Derivation and verification of governing equations in a THM model considering thermo-osmosis and thermo-filtration effects

Zhigang Ye 2020.10.19

1 Governing equations

Pore water pressure (PWP) p, temperature T and displacements $\mathbf{u}$ are treated as independent variables, leading to the governing equations presented below.

1.1 Mass balance equation

Considering the flow of water in porous media, mass equation is given as follows:

$$\frac{\mathrm{d_s}}{\mathrm{d}t}(\rho^{\mathrm{w}}\varphi)+\nabla\cdot\mathbf{J}^{\mathrm{w}}+\rho^{\mathrm{w}}\varphi\nabla\cdot\frac{\mathrm{d_s}\mathbf{u}}{\mathrm{d}t}=q_{\mathrm{w}} \tag{1}$$

where ρ^{w} is the water density, φ is the porosity, ∇ is the Nabla operator, q_{w} is the source term, and the hydraulic flux $\mathbf{J}^{\mathrm{w}}$ refers to the modifying Darcy's law (Zhou *et al.*, 1998):

$$\mathbf{J}^{\mathrm{w}}=-\rho^{\mathrm{w}}\frac{\mathbf{k}_p}{\mu}(\nabla p-\rho^{\mathrm{w}}\mathbf{g})-\rho^{\mathrm{w}}\mathbf{k}_{pT}\nabla T \tag{2}$$

where $\mathbf{J}^{\mathrm{w}}$ is the hydraulic flux, $\mathbf{k}_p$ is the intrinsic permeability tensor, μ is the viscosity, $\mathbf{g}$ is the gravitational acceleration, $\mathbf{k}_{pT}$ is the T-O tensor, and T is the temperature.

Through introducing the mass balance of the soil skeleton and auxiliary equations, the mass balance of water coupled with T and $\mathbf{u}$ can be obtained (details of the derivation refer to **Appendix I**):

$$\left(\varphi\beta_{\mathrm{w}p}\frac{\mathrm{d_s}p}{\mathrm{d}t}-\varphi\beta_{\mathrm{w}T}\frac{\mathrm{d_s}T}{\mathrm{d}t}\right)+\left((\alpha-\varphi)\beta_{\mathrm{s}p}\frac{\mathrm{d_s}p}{\mathrm{d}t}-(\alpha-\varphi)\beta_{\mathrm{s}T}\frac{\mathrm{d_s}T}{\mathrm{d}t}+(\alpha-\varphi)\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}\right)+$$
$$\nabla\cdot\left(-\frac{\mathbf{k}_p}{\mu}(\nabla p-\rho^{\mathrm{w}}\mathbf{g})-\mathbf{k}_{pT}\nabla T\right)+\varphi\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}=0 \tag{3}$$

We have:

$$\left((\alpha-\varphi)\beta_{sp}+\varphi\beta_{wp}\right)\frac{\mathrm{d_s}p}{\mathrm{d}t}-\left((\alpha-\varphi)\beta_{sT}+\varphi\beta_{wT}\right)\frac{\mathrm{d_s}T}{\mathrm{d}t}+\alpha\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}-\nabla\cdot\left(\frac{\mathbf{k}_p}{\mu}\nabla p-\rho^{\mathrm{w}}\frac{\mathbf{k}_p}{\mu}\mathbf{g}\right)-$$

$$\nabla\cdot\left(\mathbf{k}_{pT}\nabla T\right)=0 \tag{4}$$

where β_{sp} and β_{wp} are separately the compressibility for soil skeleton and pore water, and β_{sT} and β_{wT} are separately the volumetric thermal expansion coefficient for soil skeleton and pore water.

1.2 Energy conservation equation

Heat conduction and heat advection induced by the flow of pore water are considered in the energy balance equation:

$$\frac{\mathrm{d_s}}{\mathrm{d}t}\left(\left(C^{\mathrm{s}}\rho^{\mathrm{s}}(1-\varphi)+C^{\mathrm{w}}\rho^{\mathrm{w}}\varphi\right)T\right)+\nabla\cdot\mathbf{i}+\nabla\cdot\mathbf{J}_{\mathrm{E}}^{\mathrm{w}}-\frac{\beta_{wT}}{\beta_{wp}}T_{\mathrm{ref}}\nabla\cdot\left(-\frac{\mathbf{k}_p}{\mu}(\nabla p-\rho^{\mathrm{w}}\mathbf{g})-\mathbf{k}_{pT}\nabla T\right)-$$

$$\frac{\beta_{sT}}{\beta_{skp}}T_{\mathrm{ref}}\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}=q_{\mathrm{E}} \tag{5}$$

where C^{s} and C^{w} are the specific heat capacities of solid phase and pore water respectively, $\mathbf{i}$ is the conductive heat flux, describing as follows by modifying Fourier's law (Zhou *et al.*, 1998), $\mathbf{J}_{\mathrm{E}}^{\mathrm{w}}$ is the advective heat flux, T_{ref} equals to T in commom situation, and q_{E} is the source term, which equals to zero if there is no heat source. The fourth term on the left-hand side approximates the heat sink due to thermal dilatation of the fluid and the fifth term the heat sink due to thermal expansion of the medium (Bear and Bachmat 1990; Zhou *et al.*, 1998).

$$\mathbf{i}=-(\mathbf{k}^{\mathrm{s}}(1-\varphi)+\mathbf{k}^{\mathrm{w}}\varphi)\nabla T-\mathbf{k}_{Tp}\nabla p \tag{6a}$$

$$\mathbf{J}_{\mathrm{E}}^{\mathrm{w}}=C^{\mathrm{w}}\rho^{\mathrm{w}}\mathbf{v}^{\mathrm{w}} \tag{6b}$$

where $\mathbf{k}^{\mathrm{s}}$ and $\mathbf{k}^{\mathrm{w}}$ are the thermal conductivity tensors of solid phase and pore water, respectively, $\mathbf{k}_{Tp}$ is the T-F coefficient tensor, and $\mathbf{v}^{\mathrm{w}}$ is the velocity of pore water.

Considering Eqs. (5a) and (5b), Eq. (4) can be rewritten as:

$$(\rho C)\frac{\mathrm{d_s}T}{\mathrm{d}t}+C^{\mathrm{w}}\rho^{\mathrm{w}}\mathbf{v}^{\mathrm{w}}\cdot\nabla T-\nabla\cdot(\mathbf{k}_T\nabla T)-\nabla\cdot\left(\mathbf{k}_{Tp}\nabla p\right)-\frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}}T_{\mathrm{ref}}\nabla\cdot\left(-\frac{\mathbf{k}_p}{\mu}(\nabla p-\rho^{\mathrm{w}}\mathbf{g})\right)-$$
$$\frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}}T_{\mathrm{ref}}\nabla\cdot\left(-\mathbf{k}_{pT}\nabla T\right)-\frac{\beta_{\mathrm{s}T}}{\beta_{\mathrm{sk}p}}T_{\mathrm{ref}}\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}=0 \tag{7}$$

where:

$$\mathbf{k}_{Tp}=T_{\mathrm{ref}}\mathbf{k}_{pT} \tag{8}$$

That is:

$$(\rho C)\frac{\mathrm{d_s}T}{\mathrm{d}t}+C^{\mathrm{w}}\rho^{\mathrm{w}}\mathbf{v}^{\mathrm{w}}\cdot\nabla T-\nabla\cdot(\mathbf{k}_T\nabla T)-\nabla\cdot\left(\mathbf{k}_{Tp}\nabla p\right)-\frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}}T_{\mathrm{ref}}\nabla\cdot\left(-\frac{\mathbf{k}_p}{\mu}(\nabla p-\rho^{\mathrm{w}}\mathbf{g})\right)-$$
$$\frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}}T_{\mathrm{ref}}\nabla\cdot\left(-\mathbf{k}_{pT}\nabla T\right)-\frac{\beta_{\mathrm{s}T}}{\beta_{\mathrm{sk}p}}T_{\mathrm{ref}}\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}=0 \tag{9}$$

where ρC and $\mathbf{k}_T$ are the equivalent specific heat capacity and thermal conductivity of the saturated soil, here $\rho C=C^{\mathrm{s}}\rho^{\mathrm{s}}(1-\varphi)+C^{\mathrm{w}}\rho^{\mathrm{w}}\varphi$ and $\mathbf{k}_T=\mathbf{k}^{\mathrm{s}}(1-\varphi)+\mathbf{k}^{\mathrm{w}}\varphi$, and T_0 is the initial temperature.

1.3 Mechanical equilibrium equation

Neglecting the inertial forces, the linear momentum balance equation of saturated soils can be expressed as:

$$\nabla\cdot\boldsymbol{\sigma}+(\rho^{\mathrm{s}}(1-\varphi)+\rho^{\mathrm{w}}\varphi)\mathbf{g}=\mathbf{0} \tag{8}$$

where $\boldsymbol{\sigma}$ is the total stress tensor, which relates to the effective stress $\boldsymbol{\sigma}'$ by:

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}'-\alpha p\mathbf{1} \tag{9a}$$

$$\boldsymbol{\sigma}'=\mathbf{D}^{\mathbf{e}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_T)=\mathbf{D}^{\mathbf{e}}:\frac{\nabla\mathbf{u}+\nabla^{\mathrm{T}}\mathbf{u}}{2}-\mathbf{D}^{\mathbf{e}}:\mathbf{1}\alpha^{\mathrm{s}}(T-T_0) \tag{9b}$$

where $\mathbf{D}^{\mathbf{e}}$ is the linear elastic tensor, which can be determined by Young's modulus E, and Poisson's ratio v in the linear elasticity, $\mathbf{1}$ is the unit tensor, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_T$ is the total and thermal strain tensors, and α^{s} is thermal expansion coefficient of solid.

2 Numerical implementation of T-O and T-F effects

The flux boundary conditions for mass and energy equations, and traction boundary condition for the stresses are:

$$\mathbf{v}^{\mathrm{w}} \cdot \mathbf{n} = \frac{q^{\mathrm{w}}}{\rho^{\mathrm{w}}} \text{ on } \Gamma_{\mathrm{w}}^{q} \tag{10a}$$

$$\mathbf{i} \cdot \mathbf{n} = q^{\mathrm{E}} \text{ on } \Gamma_{\mathrm{E}}^{q} \tag{10b}$$

$$\nabla \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \text{ on } \Gamma_{\mathbf{u}}^{q} \tag{10c}$$

where q^{w} and q^{E} are the mass flux and energy flux normal to the boundaries, Γ_{w}^{q}, Γ_{E}^{q} and $\Gamma_{\mathbf{u}}^{q}$ are the mass, energy and stress boundaries respectively, $\bar{\mathbf{t}}$ is the stresses of $\Gamma_{\mathbf{u}}^{q}$, and $\mathbf{n}$ is the unit normal vector.

In the finite element setting, the expressions for p, T and $\mathbf{u}$ are related to the vectors ($\overline{\mathbf{p}}$, $\overline{\mathbf{T}}$, $\overline{\mathbf{u}}$) of nodal values of the independent variables and the shape function matrices $\mathbf{N}$ used for interpolation between the nodes:

$$p = \mathbf{N}_{p}\overline{\mathbf{p}} \tag{11a}$$

$$T = \mathbf{N}_{T}\overline{\mathbf{T}} \tag{11b}$$

$$\mathbf{u} = \mathbf{N}_{\mathbf{u}}\overline{\mathbf{u}} \tag{11c}$$

The Bubnov-Galerkin procedure for the construction of the weak form of Eqs. (3), (6) and mechanical equilibrium equation is used as the basis for the finite element model. Eventually, the following asymmetric, non-linear and coupling equation is obtained:

$$\begin{bmatrix} \mathbf{M}_{\mathrm{pp}} & \mathbf{M}_{\mathrm{pT}} & \mathbf{M}_{\mathrm{pu}} \\ \mathbf{M}_{\mathrm{Tp}} & \mathbf{M}_{\mathrm{TT}} & \mathbf{M}_{\mathrm{Tu}} \\ \mathbf{M}_{\mathrm{up}} & \mathrm{M}_{\mathrm{uT}} & \mathbf{M}_{\mathrm{uu}} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \overline{\mathbf{p}} \\ \overline{\mathbf{T}} \\ \overline{\mathbf{u}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{\mathrm{pp}} & \mathbf{K}_{\mathrm{pT}} & \mathbf{K}_{\mathrm{pu}} \\ \mathbf{K}_{\mathrm{Tp}} & \mathbf{K}_{\mathrm{TT}} & \mathbf{K}_{\mathrm{Tu}} \\ \mathbf{K}_{\mathrm{up}} & \mathbf{K}_{\mathrm{uT}} & \mathbf{K}_{\mathrm{uu}} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{p}} \\ \overline{\mathbf{T}} \\ \overline{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathrm{p}} \\ \mathbf{f}_{\mathrm{T}} \\ \mathbf{f}_{\mathrm{u}} \end{bmatrix} \tag{12}$$

where expressions of matrices, $\mathbf{M}_{ij}$, $\mathbf{K}_{ij}$ and $\mathbf{f}_i$ are provided in **Appendix II**, here i and j represent the subscript p, T and u.

The modified Darcy's law and modified Fourier's law are introduced by embedding matrix $\mathbf{K}_{pT}$ of the mass equation and matrices $\mathbf{K}_{Tp}$, $\mathbf{K}_{TT}$ and $\mathbf{f}_T$ of the energy equation into the source code respectively. The coupling process is realized by a monolithic method. Non-linearities are treated by means of Newton-Raphson method.

3 Verification and comparison

The THM coupled problem is difficult to fully understand for the simultaneous processes of fluid flow, heat transfer and soil skeleton deformation, especially considering T-O and T-F effects. To investigate the seepage flow effects, driven by temperature gradient, on the THM coupling behavior comprehensively, an analytical solution has been developed by Zhou *et al.* (1998) based on the integral-transform method.

For comparison, a cylindrical model with radial length 100 m was established, where temperature increment with 50 K and constant PWP boundaries were applied to the left side of the soil domain. Porosity φ, permeability $\mathbf{k}_p$ and T-O coefficient $\mathbf{k}_{pT}$ were set as 0.375, 5.0×10^{-17} m^2 and 2.7×10^{-10} m^2/(s·K), respectively.

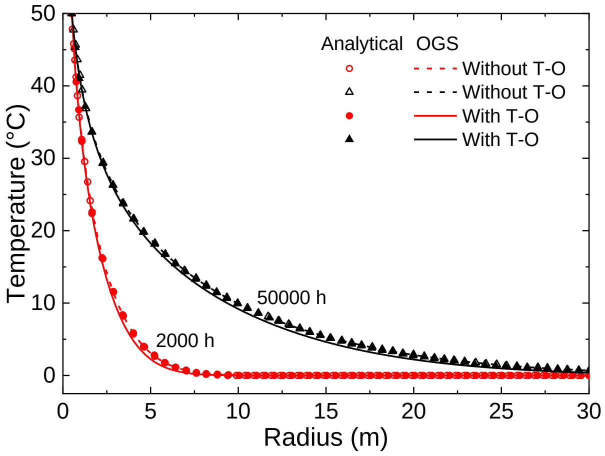
Table 1 Parameters used in the numerical model

Parameters	Solid	Water
Density, ρ^{π} (kg/m^3)	2610	1000
Thermal conductivity, k^{π} (W/(m·K))	3.290	0.582
Specific heat capacity, C^{π} (J/(kg·K))	937	4186
Viscosity, μ (Pa·s)	-	1.0×10^{-3}
Thermal expansion coefficient, α^{π} (1/K)	1.0×10^{-6}	1.0×10^{-4}

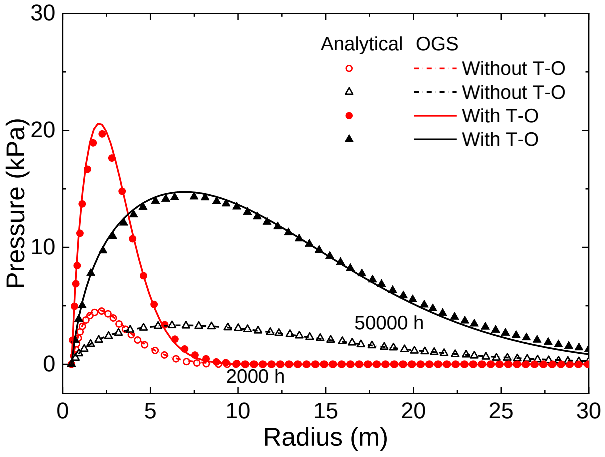
Bulk modulus K^{π} (GPa)	59.0	3.3
Young's modulus, E (kPa)	2880	-
Poisson's ratio, ν	0.2	-

Note: π represents s or w.

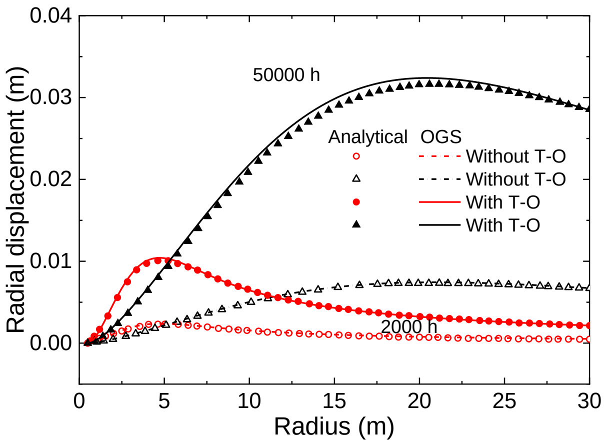
As is shown in Fig. 1, reasonably good agreement can be observed between the present numerical results and the analytical solution by Zhou *et al*. (1998). The slight difference may be due to the fact that the hydraulic flux and advective heat flux are ignored in Zhou's study (1998).



(a)



(b)



(c)

Fig. 1 Axisymmetric THM coupling behavior of saturated soil considering T-O and T-F effects: (a) temperature; (b) PWP; (c) radial displacement

Appendix I

The mass equation of soil skeleton is:

$$\frac{\mathrm{d_s}}{\mathrm{d}t}\left(\rho^{\mathrm{s}}(1-\varphi)\right)+\nabla\cdot(\rho^{\mathrm{s}}(1-\varphi)\mathbf{v}^{\mathrm{s}})=0 \tag{I-1}$$

where ρ^{s} is the solid phase density and $\mathbf{v}^{\mathrm{s}}$ is the velocity of soil skeleton.

Eq. (I-1) can be expressed as:

$$\frac{\mathrm{d_s}\varphi}{\mathrm{d}t}=\frac{1-\varphi}{\rho^{\mathrm{s}}}\frac{\mathrm{d_s}\rho^{\mathrm{s}}}{\mathrm{d}t}+(1-\varphi)\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t} \tag{I-2}$$

$$\frac{1}{\rho^{\mathrm{s}}}\frac{\mathrm{d_s}\rho^{\mathrm{s}}}{\mathrm{d}t}=\frac{1}{(1-\varphi)}\left((\alpha-\varphi)\beta_{\mathrm{s}p}\frac{\mathrm{d_s}p}{\mathrm{d}t}-(\alpha-\varphi)\beta_{\mathrm{s}T}\frac{\mathrm{d_s}T}{\mathrm{d}t}+(1-\alpha)\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t}\right) \tag{I-3}$$

$$\beta_{\mathrm{s}p}=(1-\alpha)\beta_{\mathrm{sk}p} \tag{I-4}$$

where $\beta_{\mathrm{s}p}$ is the compressibility of soil phase, $\beta_{\mathrm{s}T}$ is the volumetric thermal expansion coefficient of soil skeleton, $\beta_{\mathrm{sk}p}$ is the compressibility of soil skeleton, which is expressed as $\beta_{\mathrm{sk}p}=\frac{1}{\mathrm{K}^{\mathrm{sk}}}=\frac{3(1\text{-}2\nu)}{E}$, K^{sk} is the bulk modulus of the soil skeleton, and α is the Biot coefficient.

Combining with Eq. (I-3), Eq. (I-2) can be transformed to:

$$\frac{\mathrm{d_s}\varphi}{\mathrm{d}t} = (\alpha - \varphi)\beta_{sp}\frac{\mathrm{d_s}p}{\mathrm{d}t} - (\alpha - \varphi)\beta_{sT}\frac{\mathrm{d_s}T}{\mathrm{d}t} + (\alpha - \varphi)\frac{\mathrm{d_s}(\nabla\cdot\mathbf{u})}{\mathrm{d}t} \tag{I-5}$$

With considering compression of water, it yields:

$$\frac{1}{\rho^{\mathrm{w}}}\frac{\mathrm{d_s}\rho^{\mathrm{w}}}{\mathrm{d}t} = \beta_{\mathrm{w}p}\frac{\mathrm{d_s}p}{\mathrm{d}t} - \beta_{\mathrm{w}T}\frac{\mathrm{d_s}T}{\mathrm{d}t} \tag{I-6}$$

where $\beta_{\mathrm{w}p}$ is the compressibility of liquid phase and $\beta_{\mathrm{w}T}$ is the volumetric thermal expansion coefficient of pore water.

Appendix II

The matrices in Eq. (12) are defined as:

$$\mathbf{M}_{\mathrm{pp}} = \int \mathbf{N}_p^{\mathrm{T}}\left((\alpha - \varphi)\beta_{sp} + \varphi\beta_{\mathrm{w}p}\right)\mathbf{N}_p\,\mathrm{d}\Omega \tag{II-1}$$

$$\mathbf{M}_{\mathrm{pT}} = -\int \mathbf{N}_p^{\mathrm{T}}\left((\alpha - \varphi)\beta_{\mathrm{s}T} + \varphi\beta_{\mathrm{w}T}\right)\mathbf{N}_T\,\mathrm{d}\Omega \tag{II-2}$$

$$\mathbf{M}_{\mathrm{pu}} = \int \mathbf{N}_p^{\mathrm{T}}\alpha\nabla\mathbf{N}_{\mathbf{u}}\,\mathrm{d}\Omega \tag{II-3}$$

$$\mathbf{K}_{\mathrm{pp}} = \int \nabla\mathbf{N}_p^{\mathrm{T}}\frac{\mathbf{k}_p}{\mu}\nabla\mathbf{N}_p\,\mathrm{d}\Omega \tag{II-4}$$

$$\mathbf{K}_{\mathrm{pT}} = \int \nabla\mathbf{N}_p^{\mathrm{T}}\mathbf{k}_{pT}\nabla\mathbf{N}_T\,\mathrm{d}\Omega \tag{II-5}$$

$$\mathbf{K}_{\mathrm{pu}} = \mathbf{0} \tag{II-6}$$

$$\mathbf{f}_{\mathrm{p}} = -\int \mathbf{N}_p^{\mathrm{T}}\frac{\mathrm{q^w}}{\mathrm{\rho^w}}\mathrm{d}\Gamma + \int \nabla\mathbf{N}_p^{\mathrm{T}}\left(\rho^{\mathrm{w}}\frac{\mathbf{k}_p}{\mu}\mathbf{g}\right)\mathrm{d}\Omega \tag{II-7}$$

$$\mathbf{M}_{\mathrm{Tp}} = \mathbf{0} \tag{II-8}$$

$$\mathbf{M}_{\mathrm{TT}} = \int \mathbf{N}_T^{\mathrm{T}}(\rho C)\mathbf{N}_T\,\mathrm{d}\Omega \tag{II-9}$$

$$\mathbf{M}_{\mathrm{Tu}} = \int \mathbf{N}_T^{\mathrm{T}}\left(-\frac{\beta_{\mathrm{s}T}}{\beta_{\mathrm{sk}p}}T_{\mathrm{ref}}\right)\nabla\mathbf{N}_{\mathbf{u}}\,\mathrm{d}\Omega \tag{II-10}$$

$$\mathbf{K}_{\mathrm{Tp}} = \int\left(\nabla\mathbf{N}_T^{\mathrm{T}}\left(T_{\mathrm{ref}}\mathbf{k}_{pT} - \frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}}T_{\mathrm{ref}}\frac{\mathbf{k}_p}{\mu}\right)\nabla\mathbf{N}_p - \mathbf{N}_T^{\mathrm{T}}C^{\mathrm{w}}\rho^{\mathrm{w}}\frac{\mathbf{k}_p}{\mu}\nabla\mathbf{N}_p\nabla T\right)\mathrm{d}\Omega \tag{II-11}$$

$$\mathbf{K}_{\mathrm{TT}} = \int\left(\nabla\mathbf{N}_T^{\mathrm{T}}\left(\mathbf{k}_T - \frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}}T_{\mathrm{ref}}\mathbf{k}_{pT}\right)\nabla\mathbf{N}_T + \mathbf{N}_T^{\mathrm{T}}C^{\mathrm{w}}\rho^{\mathrm{w}}\mathbf{v}^{\mathrm{w}}\nabla\mathbf{N}_T - \mathrm{N}_T^{\mathrm{T}}C^{\mathrm{w}}\rho^{\mathrm{w}}\mathbf{k}_{pT}\nabla\mathbf{N}_T\nabla T\right)\mathrm{d}\Omega \tag{II-12}$$

$$\mathbf{K}_{\mathrm{Tu}} = \mathbf{0} \tag{II-13}$$

$$\mathbf{f}_{\mathrm{T}} = \int \nabla \mathbf{N}_T^{\mathrm{T}} \left(-\frac{\beta_{\mathrm{w}T}}{\beta_{\mathrm{w}p}} T_{\mathrm{ref}} \frac{\mathbf{k}_p}{\mu} \rho^{\mathrm{w}} \mathbf{g} \right) \mathrm{d}\Omega - \int \mathbf{N}_T^{\mathrm{T}} q^{\mathrm{E}} \, \mathrm{d}\Gamma \tag{II-14}$$

$$\mathbf{M}_{\mathrm{up}} = \mathbf{0} \tag{II-15}$$

$$\mathbf{M}_{\mathrm{uT}} = \mathbf{0} \tag{II-16}$$

$$\mathbf{M}_{\mathrm{uu}} = \mathbf{0} \tag{II-17}$$

$$\mathbf{K}_{\mathrm{up}} = -\int \nabla \mathbf{N}_{\mathbf{u}}^{\mathrm{T}} \, \alpha \mathbf{1} \mathbf{N}_p \mathrm{d}\Omega \tag{II-18}$$

$$\mathbf{K}_{\mathrm{uT}} = -\int \nabla \mathbf{N}_{\mathbf{u}}^{\mathrm{T}} \, (\mathbf{D}^{\mathbf{e}} \mathbf{:1}) \alpha^{\mathrm{s}} \mathbf{N}_T \mathrm{d}\Omega \tag{II-19}$$

$$\mathbf{K}_{\mathrm{uu}} = \int \nabla \mathbf{N}_{\mathbf{u}}^{\mathrm{T}} \, \mathbf{D}^{\mathbf{e}} \mathbf{:} \nabla \mathbf{N}_{\mathbf{u}} \mathrm{d}\Omega \tag{II-20}$$

$$\mathbf{f}_{\mathrm{u}} = -\int \mathbf{N}_{\mathbf{u}}^{\mathrm{T}} \bar{\mathbf{t}} \, (\rho \mathbf{g}) \mathrm{d}\Omega + \int \mathbf{N}_{\mathbf{u}}^{\mathrm{T}} \bar{\mathbf{t}} \, \mathrm{d}\Gamma \tag{II-21}$$

References

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Bear J and Bachmat Y. (1990). The Porous Medium. In Introduction to Modeling of Transport Phenomena in Porous Media, Springer, Dordrecht.