Various simulation experiments

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1 Problem: Multiply with carry Generator

In this section we implement the multiply-with-carry generator (MWC) for $r = 2, b = 16$, and choose ($0 < a < b$) such that $ab^r - 1$ is a prime number. If we take $a = 14$ we have that $ab^r - 1 == 14 \cdot 16^2 = 3583$ is a prime number, and we let the seeds $X = (X_0, X_1) = (2, 2)$ and $c_1 = 7$. We recall that the MWC RNG is defined as follows:

$$X_n = (aX_{n-r} + c_{n-1}) \bmod b, \qquad c_n = \lfloor \frac{aX_{n-r} + c_{n-1}}{b} \rfloor, \qquad U_n = \frac{X_n}{b}. \tag{1}$$

In order to determine the period we start by generating N ($N \geq 2000$) numbers with out implementation of the MWC generator. These numbers are generated to fall within $[0, b)$. Then for all generated number X_i and carry c_i with $i \in 4, \ldots$ we check the following conditions:

- $X_i == X_3$, $X_{i-1} == X_2$
- $c_i == c_3$, $c_{i-1} == c_2$

If all the conditions are true then we let the period be $i - 2$. As a last verification we check the first row of each period and compute the module of these numbers in the row and conclude that if these modulo's are zero we have in fact spotted a period. When we implement this algorithm we find the period to be 1791. In order to check whether the numbers obtained are uniform we count the number of times each number, in $[0, 16)$, occurs within one period. We tabulate these counts in table 1. We observe that the frequencies are relatively equal, which means that the numbers are distributed evenly. This even distribution is in line with a uniform RNG. The frequencies are not totally even though, for instance $X = 13$ occurs 93 times which is 38 counts less than the most frequent number 2.

As a final test we construct overlapping pairs (U_i, U_{i+1}) of the numbers following each other over $[0, 1)$. We have that these vectors lie on equidistant parallel lines in the 2-d unit cube, which we expect to observe. Indeed if we look at the lattice in figure 2 we see that the distance between these parallel lines are equal (as far as our eyes can see) in which ever direction we look. Furthermore the distance between these equidistant parallel lines is quite small, which indicates this RNG to be good.

2 Problem: Confidence interval estimation

Suppose that we can to generate i.i.d. observations $X_1, \dots, X_n$ of X, where X is a random variable with unknown expectation μ and variance σ^2. For Gaussian data we know the following is true:

$$\frac{(n-1)S^2}{\sigma^2} \overset{d}{\sim} \chi^2_{n-1}, \tag{2}$$

For $S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \widehat{\mathbb{E}(\mathbf{X})})^2$ (sample variance). Suppose $\chi^2_{q,n-1}$ is the qth quantile of the chi-squared distribution with $n-1$ degrees of freedom. We can use equation 2 to build a confidence interval for the variance as follows:

$$\mathbb{P}\left(\chi^2_{\alpha/2,n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2,n-1}\right) = 1-\alpha \tag{3}$$

$$\Leftrightarrow \mathbb{P}\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}\right) = 1-\alpha. \tag{4}$$

We know that the above only hold if the data originates from the normal distribution but we can use it as an approximation. We furthermore assume:

$$\chi^2_{n-1} \overset{d}{\sim} \text{ Gamma}\left(\text{shape } = \frac{n-1}{2}, \text{ scale } = \frac{1}{2}\right), \tag{5}$$

Now if we let $\mathcal{G}_{q,n-1}$ be the qth quantile of the Gamma distribution with shape parameter $(n-1)/2$ and scale $1/2$, then we have that we can construct the interval for the variance as follows:

$$\mathbb{P}\left(\mathcal{G}_{\alpha/2,n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \mathcal{G}_{1-\alpha/2,n-1}\right) \approx 1-\alpha \tag{6}$$

$$\Leftrightarrow \mathbb{P}\left(\frac{(n-1)S^2}{\mathcal{G}_{1-\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\mathcal{G}_{\alpha/2,n-1}}\right) \approx 1-\alpha. \tag{7}$$

Now for large values of the shape parameters we have that the Gamma distribution can be approximated by the Normal distribution:

$$\text{Gamma}(r,\lambda)\mathcal{N}\left(\frac{r}{\lambda}, \frac{r}{\lambda^2}\right), \tag{8}$$

which gives the following:

$$\frac{(n-1)S^2}{\sigma^2} \overset{appr}{\sim} \mathcal{N}(n-1, 2(n-1)) \tag{9}$$

$$\text{which can be re-written as:} \quad \frac{\frac{(n-1)S^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} \overset{appr}{\sim} \mathcal{N}(0,1) \tag{10}$$

Using the above we can construct again the σ^2 confidence interval. Suppose Z_q is the qth quantile of the standard normal distribution, and note that we set $r = (n-1)/2$ and $\lambda = 1/2$.

Then we have that:

$$
\begin{aligned}
&\mathbb{P}\left(Z_{\alpha/2} \leq \frac{\frac{(n-1)S^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} \leq Z_{1-\alpha/2}\right) \approx 1-\alpha && (11)\\
\Leftrightarrow\ &\mathbb{P}\left((n-1) + Z_{\alpha/2}\sqrt{2(n-1)} \leq \frac{(n-1)S^2}{\sigma^2} \leq (n-1) + Z_{1-\alpha/2}\sqrt{2(n-1)}\right) \approx 1-\alpha && (12)\\
\Leftrightarrow\ &\mathbb{P}\left(\frac{(n-1)S^2}{(n-1) + Z_{1-\alpha/2}\sqrt{2(n-1)}} \leq \sigma^2 \leq \frac{(n-1)S^2}{(n-1) + Z_{\alpha/2}\sqrt{2(n-1)}}\right) \approx 1-\alpha. && (13)
\end{aligned}
$$

3 Problem: Random Number Generation

3.1 Acceptance Rejection Method

The acceptance rejection method is built on theorem 3.1. The algorithm is simple given the theorem:

1. Generate a random variable Y according to the proposal distribution $g(\cdot)$
2. Generate $U \sim U(0,1)$
3. Gf $U \leq \frac{f(Y)}{Cg(Y)}, returnX = Y$ else, repeat from step 1

Not that in the implementation of the rather than generating the numbers in a loop, we generate a batch or random numbers in an Array and then check the conditions within the array. This is more efficient. See the implementation in the code section.

We will apply the A-R method to generate random numbers form the Gamma distribution with a scale of 1, $X \sim Gamma(1, \alpha)$, with $\alpha \in \{1.5, 2.5, 5.5, 10.5, 20.5\}$. We do this using two proposal distributions, namely the exponential distribution and the normal distribution. The algorithm for the normal distribution is obtained from page 60/61 of the book by Rubinstein and Kroese (2007), with the only difference being:

$$h(y) = -(1 - 3\alpha)\log(1 + cy) - d(1 + cy)^3 + d \tag{14}$$

The algorithm with the exponential proposal is that of 3.1. If we for instance look at the results figure 3 we see that the exponential pdf line touches the gamma pdf line, thereby miniziing the distance between the two hence aiming to maximize the acceptance ratio. This is by setting the proposal distribution, $Y \sim exp(\mu)$ with $g(x) = \mu e^{\mu x}$, parameter $\mu = \frac{1}{\alpha}$. We will now show the derivation of this optimal μ.

First we find $x^{max} := \sup_x \frac{f(x)}{g(x)}$.

$$\begin{aligned}
\sup_x \frac{f(x)}{g(x)} &= \sup_x \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)\mu e^{-\mu x}} & (15)\\
{\scriptstyle \Gamma(\alpha) > 0, \forall \alpha > 1 \textbf{ and } \mu > 0} &= \Gamma(\alpha)\mu \sup_x \frac{x^{\alpha-1}e^{-x}}{e^{-\mu x}} & (16)\\
&= \Gamma(\alpha)\mu \sup_x x^{\alpha-1}e^{x(\mu-1)} & (17)
\end{aligned}$$

We can then maximize w.r.t. x using the derivative(where the variables that dont depend on x are put in the constant) as follows:

$$\frac{d}{dx}x^{\alpha-1}e^{x(\mu-1)} = (\alpha-1)x^{\alpha-2}e^{x(\mu-1)} + x^{\alpha-1}e^{x(\mu-1)}(\mu-1) \tag{18}$$

Setting the derivative to zero we get:

$$(\alpha-1)x^{\alpha-2}e^{x(\mu-1)} + x^{\alpha-1}e^{x(\mu-1)}(\mu-1) = 0 \tag{19}$$
$$x^{\alpha-2}e^{x(\mu-1)}\left((\alpha-1)+(\mu-1)x\right) = 0 \tag{20}$$
$$\left(x^{\alpha-2}e^{x(\mu-1)} > 0, \forall(\alpha,\mu) \text{ and } x^{\alpha-2}e^{x(\mu-1)} = 0 \Leftrightarrow x = 0\right) \Leftrightarrow (\mu-1)x = (1-\alpha) \tag{21}$$
$$\text{for } \mu-1\neq 0 \Leftrightarrow x = \frac{1-\alpha}{\mu-1} \tag{22}$$

Note that $x = 0$ is not feasable because then we would get $C = 0$ which is degenerate.Then under the assumption that $\frac{d^2}{dx^2}C < 0$ we have found the maximum. Now to get $\inf_\mu C = \inf_\mu \frac{f(x^{max})}{g(x^{max})}$ we take the derivative of C w.r.t. to μ:

$$C = \frac{(1-\alpha)^{\alpha-1}e^{\frac{1-\alpha}{\mu-1}(\mu-1)}}{\Gamma(\alpha)(\mu-1)^{\alpha-1}\mu} \tag{23}$$
$$= (1-\alpha)^{\alpha-1}\cdot\Gamma(\alpha)^{-1}\cdot e^{1-\alpha}\cdot((\mu-1)^{\alpha-1}\mu)^{-1} \tag{24}$$
$$= B_\alpha\cdot((\mu-1)^{\alpha-1}\mu)^{-1} \tag{25}$$
$$\frac{d}{dx}C = -((\mu-1)^{\alpha-1}\mu)^{-2}\left[(\alpha-1)(\mu-1)^{\alpha-2}\mu + (\mu-1)^{\alpha-1}\right] \tag{26}$$

Setting the derivative to zero we get:

$$(\alpha-1)(\mu-1)^{\alpha-2}\mu = -(\mu-1)^{\alpha-1} \tag{27}$$
$$\text{for } \mu-1\neq 0 \Leftrightarrow (\alpha-1) = (1-\mu) \tag{28}$$
$$\mu > 0 \Leftrightarrow \alpha-1 = 1-\frac{1}{\mu} \tag{29}$$
$$\Leftrightarrow \alpha-1 = -1+\frac{1}{\mu} \tag{30}$$
$$\alpha > 0 \Leftrightarrow \frac{1}{\alpha} = \mu \tag{31}$$

Under the assumption that $\frac{d^2}{dx^2}C > 0$, we have found a minimum.

Theorem 3.1. *Let X be a continous RV with CDF F and density f. Suppose we can find a (proposal) density g, for RV Y, and a finite constant C such that:*

$$f(x) \leq Cg(x)\forall x\in\mathbb{R}$$

Then with U a uniform RV on $(0,1)$ independent of Y we have:

$$Y|\left(U \leq \frac{f(Y)}{Cg(Y)}\right) \overset{D}{=} X$$

3.2 Simulation results

Using the normal and the exponential distributions as proposals we have simulated $N = 1000000$ i.i.d. random variables. The histograms are visible in figure 3 and 4. We clearly see from the histograms that in both cases the numbers generated by our algorithms follow the Gamma distribution. In order to verify the algorithms we check whether or not the theoretical moments $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$ are within the computed confidence intervals. For obtaining the confidence intervals for the variance we use the approximation of problem 2 as well as a bootstrap confidence interval.We have done this because the approximation of problem 2 holds for small shape parameter only. The bootstrap confidence interval is obtained as follows:

1. We simulate a vector of 1000000 i.i.d. numbers $\mathbf{X^{(i*)}}$ from $Gamma(1, \alpha)$
2. We calculate the variance $T_i^* = \widehat{\mathbb{V}\text{ar}(\mathbf{X^{(i*)}})}$
3. Repeat 1 and 2 for i = 1,..,1000
4. From the empirical distribution of T^* we get the alpha quantiles $T^*_{[\alpha]}$ $T^*_{[1-\alpha]}$
5. We get the $1 - 2\alpha$ confidence level interval $[2\widehat{\mathbb{V}\text{ar}(\mathbf{X})} - T^*_{[1-\alpha]}, 2\widehat{\mathbb{V}\text{ar}(\mathbf{X})} - T^*_{[\alpha]},]$

Where:

$$\widehat{\mathbb{E}(\mathbf{X})} = \frac{1}{N}\sum_{i=1}^{N} X_i, \widehat{\mathbb{V}\text{ar}(\mathbf{X})} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \widehat{\mathbb{E}(\mathbf{X})})^2, \tag{32}$$

These are the sample averages and variances computed for the sample generated with the algorithms. We let $\alpha = 0.025$.The derivation of the confidence interval in 5 is available on request is as follows: Let T be some estimate of θ for which we want the distribution of $T - \theta$, say distribution G, to be concentrated around zero. We then have by definition of the quantiles:

$$\mathbb{P}(G^{-1}(\alpha) \leq T - \theta \leq G^{-1}(1-\alpha)) \geq 1 - 2\alpha \tag{33}$$

$$\text{which can be written as } \mathbb{P}(T - G^{-1}(1-\alpha) \leq \theta \leq T - G^{-1}(1-\alpha)) \geq 1 - 2\alpha \tag{34}$$

$$\tag{35}$$

Then letting $Z_i^* = T_i^* - T$ where T_i^* is defined as in step 2 of the algorithm, we can estimate G^{-1} and get:

$$\mathbb{P}(T - Z^*_{[1-\alpha]} \leq \theta \leq T - Z^*_{[1-\alpha]}) \geq 1 - 2\alpha \tag{36}$$

$$\text{which by definition of } Z_i^* \text{ gives } \mathbb{P}(2T - T^*_{[1-\alpha]} \leq \theta \leq 2T - T^*_{[1-\alpha]}) \geq 1 - 2\alpha \tag{37}$$

$$\tag{38}$$

The confidence intervals for the sample mean is obtained exactly as in homework 1 and 2. The results of the confidence intervals are found in tables 2 and 3. For the exponential proposal we observe for $\alpha \in \{1.5, 2.5, 20.5\}$ the theoretical variance is not covered (although the intervals come close to covering it) and for $\alpha = 2.5$ the mean is very close but not covered by the CI. Furthermore we see that the bootstrapped confidence interval and the approximate interval (from problem 2) are quite similar. Note that to check the coverage better we should do B number of simulations and check how many times we have a hit or miss and get a percentage. We expect the theoretical moments to be covered quite often. We see similar results for the normal proposal.

3.3 Ratio of Uniforms

We also implement the ratio of uniforms method to generate our Gamma distributed random variables. We have followed the algorithm of slides 31 and 32 of week 3 exactly. The results we obtain are found in table 4. Here as well we find that sometimes the moments are covered and sometimes they are not. As additional verification we have included figure 5.

3.4 Comparison of algorithms

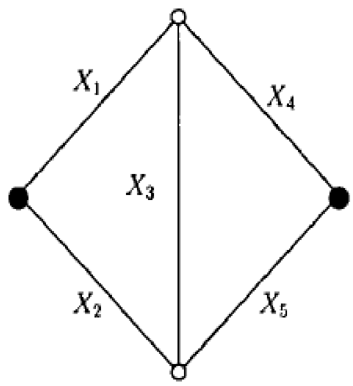
In table 5 we tabulate the acceptance ratios and computing times. In terms of computing times the ratio of uniforms method is the fastest. This is to be expected given that it only uses the uniform RNG, whereas the other algorithms are built upon other algorithms that generate non-uniform random numbers (pseudo random). In terms if acceptance ratios we see that the A-R with the standard normal proposal distribution has the highest acceptance, which is also makes sense because for $\alpha > 1$ we can approximate the gamma by shifting the normal distribution. The A-R method with performs quite well for small values of alpha, quickly deteriorates in computing time as well as acceptance ratios. This is also visible in figure 3. we observe the rejection probabilities are larger for large shape parameters, because the gamma distribution resembles the exponential very little in that case. Note please that α is used in the above interchangeably to represent the shape parameter for the Gamma distribution as well as the confidence level, but from context it should be clear which is which.

4 Assignment: Reliability estimation using simulations

4.1 Bridge System

In this problem we estimate the reliability of the bride system presented on page 98 of the book of Rubinstein and Kroese (2007). This system had 5 components and the function is said to work if the mapping $H : \mathcal{X} \to \{0,1\}$ maps to a 1, where $\mathcal{X} = \{0,1\}^5$ is the set of all binary vectors $\mathcal{X} = (X_1, \ldots, X_5)$ with independent or dependent components. We experiment with both independent and dependent components. We have that $X_i \sim \text{Ber}(p_i)$, where $X_i = 1$ represents a working link and $X_i = 0$ a defective one. We define $Y = H(\mathbf{X}) = 1 - (1 - X_1X_4)(X_2X_5)(1 - X_1X_3X_5)(1 - X_2X_3X_4)$. In figure 4.1 of the book that means that the systems is working if the black terminal nodes are connected by working links. We also add the graph below:

Figure 1: Graph of the bridge



The aim is to estimate $\ell = \mathbb{P}(Y = 1) = \mathbb{P}(H(\mathbf{X}) = 1) = \mathbb{E}[H(\mathbf{X})] = \sum_{\mathbf{X} \in \mathcal{X}} H(\mathbf{X})\mathbb{P}(\mathbf{X} = X)$ with CMC.

4.2 CMC for independent X_i

Suppose we have $(p_1, \ldots, p_5) = (0.7, 0.6, 0.5, 0.4, 0.3)$ for $X_i \sim \text{Ber}(p_i)$. In order to simulate the states of the X_i we perform the following in each run of the algorithm, for N runs.

- Using the uniform RNG `numpy.random.rand` we generate a uniformly distributed vector U on $[0, 1)$.
- Then $X_i = 1$ if $U_i < p_i$, else $X_i = 0$ for all 5 components. Thus the uniform vector represents the probabilities and the resulting vector $\mathbf{X}_n$ the system state for run $n \in N$.
- Using the structure function we compute $H(\mathbf{X_n})$, which gives the state of the system for run $n \in N$.

From here on we apply the CMC algorithm exactly as in the previous homework by the same reasoning. The only difference here is the function $H(\mathbf{X})$ and the fact the expectation is taken over a discrete space rather than a continuous one. Thus we let $\hat{\ell} = \mathbb{E}[\widehat{H(\mathbf{X})}] = \frac{1}{N}\sum_n^N H(\mathbf{X_n})$.

We compute the confidence interval exactly as in the previous homework (page 6,7) where now $\hat{W}_N = \hat{\ell}$. Similarly the relative errors and width are computed as in pages 6,7 of the previous Homework. Furthermore once more as in the both of the previous Homework's we use the relative error as a stopping criterion. The criterion $\mathbf{RE}[\hat{\ell}] < \epsilon$ is used to determine N, by for each $n \in N$ checking whether the condition is fulfilled or not and stopping when it is the case. Given that $\mathbf{RE}[\hat{\ell}]$ is stochastic itself we repeat the experiment 1000 times (with $\epsilon = 0.01$)and take the ceiling of the average of these as our N.We find this ceiling average to be $N = 11742$ which we round up to 13000 for convenience. In figure 6 a histogram is plotted of the found N.

4.3 Results

In table 6 we have tabulated the confidence interval, estimate and the statistics for ℓ for $N = 13000$. We see that the estimated reliability is less than 0.5 which is not very good. As a verification we also check the marginal moments of the links as tabulated in table 7. We clearly see that the estimated moments match the theoretical ones. Furthermore we present a the (estimated)correlation matrix in table 11 and we see that the cross terms are close to zero, indicating independent links.

4.4 CMC for dependent X_i

Now we experiment with the dependent components in the bridge system. We implement the following setting:

- X_1 and X_4 are positively correlated
- X_2 and X_5 are positively correlated
- X_3 is negatively correlated with X_1, X_2, X_4, X_5
- all other pairs independent

To implement this setting we make use of a Gaussian copula we call $\mathbf{U}$, where all marginal distributions $U_1, \ldots, U_5$ are uniform on $(0, 1)$ and dependent. We start with generating a multivariate normal random vector $\mathbf{Z} \sim \mathcal{N}(0, \Sigma)$ using the Cholesky decomposition, where we have the following covariance matrix:

$$\Sigma = \begin{bmatrix} 1 & 0 & \rho_{13} & \rho_{14} & 0 \\ 0 & 1 & \rho_{23} & 0 & \rho_{25} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} & \rho_{35} \\ \rho_{41} & 0 & \rho_{43} & 1 & 0 \\ 0 & \rho_{52} & \rho_{53} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.2 & 0.6 & 0 \\ 0 & 1 & -0.7 & 0 & 0.7 \\ -0.2 & -0.7 & 1 & -0.5 & -0.2 \\ 0.6 & 0 & -0.5 & 1 & 0 \\ 0 & 0.7 & -0.2 & 0 & 1 \end{bmatrix}. \tag{39}$$

Using the `numpy.linalg.cholesky` function, the Cholesky decomposition is readily available, mathematically we have $\Sigma = LL'$. Beforehand we check that $\lambda_i(\Sigma) > 0$ to see if we have a positive definite matrix. We obtain the following lower triangular matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.7 & 0.6855 & 0 & 0 \\ 0.6 & 0 & -0.5542 & 0.5768 & 0 \\ 0 & 0.7 & 0.4230 & 0.4064 & 0.4072 \end{bmatrix}. \tag{40}$$

Then after generating $\mathbf{Z}^* \sim \mathcal{N}(0, I_5)$ we get $\mathbf{Z} = L\mathbf{Z}^* \sim \mathcal{N}(0, LL' = \Sigma)$. Now in order to construct the copula all we do is mapping the normal components of the $\mathbf{Z}$ vector to a uniform using the Normal CDF function $\Phi(\cdot)$, that is $U = (\Phi(Y_1), \ldots, \Phi(Y_5))$. Given that we now have dependent links we can again estimate ℓ using the same method as we have done in the independent case. The difference now is in the first step:

- we generate a uniformly distributed vector U on $(0,1)$, with dependent components using a Gaussian copula.
- Then $X_i = 1$ if $U_i < p_i$, else $X_i = 0$ for all 5 components. Thus the uniform vector represents the probabilities and the resulting vector $\mathbf{X}_n$ the system state for run $n \in N$.
- Using the structure function we compute $H(\mathbf{X_n})$, which gives the state of the system for run $n \in N$.

And now again $\hat{\ell} = \mathbb{E}\widehat{[H(\mathbf{X})]} = \frac{1}{N}\sum_n^N H(\mathbf{X_n})$. We search for a suitable sample size N using the same criterion as before and find the ceiling of the average of samples sized found to be $N = 8372$ which we round up to $n = 10000$. In figure 7 we have a plot of the histogram of this.

4.5 Results

In table 9 we have tabulated the confidence interval, estimate and the statistics for ℓ for $N = 10000$. We observe not that we have a higher estimated reliability than in the dependent case. In table reftab:marg2 we see that all the marginal moments are in line with the theoretical moments. In table **??** we see that the cross-terms come close to the actual values, but they're not always accurate. The signs (which is the most important) for the non-zero cross terms are all as they should be. The cross-terms that should be zero are all close to zero. As a last results we compute $\mathbb{P}(X_1 = X_3 = X_5 = 1)$ for both the dependent and the independent links, where we count how often we have that $X_1 = X_3 = X_5 = 1$ in one experiment of N runs and dividing this by the number of runs. For the independent case we find $\mathbb{P}(X_1 = X_3 = X_5 = 1) = 0.1099$, and for the dependent case we have $\mathbb{P}(X_1 = X_3 = X_5 = 1) = 0.0742$. This is to be expected because we have that $\rho_{13} < 0$ and $\rho_{35} < 0$ for the dependent case so the number of times $X_1 = X_3 = X_5 = 1$ occurs is smaller in the dependent case.

5 Tables

5.1 Problem 1

Table 1: Method comparisons

Number occurrences and frequencies															
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
111	111	131	109	125	115	95	113	111	129	109	99	115	93	113	112

Note: The frequencies are relatively equal, which means that the numbers are distributed evenly.

5.2 Problem 3

Table 2: Confidence intervals for the sample average and variance, for A-R method with exp proposal

$\boldsymbol{\alpha}$	$\widehat{\mathbb{E}[\boldsymbol{X}]}$	**95%CI (Avg.)**		$\widehat{\mathbb{V}\boldsymbol{ar}[\boldsymbol{X}]}$	**95%CI (Var.)**		**95%CI (Var.Boot.)**	
1.5000	1.4987	1.4963	1.5011	1.4952	1.4937	1.4966	1.4829	1.4977
2.5000	2.5032	2.5001	2.5063	2.5107	2.5083	2.5131	2.5108	2.5321
5.5000	5.5011	5.4965	5.5057	5.5048	5.4996	5.5101	5.4914	5.5269
10.5000	10.4972	10.4908	10.5035	10.4999	10.4899	10.5099	10.4670	10.5309
20.5000	20.4990	20.4901	20.5079	20.4169	20.3975	20.4364	20.2756	20.3941

Note: N = 1000000, we observe for $\alpha \in \{1.5, 2.5, 20.5\}$ the theoretical variance is not covered (although the intervals come close to covering it) and for $\alpha = 2.5$ the mean is very close but not covered by the CI. Furthermore we see that the bootstrapped confidence interval and the approximate interval (from problem 2) are quite similar. Not that to check the coverage better we should do B number of simulations and check how many times we have a hit or miss and get a percentage. We expect the theoretical moments to be covered quite often.

Table 3: Confidence intervals for the sample average and variance, for A-R method with Normal proposal

$\boldsymbol{\alpha}$	$\widehat{\mathbb{E}[\boldsymbol{X}]}$	**95%CI (Avg.)**		$\widehat{\mathbb{V}\boldsymbol{ar}[\boldsymbol{X}]}$	**95%CI (Var.)**		**95%CI (Var.Boot.)**	
1.5000	1.5004	1.4980	1.5028	1.5031	1.4989	1.5073	1.4992	1.5133
2.5000	2.5009	2.4978	2.5040	2.5000	2.4931	2.5070	2.4897	2.5104
5.5000	5.4980	5.4934	5.5026	5.5100	5.4948	5.5253	5.5012	5.5383
10.5000	10.5102	10.5038	10.5165	10.5345	10.5054	10.5638	10.5339	10.6032
20.5000	20.5029	20.4940	20.5118	20.5272	20.4704	20.5842	20.4948	20.6179

Note: N = 1000000, as with the previous table, the theoretical moments are all quite close or covered. Again note that this is but one simulation.

Table 4: Confidence intervals for the sample average and variance, for the ratio of uniforms method

$\boldsymbol{\alpha}$	$\widehat{\mathbb{E}[\boldsymbol{X}]}$	**95%CI (Avg.)**		$\widehat{\mathbb{V}\boldsymbol{ar}[\boldsymbol{X}]}$	**95%CI (Var.)**		**95%CI (Var.Boot.)**	
1.5000	1.5013	1.4989	1.5037	1.5030	1.4988	1.5071	1.4992	1.5133
2.5000	2.5012	2.4981	2.5043	2.5038	2.4968	2.5107	2.4979	2.5176
5.5000	5.5042	5.4996	5.5088	5.5036	5.4884	5.5189	5.4881	5.5259
10.5000	10.4923	10.4859	10.4986	10.4712	10.4423	10.5003	10.4098	10.4762
20.5000	20.4993	20.4904	20.5082	20.5043	20.4476	20.5613	20.4511	20.5658

Note: N = 1000000

Table 5: Method comparisons

$\boldsymbol{\alpha}$	**Acc.Ratios**			**Computing Time**		
1.5000	0.7956	0.9729	0.7509	0.5921	0.7840	0.1804
2.5000	0.6033	0.9861	0.6672	0.5959	0.7737	0.1782
5.5000	0.3989	0.9944	0.5941	1.0266	0.7728	0.3670
10.5000	0.2869	0.9972	0.4458	1.3774	0.7713	0.3550
20.5000	0.2046	0.9987	0.3394	1.8713	0.7705	0.4439

Note: N = 1000000. From left to right, for the ratios and the computing times, the left column represents the A-R method with the exponential proposal, the middle A-R with the normal proposal, and the third column the ratio of uniforms method. Clearly in computing time and acceptance ratios the A-R method with the exponential proposal performs worse.

5.3 Problem 4

Table 6: Reliability estimate for the independent system

$\hat{\ell}_{13000}$	**95%CI**		**RE**	**RW**	**SE**
0.4645	0.4559	0.4731	0.0094	0.03691	0.0044

Note: We have a relative error below 0.01, and we see that that the reliability is estimated to be less than 0.5. N=13000

Table 7: Marginal means and variances for the independent system

	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$
$\widehat{\mathbb{E}[\boldsymbol{X_i}]}$	0.7038	0.6015	0.5022	0.3989	0.3060
$\mathbb{E}[\boldsymbol{X_i}]$	0.7	0.6	0.5	0.4	0.3
$\widehat{\mathbb{V}\boldsymbol{ar}[\boldsymbol{X_i}]}$	0.2084	0.2397	0.2499	0.2398	0.2123
$\mathbb{V}\boldsymbol{ar}[\boldsymbol{X_i}]$	0.21	0.24	0.25	0.24	0.21

Note: We see that the estimated marginal moments resemble the theoretical moments quite closely. Where $\mathbb{E}[X_i] = p_i, \mathbb{V}ar[X_i] = p_i - p_i^2$ for the Bernoulli RV's.N=13000

Table 8: Computed correlation matrix for the independent system

	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$
$\mathbf{X_1}$	1.0000	-0.0091	-0.0001	-0.0007	0.0125
$\mathbf{X_2}$	-0.0091	1.0000	-0.0154	0.0166	0.0121
$\mathbf{X_3}$	-0.0001	-0.0154	1.0000	0.0121	0.0027
$\mathbf{X_4}$	-0.0007	0.0166	0.0121	1.0000	0.0055
$\mathbf{X_5}$	0.0125	0.0121	0.0027	0.0055	1.0000

Note: We see that the cross term are close to zero, and zeros is what we would expect from the theoretical correlation matrix. N=13000

Table 9: Reliability estimate for the dependent system

$\hat{\ell}$	**95%CI**		**RE**	**RW**	**SE**
0.5485	0.5387	0.5582	0.0091	0.0357	0.0049

Note: We have a relative error below 0.01, and we see that that the reliability is estimated to be higher than the independent system $\hat{\ell} > 0.5$

Table 10: Marginal means and variances for the dependent system

	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$
$\widehat{\mathbb{E}[\boldsymbol{X_i}]}$	0.6982	0.6041	0.4906	0.4047	0.3030
$\mathbb{E}[\boldsymbol{X_i}]$	0.7	0.6	0.5	0.4	0.3
$\widehat{\mathbb{V}\boldsymbol{ar}[\boldsymbol{X_i}]}$	0.2107	0.2392	0.2499	0.2409	0.2111
$\mathbb{V}\boldsymbol{ar}[\boldsymbol{X_i}]$	0.21	0.24	0.25	0.24	0.21

Note: We see that the estimated marginal moments resemble the theoretical moments quite closely. Where $\mathbb{E}[X_i] = p_i, \mathbb{V}ar[X_i] = p_i - p_i^2$ for the Bernoulli RV's.

Table 11: Computed and actual correlation matrix for the dependent system

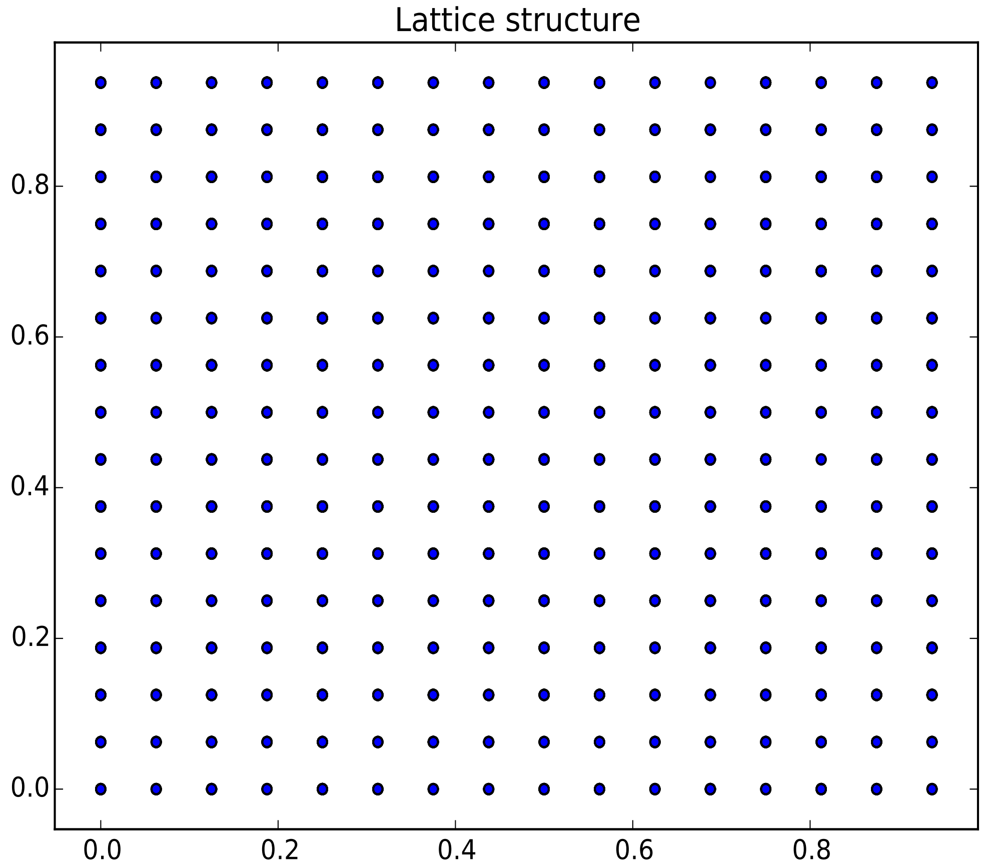
	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$
$\mathbf{X_1}$	1.0000	0.0072	-0.1226	0.3654	0.0154
$\mathbf{X_2}$	0.0072	1.0000	-0.4907	0.0113	0.4247
$\mathbf{X_3}$	-0.1226	-0.4907	1.0000	-0.3307	-0.1286
$\mathbf{X_4}$	0.3654	0.0113	-0.3307	1.0000	0.0061
$\mathbf{X_5}$	0.0154	0.4247	-0.1286	0.0061	1.0000
And the actual Σ					
$\mathbf{X_1}$	1.0000	0.0000	-0.2000	0.6000	0.0000
$\mathbf{X_2}$	0.0000	1.0000	-0.7000	0.0000	0.7000
$\mathbf{X_3}$	-0.2000	-0.7000	1.0000	-0.5000	-0.2000
$\mathbf{X_4}$	0.6000	0.0000	-0.5000	1.0000	0.0000
$\mathbf{X_5}$	0.0000	0.7000	-0.2000	0.0000	1.0000

Note: We see that the cross-term come close to the actual values, but they're not always accurate. The signs (which is the most important) for the non-zero cross terms are all as they should be. The cross-terms that should be zero are all close to zero.

6 Figures

6.1 Problem 1

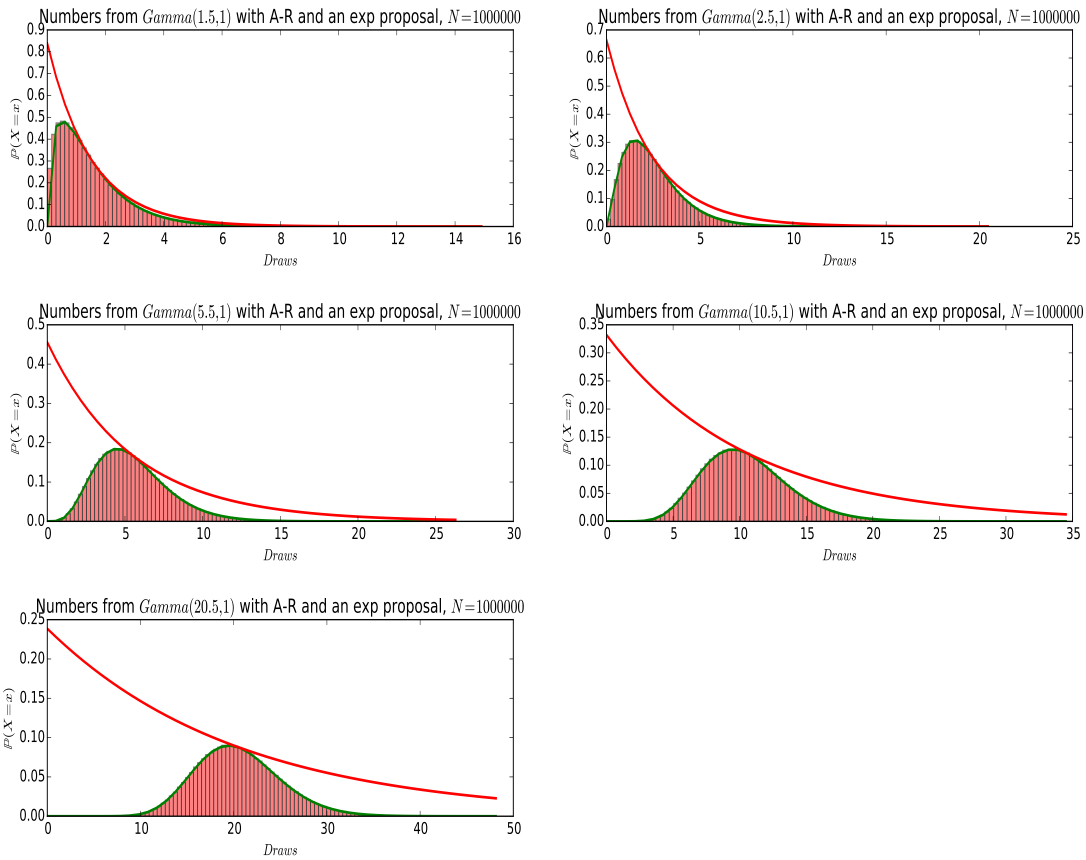
Figure 2: The lattice structure of the MWC we implemented



Note: We see that the distance between the parallel lines is the same in which ever direction we look. Furthermore the distance between these equidistant parallel lines are quite small.

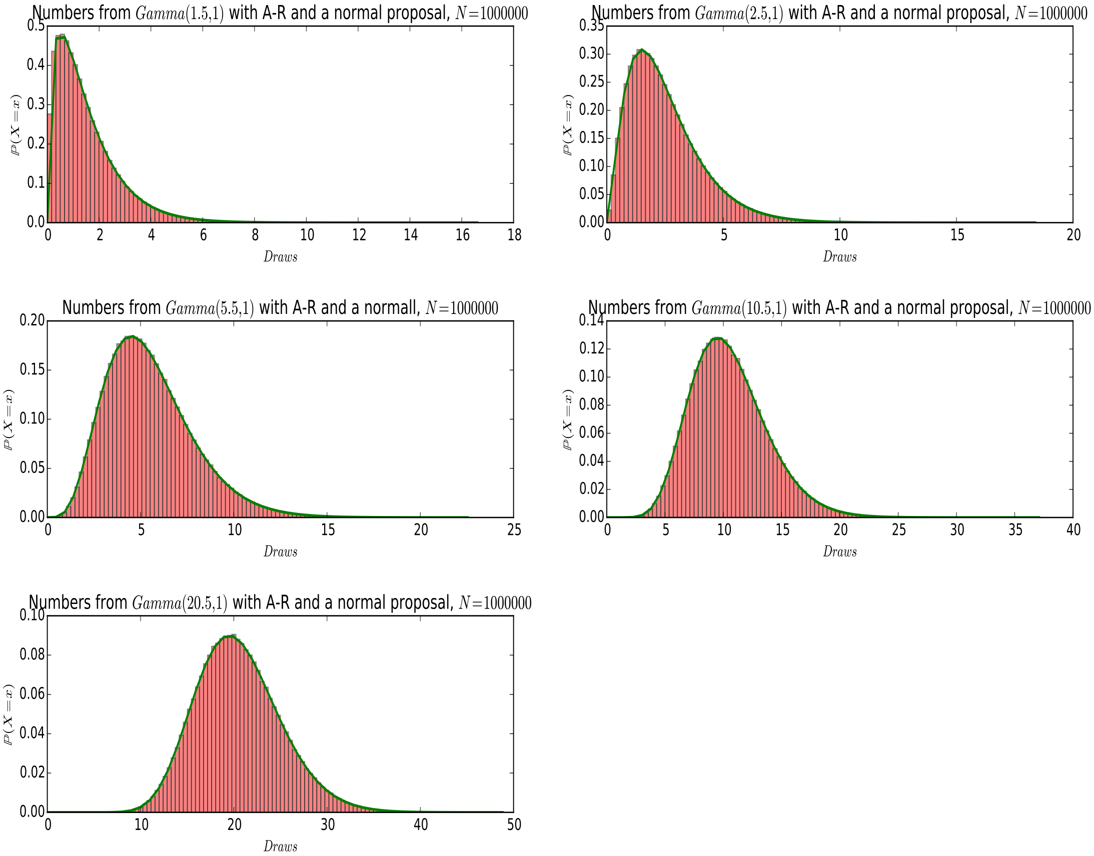
6.2 Problem 3

Figure 3: Histograms of the random numbers using A-R method with exponential proposal.



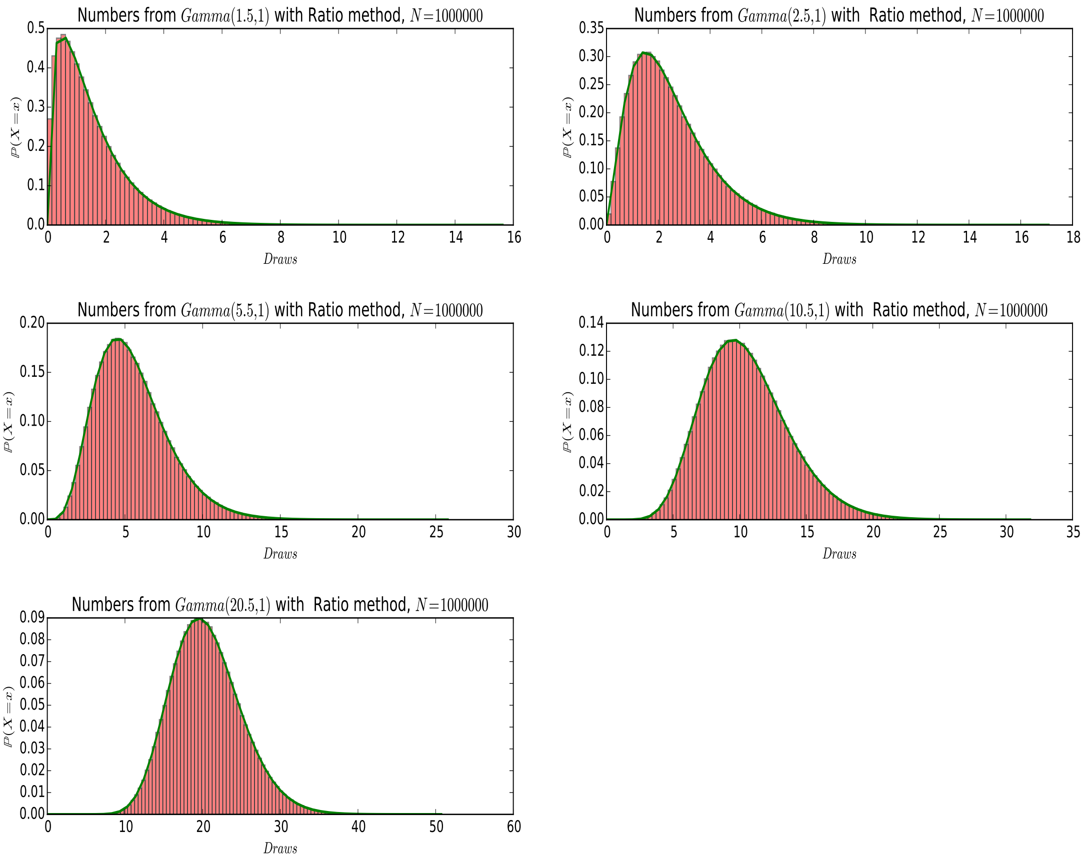
Note: The first thing we notice is that clearly from the histogram it is visible that the random numbers generated by the A-R method with the $g(y) \sim exp(\frac{1}{\alpha})$ follow the Gamma distribution (see pdf, the green line). Second we observe why it is that for large α the algorithm becomes so slow, and has such a bad acceptance ratio. The rejection probabilities are visibly larger for large shape parameters, because the gamma distribution resembles the exponential very little then.

Figure 4: Histograms of the random numbers using A-R method with normal proposal.



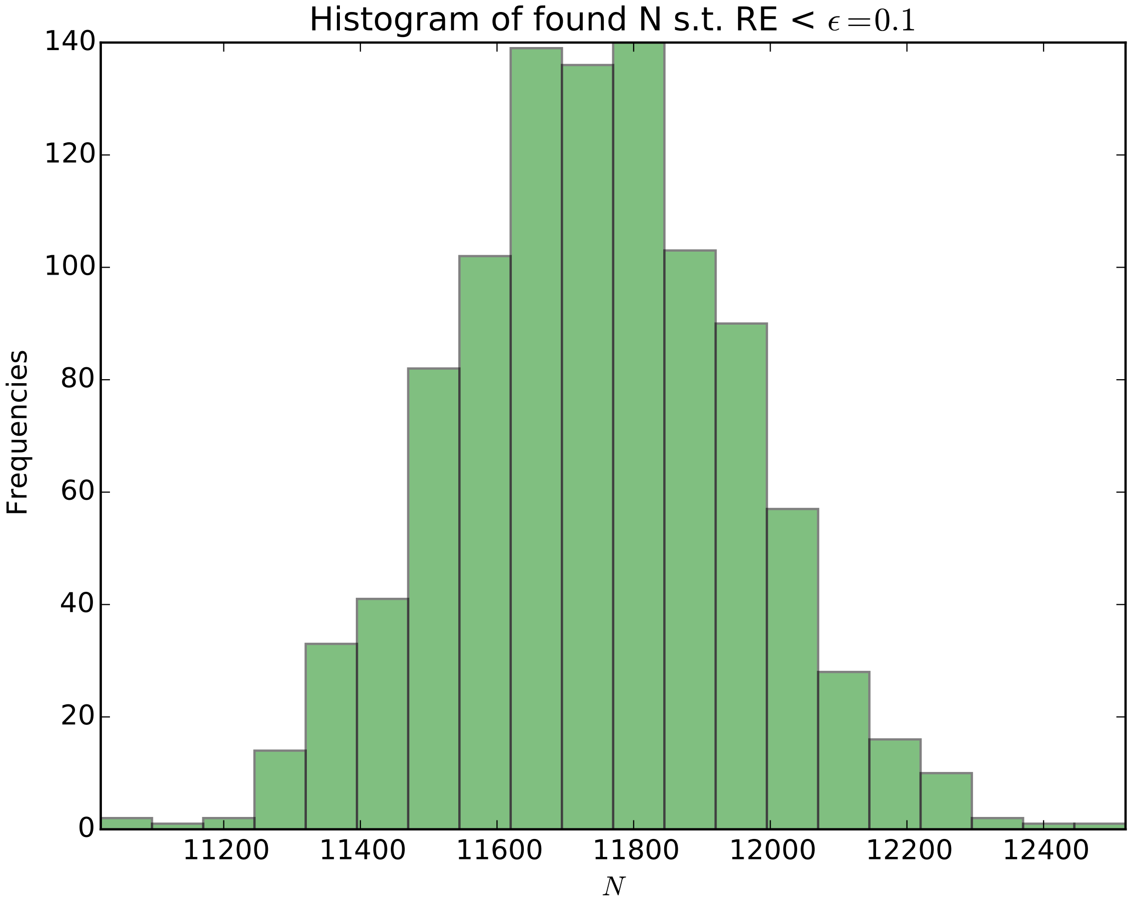
Note: It is visible that the random numbers generated by the A-R method with the standard normal proposal distribution follow the Gamma distribution.

Figure 5: Histograms of the random numbers using A-R method with normal proposal.



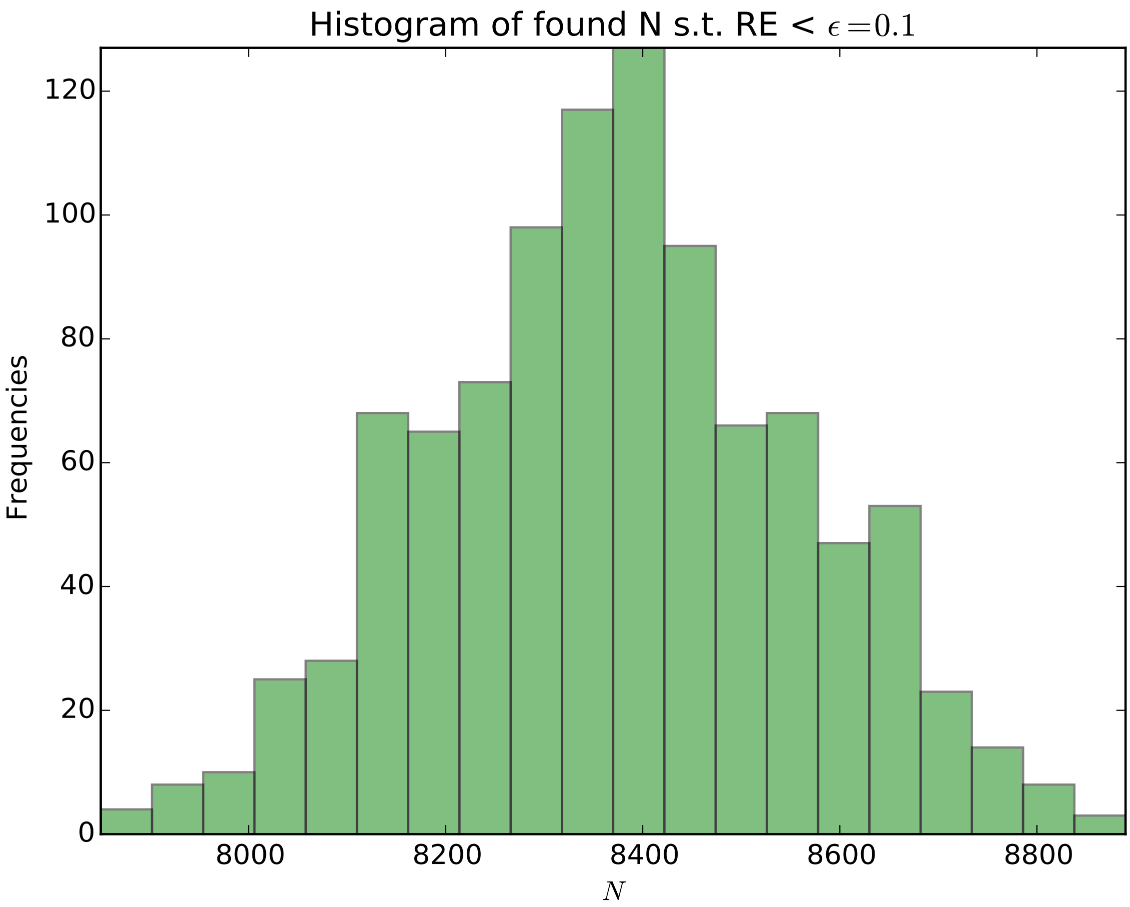
Note: It is visible that the random numbers generated by the ratio of uniforms method follow the Gamma distribution.

Figure 6: Histograms of found sample sizes with the stopping alg.



Note: We see that the found sizes are clearly centred around the ceiling average of 11742.

Figure 7: Histograms of found sample sizes with the stopping alg.



Note: We see that the found sizes are clearly centred around the ceiling average of 8372.

References

Rubinstein, R. Y. and Kroese, D. P. (2007). *Simulation and the Monte Carlo method 2nd.* John Wiley & Sons.