Problem Set 4 Microeconomics

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Problem 1. (Expected Utility) Consider the following "portfolio choice" problem. The consumer (with preferences that admit an expected utility representation) has wealth w and a Bernoulli utility $u(x) = ln(x)$ there is a safe asset that has net real return of zero. There is also a risky asset with a random net return that has only two possible returns R_1 with probability q and R_0 with probability $1 - q$.

Let A be the amount invested in the risky asset so $w - A$ is the amount invested in the safe asset.

(i) Find A as a function of w. Does the consumer buys more or less of the risky asset as w increases?

Proof. The consumer is solving the following maximization problem:

$$max_{A \in [o,w]} \{qu((1 + R_1)A + w - A) + (1 - q)u((1 + R_0)A + w - A)\}$$

Suppose the solution is interior, with $u(x) = ln(x)$, FOC is:

$$q \frac{R_1}{R_1 A + w} + (1 - q) \frac{R_0}{R_0 A + w} = 0$$

$$\implies A = -w \frac{qR_1 + (1 - q)R_0}{R_0 R_1}$$

Since the utility function is concave, so the consumer is risk averse. So, we can argue that $qR_1 + (1 - q)R_0 > 0$. Otherwise, the investor will never buy risky asset. WLOG, we assume $R_1 > 0$ and $R_0 < 0$. Otherwise, the consumer will either invest all her wealth in the risky asset ($R_0 > 0,\ R_1 > 0$), or not invest in risky asset at all ($R_0 < 0,\ R_1 < 0$). Therefore, $\frac{\partial A}{\partial w} = -\frac{qR_1 + (1-q)R_0}{R_0 R_1} > 0$.

$\square$

(ii) A second (expected utility) consumer has a Bernoulli utility function $u(x) = -exp(-x)$. How does his consumption of the risky asset changes with wealth?

Proof. We are solving the same maximization problem in part (i), with the new utility function, FOC is:

$$qR_1exp(-(1+R_1)A - w + A) + (1-q)R_0exp(-(1+R_0)A - w + A) = 0$$

$$\Longrightarrow qR_1exp(-R_1A - w) + (1-q)R_0exp(-R_0A - w) = 0$$

$$\Longrightarrow A = \frac{1}{R_1 - R_0}ln\left[-\frac{R_1q}{R_0(1-q)}\right]$$

So $\frac{\partial A}{\partial w} = 0$. That is investment in risky asset does not change with wealth.

□

(iii) Compute the coefficient of absolute risk aversion $r(x) = -\frac{u''(x)}{u'(x)}$ for the first and second consumers. How these coefficients depend on wealth? How does this coefficient explains the answers in (i) and (ii).

Proof. For $u(x) = ln(x)$, we have $u'(x) = \frac{1}{x}$ and $u''(x) = -\frac{1}{x^2}$. So $r(x) = \frac{1}{x}$. As x gets larger, $r(x)$ gets smaller. Which means the wealthier the investor is, the less risk aver she is. For $u(x) = -exp(-x)$, we have $u'(x) = exp(-x)$ and $u''(x) = -exp(-x)$. So $r(x) = 1$. Therefore, the amount that investor allocates to risky asset is independent to her wealth.

□

Problem 2. Non-expected utility theory. (Modeling)

From class we had the Allais paradox behavior:

Z is finite in fact $N = 3$,

First Prize	Second Price	Third Price
2.5 millions	0.5 millions	0

Two choice sets:

The first,

$L_1 = (0, 1, 0)$ and $L_1' = (0.10, 0.89, 0.01)$.

The second,

$L_2 = (0, 0.11, 0.89)$ and $L_2' = (0.10, 0, 0.90)$.

Imagine an individual that choose $L_1 \succ L_1'$ and $L_2' \succ L_2$.

This is inconsistent with expected utility.

If $L_1 \succ L_1'$ then

$$u_{0.5} > (0.10)u_{2.5} + (0.89)u_{.5} + (0.01)u_0.$$

Adding, $(0.89)u_0 - (0.89)u_{0.5}$ to both sides we get,

$$(0.11)u_{0.5} + (0.89)u_0 > (0.1)u_{2.5} + (0.9)u_0,$$

and thus $L_2 \succ L_2'$.

Now consider a model with weighted probabilities.

$$V(p) = \sum_{z \in Z} \phi(p(z))u(z),$$

where $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(t) \geq 0$. Find a ϕ that explains the Allais paradox and show it can explain the choices $L_1 \succ L_1'$ and $L_2' \succ L_2$ assuming that $u(z) = z$ for simplicity.

First model:

$L_1 = (0, 1, 0)$ and $L_1' = (0.10, 0.89, 0.01)$.

The second,

$L_2 = (0, 0.11, 0.89)$ and $L_2' = (0.10, 0, 0.90)$.

Imagine an individual that choose $L_1 \succ L_1'$ and $L_2' \succ L_2$.

$$\phi(x) = x^\alpha$$

$$\phi(1)(1) > \phi(.10)2.5 + \phi(.89)(1) + \phi(0.01)(0)$$

$$\phi(.10)2.5 + \phi(0)1 + \phi(.9)(0) > \phi(0)2.5 + \phi(.11)1 + \phi(.89)(0)$$

$$1 > .10^\alpha(2.5) + .89^\alpha$$

$$(0.1)^\alpha 2.5 > .11^\alpha$$

$$\alpha = \frac{29}{10}.$$

Does your new model satisfies first order stochastic dominance monotonicity, $p \geq_{FOSD} q \implies V(p) \geq V(q)$?

No, in general for $\alpha = 1$ is the only case where FOSD is respected by then we are back to expected utility.

Advanced question: How can you modify the probability weighting model to satisfy FOSD. (Optional for 10% to the Midterm).

Proof: Define

$$\phi(z) = e^{-(-\ln p)^{\alpha}}$$

where $\alpha \in (0,1)$. Set $u_{0.5} = 1$, $u_0 = 0$ and $u_{2.5} \in (1, e/(e-1))$, then we have

$$U(L_1|\phi) = 1$$

$$U(L_1'|\phi) = \phi(0.01)0 + [\phi(0.9) - \phi(0.01)] + (1 - \phi(0.9))u_{2.5}$$

$$U(L_2|\phi) = \phi(1 - \phi(0.89))$$

$$U(L_2'|\phi) = \phi(1 - \phi(0.9))u_{2.5}$$

$$\lim_{\alpha \to 0} U(L_1|\phi) = 1 > (1 - 1/e)u_{2.5} = \lim_{\alpha \to 0} U(L_2'|\phi) > 1 - 1/e = \lim_{\alpha \to 0} U(L_2|\phi)$$

Thus we have the given utility function with weighted function ϕ ensures $L_1 \succ L_1'$ and $L_2' \succ L_2$.

First order stochastic implies higher utility, i.e., $p \geq_{FOSD} q \implies V(p) \geq V(q)$?

$U(p|\phi) = \phi(p_0)u_0 + [\phi(p_0 + p_1) - \phi(p_0)]u_1 + (\phi(p_0 + p_1 + p_{2.5}) - \phi(p_0 + p_1))u_{2.5}$

$$= \phi(p_0 + p_1 + p_{2.5})u_{2.5} + \phi(p_0 + p_1)(u_1 - u_{2.5}) + \phi(p_0)(u_0 - u_1).$$

$U(q|\phi) = \phi(q_0 + q_1 + q_{2.5})u_{2.5} + \phi(q_0 + q_1)(u_1 - u_{2.5}) + \phi(q_0)(u_0 - u_1),$

if ϕ is increasing and u is strictly increasing then $p \geq_{FOSD} q \implies V(p) \geq V(q)$.

The advanced question: the order stochasitc property still holds if ϕ is increasing.