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A New Hamilton–Jacobi Differential Game Framework for Nonlinear Estimation and Output Feedback Control

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Abstract
In this paper, we develop a new framework for designing state estimators/filters and output measurement feedback controllers for affine nonlinear systems in state space. The problems are formulated as zero-sum differential games, and sufficient conditions for their solvability are given in terms of Hamilton–Jacobi–Isaacs equations (HJIEs). These HJIEs are new, in the sense that they are both state-dependent and measurement output dependent. This allows for the filter and observer gains to be optimized over all possible nonlinear gains. Examples and simulation results are also presented to support the theory.

Keywords Nonlinear system · $\mathcal{L}_2$-filtering · Zero-sum differential game · Hamilton–Jacobi–Isaacs equation · Output feedback control · Matrix inequality

1 Introduction

Many authors have considered $\mathcal{H}_\infty$ filtering techniques for nonlinear systems [18,26,35,36] because of the several advantages that they offer over Kalman filtering [2], notably robustness to noise/disturbances and model uncertainties, and the fact that the $\mathcal{H}_\infty$ filter is derived from a completely deterministic setting. It is also easier to compute and implement compared with previous statistical nonlinear filtering techniques developed using minimum variance [8], Bayesian estimation [3,4,13–16,19,21,23,29,31,34], as well as maximum-likelihood [22,24] criteria. These approaches are not very attractive since they result in filters that are infinite dimensional, and the governing partial differential equations, notably the Wong and Zakai equation, Fokker–Planck's equation or Kolmogorov equation, the Stratonovitch-Kushner equations [8,9,19,33], as well as the Mortenson equation [24], are too

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complicated to solve in general. On the other hand, the nonlinear $\mathcal{H}_\infty$ filter relies on finding a smooth solution to a Hamilton–Jacobi–Isaacs (HJI) partial differential equation (PDE) or HJIE, which can be found using numerical methods [1].

Furthermore, high-precision filters for nonlinear systems are more stringent than in the linear case [17,25,27] for the successful design of feedback controllers, since the dynamic output feedback $\mathcal{H}_\infty$-control problem is more difficult to implement, requiring the solution of two Hamilton–Jacobi–Isaacs equations (HJIEs) [11]. Thus, a better practical strategy for output feedback controller design implementation is to use state feedback in conjunction with an estimator.

The motivation behind this paper is twofold: Firstly, the realization that in [7], the nonlinear $\mathcal{H}_\infty$ filter derived cannot be implemented because the filter gain matrix $G(.)$ is a function of the original state of the system, x, and the estimated state, $\hat{x}$, i.e., $G = G(x, \hat{x})$. Therefore, instead, we construct a nonlinear filter such that the gain matrix $G = G(\hat{x}, y)$, a function of the estimated state and the measurement vector, and here lies the novelty of this new filter that we present in this paper. Clearly, this allows the filter to be practically implementable and also optimizes the gain matrix over all possible nonlinear gain matrices, as opposed to constant gain matrices as used in, e.g., the extended Kalman filter and other filters. Secondly, we also realized in [11] that the output injection gain matrix G is a constant matrix. Although this deficiency had been somewhat ameliorated in [12] with a gain matrix that is a function of the controller state, i.e., $G = G(\xi)$, it still leaves much to be desired, in the sense that it may not be optimal with respect to optimizing the gain matrix itself over all possible nonlinear gain matrices. Moreover, it is well known that a nonlinear gain matrix which is also optimal with respect to the objective function can perform much better than a constant or nonlinear sub-optimal gain matrix. Therefore, as a by-product of the results of the new filter design approach presented in the first part of the paper, we extend the results to design an observer-based measurement feedback output feedback controller. In addition, we also derive new Hamilton–Jacobi–Bellman–Isaacs equations which characterize the solution of both the filtering problem and the output feedback problem.

Accordingly, in this short paper, we first derive a nonlinear $\mathcal{L}_2$-filter (or Kalman filter) in which the gain matrix is not a function of the original state that is being estimated. Secondly, we reconsider the output feedback nonlinear control problem and resolve it to optimize the output injection gain matrix over the space of all possible nonlinear matrix functions. Both two problems are considered in the framework of zero-sum dynamic games [5,6] reminiscent of the $\mathcal{H}_\infty$ control problem. However, the problems we solve are slightly different from the $\mathcal{H}_\infty$ solution in the sense that we do not impose the disturbance attenuation condition. Therefore, it is more appropriate to call it the $\mathcal{H}_2$ solution. Nevertheless, the $\mathcal{H}_\infty$ solution is still recovered from our formulation and solution.

The rest of the paper is organized as follows. In Sect. 2, we introduce preliminaries and some assumptions. Then, in Sect. 3, we define and solve the $\mathcal{L}_2$-filtering problem. This is followed in Sect. 4 by a formulation and solution of the output measurement feedback problem. Then, in Sect. 5, we solve a couple of examples and present some simulation results. Finally, in Sect. 6, we give conclusions.

The notation is fairly standard except where otherwise stated. Moreover, $\|(.)\|$ will denote the Euclidean vector norm on $\mathbf{R}^n$. The space $\mathcal{L}_p([t_0,\infty),\mathbf{R}^n)$ is the time-domain standard Lebesgue space of pth order integrable vector-valued functions over $[t_0,\infty)$. The corresponding norm in the above space is defined accordingly as $\|(.)\|_p \stackrel{\Delta}{=} \left(\int_{t_0}^{\infty}\|(.)\|^p dt\right)^{\frac{1}{p}}$. Other notations will be defined accordingly.

2 Preliminaries

The plant is represented by an affine nonlinear causal state-space system defined on a manifold $\mathcal{X}\subseteq\Re^n$, $0\in\mathcal{X}$:

$$\Sigma_1^{\text{ans}}:\begin{cases}\dot{x}=f(x)+g_1(x)w+g_2(x)u; & x(t_0)=x_0\\ y=h_2(x)+k_{21}(x)w,\end{cases} \tag{1}$$

where $x\in\mathcal{X}$ is the state vector; $u\in\mathcal{U}\subset\mathbf{R}^p$ is the control input, which belongs to the set $\mathcal{U}=\{u:\mathbf{R}\to\mathbf{R}^p|u \text{ is measurable}\}$ of admissible controls; $w\in\mathcal{W}$ is an unknown disturbance (or noise) signal, which belongs to the set $\mathcal{W}=\{w:\mathbf{R}\to\mathbf{R}^r|w \text{ is measurable}\}\subset\mathbf{R}^r$ of admissible disturbances; $y\in\mathcal{Y}\subset\Re^m$ is the measured output (or observation) of the system and belongs to $\mathcal{Y}=\{y:\mathbf{R}\to\mathbf{R}^m|w \text{ is measurable}\}$, the set of admissible outputs. The functions $f:\mathcal{X}\to\mathcal{X}$, $g_1:\mathcal{X}\to\mathcal{M}^{n\times s}$, $g_2:\mathcal{X}\to\mathcal{M}^{n\times r}$, $k_{21}:\mathcal{X}\to\mathcal{M}^{m\times r}$ where $\mathcal{M}^{i\times j}(\mathcal{X})$ is the ring of $i\times j$ smooth matrices over $\mathcal{X}$, and $h_2:\mathcal{X}\to\mathbf{R}^m$, are all real C^∞ functions of x. Furthermore, we assume without any loss of generality that the system Σ^a has an equilibrium point at $x=0$ such that $f(0)=0$, $h_2(0)=0$, and there exists a unique solution $x(t,t_0,x_0,w)$ for the system for all initial conditions x_0, for all $w\in\mathcal{W}$ and for all $t\in\mathbf{R}$. Moreover, for simplicity, we also make the following assumption on the plant.

Assumption 2.1 The system matrices are such that

$$k_{21}(x)g_1^T(x)=0$$
$$k_{21}(x)k_{21}^T(x)=I.$$

3 Filtering Problem

In this section, we discuss the new approach to the filtering problem for the affine nonlinear system Σ_1^{ans}. We derive new HJIEs that characterize the solution to the problem. The problem can be simply defined as follows.

Definition 3.1 (Nonlinear Estimation Problem in the $\mathcal{L}_p$-norm) Find an estimator or filter, $\mathcal{F}$, for estimating the state $x(t)$ from available observations $Y_t\stackrel{\Delta}{=}\{y(\tau),\quad \tau\le t\}$ over a time period $[t_0,\infty)$, such that the $\mathcal{L}_p$-norm ($p=1,2,\infty$) of a suitable error or penalty variable z of the actual state and the estimated state, $\hat{x}(t) = \mathcal{F}(Y_t)$, is minimized for all admissible disturbances $w \in \mathcal{W} \subseteq \mathcal{L}_p$, and for all initial conditions $x_0 \in \mathcal{X}$.

To solve the above problem, we assume without any loss of generality that the system Σ_1^{ans} (with $w = 0$) is asymptotically stable (or if it is not, then there is a control law $\bar{u}$ that stabilizes it) so we can set $u = 0$. Then, we select a filter topology in the form of a Luenberger estimator:

$$\Sigma_1^{nfil} : \begin{cases} \dot{\hat{x}} = f(\hat{x}) + g_1(\hat{x})w^* + L^*(\hat{x}, y)(y - h_2(\hat{x})), & \hat{x}(t_0) = 0 \\ \tilde{z} = y - h_2(\hat{x}), \end{cases} \tag{2}$$

where $\hat{x} \in \mathcal{X}$ is the estimator state, $\tilde{z} \in \Re^m$ is the error or penalty variable, w^* is the optimal noise or disturbance, and $L^* : \mathcal{X} \times \mathcal{Y} \to \mathcal{M}^{n\times m}(\mathcal{X} \times \mathcal{Y})$ is the optimal gain matrix for the filter, where $\mathcal{M}^{n\times m}(\mathcal{X} \times \mathcal{Y})$ is the ring of $n \times m$ matrices over $\mathcal{X} \times \mathcal{Y}$. Notice here, the gain matrix L is a function of both $\hat{x}$ and y as opposed to earlier approaches in which it is a fixed constant, or a function of $\hat{x}$ and x (which does not make sense [7]), or a function of only $\hat{x}$ [18,24]. To complete the design, the pair (w^*, L^*) has to be determined.

As is well known [5], the problem can be formulated as a two-player zero-sum differential game with the cost functional:

$$\min_{\mathcal{M}^{n\times m}(\mathcal{X}\times\mathcal{Y})} \max_{\mathcal{W}} J_1(L, w) = \frac{1}{2}\|\tilde{z}\|^p_{\mathcal{L}_p[t_0,\infty)}, \tag{3}$$

where the pair (L, w) is to be optimized to determine (L^*, w^*). A saddle-point equilibrium solution to the above game is said to exist and is admissible [5], if we can find a pair of strategies (L^*, w^*), such that

$$J_1(L^*, w) \leq J_1(L^*, w^*) \leq J_1(L, w^*) \tag{4}$$

for all $w \in \mathcal{W}$, and all $L \in \mathcal{M}^{n\times m}(\mathcal{X} \times \mathcal{Y})$. In this formulation, the pair (L, w) acts non-cooperatively, and so w^* can be regarded as the worst possible disturbance level that can be suppressed. Notice also that, this formulation of the problem is different from both the $\mathcal{H}_\infty$ and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ formulations, which require disturbance attenuation or $\mathcal{L}_2$-gain less than a certain level [1]. As such, we refer to this as the nonlinear $\mathcal{L}_p$-filtering problem. Notice also that, since y is a function of w, the above objective function $J_1(.,.)$ is not unbounded above.

Lastly, there are various notions of observability; however, for our purpose, we shall adopt the following definition.

Definition 3.2 For the nonlinear system Σ_1^{ans}, we say that it is locally observable in $\mathcal{O} \subset \mathcal{X}$ if for all states $x_1,\ x_2 \in \mathcal{O}$ and input $w(.) = 0$,

$$y(., x_1, w) \equiv y(., x_2, w) \Longrightarrow x_1 = x_2,$$

where $y(., x_i, w), i = 1, 2$, is the output of the system with the initial condition $x(t_0) = x_i$. Moreover, the system is said to be observable if it is locally observable at each $x_0 \in \mathcal{X}$ or $\mathcal{O} = \mathcal{X}$.

In this paper, it is desired to solve the filtering problem for $\mathcal{L}_p$, $p = 2$, since it is more tractable.

3.1 $\mathcal{L}_2$-Filtering Problem

In this section, we consider and solve the filtering problem formulated above for $\mathcal{L}_p$, $p = 2$. This problem is traditionally known as the Kalman filtering problem or $\mathcal{H}_2$-filtering problem. Our approach to the problem is also based purely on deterministic signals. We assume in this case that the set $\mathcal{W} \subseteq \mathcal{L}_2(\mathbf{R}^r, [t_0, \infty))$.

Based on the formulation (3), (2), we proceed to determine w^* under the assumption that, the asymptotic value of $\hat{x}$ equals x. Accordingly, define the Hamiltonian function for Σ_1^{nfil} by $H_1 : T^*\mathcal{X} \times \mathbf{R}^m \times T\mathcal{Y} \times \mathcal{W} \times \Re^{n\times m} \to \mathbf{R}$:[1]

$$H_1(\hat{x}, \hat{p}, y, \dot{y}, w, L) = \hat{p}_1^T[f(\hat{x}) + g_1(\hat{x})w + L(\hat{x}, y)(y - h_2(\hat{x}))] \\ + \hat{p}_2^T \dot{y} + \frac{1}{2}\|y - h_2(\hat{x})\|^2, \tag{5}$$

where $\hat{p}_1 \in \mathbf{R}^n$, $\hat{p}_2 \in \mathbf{R}^m$ are the corresponding adjoint vectors, and $\hat{p} = (p_1^T \; p_2^T)^T$. Notice also here that the second adjoint vector $\hat{p}_2$ has been introduced to account for the variation of y, the measurement variable. Then, using the completion of squares for w, it can be shown that

$$w^* = g_1^T(\hat{x})\hat{p}_1$$

maximizes $H_1(., ., ., ., .)$. Moreover, $\dot{y}$ in (5) can be approximated from (1) as $\dot{y} = \mathcal{L}_f h_2(x) = \nabla h_2(x) f(x)$,[2] where ∇h_2 represents the Jacobian matrix of h_2. This then gives

$$H_1(\hat{x}, \hat{p}, y, w^*, L) = \hat{p}_1^T f(\hat{x}) + \hat{p}_1^T g_1(\hat{x}) g_1^T(\hat{x})\hat{p}_1 \\ + \hat{p}_2^T \nabla h_2(\hat{x}) f(\hat{x}) \\ + \hat{p}_1^T L(\hat{x}, y)(y - h_2(\hat{x})) + \frac{1}{2}\|y - h_2(\hat{x})\|^2. \tag{6}$$

Again, completing the squares now for L in the above expression for $H_1(., ., ., \hat{w}^*, L)$, we get

$$\hat{p}_1^T L^*(\hat{x}, y) = -(y - h_2(\hat{x}))^T,$$

[1] $T^*\mathcal{X}$, $T\mathcal{Y}$ represent the cotangent and tangent bundles of $\mathcal{X}$ and $\mathcal{Y}$ with coordinates $(\hat{x}, \hat{p}_1)$, $(y, \dot{y})$, respectively.

[2] Where $\mathcal{L}_f(.)$ is the Lie-derivative operator along f.

and

$$H_1(\hat{x}, \hat{p}, y, w^*, L^*) = \hat{p}_1^T f(\hat{x}) + \hat{p}_1^T g_1(\hat{x}) g_1^T(\hat{x})\hat{p}_1 + \hat{p}_2^T \nabla h_2(\hat{x}) f(\hat{x}) \\ -\frac{1}{2}\|y - h_2(\hat{x})\|^2. \tag{7}$$

It can also be checked that, the pair (w^*, L^*) satisfies the saddle-point equilibrium conditions (4). Moreover, by Pontryagin's minimum principle, the optimal filter trajectory is then governed by the following Hamiltonian system:

$$\bar{X}_1^{nfil} : \begin{cases} \dot{\hat{x}} = \frac{\partial H_1}{\partial \hat{p}_1}(\hat{x}, \hat{p}_1, y, \hat{p}_2, w^*, L^*), & \hat{x}(t_0) = 0 \\ \dot{\hat{p}}_1 = -\frac{\partial H_1}{\partial \hat{x}}(\hat{x}, \hat{p}_1, y, \hat{p}_2, w^*, L^*), & \hat{p}_1(\infty) = 0 \\ \dot{y} = \frac{\partial H_1}{\partial \hat{p}_2}(\hat{x}, \hat{p}_1, y, \hat{p}_2, w^*, L^*), & y(t_0) = y_0 \\ \dot{\hat{p}}_2 = -\frac{\partial H_1}{\partial y}(\hat{x}, \hat{p}_1, y, \hat{p}_2, w^*, L^*), & \hat{p}_2(\infty) = 0. \end{cases} \tag{8}$$

Further, it is well known from the method of characteristics [10] that there exists a solution for (8) if and only if there exists a C^1-solution to the Hamilton–Jacobi–Isaacs equation (HJIE):

$$U_{\hat{x}}(\hat{x}, y) f(\hat{x}) + U_y(\hat{x}, y)\nabla h_2(\hat{x}) f(\hat{x}) + U_{\hat{x}}(\hat{x}, y) g_1(\hat{x}) g_1^T(\hat{x}) U_{\hat{x}}^T(\hat{x}, y) \\ -\frac{1}{2}(y - h_2(\hat{x}))^T (y - h_2(\hat{x})) = 0, \quad U(0, 0) = 0, \tag{9}$$

for some smooth function $U : \mathcal{X} \times \mathcal{Y} \to \mathbf{R}$. Accordingly, the optimal decisions of the two players are determined by setting $\hat{p}_1 = U_{\hat{x}}^T(\hat{x}, y)$ and $\hat{p}_2 = U_y^T(\hat{x}, y)$ as

$$w^* = g_1(\hat{x}) U_{\hat{x}}^T(\hat{x}, y) \tag{10}$$

$$U_{\hat{x}}(\hat{x}, y) L^*(\hat{x}, y) = -(y - h_2(\hat{x}))^T, \tag{11}$$

where $U_{\hat{x}}(\hat{x}, y)$ and $U_y(\hat{x}, y)$ are the row vectors of partial derivatives of U with respect to $\hat{x}$ and y, respectively. Consequently, we have the following result.

Proposition 3.1 *Consider the nonlinear system* (1) *and the $\mathcal{L}_2$-filtering problem for this system. Suppose Assumption* 2.1 *holds, the plant Σ_1^{ans} is locally asymptotically stable about the equilibrium point $x = 0$ and observable. Further, suppose there exists a C^1 positive semidefinite function $U : N \times Y \to \mathbf{R}$ locally defined in a neighborhood $N \times Y \subset \mathcal{X} \times \mathcal{Y}$ of the origin $(\hat{x}, y) = (0, 0)$, and a matrix function $L : N \times Y \to \mathcal{M}^{n \times m}(\mathcal{X} \times \mathcal{Y})$, satisfying the HJIE* (9) *together with the side condition* (11)*. Then, the filter Σ_1^{nfil} solves the $\mathcal{L}_2$-filtering problem for the system locally in N (or for all initial conditions $x_0 \in N$).*

Proof Let $U \geq 0$ be a C^1 solution of the HJIE (9). Differentiating U along the trajectory of (2), and using (10), (11), (9), we have

$$
\begin{aligned}
\dot{U}(x(t), y(t)) &= U_{\hat{x}}(\hat{x}, y)[f(\hat{x}) + g_1(\hat{x})g_1^T(\hat{x})U_{\hat{x}}^T(\hat{x}, y) + L^*(\hat{x}, y)(y - h_2(\hat{x}))] + U_y \dot{y} \\
&= U_{\hat{x}}(\hat{x}, y)f(\hat{x}) + U_{\hat{x}}(\hat{x}, y)g_1(\hat{x})g_1^T(\hat{x})U_{\hat{x}}^T(\hat{x}, y) \\
&\quad + U_{\hat{x}}(\hat{x}, y)L^*(\hat{x}, y)(y - h_2(\hat{x})) + U_y(\hat{x}, y)\nabla h_2(\hat{x})f(\hat{x}) \\
&= -\frac{1}{2}\|y - h_2(\hat{x})\|^2.
\end{aligned}
$$

This implies that the filter dynamics (and hence the filtering error dynamics also) is stable [32]. In addition, for any $t_s > t_0$ such that

$$
\dot{U}(x(t), y(t)) \equiv 0 \;\; \forall t \geq t_s \Longrightarrow \tilde{z} \equiv 0 \Longrightarrow y = h_2(\hat{x}) \, \forall t \geq t_s.
$$

By the observability of the system Σ_1^{ans}, this implies that $x = \hat{x} \; \forall t \geq t_s$. This completes the proof of Proposition. □

Remark 3.1 To summarize on what has been achieved in the foregoing, we have introduced a new HJIE (9) the nature of which has never been used in the estimation problem before. The solution to this HJIE is a function of $\hat{x}$ and y only and does not involve the original state x. This fundamentally avoids the ambiguity of using the original state x in both the filter gain matrix and the HJIE as derived in [7]. The closest that anyone has come near the above result, and the HJIE is the result of Mortenson [24]. However, in [24], the resulting HJIE is much more complicated and in fact infinite dimensional.

For the linear system

$$
\Sigma_1^{ls} : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u; \quad x(t_0) = x_0 \\ y = C_2(x) + D_{21} w, \end{cases} \tag{12}
$$

we have the following Corollary to the proposition.

Corollary 3.1 *Consider the linear system* (12) *and the $\mathcal{L}_2$-filtering problem for this system. Suppose* Assumption 2.1 *holds, the plant Σ_1^{ls} is asymptotically (or Hurwitz) stable about the equilibrium point $x = 0$ and observable. Further, suppose there exists an $n \times n$ symmetric positive semidefinite matrix $P_1 \geq 0$ and a matrix $L \in \mathbf{R}^{n \times m}$, satisfying the linear matrix inequalities (LMIs):*

$$
\begin{bmatrix} A^T P_1 + P_1 A - C_2^T C_2 & P_1 B_1 & Q_1 C_2 A + C_2 \\ B_1^T P_1 & -2I & 0 \\ A^T C_2^T Q_1 + C_2^T & 0 & -I \end{bmatrix} \leq 0, \tag{13}
$$

$$
\begin{bmatrix} 0 & C_2^T - P_1 L \\ C_2 - L^T P_1 & -2I \end{bmatrix} \leq 0, \tag{14}
$$

then the filter

$$
\Sigma_1^{lfil} : \begin{cases} \dot{\hat{x}} = (A + B_1 B_1^T P_1)\hat{x} + P_1^{-1} C_2^T (y - C_2 \hat{x}), \quad \hat{x}(t_0) = 0 \\ \tilde{z} = y - C_2 \hat{x} \end{cases} \tag{15}
$$

solves the $\mathcal{L}_2$-filtering problem for the system for all initial conditions x_0.

Proof Take $V(\hat{x}, y) = \frac{1}{2}(x^T P_1 x + y^T Q_1 y)$ and apply the result of Proposition 4.1. □

Remark 3.2 The sensitivity of the filter Σ_1^{fil} with respect to disturbances is measured by the $\mathcal{L}_2$-gain of the filter from w to $\tilde{z}$. This can be defined as

$$\gamma_{S_{w\tilde{z}}} = \sup_{0 \neq w \in \mathcal{L}_2[0,\infty)} \frac{\|\tilde{z}\|_{\mathcal{L}_2[0,\infty)}}{\|w\|_{\mathcal{L}_2[0,\infty)}}$$

This can be shown to be equivalent [1] to finding the minimum γ that solves the following HJIE:

$$\begin{aligned} &U_{\hat{x}}(\hat{x}, y) f(\hat{x}) + U_y(\hat{x}, y)\nabla h_2(\hat{x}) f(\hat{x}) + \frac{1}{\gamma^2} U_{\hat{x}}(\hat{x}, y) g_1(\hat{x}) g_1^T(\hat{x}) U_{\hat{x}}^T(\hat{x}, y) \\ &-\frac{1}{2}(y - h_2(\hat{x}))^T (y - h_2(\hat{x})) = 0, \quad U(0,0) = 0. \end{aligned} \tag{16}$$

Similarly, the sensitivity due to possible system model uncertainties can be derived by introducing model uncertainties $\Delta f(x)$, $\Delta g_1(x)$, $\Delta h_2(x)$ and deriving the corresponding HJIE (see reference [1] for more details).

4 Output Measurement Feedback Control

In this section, we discuss the $\mathcal{L}_2$ output measurement feedback control problem for the system (1). Again, this problem is slightly different from the $\mathcal{H}_\infty$ control problem [1,11]. We consider the following representation of the system model

$$\Sigma_2^{\text{ans}} : \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u; \quad x(t_0) = x_0 \\ z_1 = h_1(x) + k_{12}(x)u \\ y = h_2(x) + k_{21}(x)w, \end{cases} \tag{17}$$

where $z_1 \in \mathbf{R}^s$ is the penalty variable, $h_1 : \mathcal{X} \to \mathbf{R}^s$, $k_{12} : \mathcal{X} \to \mathcal{M}^{s\times p}$, while all the other functions and variables are as defined previously. Further, in addition to Assumption 2.1, we also for simplicity assume the following for the system:

Assumption 4.1 The system matrices are such that

$$k_{12}^T(x) h_1(x) = 0$$
$$k_{12}^T(x) k_{12}(x) = I.$$

The objective is to synthesize a dynamic output feedback observer-based controller of the form

$$\Sigma^{ofbc} : \begin{cases} \dot{\xi} = f(\xi) + g_1(\xi)w^* + g_2(\xi)u^* + G(\xi, y)(y - h_2(\xi)) \quad \xi(t_0) = \xi_0 \\ u^* = \theta(\xi) \\ z_2 = y - h_2(\xi), \end{cases} \tag{18}$$

such that the following objective functionals are optimized simultaneously:

$$\min_{\mathcal{U}} \max_{\mathcal{W}} J_2(u, w) = \frac{1}{2}\|z_1\|_{\mathcal{L}_2[t_0,\infty)}, \tag{19}$$

$$\min_{\mathcal{M}^{n\times m}(\mathcal{X}\times\mathcal{Y})} J_3(G) = \frac{1}{2}\|z_2\|_{\mathcal{L}_2[t_0,\infty)}, \tag{20}$$

and the closed-loop system (17), (18), with $(u^*, G^*(\xi, y))$ and $w = 0$ is locally asymptotically stable for all initial conditions $(x(t_0), \xi(t_0)) \in \mathcal{O} \times \mathcal{O} \subset \mathcal{X} \times \mathcal{X}$.

Remark 4.1 There is a clear *separation principle* in the philosophy of the above problem formulation. We see that while J_2 is associated with the control problem, J_3 is associated with the estimation problem. However, unlike in linear systems and LQG control design, the two problems cannot be solved entirely separately in this case. This is also reminiscent of the nonlinear $\mathcal{H}_\infty$ control problem [1].

To solve the above problem, we first solve the state feedback sub-problem for the strategies (w^*, u^*) so as to optimize the functional J_2. After that, we solve the output feedback problem for the output injection gain matrix $G(.,.)$ to minimize J_3. The state feedback local $\mathcal{L}_2$ control problem can briefly be defined as follows:

Find an optimal feedback control law of the form $u^* = \alpha(x) \in \mathcal{L}_2[t_0, \infty)$ such that $J_3(.,.)$ is optimized for all $w \in \mathcal{L}_2[t_0, \infty)$, and the closed-loop system (17) with this control law is locally asymptotically stable (about $x = 0$) with $w = 0$, for all initial conditions $x_0 \in \mathcal{O} \subset \mathcal{X}$.

Accordingly, by again using completion of squares arguments, it is straightforward to show that the optimal feedback strategies are given by

$$\left.\begin{aligned} w^* &= g_1^T(x)V_x^T(x) \\ u^* &= -g_2^T(x)V_x^T(x) \end{aligned}\right\} \tag{21}$$

for some smooth function $V : \mathcal{X} \to \mathbf{R}_+$ which solves the HJIE:

$$\begin{aligned} &V_x(x)f(x) + V_x(x)g_1(x)g_1^T(x)V_x^T(x) - \frac{1}{2}V_x(x)g_2(x)g_2^T(x)V_x^T(x) \\ &+\frac{1}{2}h_1^T(x)h_1(x) = 0, \quad V(0) = 0. \end{aligned} \tag{22}$$

In the following Proposition, we prove that the above control law provides local asymptotic stability with $w = 0$, as well as disturbance attenuation from w to z_1, i.e., $\mathcal{L}_2$-gain ≤ 1. Note that, it is important to reprove these results since the formulation in this case is different for the ones used in the earlier references [1,11,12].

Proposition 4.1 *Consider the nonlinear system* (17) *and the* $\mathcal{L}_2$ *state feedback problem for this system. Suppose* Assumptions 2.1, 4.1 *hold for the system and the plant* Σ_2^{ans} *is observable about* $x = 0$*. Further, suppose there exists a* C^1 *positive semidefinite function* $V : N \to \mathbf{R}$*, locally defined in the neighborhood* $N \subset \mathcal{X}$ *of the origins* $x = 0$

and satisfying the HJIE (22). Then, the $\mathcal{L}_2$ state feedback problem for the system Σ_2^{ans} is locally solved in N by the control law (21), i.e., the saddle-point conditions

$$J_2(u^*, w) \leq J_2(u^*, w^*) \leq J_2(u, w^*)$$

are satisfied with the control law (21), and the closed-loop system is locally asymptotically stable.

Proof Suppose there exists a solution C^1-solution $V \geq 0$ to the HJIE (22) in $N \subset \mathcal{X}$. Then, with =$w = 0$, $u = u^*$, and along any trajectory of the system such that $x(t) \not\equiv 0$, $y(t) \not\equiv 0$,

$$\begin{aligned}
\dot{V}(x(t)) = V_x(x)[f(x) - g_2(x)g_2^T(x)V_x^T(x)] &= -\frac{1}{2}\|g_2^T(x)V_x^T(x)\|^2 \\
-\|g_1^T(x)V_x^T(x)\|^2 - \frac{1}{2}h_1^T(x)h_1(x) & \\
= -\frac{1}{2}\|u^*\|^2 - \|w^*\|^2 - \frac{1}{2}h_1^T(x)h_1(x) \leq 0. &
\end{aligned} \tag{23}$$

Thus, by Lyapunov's theorem, the trajectory is bounded or stable [32]. Moreover, the invariant set

$$\{(x(t)) \mid \dot{V}(x(t)) \equiv 0\}$$

corresponds to the set

$$\{x(t) \mid u^* \equiv 0, w^* \equiv 0, h_1(x) \equiv 0\}.$$

By the observability of the system, this in turn corresponds to the trajectory $\{(x(t) \equiv 0\}$. By LaSalle's invariance principle [32], this implies the closed-loop system with u^* is locally asymptotically stable in N about $x = 0$.

To prove disturbance attenuation, now assume $0 \neq w \in \mathcal{L}_2[t_0, \infty)$, $u = u^*$. Then,

$$\begin{aligned}
\dot{V}(x(t), y(t)) &= V_x(x)[f(x) - g_2(x)g_2^T(x)V_x^T(x)] + V_x(x)g_1(x)w \\
&= V_x(x)f(x) + V_x(x)[g_1(x)g_1^T(x) - \frac{1}{2}g_2(x)g_2^T(x)]V_x^T(x) \\
&\quad -\frac{1}{2}\|g_2^T(x)V_x^T(x)\|^2 - \frac{1}{2}\|w - g_1^T(x)V_x^T(x)\|^2 + \frac{1}{2}\|w\|^2 \\
&\quad -\frac{1}{2}\|g_1^T(x)V_x^T(x)\|^2 \\
&= -\frac{1}{2}h_1^T(x)h_1(x) - \frac{1}{2}\|u^*\|^2 - \frac{1}{2}\|w - w^*\|^2 + \frac{1}{2}\|w\|^2 \\
&\quad -\frac{1}{2}\|w^*\|^2 \\
&= -\frac{1}{2}\|z_1\|^2 - \frac{1}{2}\|w - w^*\|^2 + \frac{1}{2}\|w\|^2 - \frac{1}{2}\|w^*\|^2.
\end{aligned}$$

Integrating the above equation with respect to t from t_0 to ∞, and since $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, from above, we have

$$-V(x(t_0), y(t_0)) \leq \frac{1}{2}\int_{t_0}^{\infty} (\|w\|^2 - \|z_1\|^2)$$

(since $\lim_{t\to\infty} V(x(t), y(t)) = 0$). Therefore,

$$\|z_1\|^2_{\mathcal{L}_2[t_0,\infty)} \leq \|w\|^2_{\mathcal{L}_2[t_0,\infty)} + 2V(x(t_0), y(t_0)),$$

which implies disturbance attenuation including the effect of the initial conditions $(x(t_0), y(t_0))$.

Finally, it is easy to show that the saddle-point conditions are satisfied, since (u^*, w^*) minimize and maximize $J_2(., .)$, respectively. □

Remark 4.2 Thus, by the above proposition, we see that the above formulation of the problem in fact solves the state feedback nonlinear $\mathcal{H}_\infty$ control problem [1] as well as we shall see in the proceeding, also the output measurement feedback problem.

Next, we consider the second sub-problem. Under certainty equivalence (i.e., $\lim_{t\to\infty} \xi(t) = x(t)$), implement the feedback strategies $\hat{w}^* = g_1^T(\xi)V_x^T(\xi)$, $\hat{u}^* = -g_2^T(\xi)V_x^T(\xi)$ in (17), (18) and consider the closed-loop system

$$\Sigma^{clp}: \begin{cases} \dot{x} = f(x) + g_1(x)g_1^T(\xi)V_x^T(\xi) - g_2(x)g_2^T(\xi)V_x^T(\xi); \quad x(t_0) = x_0 \\ \dot{\xi} = f(\xi) + g_1(\xi)g_1^T(\xi)V_x^T(\xi) - g_2(\xi)g_2^T(\xi)V_x^T(\xi) + G(\xi, y)(y - h_2(\xi)), \quad \xi(t_0) = 0 \end{cases} \tag{24}$$

To minimize $J_3(.)$, we form the Hamiltonian function

$$\begin{aligned} H_2(x, \xi, W_x^T, W_\xi^T, G) = {} & W_x(x, \xi, y)[f(x) - g_2(x)g_2^T(\xi)V_x^T(\xi) + g_1(x)g_1^T(\xi)V_x^T(\xi)] \\ & + W_\xi(x, \xi, y)[f(\xi) + g_1(\xi)g_1^T(\xi)V_x^T(\xi) \\ & - g_2(\xi)g_2^T(\xi)V_x^T(\xi) + G(\xi, y)(y - h_2(\xi))] \\ & + W_y(x, \xi, y)\nabla h_2(x)f(x) + \frac{1}{2}\|y - h_2(\xi)\|^2 \end{aligned} \tag{25}$$

where $\sigma_1 = W_x^T(x, \xi, y)$, $\sigma_2 = W_\xi^T(x, \xi, y)$, $\sigma_3 = W_y^T(x, \xi, y)$ are the co-state variables, for some smooth function $W : \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \to \mathbf{R}$. Completing the squares now for $G(\xi, y)$ in the above expression, we have

$$\begin{aligned} H_2(x, \xi, W_x^T, W_\xi^T, G) = {} & W_x(x, \xi, y)[f(x) + g_1(x)g_1^T(\xi)V_x^T(\xi) - g_2(x)g_2^T(\xi)V_x^T(\xi)] \\ & + W_\xi(x, \xi, y)[f(\xi) + g_1(\xi)g_1^T(\xi)V_x^T(\xi) - g_2(\xi)g_2^T(\xi)V_x^T(\xi)] \\ & + W_y(x, \xi, y)\nabla h_2(x)f(x)\frac{1}{2}\|G^T(\xi, y)W_\xi^T(x, \xi, y) + (y - h_2(\xi))\|^2 \\ & - \frac{1}{2}W_\xi(x, \xi, y)G(\xi, y)G^T(\xi, y)W_\xi^T(x, \xi, y), \end{aligned}$$

which implies that $H_2(.,.,..,.)$ is minimized if we select $G^*(.,.,.,.,.)$ such that

$$W_\xi(x,\xi,y)G^*(\xi,y) = -(y-h_2(\xi))^T. \tag{26}$$

However, since $G^*(\xi, y)$ is a function of ξ and y only, then we cannot have x in the above expression, except only in the linear case for which G^* is a constant matrix. Therefore, it must be evaluated in the limit as $\xi \to x$, or simply replaced by

$$W_\xi(x,\xi,y)G^*(\xi,y)\big|_{x=\xi} = W_\xi^T(\xi,\xi,y)G^*(\xi,y) = -(y-h_2(\xi))^T. \tag{27}$$

Consequently, the above equation (27) must always be used in the computation of $G^*(.,.,.)$.

Finally, with the above choice of $G^*(.,.)$, to guarantee asymptotic stability of the closed-loop system (24) with $w = 0$, we assume $W(x,\xi,y) \geq 0$ and a Lyapunov function for the system. Therefore, it suffices to have that

$$\begin{aligned}
\dot{W}(x,\xi,y) &= W_x(x,\xi,y)[f(x) - g_2(x)g_2^T(\xi)V_x^T(\xi)] \\
&\quad + W_\xi(x,\xi,y)[f(\xi) + g_1(\xi)g_1^T(\xi)V_x^T(\xi) - g_2(\xi)g_2^T(\xi)V_x^T(\xi) \\
&\quad + G^*(\xi,y)(y-h_2(\xi))] + W_y(x,\xi,y)\nabla h_2(x)f(x) \\
&= H_2(x,\xi,W_x^T,W_\xi^T,G^*) - W_x(x,\xi,y)g_1(x)g_1^T(\xi)V_x^T(\xi) \\
&\quad -\frac{1}{2}(y-h_2(\xi))^T(y-h_2(\xi)) \leq 0
\end{aligned}$$

for all $x \neq 0$, $\xi \neq 0$, $y \neq 0$. Consequently, the above condition holds, if and only if, the following HJIE is satisfied

$$\begin{aligned}
&W_x(x,\xi,y)[f(x) - g_2(x)g_2^T(\xi)V_x^T(\xi)] + W_\xi(x,\xi,y)[f(\xi) \\
&\quad + g_1(\xi)g_1^T(\xi)V_x^T(\xi) - g_2(\xi)g_2^T(\xi)V_x^T(\xi)] \\
&\quad - \frac{1}{2}W_\xi(x,\xi,y)G(\xi,y)G^T(\xi,y)W_\xi^T(x,\xi,y) \\
&\quad + W_y(x,\xi,y)\nabla h_2(x)f(x) = 0, \quad W(0,0,0) = 0,
\end{aligned} \tag{28}$$

or equivalently, the following Lyapunov equation is satisfied

$$\begin{aligned}
&W_x(x,\xi,y)[f(x) - g_2(x)g_2^T(\xi)V_x^T(\xi)] \\
&\quad + W_\xi(x,\xi,y)[f(\xi) + g_1(\xi)g_1^T(\xi)V_x^T(\xi) \\
&\quad - g_2(\xi)g_2^T(\xi)V_x^T(\xi)] \\
&\quad + W_y(x,\xi,y)\nabla h_2(x)f(x) - \frac{1}{2}(y-h_2(\xi))^T(y-h_2(\xi)) = 0, \quad W(0,0,0) = 0.
\end{aligned} \tag{29}$$

We summarize this result in the following theorem.

Theorem 4.1 *Consider the nonlinear system* (17) *and the $\mathcal{L}_2$-output measurement feedback problem for this system. Suppose Assumptions* 2.1, 4.1 *hold for the system and the plant Σ^a is observable about $x = 0$. Further, suppose there exist C^1 positive semidefinite functions $V : N \times Y \to \mathbf{R}$, $W : N \times N \times Y \to \mathbf{R}$ locally defined in neighborhoods $N \times Y \subset \mathcal{X} \times \mathcal{Y}$ and $N \times N \times Y \subset \mathcal{X} \times \mathcal{X} \times \mathcal{Y}$ of the origins $(x, y) = (0, 0)$ and $(x, \xi, y) = (0, 0, 0)$, respectively, and satisfying the HJIEs* (22), (28) *or* (29)*, respectively, together with a matrix function $G^* : N \times Y \to \mathcal{M}^{n\times m}(\mathcal{X} \times \mathcal{Y})$, satisfying the HJIEs* (28)*, together with the side condition* (26)*. Then, the $\mathcal{L}_2$-output measurement feedback problem for the system Σ_2^a is locally solved in N by the controller Σ^{ofbc}.*

Proof Suppose there exist solutions $V \geq 0$ and $W \geq 0$ to the HJIEs (22), (28) together with a G^* satisfying (26), then the controller (18) is realizable. Moreover, along any trajectory of the closed-loop system (24) with $w = 0$ and any $x(t) \neq 0$, $\xi(t) \neq 0$, $y(t) \neq 0$, we have

$$\dot{W} = W_x \dot{x} + W_\xi \dot{\xi} + W_y \dot{y} = -\frac{1}{2}(y - h_2(\xi))^T (y - h_2(\xi)) \leq 0$$

Therefore, by Lyapunov's theorem, the closed-loop system is stable. Further, for any $t_s > t_0$ such that $\dot{W}(x(t_s), \xi(t_s), y(t_s)) \equiv 0$, it implies that $y(t_s) = h(\xi(t_s)) = h(x(t_s))$. By the observability of the system, this implies that $x(t_s) \equiv \xi(t_s)$. Finally, by Proposition 4.1, it proves that Σ^{ofbc} provides local asymptotic stability for the closed-loop system Σ^{clp}. □

Remark 4.3 The above output feedback control law can also be modified to an event triggered control law for use, for example, in networked control or embedded systems [28,30].

To recover the counterparts of Proposition 4.1 and Theorem 4.1, we again consider the following linear system model

$$\Sigma_2^{ls} : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u; \quad x(t_0) = x_0 \\ z_1 = C_1 x + D_{12} u \\ y = C_2 x + D_{21} w. \end{cases} \tag{30}$$

Then, we have the following results.

Corollary 4.1 *Consider the linear system* (30) *and the $\mathcal{L}_2$ state feedback problem for this system. Suppose* Assumptions 2.1, 4.1 *hold for the system, and the plant Σ_2^{ls} is observable about $x = 0$. Further, suppose there exists a symmetric positive semidefinite matrix P_2 satisfying the algebraic Riccati equation (ARE):*

$$A^T P_2 + P_2 A + P_2[2B_1 B_1^T - B_2 B_2^T] P_2 + C_1^T C_1 = 0. \tag{31}$$

Then, the $\mathcal{L}_2$ state feedback problem for the system Σ_2^{ls} is solved by the control law $u^* = -B_2^T P_2$

Proof Take $V(x) = \frac{1}{2}x^T P_2 x$ and apply the result of Proposition 4.1. □

Similarly, we have the following Corollary to Theorem 4.1.

Corollary 4.2 *Consider the linear system* (30) *and the* $\mathcal{L}_2$*-output measurement feedback problem for this system. Suppose* Assumptions 2.1, 4.1 *hold for the system and the plant* Σ_2^{ls} *is observable. Further, suppose there exist symmetric positive-definite matrices* P_2, P, $\widetilde{P} \in \mathbf{R}^{n\times n}$ *and* $Q \in \mathbf{R}^{m\times m}$*, together with a matrix* $G^* \in \mathbf{R}^{n\times m}$*, satisfying, respectively, the Riccati equation* (31) *and the matrix inequalities:*

$$\begin{bmatrix} A^T P + PA & PB_2B_2^T P_2 & A^T C_2^T Q \\ PB_2B_2^T P_2 & A^T\widetilde{P} + \widetilde{P}A - 2P_2B_2B_2^T\widetilde{P} - 2\widetilde{P}B_1B_1^T P_2 + C_2^T C_2 & C_2^T \\ QC_2A & C_2 & -I \end{bmatrix} \leq 0 \tag{32}$$

$$\begin{bmatrix} 0 & C_2^T - (P+\widetilde{P})G \\ C_2 - G^T(P+\widetilde{P}) & -2I \end{bmatrix} \leq 0. \tag{33}$$

Then, the $\mathcal{L}_2$*-output measurement feedback problem for the system* Σ_2^{ls} *is solved by the controller*

$$\Sigma_l^{ofbc} : \begin{cases} \dot{\xi} = (A + B_1B_1^T P_2 - B_2B_2^T P_2)\xi + G(y - C_2\xi); \quad \xi(t_0) = 0 \\ u = -B_2^T P_2\xi \\ z_2 = y - C_2\xi. \end{cases} \tag{34}$$

5 Examples and Simulation Results

In this section, we present some simple examples to validate the theory developed. We present results for the state estimation or filtering problem and the state feedback problem for which we are able to solve the governing HJIEs. It is hoped that the two results combined together are equivalent to solving the problem of output measurement feedback problem for which we are unable to solve the governing HJIE with the current state of computational techniques for solving HJIEs.

Example 5.1 Consider the following system and the example:

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -x_1 - x_2 + u \\ y &= x_1 + w \end{aligned}$$

with

$$g_1(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h_2(x) = x_1, \quad k_{21}(x) = I_2$$

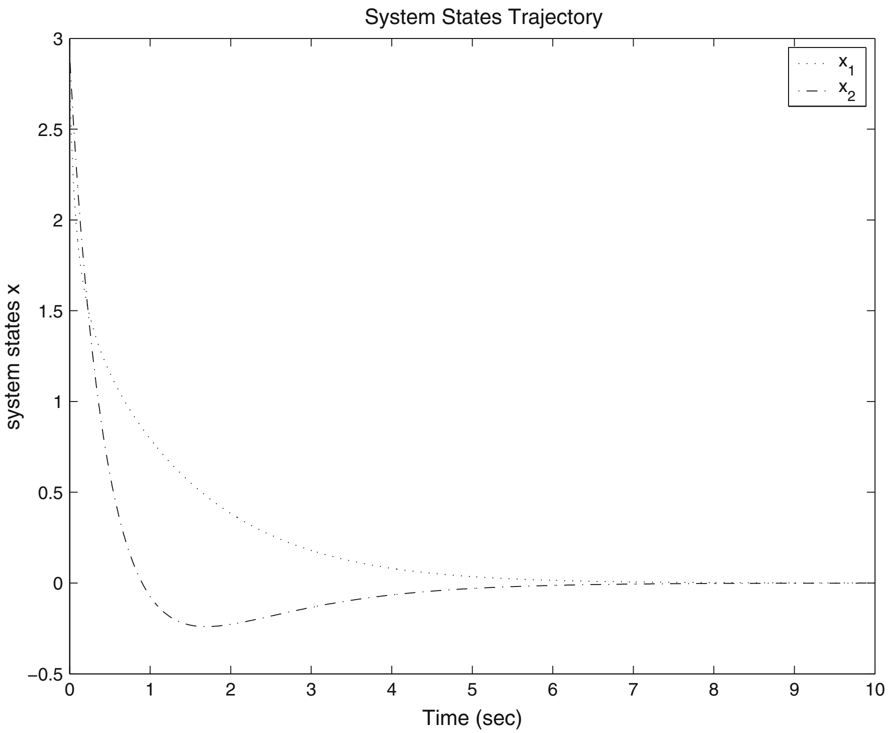

Fig. 1 System states trajectory with state feedback control

which satisfies Assumption 2.1. Define now accordingly the penalty variables for the system as discussed above

$$
\begin{aligned}
\tilde{z} &= y - \hat{x}_1 \\
z_1 &= \begin{bmatrix} h_1(x) \\ u \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \\
z_2 &= y - \xi_1
\end{aligned}
$$

For this example, we find approximate solutions to the HJIE (9) for the filtering and HJIE (22) for the state feedback control problem. We leave-out the output feedback HJIE (28) since it is more involved.

Accordingly, the HJIE (9) corresponding to the above system is given by

$$
U_{\hat{x}_1}(-\hat{x}_1^3 + \hat{x}_2) + U_{\hat{x}_2}(-\hat{x}_1 - \hat{x}_2) + U_y(-\hat{x}_1^3 + \hat{x}_2) - \frac{1}{2}(y - \hat{x}_1)^2 = 0, \quad U(0,0) = 0
$$

It can further be checked that the inequality form of the above HJIE (with ≤ 0 on the RHS, and therefore the HJIE itself) is solved by the function

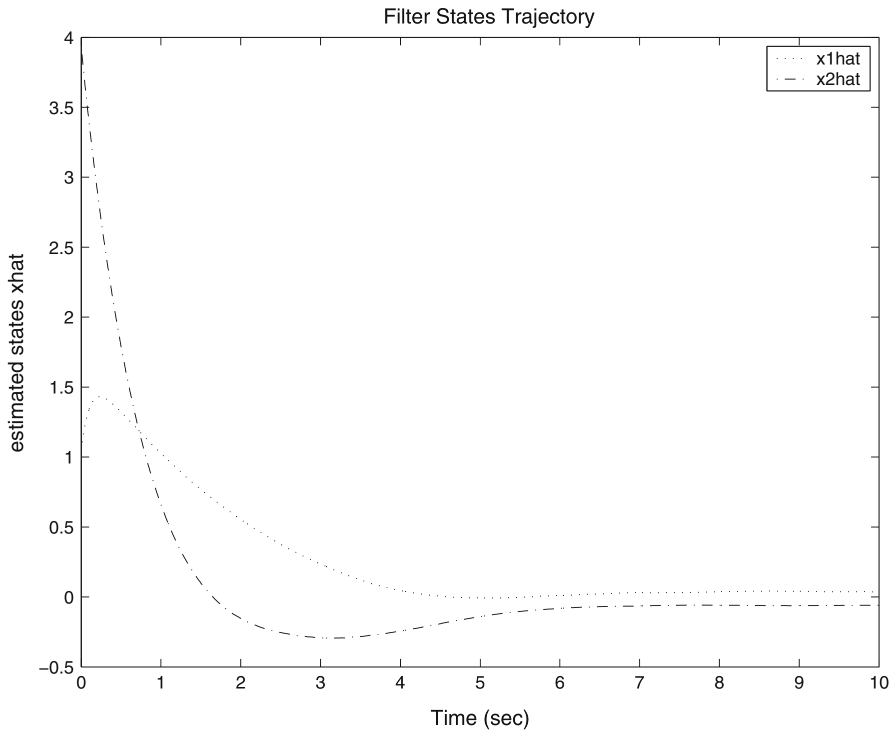

Fig. 2 Filter states trajectory

$$U(\hat{x}, y) = \frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2).$$

Thus, the filter gains can be calculated from (11) as

$$U_{\hat{x}}(\hat{x}, y)L(\hat{x}, y)] = [\hat{x}_1 \;\; \hat{x}_2]\begin{bmatrix} l_1(\hat{x}, y) \\ l_2(\hat{x}, y) \end{bmatrix} = -(y - \hat{x}_1).$$

There can be many solutions to the above equation, of which one is:

$$l_1(\hat{x}, y) = 1, \quad l_2(\hat{x}, y) = -\frac{y}{\hat{x}_2}$$

Now, consider the HJIE (22) for the state feedback problem of the system model above. It is given by

$$V_{x_1}(-x_1^3 + x_2) + V_{x_2}(-x_1 - x_2) + \frac{1}{2}(x_1^2 + x_2^2) = 0, \quad V(0) = 0$$

Again, we can take for an approximate solution to the above HJIE (which solves the inequality form of the HJIE) as :

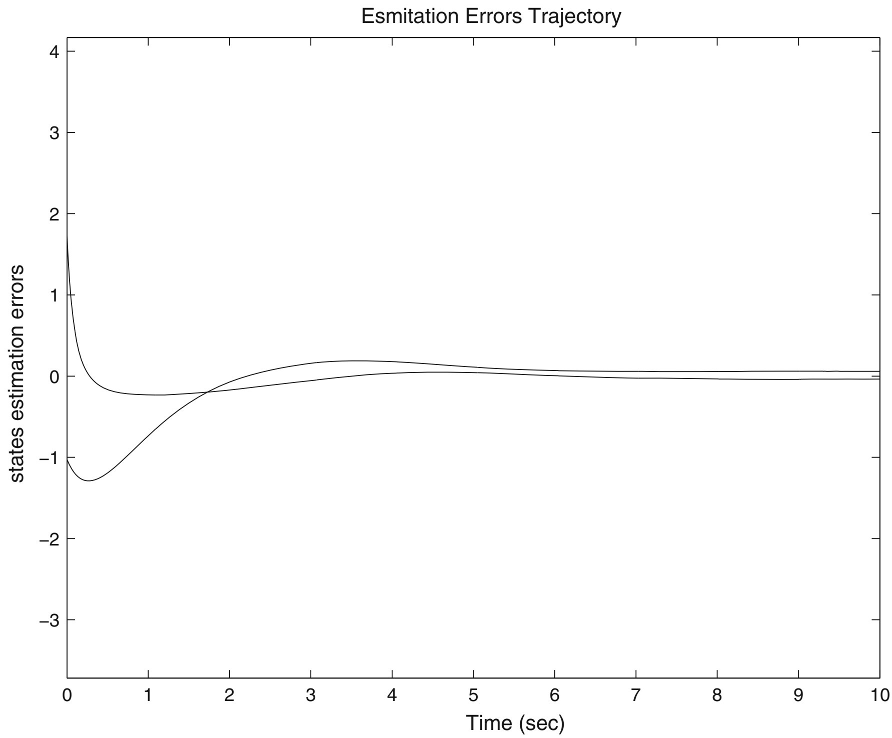

Fig. 3 State estimation errors trajectory

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2),$$

and we have the state feedback control law:

$$u = -x_2$$

Therefore, the original system and the constructed filter are simulated with the above feedback control for some arbitrary initial conditions. The measurement noise w was taken to be a zero-mean Gaussian white noise with variance 0.2 (even though this is not $\mathcal{L}_2$-bounded). The results of the simulation are shown in Figs. 1, 2 and 3. They show that the closed-loop system is asymptotically stable, and the estimation errors are well below any tolerance limit. In fact, they are almost zero.

Example 5.2 Consider the following rigid space-craft system and the example:

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{1}{2}x_2x_3\\ \dot{x}_2 &= -x_2 - x_1x_3 + w\end{aligned}$$

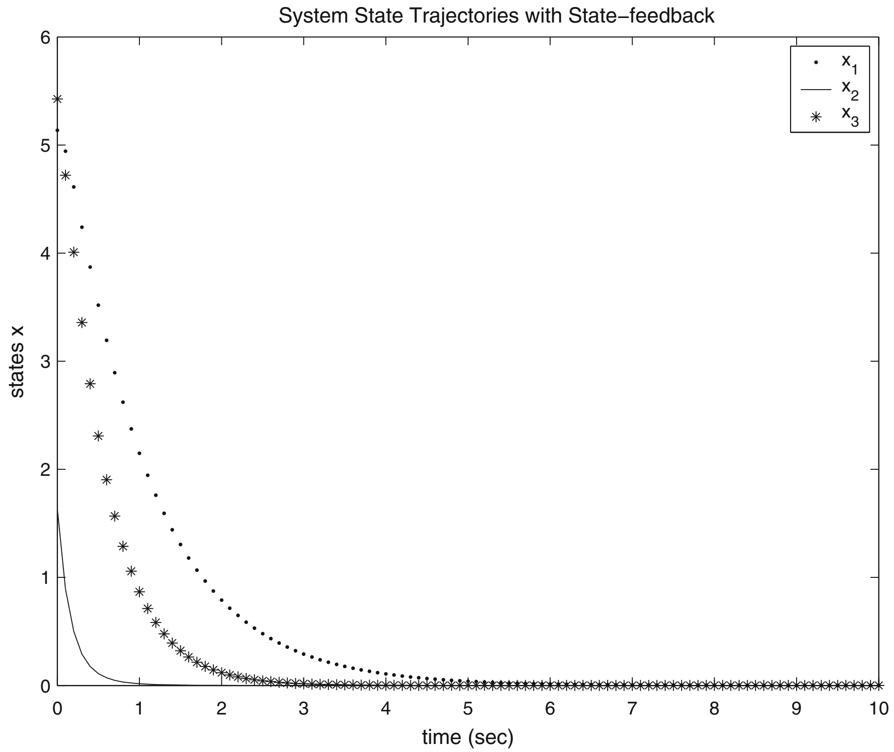

Fig. 4 System states trajectory with state feedback control

$$\begin{aligned} \dot{x}_3 &= -x_3 + \frac{1}{2}x_1x_2 + u \\ y &= x_1 + w \end{aligned}$$

with

$$g_1(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad h_2(x) = x_1, \quad k_{21}(x) = I_3$$

which satisfies Assumption 2.1. Define also accordingly the penalty variables for the system as discussed above

$$\begin{aligned} \tilde{z} &= y - \hat{x}_1 \\ z_1 &= \begin{bmatrix} h_1(x) \\ u \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ u \end{bmatrix} \\ z_2 &= y - \xi_1 \end{aligned}$$

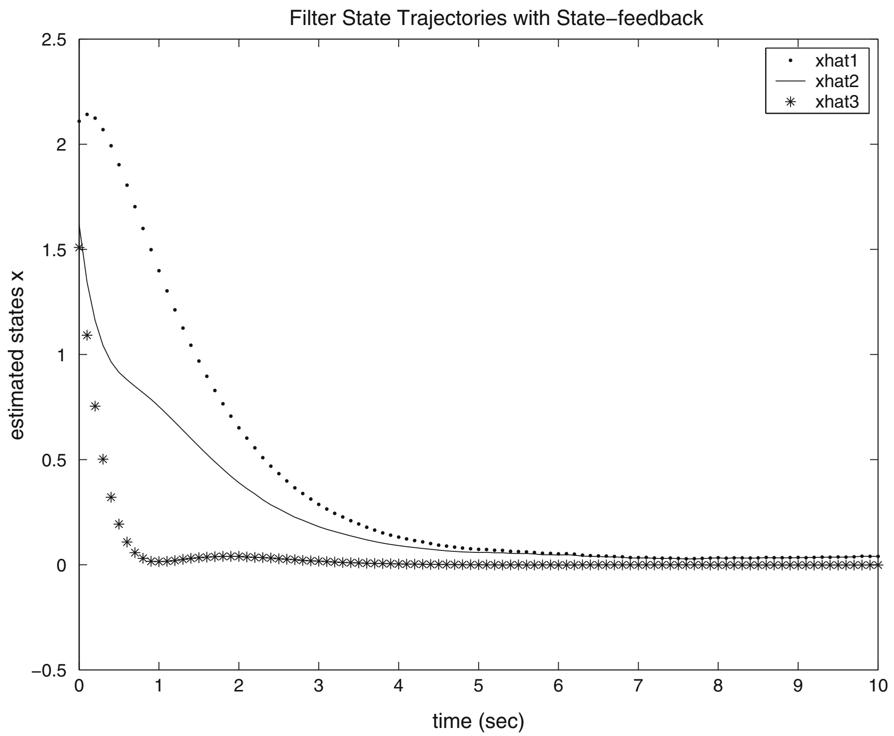

Fig. 5 Filter states trajectory

Then, the HJBIEs for the $\mathcal{L}_2$-filtering problem (9) and state feedback problem (22) are given, respectively, by

$$U_{\hat{x}_1}(-\hat{x}_1 + \frac{1}{2}\hat{x}_2\hat{x}_3) + U_{\hat{x}_2}(-\hat{x}_2 - \hat{x}_1\hat{x}_3) + U_{\hat{x}_3}(-\hat{x}_3 + \frac{1}{2}\hat{x}_1\hat{x}_2) + U_y(-\hat{x}_1 + \frac{1}{2}\hat{x}_2\hat{x}_3) +$$
$$U_{\hat{x}_2}^2 - \frac{1}{2}(y - \hat{x}_1)^2 = 0, \ U(0,0) = 0 \tag{35}$$
$$V_{x_1}(-x_1 + \frac{1}{2}x_2x_3) + V_{x_2}(-x_2 - x_1x_3) + V_{x_3}(-x_3 + \frac{1}{2}x_1x_2) + V_{x_2}^2 - \frac{1}{2}V_{x_3}^2$$
$$+\frac{1}{2}(x_1^2 + x_3^2) = 0, \ V(0) = 0. \tag{36}$$

They are also solved, respectively, by the quadratic forms

$$U(\hat{x}, y) = \frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)$$
$$V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

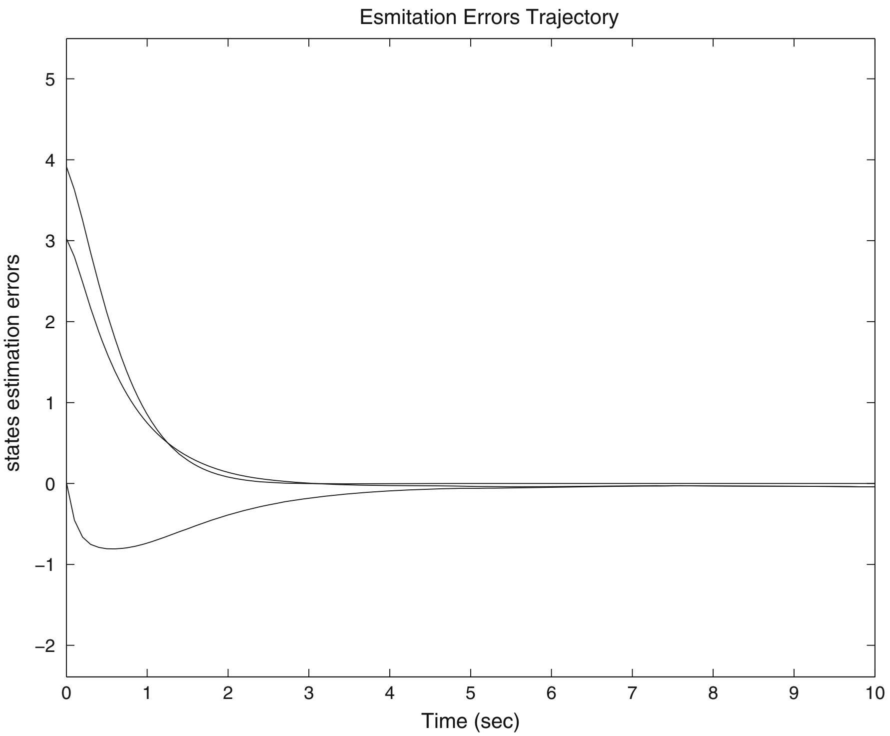

Fig. 6 State estimation errors trajectory

Similarly, the state feedback control and the filter gains can be computed as

$$u = -x_3$$
$$l_1(\hat{x}, y) = 1, \quad l_2(\hat{x}, y) = 1, \quad l_3(\hat{x}, y) = -\frac{(y + \hat{x}_2)}{\hat{x}_3}$$

The results of the simulation for the system with system feedback and while estimating the states are again shown in Figs. 4, 5 and 6. The measurement noise w was also taken to be a zero-mean Gaussian white noise with variance 0.2 The results show again very good agreement with the theory developed.

6 Conclusion

In this paper, we have presented a new formulation and solution for the Kalman, or $\mathcal{L}_2$-filtering problem for affine nonlinear systems. The solution to the problem results in the gain matrix being a function of the estimated state and the measurement information, as opposed to earlier solutions which are practically unimplementable.

Secondly, we have provided a new formulation for the output measurement feedback problem which allows for the output-injection gain matrix to be independently optimized. This formulation is based on a multi-objective design philosophy in a differential game setting, which, however, does not increase the number or dimension of the HJIEs characterizing the solution to the problem, as in the case of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with output feedback [17].

An example and simulation results have also been solved and presented to demonstrate the effectiveness of the new approach. The results have shown agreement with the theory and very promising potential for the method. However, among the limitations of the approach still includes the inability to solve the output measurement feedback HJIE (28) with the current state of the art in computational methods for nonlinear PDEs. Moreover, even solving the state feedback HJIE (22) will pose great challenges for higher-order and highly nonlinear state-space models. Therefore, future work will concentrate on overcoming this challenge and also exploring the $\mathcal{L}_1$-filtering and output feedback problems.

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