2 > \epsilon_1\alpha_2$, then $(\epsilon_1 - \frac{\beta_1}{\alpha_2}\epsilon_2)$ would be negative. As two other terms are also exponential to the power of a negative number, they are exponentially shrinking functions of time. Therefore, the fixed point would be stable. If $\beta_1\epsilon_2 < \epsilon_1\alpha_2$, then $(\epsilon_1 - \frac{\beta_1}{\alpha_2}\epsilon_2)$ would be positive, so $e^{(\epsilon_1 - \frac{\beta_1}{\alpha_2}\epsilon_2)t}$ would be exponentially growing functions of time. Thus, the fixed point would be unstable.

Interpretations in biological systems:

When we are at $x^* = 0, y^* = \frac{\epsilon_2}{\alpha_2}$, in case the impact of consuming resources of the specie times its population has greater on preventing the population increase of the other species times its population than themselves, the system would be stable. On the contrary, in case the impact of consuming resources of the specie times its population has lower on preventing the population increase of the other species times its population than themselves, the system would be unstable.

Stability analysis at $\boldsymbol{y^* = 0, x^* = \frac{\epsilon_1}{\alpha_1}}$:

$$A = \begin{bmatrix} -\epsilon_1 & -\dfrac{\beta_1}{\alpha_1}\epsilon_1 \\ 0 & \epsilon_2 - \dfrac{\beta_2}{\alpha_1}\epsilon_1 \end{bmatrix} \to \varphi(t) = \begin{bmatrix} e^{-\epsilon_1 t}u(t) & e^{-\frac{\beta_1}{\alpha_1}\epsilon_1 t}u(t) \\ 0 & e^{\epsilon_2 - \frac{\beta_2}{\alpha_1}\epsilon_1 t}u(t) \end{bmatrix}$$

If $\beta_2\epsilon_1 > \epsilon_2\alpha_1$, then $(\epsilon_2 - \frac{\beta_2}{\alpha_1}\epsilon_1)$ would be negative. As two other terms are also exponential to the power of a negative number, they are exponentially shrinking functions of time. Therefore, the fixed point would be stable. If $\beta_2\epsilon_1 < \epsilon_2\alpha_1$, then $(\epsilon_2 - \frac{\beta_2}{\alpha_1}\epsilon_1)$ would be positive, so $e^{\epsilon_2 - \frac{\beta_2}{\alpha_1}\epsilon_1 t}$ would be exponentially growing functions of time. Thus, the fixed point would be unstable.

Interpretations in biological systems:

Completely like the previous point, when we are at $y^* = 0, x^* = \frac{\epsilon_1}{\alpha_1}$, in case the impact of consuming resources of the specie times its population has greater on preventing the population increase of the other species times its population than themselves, the system would be stable. On the contrary, in case the impact of consuming resources of the specie times its population has lower on preventing the population increase of the other species times its population than themselves, the system would be unstable.

Stability analysis at $\boldsymbol{x^* = \beta_1\epsilon_2 - \beta_2\epsilon_1, y^* = \frac{\alpha_2\epsilon_1 - \alpha_1\epsilon_2}{\beta_1\alpha_2 - \beta_2\alpha_1}}$:

$$A = \begin{bmatrix} \epsilon_1 - 2\alpha_1\beta_1\epsilon_2 - \alpha_1\beta_2\epsilon_1 - \dfrac{\alpha_2\beta_1\epsilon_1 - \alpha_1\beta_1\epsilon_2}{\beta_1\alpha_2 - \beta_2\alpha_1} & -({\beta_1}^2\epsilon_2 - \beta_1\beta_2\epsilon_1) \\ \dfrac{\beta_2\alpha_1\epsilon_2 - \beta_2\alpha_2\epsilon_1}{\beta_1\alpha_2 - \beta_2\alpha_1} & \epsilon_2 - \dfrac{2\alpha_2^2\epsilon_1 - 2\alpha_1\alpha_2\epsilon_2}{\beta_1\alpha_2 - \beta_2\alpha_1} - \beta_1\beta_2\epsilon_2 - \beta_2^2\epsilon_1 \end{bmatrix}$$

$$\rightarrow \varphi(t) = \begin{bmatrix} e^{\epsilon_1 - 2\alpha_1\beta_1\epsilon_2 - \alpha_1\beta_2\epsilon_1 - \frac{\alpha_2\beta_1\epsilon_1 - \alpha_1\beta_1\epsilon_2}{\beta_1\alpha_2 - \beta_2\alpha_1}t} u(t) & e^{-({\beta_1}^2\epsilon_2 - \beta_1\beta_2\epsilon_1)t} u(t) \\ e^{\frac{\beta_2\alpha_1\epsilon_2 - \beta_2\alpha_2\epsilon_1}{\beta_1\alpha_2 - \beta_2\alpha_1}t} u(t) & e^{\epsilon_2 - \frac{2\alpha_2^2\epsilon_1 - 2\alpha_1\alpha_2\epsilon_2}{\beta_1\alpha_2 - \beta_2\alpha_1} - \beta_1\beta_2\epsilon_2 - \beta_2^2\epsilon_1 t} u(t) \end{bmatrix}$$

As the response for this fixed point is more complicated, the stability of the fixed point may occur for special relations between all the variables. Also, the interpretations in biological systems cannot easily be discussed for this fixed point.

Exercise 2:

Study the dynamic properties of the competitive Lotka–Volterra predator–prey model.

The Lotka–Volterra equations:

$$\begin{cases} \dot{x}(t) = \alpha x(t) - \beta x(t)y(t) \\ \dot{y}(t) = \delta x(t)y(t) - \gamma y(t) \end{cases}$$

Fixed point analysis ($\dot{x} = \dot{y} = 0$):

$$\rightarrow x^* = 0, y^* = 0 \qquad x^* = \frac{\gamma}{\delta}, y^* = \frac{\alpha}{\beta}$$

Linearization:

$$\begin{cases} x(t) = x_0 + \delta x(t) \\ y(t) = y_0 + \delta y(t) \end{cases}$$

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} \approx \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix},$$

$$A = \begin{bmatrix} \alpha - \beta y_0 & -\beta x_0 \\ \delta y_0 & -\gamma + \delta x_0 \end{bmatrix}$$

Stability analysis at $x^* = 0, y^* = 0$:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix} \rightarrow \varphi(t) = \begin{bmatrix} e^{\alpha t}u(t) & 0 \\ 0 & e^{-\gamma t}u(t) \end{bmatrix}$$

$e^{\alpha t}u(t)$ is exponentially growing functions of time for $\alpha > 0$, although $e^{-\gamma t}u(t)$ is exponentially shrinking functions of time for $\gamma > 0$. Therefore, $x^* = y^* = 0$ is an unstable fixed point.

Stability analysis at $x^* = \frac{\gamma}{\delta}, y^* = \frac{\alpha}{\beta}$:

$$A = \begin{bmatrix} 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{bmatrix} \rightarrow \varphi(t) = \begin{bmatrix} 0 & e^{\frac{-\beta\gamma}{\delta}t}u(t) \\ e^{\frac{\delta\alpha}{\beta}t}u(t) & 0 \end{bmatrix}$$

$e^{\frac{\delta\alpha}{\beta}t}u(t)$ is exponentially growing functions of time for $\alpha, \beta, \delta > 0$, although $e^{\frac{-\beta\gamma}{\delta}t}u(t)$ is exponentially shrinking functions of time for $\beta, \delta, \gamma > 0$. Therefore, $x^* = \frac{\gamma}{\delta}, y^* = \frac{\alpha}{\beta}$ is an unstable fixed point.

Exercise 6:

1) A) Find the fixed points of the SIR model. B) Simplify the SIR model to a susceptible-infected model. Interpret the model in terms of a competitive growth model.

A)

SIR model: $\dot{s}(t) = -\alpha s(t)i(t) + \gamma r(t)$

$\dot{i}(t) = \alpha s(t)i(t) - \beta i(t)$

$\dot{r}(t) = \beta i(t) - \gamma r(t)$

$N(t) = s(t) + i(t) + r(t)$, where N is the whole population at any given time.

Also, s, i, and r are separately $\leq$ N.

By considering $\dot{s} = \dot{i} = \dot{r} = 0$, we can calculate the fixed points:

$\dot{i}(t) = \alpha s(t)i(t) - \beta i(t) \rightarrow 0 = \alpha si - \beta i(t) \rightarrow i(\alpha s - \beta) = 0$

If i = 0:

$-\alpha si + \gamma(-s - i + N) = 0 \rightarrow s = N, r = 0$

So, the first fixed point is (s,0,0)

If $(\alpha s - \beta)$ = 0:

$\alpha si - \beta i = 0 \rightarrow S = \dfrac{\beta}{\alpha}$

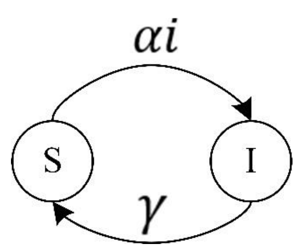
$-\alpha si + \gamma(-s - i + N) = 0 \rightarrow \gamma N - \gamma \dfrac{\beta}{\alpha} = i(\beta - \gamma) \rightarrow i = \dfrac{\gamma(\alpha N - \beta)}{\alpha(\gamma + \beta)}$

$r = N - \dfrac{\beta}{\alpha} - \dfrac{\gamma(\alpha N - \beta)}{\alpha(\gamma + \beta)} = \dfrac{\beta(\alpha N - \beta)}{\alpha(\gamma + \beta)}$

So, the first fixed point is $(\frac{\beta}{\alpha}, \frac{\gamma(\alpha N - \beta)}{\alpha(\gamma+\beta)}, \frac{\beta(\alpha N - \beta)}{\alpha(\gamma+\beta)})$

B) Simplified SIR model to represent suseptable-infected modeling.

The simplified compartmental model of the SIR model to represent suseptable-infected modeling is:



$\dot{s}(t) = -\alpha s(t)i(t) + \gamma i(t)$

$\dot{i}(t) = \alpha s(t)i(t) - \gamma i(t)$

$N(t) = s(t) + i(t)$, where N is the whole population at any given time.

$\rightarrow s = N - i$

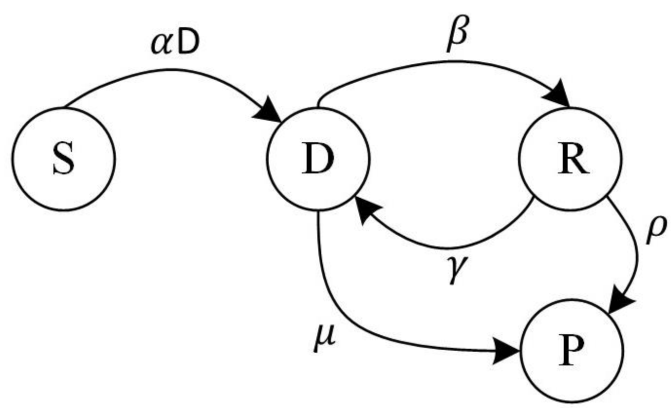
$\dot{i}(t) = \alpha i(N - i) - \gamma i(t)$

$\dot{i}(t) = (\alpha N - \gamma) - \gamma^2 i(t)$

This model can be potentially considered a competitive growth model and the two competitors are susceptible and infected cases. When the number of infected cases increases, the number of susceptible cases starts to decrease, but as the infected cases become susceptible after a while, the power of changes decreases. If there isn’t any specific impact on the system from outside, it is expected to have a converged number of susceptible and infected cases of the time after the initial changes. In a real-world scenario, for example, at first the whole population is susceptible. After the event of an infection, the number of susceptible cases starts to decrease, and subsequently, the number of infected cases starts to increase. As the infected cases would be recovered and become susceptible immediately, there would be a decrease in the number of susceptible cases and an increase in the number of infected cases. These changes continue, and there might be a convergenced number of different cases after a while.

2) Former drug addicts are known to be more prone to addiction even after recovery. Propose a model for drug addiction in a society.

My proposed compartmental model for drug addiction in a society with a focus on the fact that former drug addicts are known to addiction even after recovery is:



Where S refers to susceptible, D to drug addicted, R to recovered, and P to passed away cases.

$\dot{S}(t) = -\alpha S(t)D(t)$

$\dot{D}(t) = \alpha S(t)D(t) + \gamma R(t) - \beta D(t) - \mu D(t)$

$\dot{R}(t) = \beta D(t) - \gamma R(t) - \rho R(t)$

$\dot{P}(t) = \mu D(t) + \rho R(t)$