Study of call option pricing of ASM International NV using Monte-Carlo simulations

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Contents

1 Assignment: implied volatility estimation

Using the Black & Scholes formula as in equation 1 we compute the implied volatility σ_{impl}. With the knowledge that there is no closed form inverse (w.r.t the volatility) for the Black-Scholes formula it involves recognizing that given fixed values of all the other variables $C_{BS}(\sigma)$ is just function of the volatility and we can then solve for $C_{BS}(\sigma) = C_{market} \Rightarrow C_{BS}(\sigma) - C_{market} = 0$ where C_{market} is the current contract price. The stock and options we have chosen are that of ASM International NV.

$$C_{BS} = S(0)e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2) \text{ where:} \tag{1}$$

$$d_1 = \frac{\log(S(0)/K) + T(r - q + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T}} \tag{2}$$

$$d_2 = d_1 - \sigma\sqrt{T} \text{ and:} \tag{3}$$

$S(0) =$ **current stock price,** $K =$ **strike price,** $T =$ **maturity of option**
$q =$ **annual continuously compounded dividend yield of the stock**
$\Phi(\cdot) =$ **the CDF of the standart normal distribution** $\mathcal{N}(0, 1)$
$r =$ **current risk free interest rate (date: ...)**
$\sigma =$ **volatility,** $C =$ **price of the call option**

In order to solve for σ in $C_{BS}(\sigma) - C_{market} = 0$ we employ Newton's method. We know from mathematical finance that the B&S model 1 has a closed-form partial derivative with respect to σ called the *vega* which measures the sensitivity to the volatility. The following quick analysis of the function $\nu(\sigma) := \frac{\partial C_{BS}}{\partial \sigma}$ shows that the *vega* is a positive function of σ. We first note the following:

$$\text{By equation 3 } d_1(\sigma) - d_2(\sigma) = \sigma\sqrt{T} \Leftrightarrow \frac{\partial d_1(\sigma)}{\partial \sigma} - \frac{\partial d_2(\sigma)}{\partial \sigma} = \sqrt{T} \tag{4}$$

$$\textbf{Claim: } S(0)e^{-qT}\frac{\partial \Phi(d_1(\sigma))}{\partial \sigma} - Ke^{-rT}\frac{\partial \Phi(d_2(\sigma))}{\partial \sigma} = 0 \tag{5}$$

Proof.

$$\text{Equation 5 implies: } S(0)e^{-qT}\frac{\partial\Phi(d_1(\sigma))}{\partial\sigma} = Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma} \tag{6}$$

$$\Leftrightarrow S(0)e^{-qT}(2\pi)^{-\frac{1}{2}}e^{-\frac{d_1^2}{2}} = Ke^{-rT}(2\pi)^{-\frac{1}{2}}e^{-\frac{d_2^2}{2}} \tag{7}$$

$$\Leftrightarrow \frac{S(0)}{K}e^{T(r-q)} = e^{-\frac{d_2^2}{2}+\frac{d_1^2}{2}} \tag{8}$$

$$\Leftrightarrow \underbrace{\log\frac{S(0)}{K} + T(r-q)}_{\mathbf{A}} = \underbrace{\frac{d_1^2 - d_2^2}{2}}_{\mathbf{B}} \tag{9}$$

$$\text{Where } \mathbf{B} = \frac{1}{2}(d_1+d_2)(d_1-d_2) \underbrace{=}_{\text{by equation 3}} \frac{1}{2}(2d_1 - \sigma\sqrt{T})\sigma\sqrt{T} = d_1\sigma\sqrt{T} - \frac{1}{2}\sigma^2 T \tag{10}$$

$$\text{(by equation 2)} = \log(S(0)/K) + T(r - q + \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 T = \mathbf{A} \tag{11}$$

Thus equation 5 holds. □

Now we calculate the *vega*.

$$\nu(\sigma) = S(0)e^{-qT}\frac{\partial\Phi(d_1(\sigma))}{\partial\sigma}\frac{\partial d_1(\sigma)}{\partial\sigma} - Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma}\frac{\partial d_2(\sigma)}{\partial\sigma} \tag{12}$$

$$= \frac{\partial d_1(\sigma)}{\partial\sigma}S(0)e^{-qT}\frac{\partial\Phi(d_1(\sigma))}{\partial\sigma} - \frac{\partial d_1(\sigma)}{\partial\sigma}Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma} + \tag{13}$$

$$\frac{\partial d_1(\sigma)}{\partial\sigma}Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma} - \frac{\partial d_2(\sigma)}{\partial\sigma}Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma} \tag{14}$$

$$\text{(By equation 4)} = \frac{\partial d_1(\sigma)}{\partial\sigma}\underbrace{\left(S(0)e^{-qT}\frac{\partial\Phi(d_1(\sigma))}{\partial\sigma} - Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma}\right)}_{\text{is zero by equation 5}} + \tag{15}$$

$$\sqrt{T}Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma} \tag{16}$$

$$= \sqrt{T}Ke^{-rT}\frac{\partial\Phi(d_2(\sigma))}{\partial\sigma} \tag{17}$$

$$\text{(By equation 5)} = \sqrt{T}S(0)e^{-qT}\frac{\partial\Phi(d_1(\sigma))}{\partial\sigma} \tag{18}$$

$$= \sqrt{T}S(0)e^{-qT}(2\pi)^{-\frac{1}{2}}e^{-\frac{d_1^2}{2}} \tag{19}$$

From equation 19 it is visible the *vega* is positive, which implies that $C_{BS}(\sigma)$ is monotonically increasing in σ thus we can use Newton's method. We define $f(\sigma_{impl}) := C_{BS}(\sigma_{impl}) - C$ from which we get $f'(\sigma_{impl}) = \nu(\sigma_{impl})$ then Newton's method gives

$$\sigma_{impl}^{p+1} = \sigma_{impl}^{p} - \frac{f(\sigma_{impl}^{p})}{f'(\sigma_{impl}^{p})},\ p \in \{0, 1, 2, \ldots\} \tag{20}$$

We choose $\sigma_{impl}^0 = 0.25$ as the initial value. In section 7.1 the code is found for this, where we employ Newton's method using the Vega as derived here as well as the more robust Brent's method using the function `scipy.optimize.brentq`[1] in order to have a second estimate. The resulting estimates are found in table 1.We find that with both methods we get the same value for the implied volatility which is 17.78 %.We note that given that the values change each day (each second even) we have tabulated the used values for all variables in table 2. The sources for all options and stock related information are `http://beurs.fd.nl/derivaten/opties/?call=AEX.ASM$%$2fO` and `https://www.google.com/finance?q=asm+international$&$ei=VIizVoH8KoK7U9OKh7AJ`.

Given that we will compare various strikes and maturities in assignments 2 and 3, we compute 6 implied volatilities and take the average. We have taken all other values the same and varied the strikes, maturities, and call prices. We have estimated the prices with the single volatility of 17.78 % and the average volatility of 23.74 % in assignments 2 and 3 see table 3.

[1]See `http://mathworld.wolfram.com/BrentsMethod.html` for algorithm details

2 Assignment: Validation

To validate the estimate of σ we have found previously with the aid of the B&Sformula we simulate random paths of the stock prices, which we use to estimate the price of the Call contract and compare these with the prices reported from market sources.

2.1 GBM simulation

If we assume that the stock prices are log normally distributed, i.e. $S(t) \overset{d}{\sim} \log \mathcal{N}(\mu^*, \sigma^{2*})$, we can use the geometric Brownian motion to model the stock price process. This assumption hold only as mere approximation at best and to check this, for the ASM International NV stocks, we have plotted histograms and QQ-plots of the log transformation of these daily prices from 02-2010 to 02-2016 in figure 2. We observe that the logs are not normally distributed, however we note that for higher frequencies the assumption (while still approximately at best) might hold better. By definition the geometric Brownian motion satisfies the stochastic differential equation given in 21 with the solution as given in equation 22. From equation 23 it is visible that the process $\{\log S(t)\}_{t\in\mathbb{R}^+}$ follows a Brownian motion with $\log S(t) \overset{d}{\sim} \mathcal{N}(\log S(0) + (r - q - \frac{1}{2}\sigma)t, \sigma^2 t)$, where $S(0)$ is the initial value and $\{W(t) : t \geq 0\}$ is the standard Brownian motion with $W(t) \overset{d}{\sim} \mathcal{N}(0, t)$.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \text{where we let } \mu := r - q \tag{21}$$

$$\text{with solution: } S(t) = S(0)\exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma W(t)\right) \tag{22}$$

$$\log S(t) = \log S(0) + (r - q - \frac{1}{2}\sigma^2)t + \sigma W(t) \tag{23}$$

To simulate we use equation 22 to get $S(t-h) = S(0)\exp\left((r - q - \frac{1}{2}\sigma^2)(t - h) + \sigma W(t - h)\right)$ and use it in equation 24 to get the formula in 26.

$$S(t)/S(t-h) = S(0)\exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma W(t)\right)/S(0)\exp\left((r - q - \frac{1}{2}\sigma^2)(t - h) + \sigma W(t - h)\right) \tag{24}$$

$$= \frac{S(0)}{S(0)}\exp\left((r - q - \frac{1}{2}\sigma^2)h + \sigma(W(t) - W(t - h))\right) \tag{25}$$

$$\text{which implies: } S(t) = S(t-h)\exp\left((r - q - \frac{1}{2}\sigma^2)h + \sigma(W(t) - W(t - h))\right) \tag{26}$$

Then we note that by definition of the Brownian motion we have $W(t) - W(t-h) \overset{d}{\sim} \mathcal{N}(0, h) \overset{d}{=} \sqrt{h}\mathcal{N}(0, 1)$. If we discretize $\{S(t) : t \geq 0\}$ to get $\{S_j : j = 0, 1, 2, \ldots, M\}$ where we split the interval $[0, T]$ into M subintervals of length h (in days) thus $Mh = T$. Then we get the discrete recursive formula from equation 26:

$$S_0 = S(0) \tag{27}$$

$$S_j = S_{j-1}\exp\left((r - q - \frac{1}{2}\sigma^2)h + \sigma\sqrt{h}Z_j\right) \text{ where } \{Z_j\} \overset{d}{\sim} \mathcal{NID}(0, 1) \text{ for all } j \tag{28}$$

Using the formula in equation 28 we can simulate a geometric Brownian motion process. As an example, at the time of writing this, 03-02-2016, one possible maturity is 16-09-2016 rendering $T = ((365-106)-34)/365 = 0.616$ giving us, for $h = 1/250$ (250 trading days), an M of $\lceil \frac{1/250}{0.616} \rceil = 154$. In figure 1 we have graphed the simulated random paths. For implementation details see Python code in section 7.

2.2 Estimating the price of the Call contract

From financial mathematics we know that by the no arbitrage principle the price of the option should be set equal to the expected present value of the pay-off from the option. Thus $C = \mathbb{E}^Q[e^{-r\tau}\Phi_\tau(\{S(t), 0 \leq t \leq \tau\})]$ where τ is the exercise time, $\mathbb{E}^Q$ the expectation with respect to the risk-neutral probability measure, and $\Phi_\tau(\{S(t), 0 \leq t \leq \tau\})$ the pay-off at time τ. For a European call option τ is equal to the maturity T. The pay-off $\Phi_T(\{S(t), 0 \leq t \leq T\})$ is equal to the price of the stock at maturity minus the strike price for $S(T) - K \geq 0$ otherwise it is zero, $(S(T) - K)^+$. Thus the price is the expected present val ue of the pay-off as in equation 29.

$$\mathbb{E}^Q[e^{-rT}(S(T) - K)^+] \tag{29}$$

We simulate N paths using equation 28 and get $\{S^{(i)}(t)\}_{t=0,i=1}^{T,N}$ and from this multivariate sequence we extract the i.i.d. sequence $\{S^{(i)}(T)\}_{i=1}^N$.Let $\{Y_i\}_{i=1}^N := \{e^{-rT}(S^i(T) - K)^+\}_{i=1}^N$ then assuming that $\mathbb{E}[Y_i] < \infty$ and having that $\{Y_i\}_{i=1}^N$ is generated by the i.i.d. sequence $\{S^{(i)}(T)\}_{i=1}^N$ and is i.i.d. itself we can approximate the expectation in 29 by $\frac{1}{N}\sum_{i=1}^N Y_i$ using Theorem 2.1 as in equation 30, where we let $Y := e^{-rT}(S(T) - K)^+$.

$$\hat{C}_N := \frac{1}{N}\sum_{i=1}^N Y_i = \frac{1}{N}\sum_{i=1}^N e^{-rT}(S^i(T) - K)^+ \tag{30}$$

Theorem 2.1. *Strong Law of Large Numbers:*
(1) : *if* $Y_1, Y_2, \ldots, Y_N$ *be i.i.d. replications of* $Y \overset{d}{\sim} f$
(2) : *if* $\mathbb{E}[Y_i] := \mathbb{E}[Y] < \infty, \ \forall i$
Then : $\frac{1}{N}\sum_{i=1}^N Y_i \overset{a.s.}{\to} \mathbb{E}[Y]$ *as* $N \to \infty$

2.3 Analysis of the price estimator

For a finite sample size N, we have that $\hat{C}_N$ is random and we wish to construct a confidence interval with a certain α level. For this we use Theorem 2.2 for i.i.d. sequences (Assuming (2), and (1) is satisfied given that $\{Y_i\}_{i=1}^N$ is generated by the i.i.d. sequence $\{S^{(i)}(T)\}_{i=1}^N$ and is i.i.d.) which implies that $\hat{C}_N \overset{d}{\sim} \mathcal{N}(\mathbb{E}[Y], \sigma^2/N)$ and $\frac{\hat{C}_N - \mathbb{E}[Y]}{\sigma/\sqrt{N}} \overset{d}{\sim} \mathcal{N}(0,1)$ if σ would be known. Then using this distribution we can construct a $1-\alpha$ confidence interval as in equation 34.

Let Z_q be the q-th quantile of the normal distribution.

$$\mathbb{P}(Z_{\frac{\alpha}{2}} \leq \frac{\hat{C}_N - \mathbb{E}[Y]}{\sigma/\sqrt{N}} \leq Z_{1-\frac{\alpha}{2}}) \approx 1-\alpha \tag{31}$$

$$\text{by symmetry} \Leftrightarrow \mathbb{P}(-Z_{1-\frac{\alpha}{2}}\sigma/\sqrt{N} \leq \hat{C}_N - \mathbb{E}[Y] \leq Z_{1-\frac{\alpha}{2}}\sigma/\sqrt{N}) \approx 1-\alpha \tag{32}$$

$$\Leftrightarrow \mathbb{P}(\hat{C}_N - Z_{1-\frac{\alpha}{2}}\sigma/\sqrt{N} \leq \mathbb{E}[Y] \leq \hat{C}_N + Z_{1-\frac{\alpha}{2}}\sigma/\sqrt{N}) \approx 1-\alpha \tag{33}$$

$$\text{thus we get the } 1-\alpha \text{ confidence interval: } [\hat{C}_N - Z_{1-\frac{\alpha}{2}}\sigma/\sqrt{N}, \hat{C}_N + Z_{1-\frac{\alpha}{2}}\sigma/\sqrt{N}] \tag{34}$$

However because σ is unknown we estimate it by $\hat{\sigma} := \sqrt{\frac{N}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \hat{C}_N^2\right)}$, which by the property of the sample variance implies that $\frac{\hat{C}_N - \mathbb{E}[Y]}{\hat{\sigma}/\sqrt{N}} \overset{d}{\sim} t_{N-1}$. This changes the confidence interval from the one in equation 34 to the one in equation 35.

$$[\hat{C}_N - t_{N-1,1-\frac{\alpha}{2}}\hat{\sigma}/\sqrt{N}, \hat{C}_N + t_{N-1,1-\frac{\alpha}{2}}\hat{\sigma}/\sqrt{N}] \tag{35}$$

Note once more that in our simulations we have defined $\{Y_i\}_{i=1}^N := \{e^{-rT}(S^i(T)-K)^+\}_{i=1}^N$.The estimated price $\hat{C}_N$ depends on the N the chose sample size. In order to make a calculated choice with regards to this decision we make use of the estimator for the relative error $\mathbf{RE}[\hat{\mathbf{C}}_\mathbf{N}]$ and the relative width of the confidence interval $\mathbf{RW}[\hat{\mathbf{C}}_\mathbf{N}]$ defined in the equations below.

$$\mathbf{RE}[\hat{\mathbf{C}}_\mathbf{N}] = \frac{\hat{\sigma}/\sqrt{N}}{\hat{C}_N]} \tag{36}$$

$$\mathbf{RW}[\hat{\mathbf{C}}_\mathbf{N}] = 2 \times t_{N-1,1-\frac{\alpha}{2}} \times \mathbf{RE}[\hat{\mathbf{C}}_\mathbf{N}] \tag{37}$$

We use the criterion $\mathbf{RE}[\hat{\mathbf{C}}_\mathbf{N}] < \epsilon$ to determine N, by for each $n \in N$ checking whether the condition is fulfilled or not and stopping when it is the case. Given that $\mathbf{RE}[\hat{\mathbf{C}}_\mathbf{N}]$ is stochastic itself we repeat the experiment 200 times with $\epsilon = 0.01$)and take the ceiling of the average of these as our N. The N we find is 3973. In figure 3 a histogram is given of all the estimated samples sizes according to the stopping criterion. These estimates are based on the values as in tables 2 and 1, thus for different strikes and maturities different relative errors will be found. To be sure that the relative error remains low we set N=6000.

Theorem 2.2. *Central Limit Theorem for i.i.d. sequences*
(1) : *if* $Y_1, Y_2, \ldots, Y_N$ *be i.i.d. replications of* $Y \overset{d}{\sim} f$
(2) : *if* $\mathbb{V}ar[Y_i] := \sigma^2 < \infty\ \forall i$ *which implies a finite first moment* $\mathbb{E}[Y]$
Then: $\sqrt{N}(\hat{C}_N - \mathbb{E}[Y]) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

2.4 Verification of the generated random variables

The size of a single path depends on the the time to maturity T (in Years) and the length of the interval with which we discretize the continuous time process. As an example for the maturity as in table 2 and an interval length of 1 day, we have $|\{S_j\}| = 154$ for a single path generated using the exact method as in equation 28. We want to make sure that the generated standard normal Random Variable Z_j in fact follows a standard normal distribution. Depending on the sample size $M = T/h$ we have tabulated the mean and variance of the generated sample in table 4.In figure 4 we observe that indeed the distribution generated by `numpy.random.normal` increasingly resembles the standard normal distribution as N gets larger.

2.5 Results

In table 5 the results of the simulation based estimations of the European call prices are visible. We see that using the estimated implied volatility $\sigma_{impl} = 0.1778$ and the values of table 2 while varying the strike price K and maturity T the estimated prices are quite close to the recorded market prices. For the strikeprice $K = 28$ the estimates are the most accurate, which is to be expected given that the implied volatility was computed using that strike price (and the same values for all the other variables). If we now estimate the prices with the average of computed implied volatilities of 0.2374 we get the results in table 5. We see now that in fact the volatility of 0.1778 was more accurate (the market prices are closer to estimated prices). Furthermore some relative errors are slightly above 0.01 but by an acceptably small amount.

3 Assignment: Asian call option

3.1 Method

To estimate the price of an Asian call we employ the same estimator as with the European call option. The difference now lies in the Payoff $\Phi_T(\{S(t), 0 \leq t \leq T\})$ which is defined as $\left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+$. We approximate $\frac{1}{T}\int_0^T S(t)dt$ with $\frac{1}{T}\sum_{t=0}^T S(t)$ thus employ the Payoff $\left(\frac{1}{T}\sum_{t=0}^T S(t) - K\right)^+$ in the calculation of the price. Applying common random number we use the same simulated N paths $\{S^{(i)}(t)\}_{t=0,i=1}^{T,N}$ to get the i.i.d. sequence $\{\frac{1}{T}\sum_{t=0}^T S^i(t)\}_{i=1}^N$ from which we create the sequence of payoffs $\{Y_i\}_{i=1}^N := \{e^{-rT}(\frac{1}{T}\sum_{t=0}^T S^i(t) \ - K)^+\}_{i=1}^N$. We then employ the same estimator $\frac{1}{N}\sum_{i=1}^N Y_i$ by the arguments given in assignment 2 to estimate $\hat{C}_{6000}$. We construct the confidence intervals, relative errors and the relative widths as before.

3.2 Results

The results are found in table 7 for a single volatility estimate of 0.1778. The first thing we notice is that the estimated prices are quite close to those estimated for the European call option. We would have expected in fact that the Asian European call prices are lower given that they are based on the average of each simulated path $\frac{1}{T}\sum_{t=0}^T S(t)$ which is less volatile than the last price $S(T)$.

The fact that the estimated prices of the Asian call option are so close to the European call option prices indicates that the stock price at maturity $S(T)$ is quite close the the average of the stock prices over the period $t = 0$ to $t = T$, $\frac{1}{T}\sum_{t=0}^T S(t)$. Indeed if we look at figure 5 we see that the simulated paths are not very volatile for lower values of T thus the simulated price at $t = T$ (which should be more more volatile than the arithmetic average of the simulated stock prices) is not much different than the average price over the simulated path for lower values of T. In the case of $T = 2$ the difference between the last price $S(T)$ is greater than the average $\frac{1}{T}\sum_{t=0}^T S(t)$. If we set the maturity to $T = 2$ and as an example estimate the price of a European and a Asian call option we do see that the Asian price 95% confidence interval at $[7.30926082, 7.558774]$ is in fact noticeably lower than that of the European call option at $[7.61756813, 8.01681727]$. If the volatility we used, which was 0.1778, was higher than it is expected that we would see the same effect for lower T.

We also estimate the prices with the average of computed implied volatilities of 0.2374 in table 7. Again we see that the estimated Asian prices are quite close to the estimated European call prices, which can be due to the small T (relatively short lengths of the simulated paths).

3.3 Verification of SLLN and CLT

Now for the stock and option values in table 2 and the implied volatility of table 1 we verify theorem 2.1 and 2.2 visually. The procedure is exactly follow slide 38. In figure 6 we see that indeed the distribution of the prices by repeating the estimation experiment 400 times in fact resembles the normal distribution. The striped lines are that of the normal distribution where the green line is that of the $\mathcal{N}(\hat{C}_{6000}, \hat{\sigma}^2)$ where $\hat{C}_{6000}, \hat{\sigma}^2$ are estimations based on one experiment. The red line is that of $\mathcal{N}(mean(estimatedPrices), \hat{\sigma}^2)$ where $mean(estimatedPrices)$ means that the mean of the prices over the 400 experiments is taken.

In figures 7 and 8 we verify the SLLN. It is visible that as N get larger the estimation of the prices gets more and more stable. In the fourth picture in both figures we see that the fluctuations in estimation is very fine (notice the finegrained range of the Y-axis), thus this picture verifies the SLLN graphically.

4 Assignment: Heston model

In this section we employ the Heston model to estimate the price of the European call option of assignment 2.

4.1 Simulation and Estimation

The Heston extends the models of the previous assignment by letting the volatility be stochastic, where it follows its own random path. It is defined as in equations 38, 39 where $\{W(t) = (W_1(t), W_2(t))\}$ is a correlated Brownian motion with $dW_1(t)dW_(t) = \rho\ dt$.We let the correlation coefficient ρ be -0.6

$$dS(t) = (r - q)S(t)dt + \sqrt{V(t)}S(t)dW_1(t) \tag{38}$$
$$dV(t) = \kappa(\theta - V(t))dt + \xi\sqrt{V(t)}dW_2(t) \tag{39}$$

Before we used the exact method solution to discretize the model, and now approximatly solve the differential equation using the Euler Scheme. We first rewrite the model as in equations 42 and 43.

$$\textit{from euqation 38 we get: } S(t + dt) = S(t) + dS(t) \tag{40}$$
$$= S(t) + (r - q)S(t)dt + \sqrt{V(t)}S(t)dW_1(t) \tag{41}$$
$$= S(t)\left(1 + (r - q)dt + \sqrt{V(t)}dW_1(t)\right) \tag{42}$$
$$\textit{similarly equation 39 gives : } V(t + dt) = V(t) + \kappa(\theta - V(t))dt + \xi\sqrt{V(t)}dW_2(t) \tag{43}$$

Then we discretize as before by splitting the interval $[0, T]$ into M subintervals of length h, where $h \approx dt$ and $t \approx jh$ for $j = 0, \dots, M$. We then get the discrete recursion as in

$$S_0 = S(0)$$
$$S_{j+1} = S_j\left(1 + (r - q)h + \sqrt{V_j}\sqrt{h}Z_{1,j}\right) \tag{44}$$
$$V_0 = \sigma_{impl} \text{ the estimated implied volatility}$$
$$V_{j+1} = V_j + \kappa(\theta - V_j)h + \xi\sqrt{V_j}\sqrt{h}Z_{2,j} \tag{45}$$

In equations 44 and 45 we have by definition of the Wiener process that $\{Z_j = (Z_{1,j}, Z_{2,j})\} \overset{d}{\sim} \mathcal{N}(0, \Sigma)$ with $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.Note that ρ is equal to the covariance due to the unit variances. Note further that we have taken $V(0)$ to be the single implied volatility estimate because it was found in to be more accurate in assignment 2. $\{Z_j = (Z_{1,j}, Z_{2,j})\}$ is generated using the function `numpy.random.multivariate_normal` and to verify its effectiveness we have tabulated means, variances and covariances of various samples drawn from the multivariate normal distribution using this function in table 9. Using this model we have estimated the prices using the same estimator as in assignment 2, and the confidence intervals, relative errors and relative widths are all computed the same way. We have take the following values for the remaining Heston model parameters:

- $\kappa = 2$
- $\theta = 0.0625$
- $\xi = 0.2$

In figure 9 we have plotted 10 random paths of both the volatility and the stock prices that are simulated using the scheme presented above.

4.2 Verification

As a verification we generate an extraordinary long path of 25000000 days in order to see where the process moves towards. Certain parts of this path is graphed in figure 10. We first notice that the process looks stationary around a mean of $0.25 = \sqrt{\theta}$ (this is verifiable analytically) thus we do not expect it to move away from this mean even after a long time. Indeed in the last picture in graph 10 we see that even after 24999900 days to 25000000 days the paths moves very close to 0.25, which is the square root of the long run variance of the model. Note that in the last the path graphed over only 100 days, which is why it looks more stable. However due to the high volatility of $V(t)$, $\xi = 0.2$, the process keeps fluctuating around the square root of the long run variance.

4.3 Application: options of assignment 2

We apply the model to estimate the European Call option prices of assignment 2. Because the assignment said to implement the Heston model for the same options as in assignment 2 we have only included results for the EU call options (using the functions in the code sections Asian option prices using the Heston model are easily obtained and are available on request) as to limit the length of this report. In table 10 we can see the results. The estimated prices are all higher than in assignment 2. This means that the volatility paths generated by the Heston model using the given parameters results in a higher volatility in general than the implied volatilities used in assignments 2 and 3. In some sense this is to be expected because the volatility is a random process itself now with its own volatility. Thus we would expect a sum of volatile random processes to yield a higher volatility than a single one. However this also depends on the parameters chosen for the volatility process, which ideally should be calibrated to the stock used.

5 Tables

5.1 Assignment 1

Table 1: Computed implied volatilities

Method	**Implied Volatility:** $\boldsymbol{\sigma_{impl}}$
Newton's Method	0.1778
Brent's Method	0.1778

Note: We can see that both methods find the same value

Table 2: Used values for the Black-Scholes formula to find σ_{impl} found on 03-02-2016

Variable	**Variable**
Stockprice $S(0)$	35.77
Strikeprice K	28
Maturity T	0.616
Dividend Yield q	0.0168
Interest r	0.007
Call price C	7.6

Note: Strike date:16-09-16 gives $T = ((365 - 106) - 34)/365 = 0.616$

Table 3: Various implied volatilities for different strikes and maturities

$\mathbf{C_{market}}$	**T**	**K**	**Implied Volatility:** $\boldsymbol{\sigma_{impl}}$
10.15	0.616	26	0.3719
7.6	0.616	28	0.1778
4.5	0.616	30	0.1608
9.8	0.367	26	0.3457
7.65	0.367	28	0.1948
4.75	0.367	30	0.1736
Average			0.2374

Note: All Values recorded at 03-02-2016. $T = 0.616$ corresponds with 16-09-16 , $T = 0.367$ with 17-06-16.

5.2 Assignment 2

Table 4: Means and variances of samples drawn from the $\mathcal{N}(0,1)$ distribution

Sample Size	**Mean: μ**	**Variance: σ^2**
20.0000	0.1003	0.5269
70.0000	0.0686	0.9639
120.0000	-0.0918	0.9466
170.0000	-0.0533	0.8816
220.0000	0.0179	0.9878
270.0000	-0.0970	1.0581
320.0000	-0.0675	0.9015
370.0000	-0.0190	1.1283
420.0000	-0.0747	1.0147
470.0000	-0.0716	1.0859
520.0000	-0.0450	1.0684
570.0000	0.0550	1.0508
620.0000	0.0448	0.9670
670.0000	0.0620	1.0286
720.0000	-0.0115	1.0065
770.0000	-0.0001	0.9671
820.0000	0.0024	1.0896
870.0000	-0.0090	0.9890
920.0000	-0.0148	0.9606
970.0000	0.0342	0.9765

Note: From the table it is visible that from M=70 on the mean and variance of the random sample generated by `numpy.random.normal` function are close to the 0 and 1 respectively

Table 5: Price estimation results for the EU call options with a single implied volatility estimate.

K	**T**	$\mathbf{C_{market}}$	**95%CI**		$\mathbf{\hat{C}_{6000}}$	**SE**	**RE**$[\mathbf{\hat{C}_{6000}}]$	**RW**$[\mathbf{\hat{C}_{6000}}]$
26	0.616	10.15	9.4579	9.7096	9.5838	0.0642	0.0067	0.0263
28	0.616	7.6	7.5276	7.7742	7.6509	0.0629	0.0082	0.0322
30	0.616	4.5	5.7067	5.9417	5.8242	0.0599	0.0103	0.0404
26	0.367	9.8	9.5140	9.7079	9.6109	0.0495	0.0051	0.0202
28	0.367	7.65	7.5337	7.7261	7.6299	0.0491	0.0064	0.0252
30	0.367	4.75	5.6088	5.7953	5.7021	0.0476	0.0083	0.0327
26	0.2	9.6	9.6231	9.7664	9.6948	0.0365	0.0038	0.0148
28	0.2	7.45	7.6263	7.7695	7.6979	0.0365	0.0047	0.0186
30	0.2	4.5	5.6402	5.7822	5.7112	0.0362	0.0063	0.0249

Note: All Values recorded at 03-02-2016. $T = 0.616$ corresponds with 16-09-16 , $T = 0.367$ with 17-06-16 and $T = 0.2$ with 18-03-16.The volatility used is $\sigma_{impl} = 0.1778$.For all strikes and maturities it holds that the estimates are quite close to the recorded price with the volatility used.

Table 6: Price estimation results for the EU call options with the average of multiple implied volatilities

K	**T**	$\mathbf{C_{market}}$	**95%CI**		$\hat{\mathbf{C}}_{\mathbf{6000}}$	**SE**	**RE**$[\hat{\mathbf{C}}_{\mathbf{6000}}]$	**RW**$[\hat{\mathbf{C}}_{\mathbf{6000}}]$
26	0.616	10.15	9.5306	9.8634	9.6970	0.0849	0.0088	0.0343
28	0.616	7.6	7.7119	8.0334	7.8726	0.0820	0.0104	0.0408
30	0.616	4.5	6.0470	6.3504	6.1987	0.0774	0.0125	0.0489
26	0.367	9.8	9.5021	9.7605	9.6313	0.0659	0.0068	0.0268
28	0.367	7.65	7.5768	7.8295	7.7032	0.0645	0.0084	0.0328
30	0.367	4.75	5.7700	6.0102	5.8901	0.0613	0.0104	0.0408
26	0.2	9.6	9.6029	9.7952	9.6990	0.0490	0.0051	0.0198
28	0.2	7.45	7.6158	7.8070	7.7114	0.0488	0.0063	0.0248
30	0.367	4.75	5.6851	5.8707	5.7779	0.0473	0.0082	0.0321

Note: All Values recorded at 03-02-2016. $T = 0.616$ corresponds with 16-09-16 , $T = 0.367$ with 17-06-16 and $T = 0.2$ with 18-03-16.The volatility used is the average $\sigma_{impl} = 0.2374$. We see now that less market prices fall within the estimated confidence interval

5.3 Assignment 3

Table 7: Price estimation results for the Asian call options using a simple implied volatility

K	T	$\mathbf{C_{market}}$ (EU)	95%CI		$\mathbf{\hat{C}_{6000}}$	SE	RE[$\mathbf{\hat{C}_{6000}}$]	RW[$\mathbf{\hat{C}_{6000}}$]
26	0.616	10.15	9.5655	9.7123	9.6389	0.0375	0.0039	0.0152
28	0.616	7.6	7.5743	7.7211	7.6477	0.0374	0.0049	0.0192
30	0.616	4.5	5.5937	5.7394	5.6665	0.0374	0.0066	0.0257
26	0.367	9.8	9.6035	9.7145	9.6590	0.0283	0.0029	0.0115
28	0.367	7.65	7.6087	7.7196	7.6641	0.0283	0.0037	0.0145
30	0.367	4.75	5.6150	5.7258	5.6704	0.0283	0.0050	0.0195
26	0.2	9.6	9.6899	9.7711	9.7305	0.0207	0.0021	0.0083
28	0.2	7.45	7.6927	7.7739	7.7333	0.0207	0.0027	0.0105
30	0.2	4.5	5.6955	5.7767	5.7361	0.0207	0.0036	0.0142

Note: All Values recorded at 03-02-2016. $T = 0.616$ corresponds with 16-09-16 , $T = 0.367$ with 17-06-16 and $T = 0.2$ with 18-03-16.The volatility used is $\sigma_{impl} = 0.1778$.We see that the estimated prices are very close to the European Call prices. Furthermore the equality of certain standard errors are due to rounding.

Table 8: Price estimation results for the Asian call options using the average implied volatility

K	T	$\mathbf{C_{market}}$ (EU)	95%CI		$\mathbf{\hat{C}_{6000}}$	SE	RE[$\mathbf{\hat{C}_{6000}}$]	RW[$\mathbf{\hat{C}_{6000}}$]
26	0.616	10.15	9.5491	9.7467	9.6479	0.0504	0.0052	0.0205
28	0.616	7.6	7.5670	7.7637	7.6653	0.0502	0.0065	0.0257
30	0.616	4.5	5.6400	5.8314	5.7357	0.0488	0.0085	0.0334
26	0.367	9.8	9.5772	9.7261	9.6517	0.0380	0.0039	0.0154
28	0.367	7.65	7.5833	7.7321	7.6577	0.0380	0.0050	0.0194
30	0.367	4.75	5.6034	5.7507	5.6771	0.0376	0.0066	0.0259
26	0.2	9.6	9.6790	9.7880	9.7335	0.0278	0.0029	0.0112
28	0.2	7.45	7.6818	7.7908	7.7363	0.0278	0.0036	0.0141
30	0.2	4.5	5.6855	5.7944	5.7400	0.0278	0.0048	0.0190

Note: All Values recorded at 03-02-2016. $T = 0.616$ corresponds with 16-09-16 , $T = 0.367$ with 17-06-16 and $T = 0.2$ with 18-03-16.The volatility used is the average of implied volatilities of $\sigma_{impl} = 0.2374$. Again we see that the estimated prices are very close to the European Call prices.

5.4 Assignment 4

Table 9: Means, variances and covariances of samples drawn from the multivariate $\mathcal{N}(0, \Sigma)$ distribution

Sample Size	μ_{X_1}	μ_{X_2}	$\sigma^2_{X_1}$	$\sigma^2_{X_2}$	$\mathbb{C}ov(X_1, X_2)$
20.0000	0.1223	0.2180	0.9181	1.2393	-0.6690
70.0000	0.0260	-0.1689	1.0738	0.7548	-0.3112
120.0000	-0.0370	-0.1152	1.0656	0.9886	-0.7383
170.0000	0.0358	0.0192	1.1689	0.8956	-0.5715
220.0000	0.1173	-0.0590	0.9342	0.9921	-0.5636
270.0000	-0.0229	0.0327	0.9722	0.9582	-0.6126
320.0000	-0.0003	0.0479	1.0431	1.0711	-0.6062
370.0000	-0.0452	0.0341	1.0766	1.0575	-0.6882
420.0000	-0.0012	-0.0160	1.0381	0.9509	-0.5354
470.0000	0.0436	0.0153	0.9387	0.8841	-0.5055
520.0000	0.0439	-0.0965	1.1541	1.1011	-0.7484
570.0000	-0.0147	0.0163	1.0335	1.0370	-0.6368
620.0000	0.1093	-0.0379	0.9847	0.9403	-0.4950
670.0000	-0.0090	-0.0195	0.9538	1.0597	-0.6201
720.0000	0.0402	0.0152	0.9993	0.9945	-0.5855
770.0000	-0.0406	-0.0090	0.9399	0.9758	-0.5529
820.0000	0.0363	-0.0254	1.0383	0.9445	-0.6046
870.0000	-0.0365	0.0312	1.0018	1.0248	-0.6165
920.0000	-0.0323	0.0051	0.9949	1.0030	-0.5987
970.0000	-0.0046	-0.0282	1.0011	1.0213	-0.6276

Note: From the table it is visible that from N=170 on the means, variances and covariances of the random sample generated by `numpy.random.multivariate_normal` function are close to the 0,1 and -0.6 respectively

Table 10: Price estimation results for the EU call options with the Heston Model

K	T	$\mathbf{C_{market}}$	95%CI		$\mathbf{\hat{C}_{6000}}$	SE	$\mathbf{RE}[\mathbf{\hat{C}_{6000}}]$	$\mathbf{RW}[\mathbf{\hat{C}_{6000}}]$
26	0.616	10.15	10.4656	10.7507	10.6081	0.0727	0.0069	0.0269
28	0.616	7.6	8.6399	8.9175	8.7787	0.0708	0.0081	0.0316
30	0.616	4.5	6.9084	7.1729	7.0406	0.0675	0.0096	0.0376
26	0.367	9.8	10.0119	10.2277	10.1198	0.0550	0.0054	0.0213
28	0.367	7.65	8.1048	8.3162	8.2105	0.0539	0.0066	0.0258
30	0.367	4.75	6.2619	6.4643	6.3631	0.0516	0.0081	0.0318
26	0.2	9.6	9.9248	10.0770	10.0009	0.0388	0.0039	0.0152
28	0.2	7.45	7.9564	8.1080	8.0322	0.0387	0.0048	0.0189
30	0.2	4.5	6.0073	6.1565	6.0819	0.0380	0.0063	0.0245

Note: All Values recorded at 03-02-2016. $T = 0.616$ corresponds with 16-09-16 , $T = 0.367$ with 17-06-16 and $T = 0.2$ with 18-03-16.The volatility used is $\sigma_{impl} = 0.1778$. We see the estimated prices are higher than with the previous models. This makes sense given that this model is more volatile.

6 Figures

6.1 Assignment 2

Figure 1: Plot of ten simulated sample paths

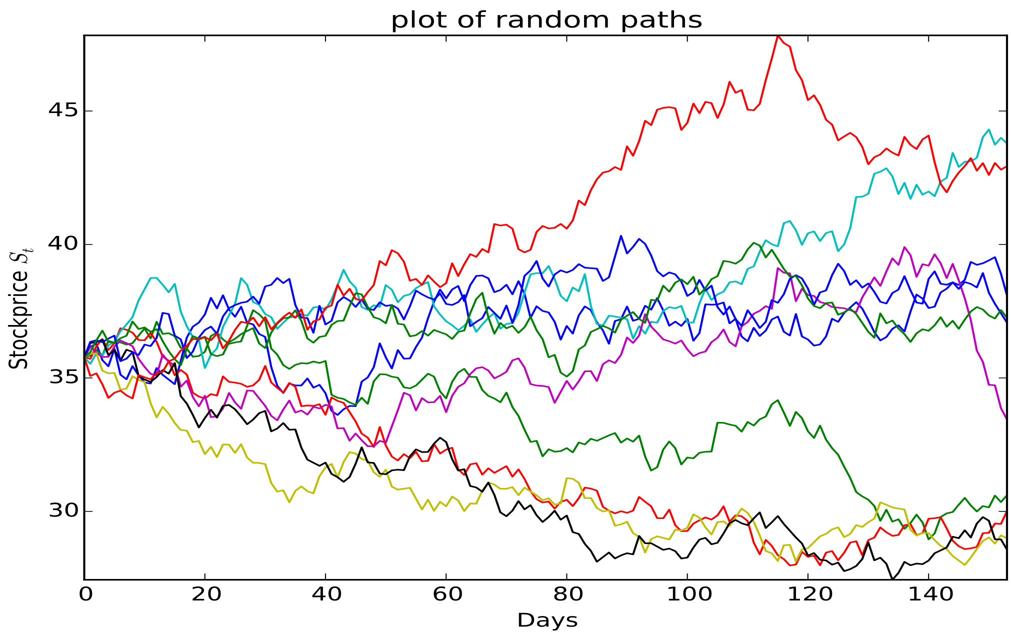
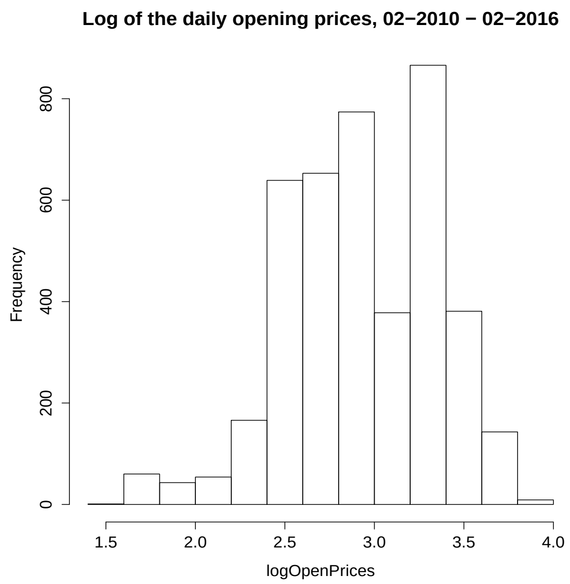
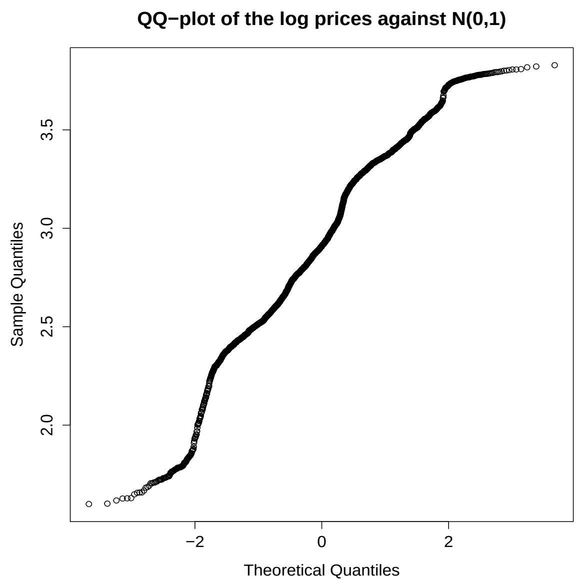


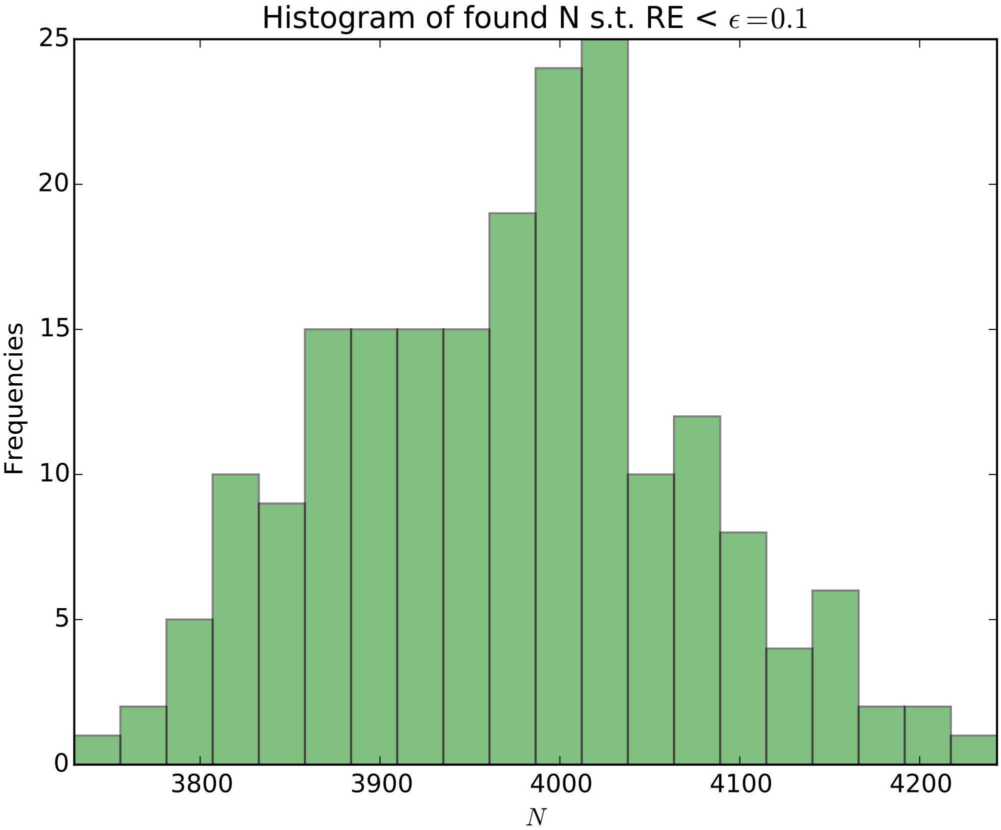
Figure 2: Histogram and QQ-plot of the log of the opening prices from 02-2010 to 02-2016





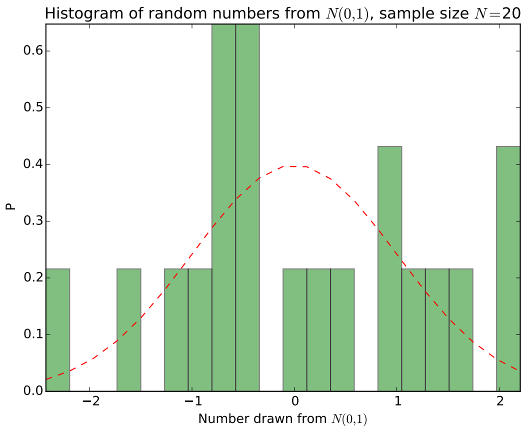
Note: From these we observe that the daily prices are not log-normally distributed. The distribution of the logs is skewed and has more mass in the tails than the normal distribution. However at higher frequencies the situation could be different.

Figure 3: Histogram of N's found using the stopping criterion $\mathbf{RE}[\hat{\mathbf{C}}_\mathbf{N}] < 0.1$

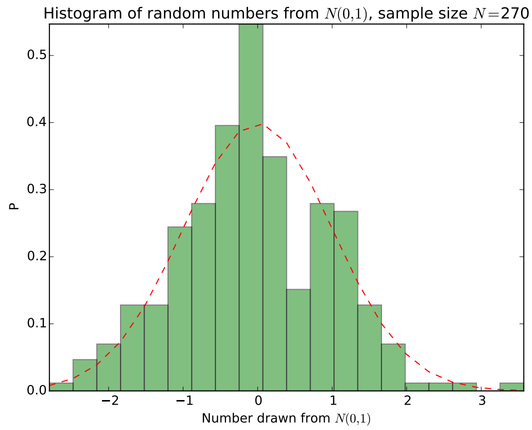


Note: We see that the estimated choices for the sample size have a hdistribution around the found mean of $\lceil 3972.415 \rceil$

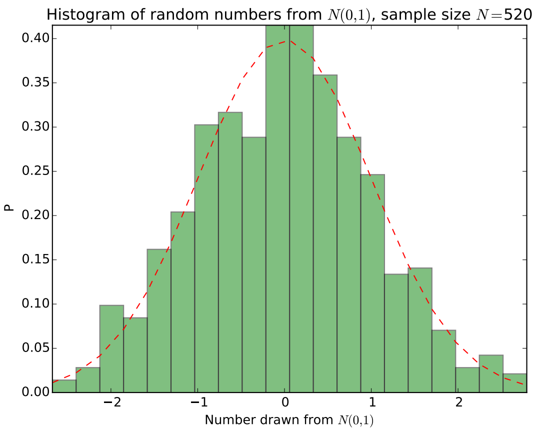
Figure 4: Verification normal random number generator



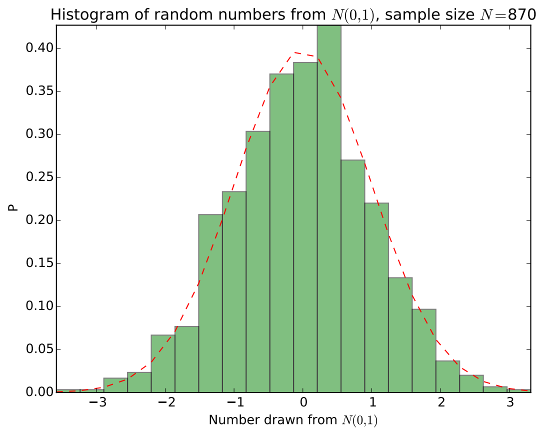
(a) Random numbers drawn from $\sim \mathcal{N}(0,1)$ for $N = 20$



(b) Random numbers drawn from $\sim \mathcal{N}(0,1)$ for $N = 270$



(c) Random numbers drawn from $\sim \mathcal{N}(0,1)$ for $N = 520$

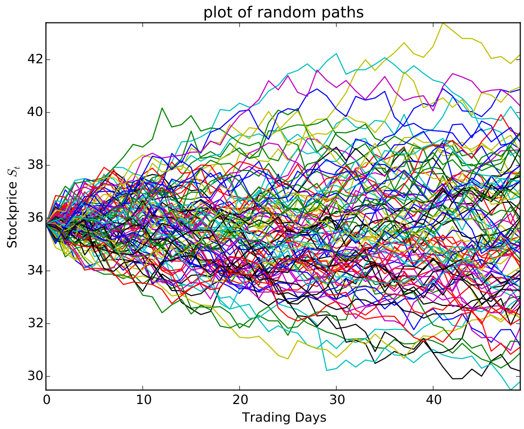


(d) Random numbers drawn from $\sim \mathcal{N}(0,1)$ for $N = 870$

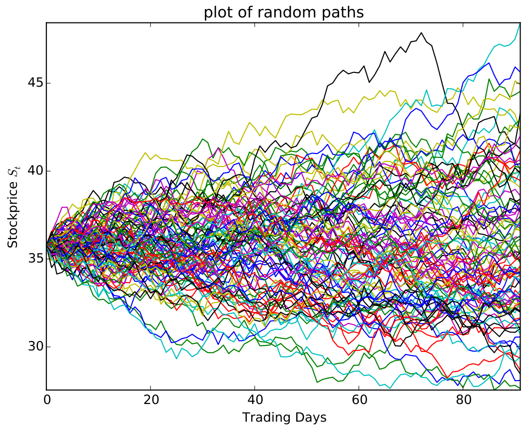
Note: We see that the histograms resemble the normal distribution increasingly well as N increases.

6.2 Assignment 3

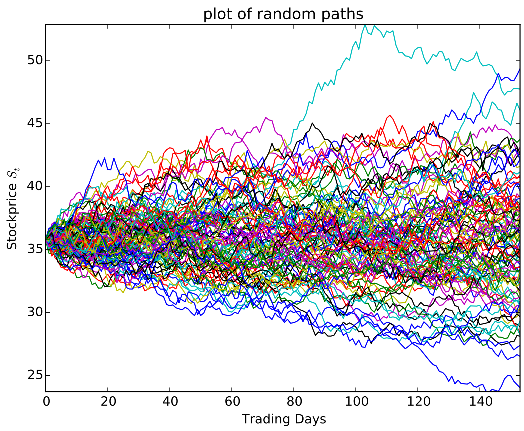
Figure 5: Simulated random Paths for various maturities



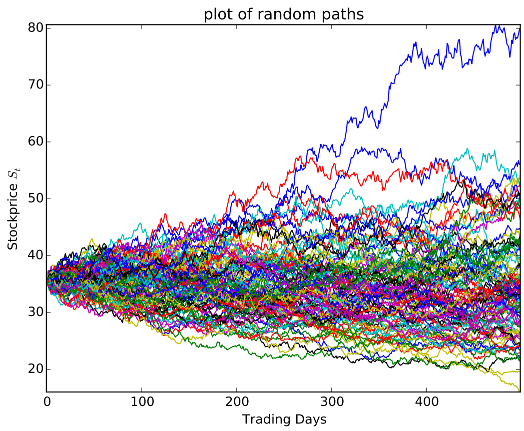
(a) 100 paths at $T = 0.2$



(b) 100 paths at $T = 0.367$



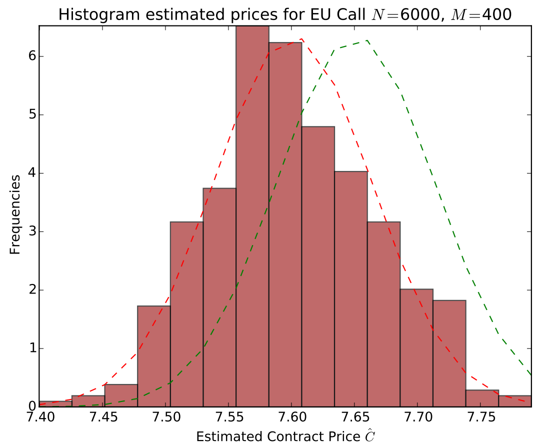
(c) 100 paths at $T = 0.616$



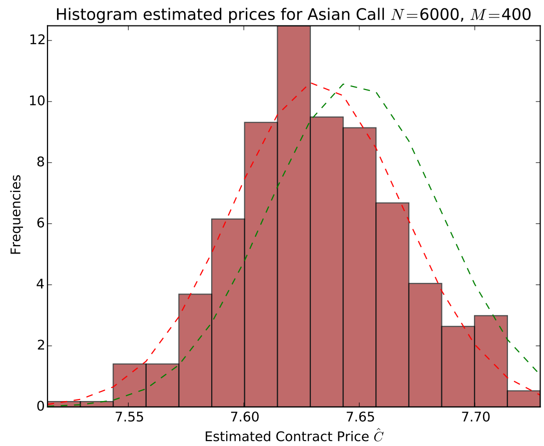
(d) 100 paths at $T = 2$

Note: We see that as T gets larger the difference between the average of the path and the end of the path becomes larger. Note the different scales in each figure. All other variables are set to those of table 1 and 2.

Figure 6: Verification CLT



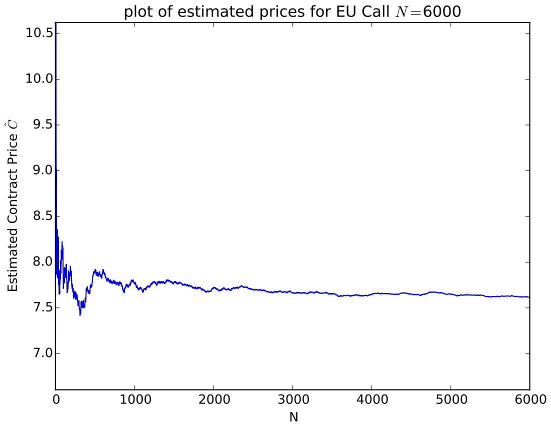
(a) Estimated prices of the European call option



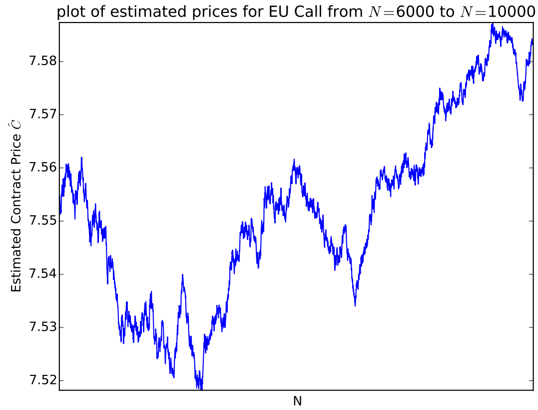
(b) Estimated prices of the Asian call option

Note: The striped lines are that of the normal distribution where the green line is that of the $\mathcal{N}(\hat{C}_{6000}, \hat{\sigma}^2)$ where $\hat{C}_{6000}, \hat{\sigma}^2$ are estimations based on one experiment. The red line is that of $\mathcal{N}(mean(estimatedPrices), \hat{\sigma}^2)$.

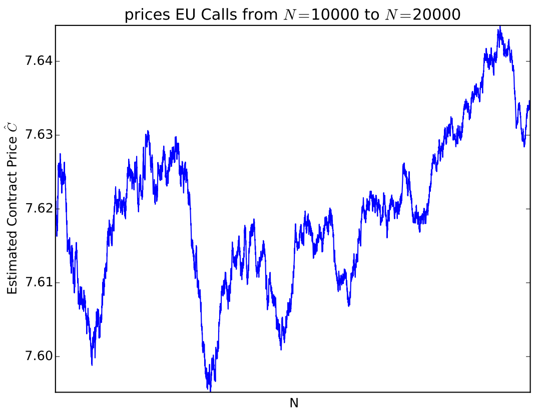
Figure 7: Verification SLLN European Call Options



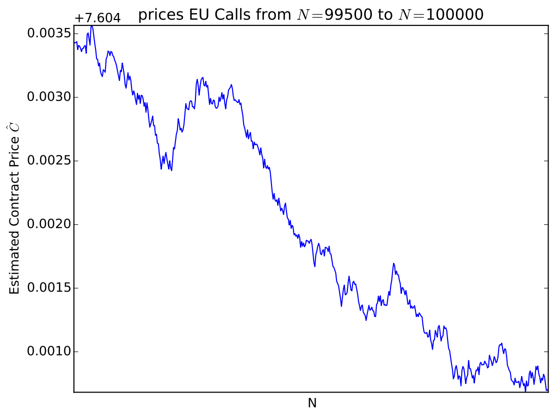
(a) Estimated prices for $N = 0$ to $N = 6000$



(b) Estimated prices for $N = 6000$ to $N = 10000$



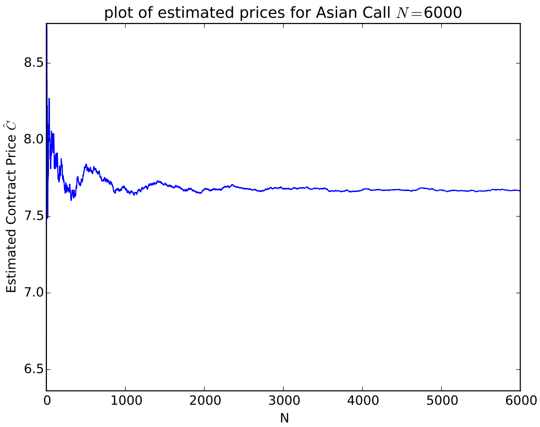
(c) Estimated prices for $N = 10000$ to $N = 20000$



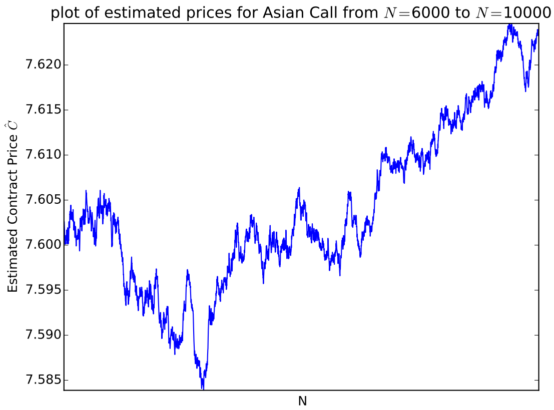
(d) Estimated prices for $N = 99500$ to $N = 100000$

Note: It is very clearly visible that the estimation stabilized for large N, note that as N gets larger the range of the Y axis is much finer.

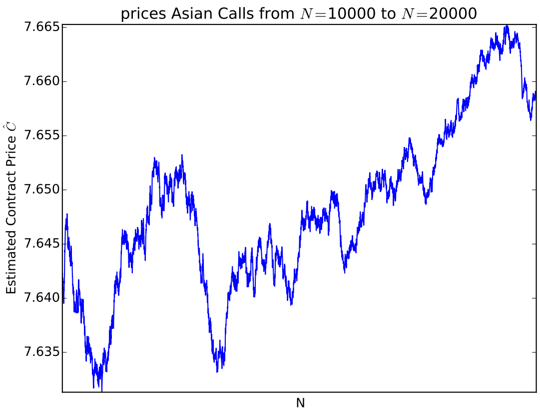
Figure 8: Verification SLLN Asian Call Options



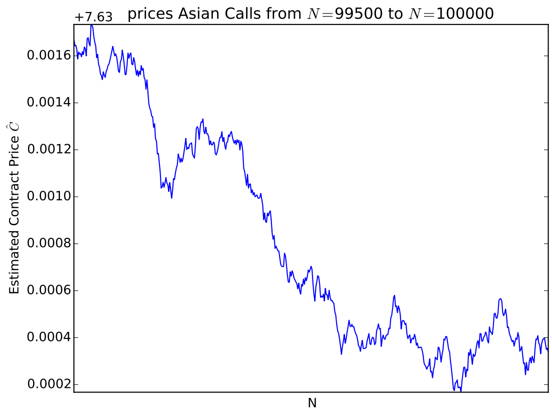
(a) Estimated prices for $N = 0$ to $N = 6000$



(b) Estimated prices for $N = 6000$ to $N = 10000$



(c) Estimated prices for $N = 10000$ to $N = 20000$

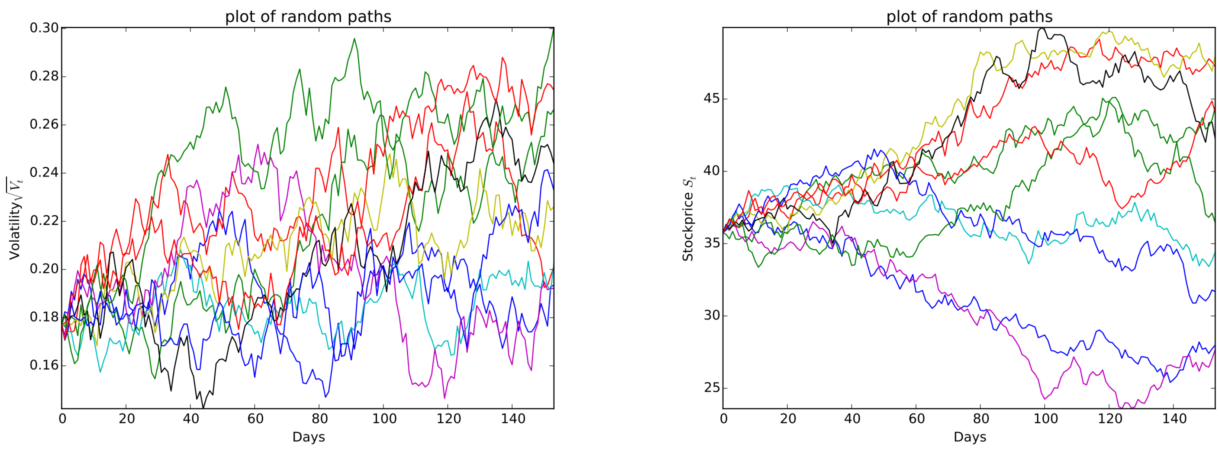


(d) Estimated prices for $N = 99500$ to $N = 100000$

Note: It is very clearly visible that the estimation stabilized for large N, note that as N gets larger the range of the Y axis is much finer.

6.3 Assignment 4

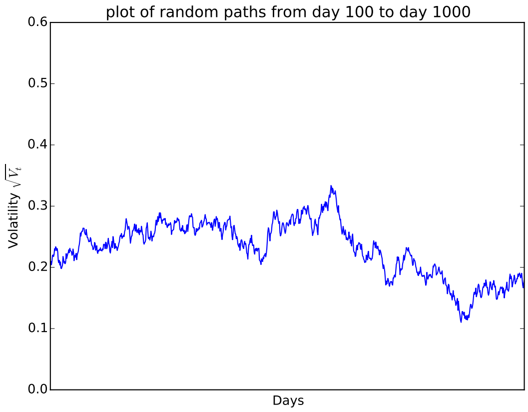
Figure 9: Simulated random Paths according to the Heston model



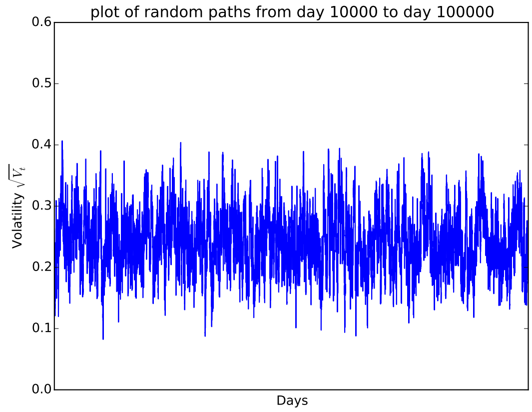
(a) 10 $\{V(t)\}$ paths at $T = 0.616$

(b) 10 $\{S(t)\}$ paths at $T = 0.616$

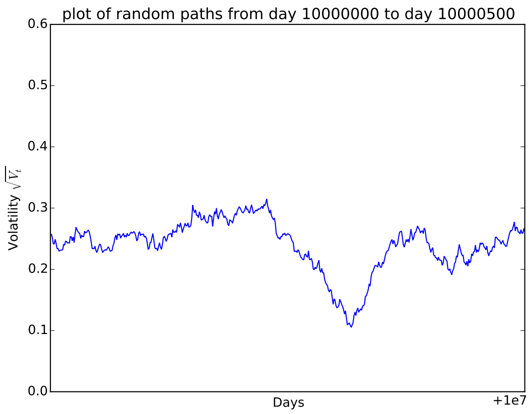
Figure 10: Simulated random Paths for various maturities



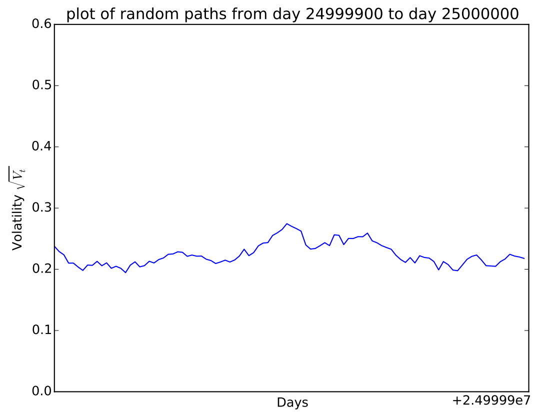
(a) volatility path from day 100 to 1000



(b) volatility path from day 10000 to 100000



(c) volatility path from day 10000000 to 10000500



(d) volatility path from day 24999900 to 25000000

Note: From the second picture the process looks stationary around a mean of $\sqrt{\theta} = 0.25$, and from the last figure we see that indeed after 24999900 days the volatility fluctuates around the square root of the long run variance 0.25.