Micro Summer: Problem Set 1.

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Every answer is 25 points.

Exercise 1. Solve the Utility Maximization Problem for the following utility functions, for $x \in B(p,w) = \{x \in X | p'x = w\}$.

(a) Cobb-Douglas: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

(b) Constant Elasticity of Substitution (CES): $u(x_1, x_2) = \left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}}\right)$.

Proof. a) $x_1 = \frac{\alpha w}{p_1}$, $x_2 = \frac{(1-\alpha)w}{p_2}$

b)$\mathcal{L} = \left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}}\right) + \lambda(w - p_1 x_2 - p_2 x_2)$

This function has nice properties, in fact it has an interior solution. It is strictly quasi-concave and the budget set convex.

The first order conditions give me:

$\frac{\sigma-1}{\sigma} x_1^{\frac{\sigma-1}{\sigma} - \frac{\sigma}{\sigma}} = \lambda p_1$

$\frac{\sigma-1}{\sigma} x_2^{\frac{\sigma-1}{\sigma} - \frac{\sigma}{\sigma}} = \lambda p_2$

Dividing:

$\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}} = \frac{p_1}{p_2}$

$x_2 = \left(\frac{p_1}{p_2}\right)^{\sigma} x_1$

Replacing in the budget constraint:

$p_1 x_1 + p_2 \left(\frac{p_1}{p_2}\right)^{\sigma} x_1 = w$

$x_1 = w(p_1 + p_1^{\sigma} p_2^{1-\sigma})^{-1} = \frac{w}{p_1^{\sigma}(p_1^{1-\sigma} + p_2^{1-\sigma})}$

$x_2 = \left(\frac{p_1}{p_2}\right)^{\sigma} x_1 = \left(\frac{p_1}{p_2}\right)^{\sigma} \frac{w}{p_1^{\sigma}(p_1^{1-\sigma} + p_2^{1-\sigma})} = \frac{w}{p_2^{\sigma}(p_1^{1-\sigma} + p_2^{1-\sigma})}$.

□

Exercise 2. For the CES demand function derived in Exercise 1.c compute its Slutsky matrix $S(p,w) = D_p x^{CES}(p,w) + D_w x^{CES}(p,w) x^{CES}(p,w)'$.

a) Check that $S(p,w)$ is symmetric, negative semi-definite, and $S(p,w)p = 0$ (singular in prices).

b) Verify the Homogeneity of degree zero $x^{CES}(p,w)$.

Proof. $x_i = w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-1}$ for $i,j=1,2$

$\frac{\partial x_i}{\partial p_j} = -w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[(1-\sigma)p_i^\sigma p_j^{-\sigma}]$

$\frac{\partial x_j}{\partial p_i} = -w[p_j^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[(1-\sigma)p_j^\sigma p_i^{-\sigma}]$

$\frac{\partial x_i}{\partial p_i} = -w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[1+\sigma p_j^{1-\sigma}p_i^{\sigma-1}]$

$\frac{\partial x_i}{\partial w} = [p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-1}$

$\frac{\partial x_i}{\partial w}\cdot x_i = [p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-1}w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-1} = w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}$

$\frac{\partial x_i}{\partial w}\cdot x_j = [p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-1}w[p_j^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-1} = wp_i^{-\sigma}p_j^{-\sigma}[p_i^{1-\sigma}+p_j^{1-\sigma}]^{-2}$

$S(p,w)_{ij} = -w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[(1-\sigma)p_i^\sigma p_j^{-\sigma}] + wp_i^{-\sigma}p_j^{-\sigma}[p_i^{1-\sigma}+p_j^{1-\sigma}]^{-2} = w[p_i^{1-\sigma}+p_j^{1-\sigma}]^{-2}[-p_i^{-2\sigma}(1-\sigma)p_i^\sigma p_j^{-\sigma}+p_i^{-\sigma}p_j^{-\sigma}]$

$S(p,w)_{ij} = w[p_i^{1-\sigma}+p_j^{1-\sigma}]^{-2}[p_i^{-\sigma}p_j^{-\sigma}(\sigma)]$

$S(p,w)_{ji} = -w[p_j^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[(1-\sigma)p_j^\sigma p_i^{-\sigma}] + wp_j^{-\sigma}p_i^{-\sigma}[p_j^{1-\sigma}+p_i^{1-\sigma}]^{-2} = w[p_i^{1-\sigma}+p_j^{1-\sigma}]^{-2}[p_i^{-\sigma}p_j^{-\sigma}(\sigma)]$

$S(p,w)_{ii} = -w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[1+\sigma p_j^{1-\sigma}p_i^{\sigma-1}] + w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2} = w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[-\sigma p_j^{1-\sigma}p_i^{\sigma-1}]$

a) Given that $S(p,w)_{ij} = S(p,w)_{ji}$ then $S(p,w)$ is symmetric. We verify now $S(p,w)$ is singular in prices

$S(p,w)_{ii}p_i + S(p,w)_{ij}p_j = 0$,

$S(p,w)_{ii}p_i + S(p,w)_{ij}p_j = p_i w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[-\sigma p_j^{1-\sigma}p_i^{\sigma-1}] + p_j w[p_i^{1-\sigma}+p_j^{1-\sigma}]^{-2}[p_i^{-\sigma}p_j^{-\sigma}(\sigma)] = 0$.

Finally, given that $S(p,w)p = 0$ we can delete one row and one column from $S(p,w)$ to obtain:

$S(p,w)_{-j,-j} = S(p,w)_{i,i} = w[p_i^\sigma[p_i^{1-\sigma}+p_j^{1-\sigma}]]^{-2}[-\sigma p_j^{1-\sigma}p_i^{\sigma-1}] \leq 0$ for $\sigma \geq 0$.

b) This is direct and omitted.

$\square$

Exercise 3. Read the following chapter from Predictabily Irrational from Dan Ariely:

http://christophe.heintz.free.fr/bgt/Ariely-Predicably-Irrational-Ch3.pdf

from pages 55-60. Then answer: What is the zero price effect, and why it is not compatible with the Utility Maximization Problem.

Grading: The answer was required to be short, about 2 paragraphs long. The first one should contain a clear answer to the question of what is the zero price effect. The second paragraph should contain a clear revealed preference reasoning to justify the answer. The first paragraph is worth 10 points the second paragraph is worth 15 points.

The zero price effect is an experimental finding that shows that even when relative price differences are the same among two goods, choices change significantly when one of the prices is zero. In particular, consumers may choose one good when both prices are positive, to a free good while keeping the price difference the same.

More formally, let there be two items $a, b \in X$, where a is a Lindt truffle and b is a Hershey Kiss.

- In the first experiment, the price of a, was $p_a^1 = 15$ cents per unit and $p_b^1 = 1$ cent per unit. In that case most consumers chose a over b.
- Then in the second experiment the prices where changed, keeping the price difference constant, to $p_a^2 = 14$ and $p_b^2 = 0$. In that case, most consumer chose b over a.

We can use the tools of revealed preferences to answer the question of whether this experimental findings are consistent with rationality.

- In particular, note that in experiment 1, given that a subject bought a at $p_a^1 = 15$, she could have buy b at $p_b^1 = 1$. That means that a is revealed preferred to b strictly.
- In experiment 2, given the experiment 1, the subject can afford both the a and b, because both are cheaper, moreover the price difference is the same as in experiment 1, however b is chosen over a.

This means that the subject cannot be rational. We can add a bit more detail to the model, consider $u(a) = g(a) - c(a)$ where the utility of a is the benefit of a versus the cost of a, for simplicity let $c(a) = p_a$. In experiment 1, we get that $u(a) = g(a) - 15$ and $u(b) = g(b) - 1$, then when the subject chooses a over b, we conclude that $u(a) > u(b)$ which implies $g(a) - g(b) \geq 14$.

Then in experiment 2, we get that $u(a) = g(a) - 14$ and $u(b) = g(b) - 0$, notice that the main idea behind a rational consumer is that preferences remain stable, so the benefits are the same.

Then it must be that $u(a) > u(b)$ because $g(a) - g(b) \geq 14$ by the first experiment.