# Complete Mathematical Formulation of PCA for RGB Image Compression

## Notation recap

Let $m$ and $n$ denote the image height and width, respectively. We use the following conventions for matrices and vectors:

$$I \in \mathbb{R}^{m \times n \times 3}$$

is the original RGB image tensor; the third dimension indexes the color channels. For a matrix $X \in \mathbb{R}^{m \times n}$, $X_{:,j}$ denotes its $j$-th column and $\mathbf{1}_n \in \mathbb{R}^n$ denotes the all-ones column vector. Superscripts and subscripts such as $(k)$, $\widehat{\cdot}$, and $\widetilde{\cdot}$ are used as described below.

## 1. Image representation

Let the input RGB image be represented as a tensor

$$I \in \mathbb{R}^{m \times n \times 3},$$

where the three channels correspond to Red, Green and Blue.

## 2. Normalization

Define the maximum pixel intensity

$$M = \max_{1 \le i \le m,\, 1 \le j \le n,\, 1 \le c \le 3} I_{i,j,c}.$$

The normalized image is

$$\tilde{I}_{i,j,c} = \frac{I_{i,j,c}}{M}, \qquad \text{for all } i, j, c,$$

or compactly $\tilde{I} = \frac{1}{M} I$.

# 3. Channel extraction

Extract the three color-channel matrices:

$$B = \tilde{I}(:,:,1), \qquad G = \tilde{I}(:,:,2), \qquad R = \tilde{I}(:,:,3),$$

i.e.

$$B_{i,j} = \tilde{I}_{i,j,1}, \quad G_{i,j} = \tilde{I}_{i,j,2}, \quad R_{i,j} = \tilde{I}_{i,j,3}.$$

Each of $R, G, B$ is in $\mathbb{R}^{m\times n}$.

# 4. Mean centering

For a general channel matrix $X \in \{R, G, B\}$ define the column mean

$$\mu_X = \frac{1}{n}\sum_{j=1}^{n} X_{:,j} \in \mathbb{R}^m.$$

Define the centered matrix

$$\widehat{X}_{:,j} = X_{:,j} - \mu_X,$$

so that

$$\widehat{R},\ \widehat{G},\ \widehat{B} \in \mathbb{R}^{m\times n}$$

are the centered channel matrices.

# 5. Covariance matrices

For any centered channel $\widehat{X}$ the sample covariance matrix (columns as observations) is

$$\Sigma_X = \frac{1}{n-1}\,\widehat{X}\,\widehat{X}^\top \in \mathbb{R}^{m\times m}.$$

Hence

$$\Sigma_R = \frac{1}{n-1}\widehat{R}\widehat{R}^\top, \quad \Sigma_G = \frac{1}{n-1}\widehat{G}\widehat{G}^\top, \quad \Sigma_B = \frac{1}{n-1}\widehat{B}\widehat{B}^\top.$$

# 6. Eigenvalue decomposition

For each channel compute the eigen-decomposition:

$$\Sigma_R V_R = V_R \Lambda_R, \qquad \Sigma_G V_G = V_G \Lambda_G, \qquad \Sigma_B V_B = V_B \Lambda_B,$$

where $\Lambda_R = \mathrm{diag}(\lambda_{R,1}, \ldots, \lambda_{R,m})$ (similarly for $\Lambda_G, \Lambda_B$) and the columns of $V_R, V_G, V_B$ are the corresponding eigenvectors $v_{R,i}, v_{G,i}, v_{B,i}$.

# 7. Sorting eigenvalues and eigenvectors

Sort eigenvalues in descending order and reorder eigenvectors accordingly. Denote the sorted eigenvalues by

$$\lambda_{R,(1)} \geq \lambda_{R,(2)} \geq \cdots \geq \lambda_{R,(m)}$$

and let $\pi_R$ be the permutation satisfying $\lambda_{R,(k)} = \lambda_{R,\pi_R(k)}$. Define sorted eigenvectors

$$v_{R,(k)} = v_{R,\pi_R(k)}, \qquad V_R^{(\text{sorted})} = \left[v_{R,(1)}, v_{R,(2)}, \ldots, v_{R,(m)}\right].$$

Similarly define $V_G^{(\text{sorted})}, V_B^{(\text{sorted})}$. From now on we use $V_R, V_G, V_B$ to denote the sorted-eigenvector matrices for brevity.

# 8. Select top-$k$ principal components

For a given $k$ with $1 \leq k \leq m$ define the top-$k$ eigenvector matrices

$$V_{R,k} = \left[v_{R,(1)}, \ldots, v_{R,(k)}\right] \in \mathbb{R}^{m \times k},$$

and analogously $V_{G,k}, V_{B,k} \in \mathbb{R}^{m \times k}$.

# 9. Projection (dimensionality reduction)

Project the centered channel data onto the $k$-dimensional subspace:

$$C_R^{(k)} = V_{R,k}^\top \widehat{R} \in \mathbb{R}^{k \times n}, \qquad C_G^{(k)} = V_{G,k}^\top \widehat{G}, \qquad C_B^{(k)} = V_{B,k}^\top \widehat{B}.$$

# 10. Inverse transformation (reconstruction)

Reconstruct the centered channels from reduced coordinates:

$$\widehat{R}^{(k)} = V_{R,k} C_R^{(k)} = V_{R,k} V_{R,k}^\top \widehat{R} \in \mathbb{R}^{m \times n},$$

$$\widehat{G}^{(k)} = V_{G,k} C_G^{(k)} = V_{G,k} V_{G,k}^\top \widehat{G},$$

$$\widehat{B}^{(k)} = V_{B,k} C_B^{(k)} = V_{B,k} V_{B,k}^\top \widehat{B}.$$

Add the column means back to obtain reconstructed (non-centered) channels:

$$\widetilde{R}^{(k)} = \widehat{R}^{(k)} + \mu_R \mathbf{1}_n^\top, \qquad \widetilde{G}^{(k)} = \widehat{G}^{(k)} + \mu_G \mathbf{1}_n^\top, \qquad \widetilde{B}^{(k)} = \widehat{B}^{(k)} + \mu_B \mathbf{1}_n^\top.$$

# 11. Combine reconstructed channels into RGB image

Stack the reconstructed channels to form the rank-$k$ reconstructed image:

$$\widetilde{I}^{(k)} = \text{stack}\left(\widetilde{B}^{(k)}, \widetilde{G}^{(k)}, \widetilde{R}^{(k)}\right) \in \mathbb{R}^{m \times n \times 3},$$

## 12. Processing multiple values of $k$ (visualization loop)

Let
$$\mathcal{K} = \{10, 20, 50, 100, 500, 1400\}.$$

For each $k \in \mathcal{K}$ perform:

1. Select $V_{R,k}, V_{G,k}, V_{B,k}$.
2. Compute $C_R^{(k)}, C_G^{(k)}, C_B^{(k)}$.
3. Form $\widehat{R}^{(k)}, \widehat{G}^{(k)}, \widehat{B}^{(k)}$.
4. Add means to obtain $\widetilde{R}^{(k)}, \widetilde{G}^{(k)}, \widetilde{B}^{(k)}$.
5. Stack to obtain $\widetilde{I}^{(k)}$ and display as a subplot.

## 13. Shapes and dimension checks

$$V_{R,k} \in \mathbb{R}^{m\times k}, \quad \widehat{R} \in \mathbb{R}^{m\times n} \quad \Rightarrow \quad C_R^{(k)} = V_{R,k}^{\top}\widehat{R} \in \mathbb{R}^{k\times n},$$
$$\widehat{R}^{(k)} = V_{R,k}C_R^{(k)} \in \mathbb{R}^{m\times n}.$$

Analogous relations hold for $G$ and $B$.

## 14. Optional: Variance interpretation and reconstruction error

Let $\{\lambda_{(i)}\}_{i=1}^{m}$ be the sorted eigenvalues of a channel covariance matrix. Total variance:

$$\mathrm{Var}_{\text{total}} = \sum_{i=1}^{m} \lambda_{(i)}.$$

Variance captured by first $k$:

$$\mathrm{Var}_k = \sum_{i=1}^{k} \lambda_{(i)}.$$

Fraction of variance explained (FVE):

$$\mathrm{FVE}(k) = \frac{\mathrm{Var}_k}{\mathrm{Var}_{\text{total}}}.$$

When covariance uses the factor $\frac{1}{n-1}$, the squared Frobenius reconstruction error satisfies

$$\left\|\widehat{R} - \widehat{R}^{(k)}\right\|_F^2 = (n-1) \sum_{i=k+1}^{m} \lambda_{(i)}.$$

# 15. Compact mathematical pseudocode

For each $k \in \mathcal{K}$:

$$\begin{aligned}
&V_{*,k} \leftarrow \text{first } k \text{ columns of } V_*,\\
&C_*^{(k)} \leftarrow V_{*,k}^\top \widehat{*},\\
&\widehat{*}^{(k)} \leftarrow V_{*,k} C_*^{(k)},\\
&\widetilde{*}^{(k)} \leftarrow \widehat{*}^{(k)} + \mu_* \mathbf{1}_n^\top,\\
&\widetilde{I}^{(k)} \leftarrow \text{stack}(\widetilde{B}^{(k)}, \widetilde{G}^{(k)}, \widetilde{R}^{(k)}),
\end{aligned}$$

where "$*$" stands for each color channel $R, G, B$.

This document unifies the normalization, channel extraction, covariance computation, eigendecomposition, top-$k$ selection, projection, reconstruction and visualization steps into a single.