Fields and Waves Project

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1.1

$$
\begin{aligned}
k_0 &= \omega\sqrt{\mu_0\varepsilon_0}, \\
k_1 &= n_1 k_0, \qquad k_2 = n_2 k_0, \\
k_{1x} &= k_1 \sin\theta_1, \qquad k_{1z} = k_1\cos\theta_1, \qquad \eta_1 = \frac{\eta_0}{n_1},\ \eta_2 = \frac{\eta_0}{n_2}, \qquad \eta_0 \simeq 377\,\Omega. \\
k_{2x} &= k_2 \sin\theta_2, \qquad k_{2z} = k_2\cos\theta_2, \\
&\text{(Snell)} \ : \quad n_1\sin\theta_1 = n_2\sin\theta_2,
\end{aligned}
$$

(Throughout, the common factor $e^{j\omega t}$ is suppressed.)
Electric–field phasors

$$
\boxed{
\begin{aligned}
\mathbf{E}_i(\mathbf{r}) &= E_0\big(\cos\theta_1\,\hat{\mathbf{x}} - \sin\theta_1\,\hat{\mathbf{z}}\big)e^{-j\,(k_{1x}x + k_{1z}z)}, \qquad z<0, \\
\mathbf{E}_r(\mathbf{r}) &= \Gamma_{\text{TM}}\,E_0\big(\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}\big)e^{-j\,(k_{1x}x - k_{1z}z)}, \qquad z<0, \\
\mathbf{E}_t(\mathbf{r}) &= T_{\text{TM}}\,E_0\big(\cos\theta_2\,\hat{\mathbf{x}} - \sin\theta_2\,\hat{\mathbf{z}}\big)e^{-j\,(k_{2x}x + k_{2z}z)}, \qquad z>0.
\end{aligned}
}
$$

Reflection transmission coefficients (TM / p–polarization)

$$
\boxed{\Gamma_{\text{TM}} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1}}, \qquad \boxed{T_{\text{TM}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1}}.
$$

Magnetic–field phasors

$$
\boxed{
\begin{aligned}
\mathbf{H}_i(\mathbf{r}) &= \frac{E_0}{\eta_1}\,\hat{\mathbf{y}}\;e^{-j\,(k_{1x}x + k_{1z}z)}, \\
\mathbf{H}_r(\mathbf{r}) &= -\frac{\Gamma_{\text{TM}}E_0}{\eta_1}\,\hat{\mathbf{y}}\;e^{-j\,(k_{1x}x - k_{1z}z)}, \\
\mathbf{H}_t(\mathbf{r}) &= \frac{T_{\text{TM}}E_0}{\eta_2}\,\hat{\mathbf{y}}\;e^{-j\,(k_{2x}x + k_{2z}z)}.
\end{aligned}
}
$$

All fields are written in phasor form and satisfy the boundary conditions at $z = 0$.

1.2

$$
\begin{aligned}
k_0 &= \omega\sqrt{\mu_0\varepsilon_0}, \\
k_1 &= n_1 k_0, \qquad k_2 = n_2 k_0, \\
k_{2x} &= k_2 \sin\theta_2, \quad k_{2z} = k_2\cos\theta_2, \qquad \eta_1 = \frac{\eta_0}{n_1},\ \eta_2 = \frac{\eta_0}{n_2}, \qquad \eta_0 \simeq 377\ \Omega. \\
k_{1x} &= k_1 \sin\theta_1, \quad k_{1z} = k_1\cos\theta_1,
\end{aligned}
$$

$$
\text{(Snell)} : \ n_2\sin\theta_2 = n_1\sin\theta_1.
$$

(The common time factor $e^{j\omega t}$ has been suppressed.)

$$
\boxed{
\begin{aligned}
\mathbf{E}_i(\mathbf{r}) &= E_0\big(-\cos\theta_2\,\hat{\mathbf{x}} - \sin\theta_2\,\hat{\mathbf{z}}\big)\,e^{-j\,(k_{2x}x - k_{2z}z)}, \qquad z>0, \\
\mathbf{E}_r(\mathbf{r}) &= \Gamma_{\text{TM}}\,E_0\big(\cos\theta_2\,\hat{\mathbf{x}} - \sin\theta_2\,\hat{\mathbf{z}}\big)\,e^{-j\,(k_{2x}x + k_{2z}z)}, \quad z>0, \\
\mathbf{E}_t(\mathbf{r}) &= T_{\text{TM}}\,E_0\big(-\cos\theta_1\,\hat{\mathbf{x}} - \sin\theta_1\,\hat{\mathbf{z}}\big)\,e^{-j\,(k_{1x}x - k_{1z}z)}, \quad z<0.
\end{aligned}
}
$$

$$\boxed{\Gamma_{\text{TM}} = \frac{\eta_1 \cos\theta_1 - \eta_2 \cos\theta_2}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}}, \qquad \boxed{T_{\text{TM}} = \frac{2\,\eta_1 \cos\theta_2}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}}.$$

$$\boxed{\begin{aligned} \mathbf{H}_i(\mathbf{r}) &= \frac{E_0}{\eta_2}\,\hat{\mathbf{y}}\,e^{-j\,(k_{2x}x - k_{2z}z)}, \\ \mathbf{H}_r(\mathbf{r}) &= -\frac{\Gamma_{\text{TM}} E_0}{\eta_2}\,\hat{\mathbf{y}}\,e^{-j\,(k_{2x}x + k_{2z}z)}, \\ \mathbf{H}_t(\mathbf{r}) &= \frac{T_{\text{TM}} E_0}{\eta_1}\,\hat{\mathbf{y}}\,e^{-j\,(k_{1x}x - k_{1z}z)}. \end{aligned}}$$

1.3

$$\begin{pmatrix} B \\ C \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{12}^{\text{TM}} & T_{21}^{\text{TM}} \\ T_{12}^{\text{TM}} & \Gamma_{21}^{\text{TM}} \end{pmatrix}}_{S_{\text{TM}}} \begin{pmatrix} A \\ D \end{pmatrix}, \qquad S_{\text{TM}} \;:\; \text{dispersion (scattering) matrix.}$$

1.4

$$\Gamma_{ij}^{\text{TM}} = \frac{\eta_j \cos\theta_j - \eta_i \cos\theta_i}{\eta_j \cos\theta_j + \eta_i \cos\theta_i}, \qquad T_{ij}^{\text{TM}} = \frac{2\,\eta_j \cos\theta_i}{\eta_j \cos\theta_j + \eta_i \cos\theta_i},$$

where $\eta_k = \eta_0/n_k,\ k = 1,2,3$ (for non-magnetic dielectrics) and $n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_3 \sin\theta_3$ (Snell).

$$\beta_2 = k_2 \cos\theta_2, \qquad \delta = \beta_2 d = k_0 n_2 \cos\theta_2\, d.$$

$$\Gamma_{12} \equiv \Gamma_{12}^{\text{TM}}, \qquad \Gamma_{21} \equiv \Gamma_{21}^{\text{TM}}, \qquad \Gamma_{23} \equiv \Gamma_{23}^{\text{TM}}, \qquad \Gamma_{32} \equiv \Gamma_{32}^{\text{TM}},$$
$$T_{12} \equiv T_{12}^{\text{TM}},\ T_{21} \equiv T_{21}^{\text{TM}},\ T_{23} \equiv T_{23}^{\text{TM}},\ T_{32} \equiv T_{32}^{\text{TM}}.$$

With the above definitions, the TM–polarised scattering matrix reads

$$\boxed{S_{\text{TM}} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}}$$

$$S_{11} = \Gamma_{12} + \frac{T_{12}\,T_{21}\,\Gamma_{23}\,e^{-2j\delta}}{1 - \Gamma_{21}\,\Gamma_{23}\,e^{-2j\delta}},$$

$$S_{22} = \Gamma_{32} + \frac{T_{32}\,T_{23}\,\Gamma_{21}\,e^{-2j\delta}}{1 - \Gamma_{21}\,\Gamma_{23}\,e^{-2j\delta}},$$

$$S_{12} = \frac{T_{12}\,T_{23}\,e^{-j\delta}}{1 - \Gamma_{21}\,\Gamma_{23}\,e^{-2j\delta}},$$

$$S_{21} = \frac{T_{21}\,T_{32}\,e^{-j\delta}}{1 - \Gamma_{21}\,\Gamma_{23}\,e^{-2j\delta}}.$$

Hence the desired relations between the external wave amplitudes are

$$\boxed{\begin{pmatrix} B \\ E \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ F \end{pmatrix}}$$

2.1

Transfer matrix The 2×2 transfer matrix that carries the state vector $\Psi = (E^+, E^-)^{\mathrm{T}}$ from medium 2 back into medium 1 is

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$m_{11} = \frac{1}{T_{10}}, \qquad m_{12} = \frac{\Gamma_{10}}{T_{10}}, \qquad m_{21} = \frac{\Gamma_{10}}{T_{10}}, \qquad m_{22} = \frac{1}{T_{10}}$$

2.2

$$\begin{bmatrix} A \\ B \end{bmatrix} = \underbrace{\frac{1}{T_{12}} \begin{bmatrix} 1 & \Gamma_{12} \\ \Gamma_{12} & 1 \end{bmatrix}}_{\mathbf{T}_{12}} \begin{bmatrix} C \\ D \end{bmatrix},$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} e^{+jk_{z2}d} & 0 \\ 0 & e^{-jk_{z2}d} \end{bmatrix}}_{\mathbf{P}_2(d)} \begin{bmatrix} C_0 \\ D_0 \end{bmatrix}, \qquad k_{z2} = k_0\, n_2 \cos\theta_2.$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\frac{1}{T_{23}} \begin{bmatrix} 1 & \Gamma_{23} \\ \Gamma_{23} & 1 \end{bmatrix}}_{\mathbf{T}_{23}} \begin{bmatrix} E \\ F \end{bmatrix}.$$

$$\boxed{\begin{bmatrix} A \\ B \end{bmatrix} = \mathbf{T}_{12}\, \mathbf{P}_2(d)\, \mathbf{T}_{23} \begin{bmatrix} E \\ F \end{bmatrix}}$$

2.3

Basic data

$$\boxed{n_1 = n_3 = 3.4, \qquad n_2 = 1.5, \qquad \lambda_0 = 4\ \mu\text{m}, \qquad d = 1\ \mu\text{m}, \qquad \theta_1 = 45^\circ} \tag{1}$$

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\pi}{2}\ \mu\text{m}^{-1} \tag{2}$$

Snell's law and longitudinal wavenumbers

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = 1.604\ (>1) \implies \cos\theta_2 = j\,\alpha, \qquad \alpha = \sqrt{\sin^2\theta_2 - 1} = 1.254. \tag{3}$$

$$\cos\theta_1 = \cos\theta_3 = 0.7071 \tag{4}$$

$$\begin{aligned} k_{z1} = k_{z3} = k_0 n_1 \cos\theta_1 = 3.776\ \text{rad}\,\mu\text{m}^{-1}, \\ k_{z2} = k_0 n_2 \cos\theta_2 = j\,2.954\ \text{rad}\,\mu\text{m}^{-1}, \qquad \beta = k_{z2} d = j\,2.954. \end{aligned} \tag{5}$$

Fresnel coefficients (TM)

$$\eta_i = \frac{\eta_0}{n_i} \quad \Longrightarrow \quad \eta_1 = 0.2941\,\eta_0,\ \eta_2 = 0.6667\,\eta_0,\ \eta_3 = 0.2941\,\eta_0. \tag{6}$$

$$\Gamma_{12}^{\mathrm{TM}} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = 0.8836 + j\,0.4692, \tag{7}$$

$$T_{12}^{\mathrm{TM}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = 0.2654 - j\,1.0621, \tag{8}$$

$$\Gamma_{23}^{\mathrm{TM}} = -0.8836 - j\,0.4692, \qquad T_{23}^{\mathrm{TM}} = 0.8310 + j\,0.2070. \tag{9}$$

Transfer matrices

$$\mathbf{T}_{12} = \frac{1}{T_{12}}\begin{bmatrix} 1 & \Gamma_{12} \\ \Gamma_{12} & 1 \end{bmatrix}, \quad \mathbf{P}_2(d) = \begin{bmatrix} e^{+j\beta} & 0 \\ 0 & e^{-j\beta} \end{bmatrix}, \quad \mathbf{T}_{23} = \frac{1}{T_{23}}\begin{bmatrix} 1 & \Gamma_{23} \\ \Gamma_{23} & 1 \end{bmatrix}. \tag{10}$$

$$\mathbf{M} = \mathbf{T}_{12}\,\mathbf{P}_2(d)\,\mathbf{T}_{23} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 9.581 - j\,18.047 & * \\ -0.0179 - j\,20.403 & * \end{bmatrix}. \tag{11}$$

Outgoing waves ($A = 1,\ F = 0$)

$$E = \frac{1}{m_{11}}A = 0.02296 + j\,0.04325 = 0.0490\,e^{j\,62.9^\circ}, \tag{12}$$

$$B = \frac{m_{21}}{m_{11}}A = 0.882 - j\,0.468 = 0.998\,e^{-j\,27.9^\circ}. \tag{13}$$

Reflection / transmission

$$r = \frac{B}{A} = 0.998\,e^{-j\,27.9^\circ} \qquad (|r| \simeq 1 \text{ — total internal reflection}), \tag{14}$$

$$t = \frac{E}{A} = 0.0490\,e^{j\,62.9^\circ} \qquad (|t| \simeq 4.9\%), \tag{15}$$

$$\boxed{|r|^2 + |t|^2 \ \approx\ 1 \quad \text{(power balance)}}. \tag{16}$$

2.4

the written codes have been used in the following sections.

section 3

Purpose

The program evaluates, for transverse–magnetic (TM) polarization, the amplitude reflection $|R|$ and transmission $|T|$ of a five-layer dielectric stack as functions of the incidence angle θ_1. Three different values of the outer-layer thickness d_1 are examined.

Structure

air $\mid n_1 = 3.4$ $\quad$ $n_2 = 1.5$ $\quad$ $n_3 = 3.4$ $\quad$ $n_4 = 1.5$ $\quad$ $n_5 = 3.4 \mid$ air

The central layer (n_3) has a fixed thickness $d_0 = 0.5\ \mu$m, whereas the two identical neighborhood layers (n_2 and n_4) are scanned over

$$d_1 \in \{1,\ 2,\ 2.5\}\ \mu\text{m}.$$

Main computational steps

1. **Initialization**
   The script clears the workspace and defines $\lambda_0 = 4\ \mu$m, $k_0 = 2\pi/\lambda_0$ and the free-space impedance η_0.
2. **Angle grid**
   1000 incident angles are generated from 0° to 89.999° (the extremes are avoided to prevent division by zero).
3. **Double loop**
   For every thickness choice d_1 and for every angle θ_1 the code
   (a) computes the transverse component k_x and therefore every longitudinal component $k_{z,i}$ inside the layers,
   (b) builds the global 2×2 transfer matrix M by cascading interface matrices $T_{i,i+1}$ with the propagation matrices P_i,
   (c) extracts the overall Fresnel coefficients $r = M_{21}/M_{11}$ and $t = 1/M_{11}$,
   (d) stores $|R| = |r|$ and $|T| = |t|$.
4. **Visualization**
   Two figures are drawn:
   - $|R|(\theta_1)$ for the three thicknesses,
   - $|T|(\theta_1)$ for the same cases,

   both spanning $0° \leq \theta_1 \leq 90°$.

Results

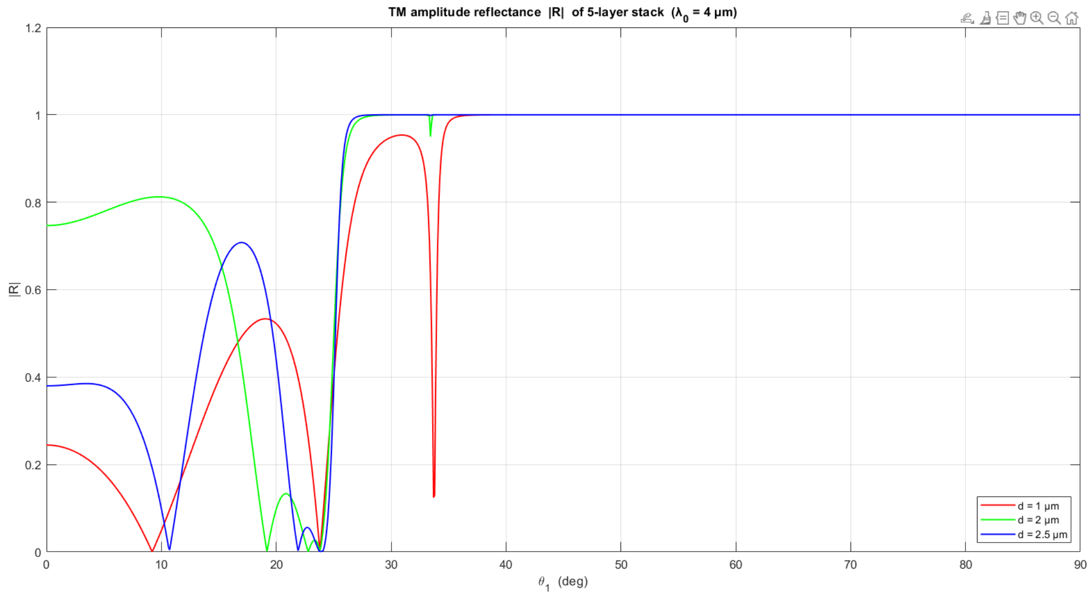

Figure 1: absolute value of reflection coefficient

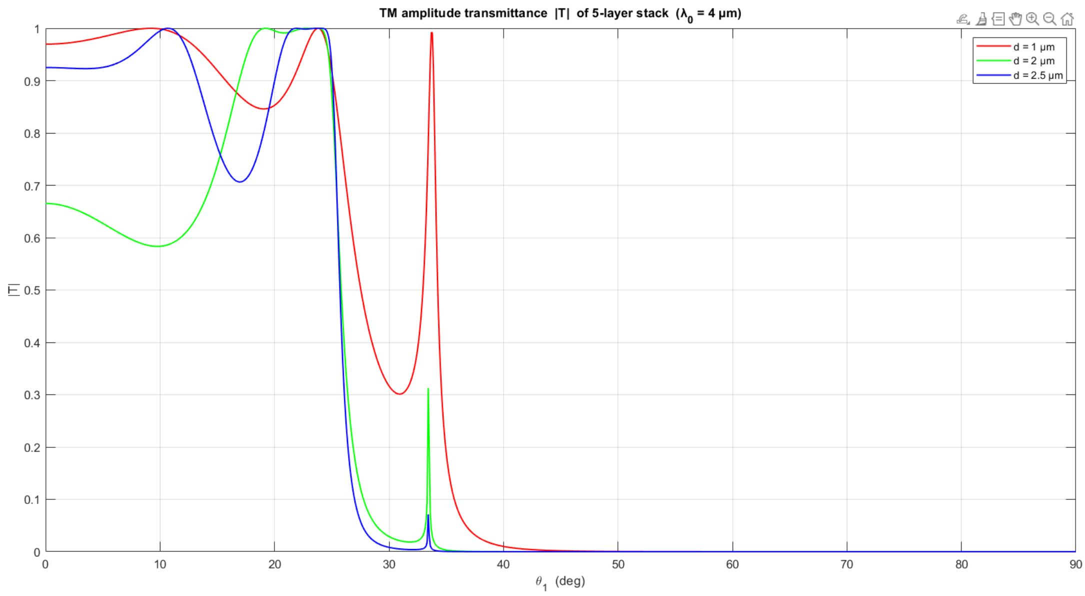

Figure 2: absolute value of transmission coefficient

section 4

Aim

The code finds the smallest number of (HL) pairs that gives a power reflection $|R|^2 \geq 0.99$ at $\lambda_0 = 1\,\mu$m and an incidence angle of 5° (TM). It then plots $|R|^2(\theta_1)$ for four stacks: 16, 26, 46 and the just-found optimum layer count.

Design data

High-index layers have $n_\mathrm{H} = 1.5$ and thickness $d_\mathrm{H} = 100\,\mu$m; low-index layers have $n_\mathrm{L} = 1.4$ and $d_\mathrm{L} = 90\,\mu$m; both external media are air.

Algorithm

A while-loop increases the pair number until the transfer-matrix routine `reflectionTM` returns $|R|^2 \geq 0.99$ at 5°. Afterwards, a sweep from 0° to 89.9° (step 0.05°) is executed for each chosen layer count and a separate figure is drawn.

Functions

`buildStack` creates the refractive-index and thickness vectors, including the two air half-spaces. `reflection TM` evaluates the global TM reflection with the 2×2 transfer-matrix method.

Results

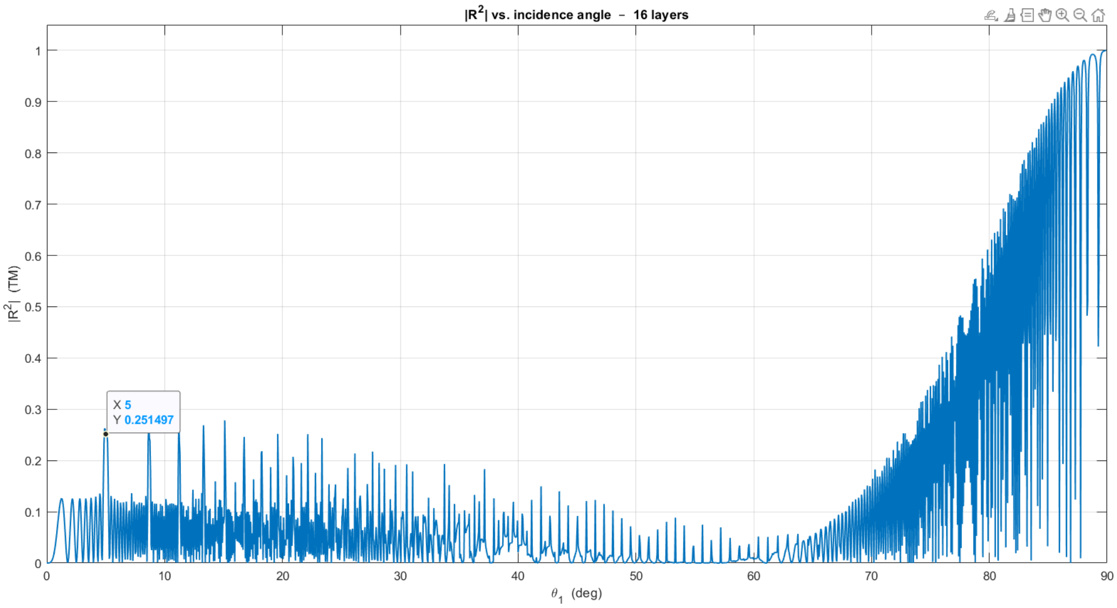

Figure 3: for 16 layers

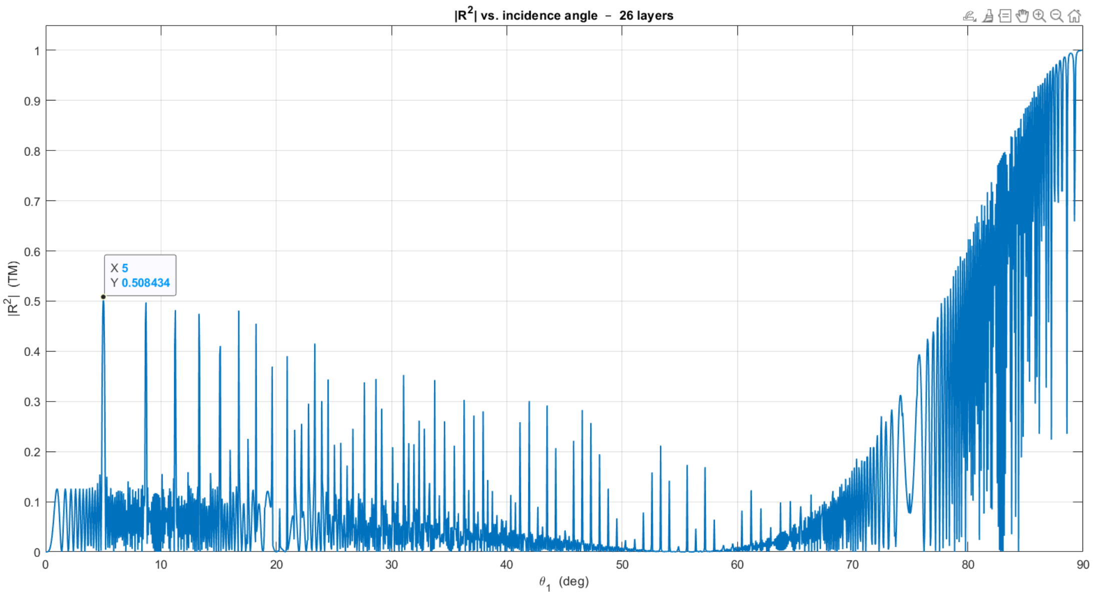

Figure 4: for 26 layers

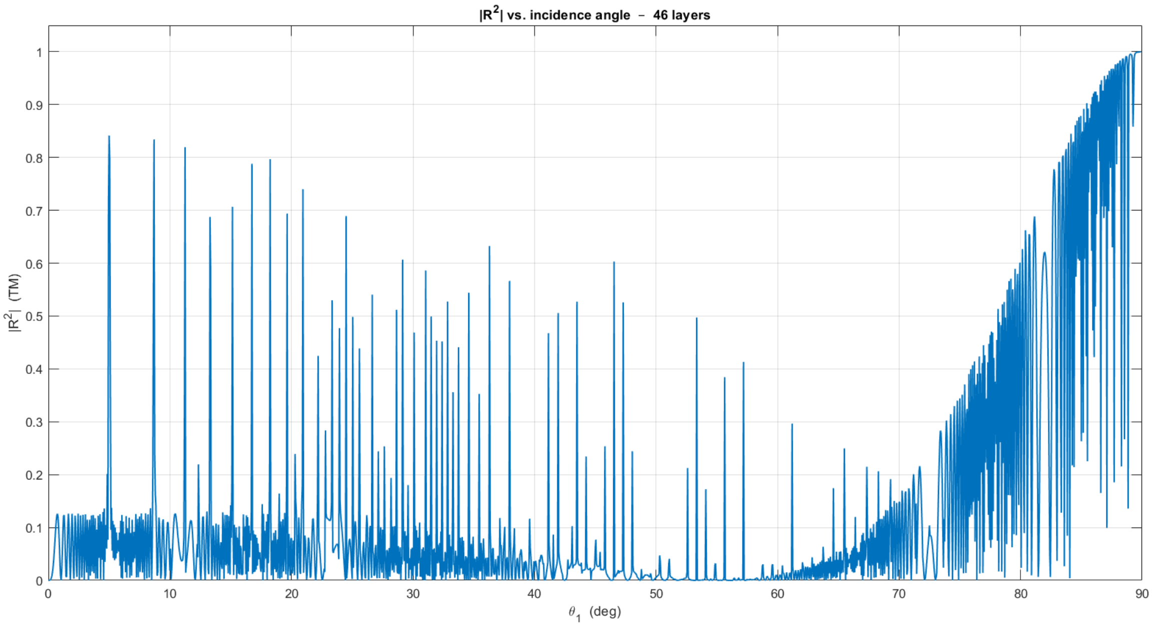

Figure 5: for 46 layers

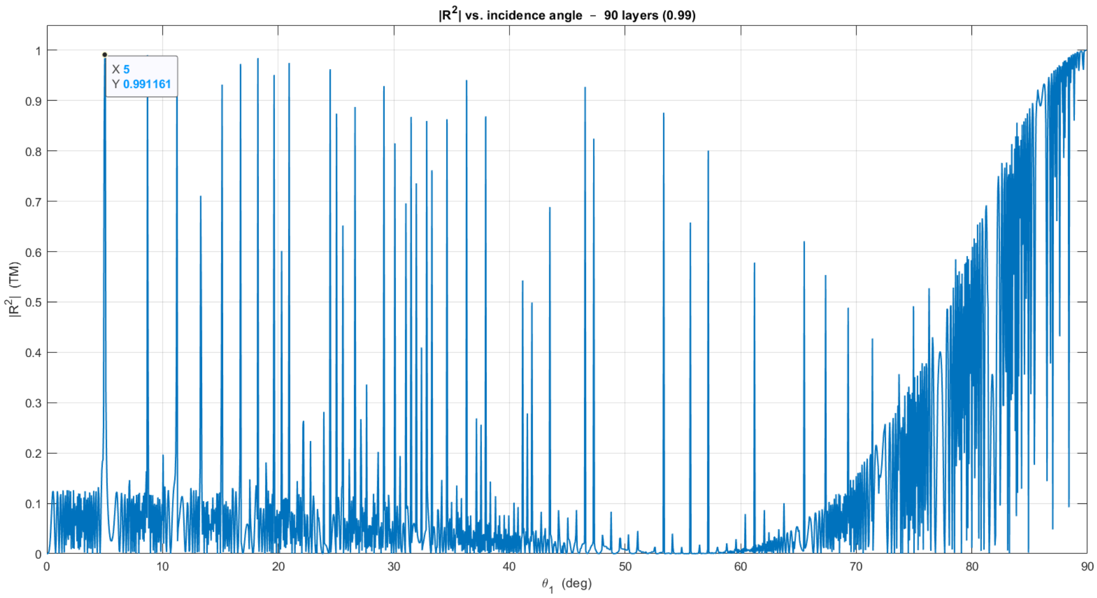

Figure 6: for 90 layers and desired result

Thanks for your attention