Capacity Results for Erasure Broadcast Channels with Intermittent Feedback

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***Abstract*—Recently, we showed that, rather surprisingly, the capacity region of the two-user erasure broadcast channel with global delayed channel state information (CSI) can be achieved with single-user delayed CSI only. More precisely, we assumed one receiver does not provide its channel state to the other two nodes (the other receiver and the transmitter), while the other receiver reveals its state globally with unit delay. In this work, we consider a more general setting in which feedback links are intermittent. To be precise, at any time instant, each receiver broadcasts its CSI, and this information either becomes available to the other two nodes or gets erased. For this setting, we develop a new set of outer bounds to capture the intermittent nature of the feedback links. These outer bounds depend on the probability that both feedback links are erased rather than the individual erasure probability of each feedback link. This result matches our earlier findings for the single-user delayed CSI scenario. We also provide a capacity-achieving recursive communication protocol for the scenario in which feedback links are fully correlated.**

***Index Terms*—Erasure broadcast channel, intermittent feedback, capacity region, delayed CSIT.**

I. Introduction

Feedback links are typically scarce, bandlimited, unreliable, and subject to security threats and jamming. These limitations are exasperated by the ever-increasing size of dynamic networks, the emergence of new applications such as Internet of Things and distributed computing, and the growing cyber-security threats. Thus, we ought to understand the behavior of wireless networks when feedback links are dropped and channel state information (CSI) becomes unavailable occasionally.

As a stepping stone, in this work, we consider such a scenario in the context of the canonical two-user erasure broadcast channels (BCs) with intermittent output[1] feedback. More precisely, we consider an erasure BC with a transmitter and two receivers. The channel model from the transmitter to the two receivers follows the standard erasure BC model. For the feedback model, at any time instant, each receiver i, $i = 1, 2$, broadcasts its CSI to the other two nodes, that is, whether or not the symbol sent by the transmitter successfully arrives. Then, with some probability, this broadcast of CSI is successful, and the transmitter and the other receiver will learn the CSI of receiver i with unit delay; otherwise the feedback signal is erased. We refer to this model as the two-user erasure BC with *intermittent feedback*. This model generalizes those in prior works which either assume all receivers can provide delayed state feedback [1]–[3], or only a single user provides its CSI [4]–[7]. In particular, in [7], we showed that, rather surprisingly, the capacity region of the two-user erasure BC with global delayed CSI can be achieved with single-user delayed CSI only.

Our contributions in this work are two-fold. First, we derive a new set of outer bounds the two-user erasure BC with intermittent feedback. The derivation has two stages. In the first stage, we create a modified BC in which forward links are fully correlated across users when both feedback links are erased. We show that the capacity region of this modified problem is the same as that of the original problem. In the second stage, we derive the outer bounds for this modified problem using an extremal entropy inequality for erasure links with delayed CSI. Interestingly, these outer bounds are governed by the probability of missing both feedback links. In other words, as long as one feedback link is active at any given time, the outer-bound region does not degrade compared to the one with global delayed CSI. This observation matches our earlier findings in [7].

We also show that under certain conditions these outer bounds can be achieved. One such scenario is when the feedback links from the two receivers are fully correlated, that is, they are either both available or both erased. We propose a recursive transmission strategy where the first iteration has three phases resembling the three phase communication protocol of the global feedback case [1]. After these three phases, we create recycled bits that we feed to the same transmission protocol as the recursive step.

It is worth comparing our results to another line of work in which the availability of CSI alternates between various states [8]–[10]. However, these results assume at each time the CSI availability structure is known to the transmitter, whereas in our work, the transmitter does not know whether or not the CSI of the current transmission will be available since the feedback links are erased randomly at each time.

The rest of the paper is organized as follows. We describe the problem formulation in Section II. We state our main contributions in Section III. The proofs are presented in Sections IV and V. Section VI concludes the paper.

[1]For erasure BCs, channel output feedback is equivalent to CSI feedback.

II. Problem Formulation

We consider the two-user erasure broadcast channel in which a single-antenna transmitter, Tx, wishes to transmit two independent messages, W_1 and W_2, to two single-antenna receiving terminals Rx_1 and Rx_2, respectively, over n channel uses. Each message, W_i, is uniformly distributed over $\{1, 2, \ldots, 2^{nR_i}\}$, for $i = 1, 2$. At time instant t, the messages are mapped to channel input $X[t] \in \mathbb{F}_2$, the binary field, and the corresponding received signals at Rx_1 and Rx_2 are

$$Y_1[t] = S_1[t]X[t] \quad \text{and} \quad Y_2[t] = S_2[t]X[t], \tag{1}$$

respectively, where $\{S_i[t]\}$ denotes the Bernoulli $(1-\delta_i)$ process that governs the erasure at Rx_i, and it is distributed i.i.d. over time. When $S_i[t] = 1$, Rx_i receives $X[t]$ noiselessly; and when $S_i[t] = 0$, it receives an erasure. We also assume $\delta_{12} = P\{S_1[t] = 0, S_2[t] = 0\}$.

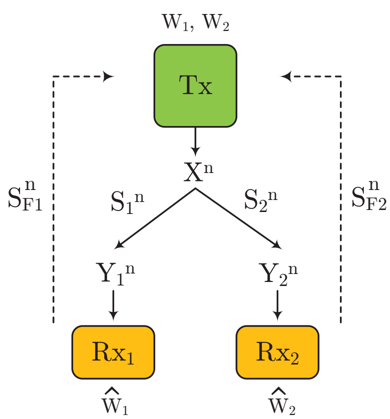

Fig. 1. Two-user erasure Broadcast Channel with intermittent feedback.

We further assume that at time instant t, Rx_i broadcasts $S_i[t]$, and the successful delivery of this information is governed by the Bernoulli $(1-\delta_{Fi})$ process $\{S_{Fi}[t]\}$ that is distributed i.i.d. over time, $i = 1, 2$. More precisely, if $S_{Fi}[t] = 1$, then the transmitter and the other receiver, $\text{Rx}_{\bar{i}}$, will learn $S_i[t]$ with unit delay, $\bar{i} = 3 - i$; otherwise the feedback signal is erased. See Fig. 1 for illustration. We also assume $\delta_{FF} = P\{S_{F1}[t] = 0, S_{F2}[t] = 0\}$. We also assume that the forward channels are distributed independently from the feedback links.

The constraint imposed at the encoding function $f_t(.)$ at time index t is

$$\begin{aligned} &X[t] \\ &= f_t\left(W_1, W_2, S_{F1}^{t-1}, S_{F2}^{t-1}, \{S_{F1}S_1\}^{t-1}, \{S_{F2}S_2\}^{t-1}\right), \end{aligned} \tag{2}$$

where

$$\begin{aligned} S_{Fi}^{t-1} &= (S_{Fi}[1], \ldots, S_{Fi}[t-1]), \\ \{S_{Fi}S_i\}^{t-1} &= (S_{Fi}[1]S_i[1], \ldots, S_{Fi}[t-1]S_i[t-1]). \end{aligned}$$

We also set $S[t] = (S_1[t], S_2[t])$ and $S_F[t] = (S_{F1}[t], S_{F2}[t])$.

Each receiver Rx_i, $i = 1, 2$, uses a decoding function $\varphi_{i,n}\left(Y_i^n, S_i^n, S_{Fi}^n, \{S_{F\bar{i}}S_{\bar{i}}\}^n\right)$ to get an estimate $\widehat{W}_i$ of W_i. An error occurs whenever $\widehat{W}_i \neq W_i$. The average probability of error is given by

$$\lambda_{i,n} = \mathbb{E}[P(\widehat{W}_i \neq W_i)], \tag{3}$$

where the expectation is taken with respect to the random choice of the transmitted messages.

We say that a rate pair (R_1, R_2) is achievable, if there exists a block encoder at the transmitter, and a block decoder at each receiver, such that $\lambda_{i,n}$ goes to zero as the block length n goes to infinity. The capacity region, $\mathcal{C}$, is the closure of the set of the achievable rate pairs.

III. Statement of the Main Results

The following theorem establishes an outer bound on the capacity region of the two-user erasure BC with intermittent feedback.

Theorem 3.1: The capacity region, $\mathcal{C}$, of the two-user erasure BC with intermittent feedback as described in Section II is included in:

$$\mathcal{C}_{\text{out}} = \left\{ (R_1, R_2) \left| \begin{array}{l} R_1 + \beta_2 R_2 \leq \beta_2 (1-\delta_2) \\ \beta_1 R_1 + R_2 \leq \beta_1 (1-\delta_1) \end{array} \right. \right\} \tag{4}$$

where for $i = 1, 2$,

$$\beta_i = \frac{\delta_{FF}\left(1 - \min_j \delta_j\right) + (1-\delta_{FF})(1-\delta_{12})}{(1-\delta_i)}. \tag{5}$$

We derive the outer bounds in two steps. First, we create a modified BC in which forward links are correlated across users when both feedback links are erased. We show that the capacity region of this modified BC is the same as that of the original BC. Next, we derive the outer bounds for this modified problem which in turn serve as outer bounds on $\mathcal{C}$. The details are provided in Section IV.

An interesting observation is that δ_{F1} and δ_{F2} do not appear in these outer bounds, but rather these outer bounds are governed by the probability of missing both feedback links, that is, δ_{FF}. In other words, as long as one feedback link is active at any given time, the outer-bound region in Theorem 3.1 matches the one with global delayed CSI. This observation is in agreement with our earlier result [7] where we showed that the capacity region of the two-user erasure broadcast channel with global delayed CSI can be achieved with single-user delayed CSI only.

Next, we show that under certain conditions, the outer-bound region $\mathcal{C}_{\text{out}}$ can be achieved.

Theorem 3.2: The outer-bound region, $\mathcal{C}_{\text{out}}$, on the capacity region of the two-user erasure BC with intermittent feedback as given in Theorem 3.1, equals the capacity region, $\mathcal{C}$, when:

1) $\delta_{F1}\delta_{F2} = 0$, or
2) $\delta_1 = \delta_2$, $\delta_{F1} = \delta_{F2}$, and $\Pr\left(S_{F1}[t] \neq S_{F2}[t]\right) = 0$.

To prove the achievability, we propose a transmission strategy that has a recursive form. The first iteration has three phases which resemble the three phase communication of two-user BC with global feedback [1]–[3]. After these three phases, we create recycled bits that we feed to the same transmission protocol as the recursive step. We show that the achievable rate matches the outer bounds of Theorem 3.1. The details are provided in Section V.

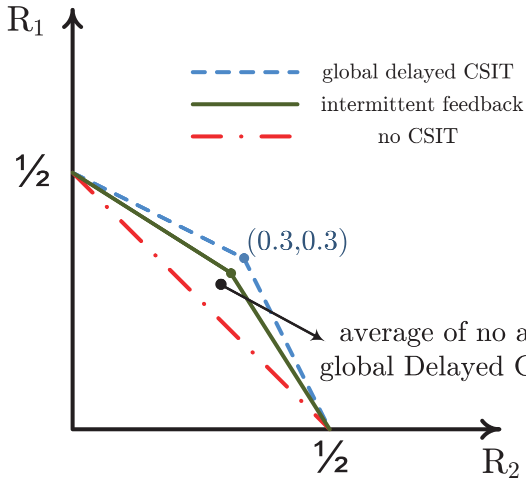

Fig. 2. The maximum achievable sum-rate of the erasure BC with intermittent feedback for parameters given in (6) is $5/9$ which is greater than the average of the no feedback and the global feedback scenarios.

The above two theorems demonstrate how the capacity region degrades as the quality of feedback channel diminishes. However, an interesting observation is that the capacity region of the two-user erasure BC with intermittent feedback is larger than the average of the one with no feedback and the one with global feedback. To clarify this, consider an example in which

$$\begin{aligned} &\delta_1 = \delta_2 = 0.5, \\ &\delta_{12} = 0.25, \\ &\delta_{F1} = \delta_{F2} = \delta_{FF} = 0.5. \end{aligned} \tag{6}$$

Note that this example falls under Case 2 of Theorem 3.2, and hence, the capacity region is characterized by the outer-bound region in Theorem 3.1. For these parameters, the maximum achievable sum-rate without feedback is 0.5, while that with global delayed CSIT is 0.6 as depicted in Fig. 2. Moreover, half of the times at least one feedback link is active, that is, $\delta_{FF} = 0.5$. Averaging the maximum achievable sum-rates of the no feedback and the global feedback scenarios gives us 0.55. Interestingly, the maximum achievable sum-rate of this problem with intermittent feedback is $5/9$ which is greater than 0.55, the sum-rate achieved by time sharing (see Fig. 2 for illustration). The intuition is as follows. For the no feedback case with $\delta_1 = \delta_2$, both receivers can decode both messages since the two receivers are stochastically equivalent. However, as we will show in Section V, our recursive transmission protocol efficiently exploits the available feedback, and prevents the suboptimal decoding of messages by both receivers in the naive time-sharing scheme.

IV. Proof of Theorem 3.1: Outer Bounds

We derive the outer bounds in two steps. First, we introduce a modified BC whose capacity region is the same as that of the BC introduced in Section II. Thus, any outer bound on the capacity region of this modified channel will serve as an outer bound on $\mathcal{C}$. Then, we obtain the outer bounds for this modified BC.

Step 1: Consider a two-user erasure BC as defined in Section II with only one difference in the joint distribution of $(S_1[t], S_2[t], S_{F1}[t], S_{F2}[t])$ as described below and with more details in Appendix A. When both feedback links are erased, *i.e.* $S_{F1}[t] = 0$ and $S_{F2}[t] = 0$, then the forward channels are fully correlated, see Table I in Appendix A. More precisely, for $\delta_1 \geq \delta_2$ and when $S_{F1}[t] = S_{F2}[t] = 0$, we change the state sequence at Rx_2 such that $S_1[t] = 1$ implies $S_2[t] = 1$ while keeping $\Pr(S_2[t] = 1) = 1 - \delta_2$. Note that the links are still distributed independently over time.

Claim 4.1: The capacity region of the modified two-user erasure BC with intermittent feedback as described above is the same as that of the two-user erasure BC with intermittent feedback described in Section II.

Proof: The idea is to show that any scheme that achieves rates (R_1, R_2) in the original problem can be used in the modified BC described above and achieves the same rates, and vice versa.

We know that the capacity region of a BC is only a function of the marginal multi-letter distributions at the receivers [11]. Suppose in the original problem rates (R_1, R_2) are achievable, and W_1 and W_2 are encoded as X^n. From (2), we know that $X[t]$ only depends on

$$\left(W_1, W_2, S_{F1}^{t-1}, S_{F2}^{t-1}, \{S_{F1}S_1\}^{t-1}, \{S_{F2}S_2\}^{t-1}\right) \tag{7}$$

which is the same in both BCs since we modify the states only when they are not fed back. Suppose we use X^n as the input for the modified BC. The available information at Rx_i is given by $(Y_i^n, S_i^n, S_{Fi}^n, \{S_{F\bar{i}}S_{\bar{i}}\}^n)$ which remains identical for both problems. In particular, the links are still distributed independently over time, and in Appendix A, we prove that the joint distribution of $(S_i[t], S_{Fi}[t], S_{F\bar{i}}[t]S_{\bar{i}}[t])$ remains unchanged for the modified BC, $i = 1, 2$. As a result, the marginal multi-letter distributions are the same in both cases, and rates (R_1, R_2) are achievable in the modified BC as well. Similarly, we can show that any scheme that achieves rates (R_1, R_2) in the modified BC works in the original problem. This completes the proof. ■

Claim 4.1 implies that any outer bound on the capacity region of the modified BC serves as an outer bound on that of the problem described in Section II. In Step 2, we derive the outer bounds on the capacity region of the modified BC.

Step 2: Suppose rate-tuple (R_1, R_2) is achievable in the modified BC. For

$$\beta_2 = \frac{\delta_{FF}\left(1 - \min_j \delta_j\right) + \left(1 - \delta_{FF}\right)\left(1 - \delta_{12}\right)}{\left(1 - \delta_2\right)}, \tag{8}$$

we have

$$n\left(R_1 + \beta_2 R_2\right) = H\left(W_1\right) + \beta_2 H\left(W_2\right)$$

$$= H\left(W_1|W_2,S^n,S_F^n\right)+\beta_2 H\left(W_2|S^n,S_F^n\right)$$
$$\stackrel{\text{Fano}}{\leq} I\left(W_1;Y_1^n|W_2,S^n,S_F^n\right)+\beta_2 I\left(W_2;Y_2^n|S^n,S_F^n\right)+n\epsilon_n$$
$$= H\left(Y_1^n|W_2,S^n,S_F^n\right)-\underbrace{H\left(Y_1^n|W_1,W_2,S^n,S_F^n\right)}_{=\,0}$$
$$+\beta_2 H\left(Y_2^n|S^n,S_F^n\right)-\beta_2 H\left(Y_2^n|W_2,S^n,S_F^n\right)+n\epsilon_n$$
$$\stackrel{(a)}{\leq} \beta_2 H\left(Y_2^n|S^n,S_F^n\right)+n\epsilon_n$$
$$\stackrel{(b)}{\leq} n\beta_2(1-\delta_2)+n\epsilon_n, \tag{9}$$

where $\epsilon_n \to 0$ when $n \to \infty$, (a) follows from Claim 4.2 below, and (b) holds since $S_2[t]$ is Bernoulli $(1-\delta_2)$.

Claim 4.2: For the two-user erasure BC with intermittent feedback as described in Section II, and for any input distribution, we have

$$H\left(Y_1^n|W_2,S^n,S_F^n\right)-\beta_2 H\left(Y_2^n|W_2,S^n,S_F^n\right)\leq 0. \tag{10}$$

Proof:

$$H\left(Y_2^n|W_2,S^n,S_F^n\right)$$
$$=\sum_{t=1}^{n} H\left(Y_2[t]|Y_2^{t-1},W_2,S^n,S_F^n\right)$$
$$\stackrel{(a)}{=}\sum_{t=1}^{n} H\left(Y_2[t]|Y_2^{t-1},W_2,S^t,S_F^t\right)$$
$$\stackrel{(b)}{=}\sum_{t=1}^{n}(1-\delta_2)H\left(X[t]|Y_2^{t-1},W_2,S_1[t],S_2[t]=1,S^{t-1},S_F^t\right)$$
$$\stackrel{(c)}{=}\sum_{t=1}^{n}(1-\delta_2)H\left(X[t]|Y_2^{t-1},W_2,S^t,S_F^t\right)$$
$$\stackrel{(d)}{\geq}\sum_{t=1}^{n}(1-\delta_2)H\left(X[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^t,S_F^t\right)$$
$$\stackrel{(e)}{=}\sum_{t=1}^{n}\frac{(1-\delta_2)H\left(Y_1[t],Y_2[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^t,S_F^t\right)}{\delta_{FF}\left(1-\min_j\delta_j\right)+\left(1-\delta_{FF}\right)\left(1-\delta_{12}\right)}$$
$$\stackrel{(8)}{=}\sum_{t=1}^{n}\frac{1}{\beta_2}H\left(Y_1[t],Y_2[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^t,S_F^t\right)$$
$$\stackrel{(f)}{=}\sum_{t=1}^{n}\frac{1}{\beta_2}H\left(Y_1[t],Y_2[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^n,S_F^n\right)$$
$$=\sum_{t=1}^{n}\frac{1}{\beta_2}H\left(Y_1^n,Y_2^n|W_2,S^n,S_F^n\right)$$
$$\stackrel{(g)}{\geq}\sum_{t=1}^{n}\frac{1}{\beta_2}H\left(Y_1^n|W_2,S^n,S_F^n\right), \tag{11}$$

where (a) follows since $X[t]$ is independent of future channel realizations and the channel gains are distributed as i.i.d. random variables over time, (b) holds since $\Pr\left(S_2[t]=1\right)=(1-\delta_2)$, (c) follows from the same logic as step (a), (d) holds since conditioning reduces entropy, (e) follows from the fact that $X[t]$ is independent from $S[t]$ and $S_F[t]$, and applying the total probability law as below

$$\Pr\left(\{S_1[t]=S_2[t]=0\}^c\right)$$
$$=1-\Pr\left(S_1[t]=S_2[t]=0\right)$$
$$=1-\delta_{FF}\Pr\left(S_1[t]=S_2[t]=0|\{S_{F1}[t]=S_{F2}[t]=0\}\right)$$
$$-\left(1-\delta_{FF}\right)\Pr\left(S_1[t]=S_2[t]=0|\{S_{F1}[t]=S_{F2}[t]=0\}^c\right)$$
$$=1-\delta_{FF}\min_j\delta_j-(1-\delta_{FF})\delta_{12}, \tag{12}$$

(f) follows from the same logic as step (a), and (g) follows from the non-negativity of differential entropy. ■

V. Proof of Theorem 3.2: Transmission Protocol

Case 1: If $\delta_{F1}=0$ and $\delta_{F2}=0$, then the capacity region of the two-user BC with intermittent FB collapses to that of the no CSIT assumption [12]. On the other hand, if one of the δ_{Fi}'s is non-zero, then the problem becomes equivalent to a two-user erasure BC with one-sided feedback ("DN" assumption) for which we recently presented the capacity region in [7].

Case 2: $\delta_{F1}=\delta_{F2}\stackrel{\triangle}{=}\delta_F$ and $\Pr\left(S_{F1}[t]\neq S_{F2}[t]\right)=0$. This also implies that $\delta_{FF}=\delta_F$. Moreover, we have $\delta_1=\delta_2\stackrel{\triangle}{=}\delta$.

In this case, we can simplify some expressions as follows.

$$\beta_i=\frac{(1-\delta_{12})-\delta_F(\delta-\delta_{12})}{1-\delta}\stackrel{\triangle}{=}\frac{A}{1-\delta}. \tag{13}$$

Then, from Theorem 3.1, we obtain the maximum sum-rate corner point:

$$R_1=R_2=\frac{A}{1+\frac{A}{(1-\delta)}}. \tag{14}$$

Transmission Protocol: The transmission strategy has a recursive format. We start with m bits for each receiver. We start by sending the bits for each receiver and based on the available feedback, we create two sets of recycled equations. For the first set, feedback bits were available whereas for the second set, no feedback was available. We then send the XOR of the bits in first set. Whatever is left will be fed as the input to the transmission protocol again.

Phase 1: The transmitter creates

$$\frac{m}{1-\delta_{12}} \tag{15}$$

linearly independent equations of the m bits for receiver 1 and sends them out. For $(1-\delta_F)$ fraction of the time, feedback is available and

$$(1-\delta_F)\frac{\delta-\delta_{12}}{1-\delta_{12}}m \tag{16}$$

bits are available at Rx_2 and needed at Rx_1. Denote these bits by $v_{1|2}$. Moreover, for δ_F fraction of the time, no feedback is available. But due to the statistics of the channel, we know

$$\delta_F\frac{\delta-\delta_{12}}{1-\delta_{12}}m=\frac{\delta_F}{1-\delta_F}\left|v_{1|2}\right| \tag{17}$$

bits are available at Rx_2 and needed at Rx_1. The transmitter creates the same number of linearly independent equations as (17) from the transmitted bits $X[t]$ when feedback links were not available. Denote these equations by $v_{1|2}^{\text{noFB}}$.

Remark 5.1: To keep the description of the protocol simple, we use the expected value of the number of bits in different states, *e.g.*, (16) and (17). A more precise statement would use a concentration theorem result such as the Bernstein inequality to show the omitted terms do not affect the overall result and the achievable rates. Moreover, when talking about the number of bits or equations, we are limited to integer numbers. If a ratio is not an integer number, we can use $\lceil \cdot \rceil$, the ceiling function, and since at the end we take the limit for $m \rightarrow \infty$, the results remain unchanged.

Phase 2: This phase is similar to the previous one. The transmitter creates

$$\frac{m}{1-\delta_{12}} \tag{18}$$

linearly independent equations of the m bits for receiver 2 and sends them out. For $(1-\delta_F)$ fraction of the time, feedback is available and

$$(1-\delta_F)\frac{\delta-\delta_{12}}{1-\delta_{12}}m \tag{19}$$

bits are available at Rx_1 and needed at Rx_2. Denote these bits by $v_{2|1}$. Moreover, for δ_F fraction of the time, no feedback is available. But due to the statistics of the channel, we know

$$\delta_F\frac{\delta-\delta_{12}}{1-\delta_{12}}m = \frac{\delta_F}{1-\delta_F}\left|v_{2|1}\right| \tag{20}$$

bits are available at Rx_1 and needed at Rx_2. The transmitter creates the same number of linearly independent equations as (20) from the transmitted signal when feedback links were not available. Denote these equations by $v_{2|1}^{\mathrm{noFB}}$.

Phase 3: In this phase, the transmitter encodes $v_{1|2}$ and $v_{2|1}$ using erasure codes of rate $(1-\delta)$. Note that $\left|v_{1|2}\right| = \left|v_{2|1}\right|$. The transmitter creates the XOR of the encoded bits for Rx_1 and Rx_2 and sends them out. This Phase takes

$$\frac{\left|v_{1|2}\right|}{(1-\delta)} = \frac{(1-\delta_F)(\delta-\delta_{12})}{(1-\delta)(1-\delta_{12})}m. \tag{21}$$

time instants.

Recursive step: Consider $v_{1|2}^{\mathrm{noFB}}$ and $v_{2|1}^{\mathrm{noFB}}$ as the input messages in new Phase 1 and Phase 2, respectively, and repeat the communication strategy.

Termination: To simplify the protocol, when the remaining bits in $v_{1|2}^{\mathrm{noFB}}$ and $v_{2|1}^{\mathrm{noFB}}$ is $o\left(m^{1/3}\right)$ we stop the recursion and send the remaining bits using time sharing between two erasure codes. Note that while we used $o\left(m^{1/3}\right)$ as our termination threshold, any threshold with vanishing (as $m \rightarrow \infty$) impact would work.

Decoding: Our transmission protocol is built upon that of global feedback [1]–[3] with the addition of the recursive step. Decoding starts with the last recursive step and goes backwards to the first iteration. A subtle point worth noting is that in each iteration the transmitter creates linearly independent equations similar to $v_{1|2}^{\mathrm{noFB}}$ and $v_{2|1}^{\mathrm{noFB}}$, and we need to guarantee that this task is feasible as we have many iterations. It is easy to verify that the geometric sum of the number of linearly independent equations created for each receiver during all iterations is smaller than the total number of unknown bits we start with, *i.e.* m. Thus, the transmitter is able to carry out its task as needed.

Achievable rates:

$$R_{\mathrm{SUM}} = \frac{2m}{\frac{2m}{1-\delta_{12}} + \frac{(1-\delta_F)(\delta-\delta_{12})}{(1-\delta)(1-\delta_{12})}m + \frac{2\left|\bar{v}_{2|1}^{\mathrm{noFB}}\right|}{R_{\mathrm{SUM}}}}. \tag{22}$$

where

$$\left|\bar{v}_{2|1}^{\mathrm{noFB}}\right| = \delta_F\frac{\delta-\delta_{12}}{1-\delta_{12}}m. \tag{23}$$

Solving the equation, we obtain

$$R_{\mathrm{SUM}} = R_1 + R_2 = \frac{2A(1-\delta)}{(1-\delta+A)}, \tag{24}$$

where R_1 and R_2 are given in (14). This completes the proof of Theorem 3.2.

VI. Conclusion

In this work, we developed new outer bounds on the capacity region of two-user erasure BCs with intermittent feedback. We also showed that these outer bounds are achievable under certain assumptions on channel parameters. The next step is to characterize the capacity region for all channel parameters. We conjecture the outer bounds are tight, and the region given in Theorem 3.1 is in fact the capacity region.

Appendix

A. Joint PMF for the modified BC

For the modified BC, when both feedback links are erased, *i.e.* $S_{F1}[t] = 0$ and $S_{F2}[t] = 0$, then the forward channels are fully correlated. More precisely, for $\delta_1 \geq \delta_2$, the joint probability mass function (PMF) for the modified BC is given in Table I.

From Table I, we can evaluate all other joint PMFs for the forward channel and the feedback links. For instance, as described below, we can use Table I to show that $(S_i[t], S_{Fi}[t], S_{F\bar{i}}[t]S_{\bar{i}}[t])$ remains unchanged with the modified BC. Given the correlation, for receiver 1, we need to verify only

$$P\left(S_1[t]=0, S_{F1}[t]=0, S_{F2}[t]S_2[t]=0\right), \tag{25}$$

and

$$P\left(S_1[t]=1, S_{F1}[t]=0, S_{F2}[t]S_2[t]=0\right), \tag{26}$$

are the same in both problems. Moreover, for receiver 2, we need to verify only

$$P\left(S_2[t]=0, S_{F2}[t]=0, S_{F1}[t]S_1[t]=0\right), \tag{27}$$

and

$$P\left(S_2[t]=1, S_{F2}[t]=0, S_{F1}[t]S_1[t]=0\right), \tag{28}$$

are the same in both problems.

For (25), the event corresponds to cases 1, 2, and 5 in Table I. It is easy to verify that the sum of the probabilities in the aforementioned cases is the same for both BCs. Then, for (26), the event corresponds to cases 9, 10, and 13 in Table I. Again, it is easy to verify that the sum of the probabilities in these cases is the same for both BCs.

TABLE I

JOINT PMF FOR THE MODIFIED AND THE ORIGINAL BCS

ID	$S_1[t]$	$S_2[t]$	$S_{F1}[t]$	$S_{F2}[t]$	modified BC	original BC
(1)	0	0	0	0	$\delta_2\delta_{FF}$	$\delta_{12}\delta_{FF}$
(2)	0	0	0	1	$\delta_{12}(\delta_{F1}-\delta_{FF})$	$\delta_{12}(\delta_{F1}-\delta_{FF})$
(3)	0	0	1	0	$\delta_{12}(\delta_{F2}-\delta_{FF})$	$\delta_{12}(\delta_{F2}-\delta_{FF})$
(4)	0	0	1	1	$\delta_{12}(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$	$\delta_{12}(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$
(5)	0	1	0	0	$(\delta_1-\delta_2)\delta_{FF}$	$(\delta_1-\delta_{12})\delta_{FF}$
(6)	0	1	0	1	$(\delta_1-\delta_{12})(\delta_{F1}-\delta_{FF})$	$(\delta_1-\delta_{12})(\delta_{F1}-\delta_{FF})$
(7)	0	1	1	0	$(\delta_1-\delta_{12})(\delta_{F2}-\delta_{FF})$	$(\delta_1-\delta_{12})(\delta_{F2}-\delta_{FF})$
(8)	0	1	1	1	$(\delta_1-\delta_{12})(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$	$(\delta_1-\delta_{12})(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$
(9)	1	0	0	0	0	$(\delta_2-\delta_{12})\delta_{FF}$
(10)	1	0	0	1	$(\delta_2-\delta_{12})(\delta_{F1}-\delta_{FF})$	$(\delta_2-\delta_{12})(\delta_{F1}-\delta_{FF})$
(11)	1	0	1	0	$(\delta_2-\delta_{12})(\delta_{F2}-\delta_{FF})$	$(\delta_2-\delta_{12})(\delta_{F2}-\delta_{FF})$
(12)	1	0	1	1	$(\delta_2-\delta_{12})(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$	$(\delta_2-\delta_{12})(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$
(13)	1	1	0	0	$(1-\delta_1)\delta_{FF}$	$(1-\delta_1-\delta_2+\delta_{12})\delta_{FF}$
(14)	1	1	0	1	$(1-\delta_1-\delta_2+\delta_{12})(\delta_{F1}-\delta_{FF})$	$(1-\delta_1-\delta_2+\delta_{12})(\delta_{F1}-\delta_{FF})$
(15)	1	1	1	0	$(1-\delta_1-\delta_2+\delta_{12})(\delta_{F2}-\delta_{FF})$	$(1-\delta_1-\delta_2+\delta_{12})(\delta_{F2}-\delta_{FF})$
(16)	1	1	1	1	$(1-\delta_1-\delta_2+\delta_{12})(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$	$(1-\delta_1-\delta_2+\delta_{12})(1-\delta_{F1}-\delta_{F2}+\delta_{FF})$

Similarly, for (27), the event corresponds to cases 1, 3, and 9 in Table I. Finally, for (28), the event corresponds to cases 5, 7, and 13 in Table I. It is easy to verify that the sum of the probabilities in these cases is the same for both BCs.

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