Micro Summer: Problem Set 1.

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Exercise 1. Say X is finite and $\mathcal{B} \subseteq 2^X \setminus \emptyset$ does not contain some pairs and triples (i.e., there are some missing menus that of cardinality 2 and 3). Prove that there exists some choice correspondences $c : \mathcal{B} \to 2^X \setminus \emptyset$ that cannot be rationalized. (Hint: Do it by means of a counterexample).

Proof. Consider the following counter example:

$C(a, b) = a, C(b, d) = b, C(a, d) = d.$

It is easy to check that WARP is satisfied. Intuitively, that is because there is no two distinct menus, namely B and B', such that $x, y \in B$, $x, y \in B'$,and $x \neq y$,

However the transitivity fails, which indicates that it cannot be rationalized.

□

Exercise 2. Prove that a demand function x that satisfies Walras' law, HD0 and the Compensated Law of Demand has to satisfy WARP.

Proof. Assume WARP fails, that means that there exists a compensated price change from some (p', w') to some (p, w) such that $x(p, w) \neq x(p', w')$,$p * x(p', w') = w$,and $p' * x(p, w) \leq w'$.

But since x(.,.) satisfies Walras' law, then these two inequalities imply:

$p * [x(p', w') - x(p, w)] = 0$,and $p' * [x(p', w') - x(p, w)] \geq 0$.

Then we have $(p' - p) * [x(p', w') - x(p, w)] \geq 0$ and $x(p, w) \neq x(p', w')$

This contradicts to the Compensated Law of Demand.

□

Exercise 3. Consider a choice set X, and a consumer with preference defined over X, $\succeq \subseteq X \times X$, such that the preferences are a preorder (but not necessarily complete).

a) Assume in this literal that X is finite, prove that the preferences $\succeq \subseteq X \times X$ defined above, in general, cannot be represented by a utility function $u : X \mapsto \mathbb{R}$. Formally, find a counterexample to the statement: There exists a $u : X \mapsto \mathbb{R}$ such that for any $a, b \in X$, $a \succeq b \iff u(a) \geq u(b)$.

b) Assume in this literal that X is finite, prove that the preferences $\succeq \subseteq X \times X$ defined above, have a multiple utility representation, i.e., there is a set of utility functions $\mathcal{U}$ such that for any $a, b \in X$ with $a \succeq b \iff u(a) \geq u(b)$ for all $u \in \mathcal{U}$. (Hint: the set of utilities $\mathcal{U}$ can be finite. Also you could use the idea of a binary relation closure).

c) Maintain the assumptions in b). Let $C^{\succeq}(A) = \{a \in A | \text{there is no } b \in A, b \succ a\}$. Consider the data set $\{C^{\succeq}(A)\}_{A \in \mathcal{A}}$, where $\mathcal{A} \equiv 2^X \setminus \emptyset$, show (by means of a counterexample) that the data set may fail GARP.

d) Maintain the assumptions in b), and consider the data set $\{C(A)\}_{A \in \mathcal{A}}$, where $\mathcal{A} \equiv 2^X \setminus \emptyset$. Prove that the following axiom is a necessary condition for the data set $\{C(A)\}_{A \in \mathcal{A}}$ to be generated by the preferences in b) (i.e., $C(A) \equiv \{a \in A | \text{there is no } b \in A, b \succ a\}\}$.

Weak Axiom of Revealed Non-Inferiority (WARNI). For any $A \in \mathcal{A}$ and $y \in A$, if for every $x \in C(A)$ there exists a $B \in \mathcal{A}$ with $y \in C(B)$ and $x \in B$, then $y \in C(A)$.

Proof. a) When X is finite we can let $X = \{a, b, c\}$ and $\succeq$ be such that $a \succeq a, b \succeq b, c \succeq c$, $a \succeq b$, $c \succeq a$. This ordering is transitive and but not complete. We are going to prove that there is no utility that represents that ordering. Assume contrary to the fact that there is a utility function $u : X \mapsto \mathbb{R}$ that represents $\succeq$. We observe that any $u : X \mapsto \mathbb{R}$ that o represent $\succeq$, must be such that $u(a) \geq u(b)$ and $u(c) \geq u(a)$. This implies that, $u(c) \geq u(a) \geq u(b)$, but that means that $c \succeq b$, clearly $c \succeq b$ is not true, this is a contradiction.

b) Take $\succeq$ as defined in the literal, that is a (possibly incomplete) preorder, we take the transitive and complete closure of the order $\succeq$ or its complete closure. Let's call the set of complete transitive closures $T(\succeq)$. Formally, for every $\succeq^* \in T(\succeq)$, it follows that if $a, b \in X$, $a \succeq b$ then $a \succeq^* b$ and $\succeq \subseteq \succeq^*$ (and any $\succeq'$ binary relation such that $\succeq' \subset \succeq$ is either intransitive/incomplete or does not contain $\succeq$).

By the finiteness of X we know that $T(\succeq)$ is finite and of cardinality $|T(\succeq)| = J$. Take $\succeq^j \in T(\succeq)$ for $j = \{1, \cdots, J\}$, it has to be a complete preorder (reflexive, transitive and complete). By the notes in the course, in particular, the utility representation Theorem, for every jth complete preorders on a finite set X, we know that there is a utility function $u^j : X \mapsto \mathbb{R}$ that represents it (i.e, $a, b \in X$, $a \succeq^j b \iff u^j(a) \geq u^j(b)$). This implies that there is a set of utility functions

$\mathcal{U} = \{u^1, \cdots u^J\}$ such that for any $a, b \in X$, $a \succeq b \iff u^j(a) \geq u^j(b)$ for every $j \in \{1, \cdots, J\}$. To see that this is true, notice that by construction if $a \succeq b \implies a \succeq^j b \implies u^j(a) \geq u^j(b)$ for all j.

If $u^j(a) \geq u^j(b)$ for all $j \in \{1, \cdots, J\}$ it has to be that for all $j \in \{1, \cdots, J\} \succeq^j \in T(\succeq)$ and $a \succeq^j b$. By the definition of $T(\succeq)$, it must be the case that $\cap_{j=1}^{J} \succeq^j = \succeq$, this implies that $a \succeq b$.

c) Consider $C^{\succeq}(A) = \{a \in A | \text{there is no } b \in A, b \succ a\}$ and $X = \{a, b, c\}$ and $\succeq$ be such that $a \succeq a$ $a \succ b$, $c \succ a$. Then $C^{\succeq}(\{a, b\}) = \{a\}$, $C^{\succeq}(\{b, c\}) = \{b, c\}$, and $C^{\succeq}(\{a, c\}) = \{c\}$, this means in terms of revealed preferences that aPb (stricty), cPa and bIc (indifferent), but that means there is a cycle since $aPbIc$, and cPa. This is a violation of GARP.

d) For any $A \in \mathcal{A}$ and $y \in A$, if for every $x \in C(A)$ there exists a $B \in \mathcal{A}$ with $y \in C(B)$ and $x \in B$, it means that there is no $b \in A \setminus \{x\}$ such that $b \succ x$, and at the same time there is a $B \in \mathcal{A}$ for each $x \in B \cap A$ such that $y \in C(B)$. This means in turn that there is no $c \in B \setminus \{y\}$ such that $c \succ y$. This implies that $\neg x \succ y$. Now given the fact that $\succeq$ is a partial order, we know that $y \in A$ is such that there is no $d \in A \setminus \{y\}$ such that $d \succ y$. It follows that $y \in C(A)$.

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