Micro I: Problem Set 2.

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Exercise 1. Towards demand estimation (the implicit Marshallian demand systems):

Part (I)

(i) Let p, w be the logarithm of prices and the logarithm of wealth.

First, let log denote the natural logarithm.

Now define $e(p,u) = log(exp(p)'h(p,u))$, the log expenditure function where $exp(p) = (exp(p_l))_{l\in\{1,\cdots,L\}}$ is the vector of prices (in levels) and $h(p,u)$ is the Hicksian demand function ($h(p,u) = argmin_{x\geq 0} exp(p)'x \quad s.t. u(x) \geq u$).

Prove that $(\nabla_p e(p,u))|_{u=v(p,w)} = b(p,w)$, where $v(p,w)$ is the indirect utility associated with e, such that $v(p, e(p,u)) = u$ and $e(p, v(p,w)) = w$ and $b(p,w) = (b_l(p,w))_{l\in\{1,\cdots,L\}}$ with $b_l(p,w) = exp(p_l)x_l(p,w)/exp(w)$ the budget share of good l, and $x(p,w) = argmax_{x\geq 0} u(x) \quad exp(p)'x = exp(w)$ is the Marshallian demand expressed in log prices and log wealth.

Part (II)

In an ideal world, estimating Hicksian demands would be much easier than estimating the Marshallian demands. The reason is that Hicksian demands are compensated and depend directly on the utility level which makes it very easy to adapt to model preference heterogeneity when dealing with Survey data.

We are going to see, that with some tricks we can actually estimate Hicksian demands, even when the utility is unobservable.

(i) Let's postulate/propose the following candidate for log expenditure function, for a population of consumers:

$e(p,u) = u + p'g(u) + \frac{1}{2}p'Ap$,

$A \in \mathbb{R}^{L\times L}$ is a matrix of coefficients (a parameter) (and p is log prices, and u is a utility level that every consumers wants to achieve), the function $g : \mathbb{R} \mapsto \mathbb{R}^L$ is any continuous vector valued function that relates the common utility level u with the log expenditure level (i.e., $g(u) \in \mathbb{R}^L$).

Use Part (I) result, $(\nabla_p e(p,u))|_{u=v(p,w)} = b(p,w)$ together with the identity of duality theory $e(p, v(p,w)) = w$, to prove that we can find the utility index in terms of the observables:

$u = w - p'b$, where b is the observed budget share vector.

(ii) Using Part II (i) prove that, we can write the budget share as:

$$b(p,w) = g(w - p'b) + Ap.$$

This model, as you saw, is much easier to deal with that the AIDS, one way to see its advantages is that we can impose very easily the restriction of rationality.

Note that $b(\cdot)$ is a function of prices and wealth, while $b \in [0,1]^L$ is a numerical vector of budget shares, and $b(p,w) = b$. Take this into account when deriving.

(iii) Compute the normalized Slutsky matrix $\overline{S}(p,w) = D_p b(p,w) + D_w b(p,w) b(p,w)' - Diag(b(p,w)) + b(p,w)b(p,w)'$ and show that a sufficient condition for $b(p,w)$ to be integrable (rationalizable) is that A is symmetric and NSD.

(Note that these same conditions are sufficient to conclude that the function $e(p,u) = u + p'g(u) + \frac{1}{2}p'Ap$, is the logarithm of a concave expenditure function).

(iv) Note that for a fixed level of utility u, we can write the budget share above as $b(p,u) = g(u) + Ap$, show that the semi-elasticity with respect to prices $D_p b(p,u) = D_p b(p,w) + D_w b(p,w) b(p,w)'$. What is the meaning of this quantity (explain).

(v) What sufficient restrictions can you imposed in g and in A for the budget constraints to add up: $\sum_l b_l(p,w) = 1$ or $1'_L b(p,u) = 0$ in matrix notation.

Now, we conclude with a very easy question that will show you the benefits of this model.

(vi)(Welfare Analysis) Say we have a original log price situation p^0 and there is a policy that changes it to p^1, what is the change in log expenditure that keeps the consumer with the same level of satisfaction as before? Compute $e(p^1,u) - e(p^0,u)$ with the help of Part (II) (i).

(The key is that using this model, we do not need to integrate to do Welfare analysis, we can do it almost directly).

Exercise 2. (SARP) Consider a limited data choice environment. Let X be a finite choice set and let $(\mathcal{A}, c)$ be a choice structure where $\mathcal{A} \subset 2^X \setminus \emptyset$ is a strict subset of the power set and c is a choice function (singled valued).

-Define $xR^D y$ if and only if $x \in C(A)$ for some $A \in \mathcal{A}$ and $y \in A$.

-Let R be the transitive closure of R^D. xRy if there exists a sequence $xR^D x^1 R^D x^2 \cdots R^D x^n R^D y$.

-(SARP) We say that the dataset satisfies SARP if and only if R is acyclic, i.e., there cannot be cycles, that is there cannot be a sequence $x^1 R x^2 \cdots R x^n R x^1$.

(i) Show that in general if c satisfies the SARP then there is a rational order $\succeq$ on X such that there is a choice function $(2^X \backslash \emptyset, c^*(\succeq))$ defined over the whole power set such that $c^*(\succeq, A) = c(A)$ for all $A \in \mathcal{A}$.

(ii) Prove that SARP implies WARP.

Exercise 3. (Afriat's Inequalities) (i) Verify that for $L = 1$, a dataset of prices and consumption $O = \{p^k, x^k\}_{k=1}^K$ will always satisfy the Afriat's inequalities with $p^k > 0$. That means that there exists positive numbers $v_i - v_j \geq \lambda_i p_i [x_i - x_j]$ for all $i, j = 1, \cdots, K$.

(ii) (Advanced Question, Optional) Prove that if a demand function $x(p, w)$ satisfies the Waras' law, WARP, and the number of commodities is $L = 2$, then $x(p, w)$ satisfies SARP (Hint. you can use the fact that WARP implies Homogeneity of Degree Zero).