Microeconomics I.

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Chapter 1

Order Theory

1.1 Preliminaries

Order theory is essential to understand some decision theoretical aspects of consumer theory. In particular, the modelling preferences is done using order theory tools. However, the applicability of order theory in economics is not limited to this.

1.1.1 Binary Relations

Let X be a non-empty set. A **binary relation on** X is a subset R of $X \times X$. If $(x, y) \in R$ we will write xRy and $xRyRz$ will mean that $(x, y) \in R$ and $(y, z) \in R$ and so on.

Example 1. $X = \mathbb{R}$, R be defined by for all $a, b \in X$, $aRb \iff a \geq b$.

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Example 2. $X = \mathbb{R}^2$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$, the Lexicographic ordering says that $a, b \in X$, $aRb \iff a_1 > b_1$ or $a_1 = b_1$ and $a_2 \geq b_2$.

WTS: R lexicographic is complete.

xRy or yRx, take any arbitrary vectors $v, w \in X$, notice that

P: either $v_1 \geq w_1$ or $w_1 \geq v_1$, either $v_2 \geq w_2$ or $w_2 \geq v_2$.

Q: $vRw \iff v_1 > w_1$ or $v_1 = w_1$ and $v_2 \geq w_2$, or

$wRv \iff w_1 > v_1$ or $w_1 = v_1$ and $w_2 \geq v_2$,

$P \implies Q$.

dd

Example 3. $X = \mathbb{R}^2$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$, the canonical ordering says that $a, b \in X$, $aRb \iff a_i \geq b_i$ for all $i = 1, 2$.

Lemma: R canonical vector order is not complete.

I want to construct a counter-example, take $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, notice that neither $(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}) \in R$ nor $(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}) \in R$.

Properties of Binary Relations

- Reflexivity: We say R is reflexive if xRx for every $x \in X$.
- Completeness (Totality): We say that R is complete if either xRy or yRx holds for every $x, y \in X$.
- Symmetry: We say that R is symmetric if xRy implies yRx.
- Asymmetry: We say that R is asymmetric if xRy implies $\neg yRx$.
- Antisymmetric $xRyRx$ implies $x = y$
- Transitiviy: We say that R is transitive if xRy and yRz implies xRz.
- Acyclic: $x_1 R x_2 R \cdots R x_n$ implies that $x_1 \neq x_n$.

Example 4. Asymmetric Relation. $X = \mathbb{R}$, and $R =>$.

Classes of Relations

We can use this properties to define useful relations that are usually applied to describe preferences.

Definition 1. (Equivalence Relations) We say that $I \subseteq X \times X$ is an equivalence relation if and only if it is reflexive, symmetric and transitive.

The equivalence relation is an important concept in economics, usually we use it to model **indifference**.

Definition 2. (Preorder) We say that $R \subseteq X \times X$ is a preorder if and only if R is reflexive and transitive.

The most known application of a preorder to economics is modelling decision maker preferences. But usually we impose also completeness or totality. We say that a consumer has a rational preference relation $\succeq \in X \times X$ if $\succeq$ is a total/complete preorder.

Definition 3. (Partial Order) We say that $R \subseteq X \times X$ is a partial order if and only if R is reflexive, transitive, and antisymmetric.

Observe that the partial order rules out the existence of an equivalence relation.

Finally we define a linear order.

Definition 4. (Linear Order) We say that $R \subseteq X \times X$ is a linear order if and only if it is a complete partial order.

Ordered Sets

We are going to study now some tuples of sets and binary relations that are also very useful in economic modelling.

Definition 5. (Poset) We say that the pair (X, R) is a poset if and only if R is a partial order.

The prime example of a Poset for economists is a consumer environment where X is the choice set and R is a strict preference (albeit possibly incomplete).

Definition 6. (Loset) We say that the pair (X, R) is a loset if and only if R is a linear order.

Again, we can think of a consumer environment where X is the choice set and R is a (complete) strict preference.

Definition 7. (Equivalent class) An equivalence class of any $x \in X$ with respect to an equivalence relation $I \subseteq X \times X$ is $[x]_I = \{y \in X | yIx\}$.

The equivalence class of $x \in X$ under I is a subset of X where all items are considered "equivalent" under I to the element x. The typical example of an equivalent class in economics is an **indifference curve.** Let $x \in X$ be a consumption bundle and let $\sim$ be an indifference preference relation for a consumer then $[x]_\sim$ is the set of consumption bundles that the consumer is indifferent about. In the eyes of this particular consumer $y \in [x]_\sim$ is exactly the same as x.

Now we define an important concept that is derived from the equivalence relation notion.

Definition 8. (Quotient Set) Let I be an equivalence relation. The set of equivalence classes of X with respect to I is called the quotient set of X by I and is denoted as X/I.

This requires an example to fix ideas.

Example 5. Let $X = \{a, b, c\}$ and let I be an equivalence relation such that aIb, now $X/I = \{[a]_I, [c]_I\}$ where $[a]_I = \{a, b\}$ and $[c]_I = \{c\}$ by reflexivity.

dd

Example 6. Quotient Set. $\mathbb{Q}$ set of rational numbers, endowed with the $\geq$ (same as the reals). Notice that $\frac{2}{4} = \frac{1}{2}$. $\mathbb{Q}/=$ (to do). Application: Well-defined functions.

$f(\frac{p}{q}) = p - q$. Well-defined means that if $x = y$ then $f(x) = f(y)$.

Important Concepts in Ordered Sets

First let's note that any binary relation can be separated into its symmetric and asymmetric part. Formally for every R, there is a symmetric part called I and a asymmetric part called P. If R is a preorder order (preference relation), I is an indifference relation and P is a partial order (strict preferences).

For binary relations we can define the concepts of maximal element and upper bound.

The symmetric part of a relation I is defined as xIy if and only if xRy and yRx. The asymmetric part of a relation P is defined as xPy if and only if xRy and not yRx.

Definition 9. (Maximum element) Let R be defined as a binary relation on X: the maximal elements of X according to R are defined as

$$max(X, R) = \{x \in X | \text{there is no} \quad \text{y}\in\text{X:yPx}\},$$

where P its the asymmetric part of R.

Also we can define an upper bound of an ordered set as:

Definition 10. (Upper bound/maximal element) Let R be defined as a binary relation on X: the upper bound elements of X according to R are defined as

$$M(X, R) = \{x \in X | xRy \forall y \in X\}.$$

Example 7. $X = \{a, b, c\}$ $a \succ b$,

$M(X, \succ) = \emptyset$, $Max(X, \succ) = \{a, c\}$.

1.2 Transitive Closure and Extensions.

First we define what is a **closure** of a relation R.

Definition 11. ($\mathfrak{P}$-Closure) Let R be a binary relation, let $\mathfrak{P}$ be a binary relation property. $C(R) \subseteq X \times X$ is a closure of R if and only if xRy implies $xC(R)y$ and such that $C(R)$ satisfies $\mathfrak{P}$ and is the smallest relation that contains R and satisfies $\mathfrak{P}$ (i.e., $C(R) = \cap_{i \in \mathcal{C}} C_i(R)$ it is the intersection of all relations $C_i(R) \subseteq X \times X$ such that xRy implies $xC_i(R)y$ in the set $\mathcal{C} = \{M \subseteq X \times X | xRy \implies xMy|; M satisfies \mathfrak{P}\}$).

It should be clear that a closure of R is a binary relation that usually adds a property such as transitivity, symmetry or reflexivity.

Definition 12. (Transitive Closure) Let R be a binary relation. $T(R)$ is a transitive closure of R if and only if it is a closure of R that satisfies transitivity.

Now we are going to observe the following fact.

Fact 1. *(Existence of Transitive Closure) Every binary relation R has a transitive closure $T(R)$.*

This is trivial because there is always a transitive order that contains R, the trivial relation T such that xTy for all $x, y \in X$.

Example 8. $X = \{a, b, c\}$ $a \succ b$, aC_1bC_1c, cC_2aC_2b, cI_3aI_3b,

Another useful concept is an **extension.**

Definition 13. Let R be a preorder on X. An extension of R is a preorder $\succeq$ such that

$$R \subset \succeq$$

$$P \subset \succ .$$

Where $\succ$ and P are the antisymmetric parts of $\succeq$ and R respectively.

We have our first result at hand.

Theorem 1. *(Sziplrajn) For any non-empty set X and a partial order $\succ$ on X there exists a linear order $\succ^*$ that is an extension of $\succ$.*

Proof. Omitted. Check Efe Ok Real Analysis book for details. □

Corollary 1. *Let $(X, \succeq)$ be a preordered set. There exists a complete preorder on X that extends $\succeq$.*

Proof. We use $\succeq$ to obtain its symmetric part, $\sim$, defined as $x \sim y \iff x \succeq y$ and $y \succeq x$.

Pick any item $x \in X$, and build its equivalence class $[x]_{\sim} = \{y \in X | y \sim x\}$.

The quotient set $Y = X \backslash \sim = \{[x]_{\sim}\}_{x \in X}$ removing repetitions.

$(Y, >)$ is partially ordered set (Poset), when $a, b \in Y$, $a > b$ if for any $x \in a$ and $y \in b$ $x \succ y$. We note that $>$ is a partial order because it is reflexive, transitive and antisymmetric.

We apply Sziplrajn theorem, to conclude that there exists a linear order $>^*$ on Y that is extends $>$.

We are going to build a total or complete preorder $\succeq^*$ on X (the original set), as follows:

$$x, y \in X : x \succ^* y \iff [x]_{\sim} >^* [y]_{\sim},$$

and the symmetric part is going to be $x, y \in X : x \sim *y \iff x \sim y$.

□

1.3 Existence of Extremal Elements

Proposition 1. *Let $(X, \succeq)$ be a total preordered set and S a non-empty finite subset of X . If $\succeq \cap (S \times S) \neq \emptyset$ (complete in S) then $M(S, \succeq) = max(S, \succeq) \neq \emptyset$.*

Proof. $max(\succeq, S) = \{y \in S | \nexists x \in S : x \succ y\}$

First step: $|S| = 1$, $S = \{a\}$

$max(\succeq, S) = \{a\}$

Second step: $|S| = 2$, $S = \{a, b\}$

Assume that $max(\succeq, S) = \emptyset$, $a \succ b$ and $b \succ a$

By totality/completeness these 2 cases are not possible, because a total preorder has to satisfy that either $a \succeq b$ or $b \succeq a$. $(a, b) \in \succeq$ and not $(b, a) \in \succeq \iff a \succ b$, $(b, a) \in \succeq$ and not $(a, b) \in \succeq$

Assume that for $|S| = k$, with k fixed and $k > 2$, it follows that

$max(S, \succeq) \neq \emptyset$.

Now, I want to show that for $|S'| = k + 1$ it has to be that

$max(S', \succeq) \neq \emptyset$.

$S' = S \cup \{a\}$ for some $a \in X$ such that $a \notin S$.

By the induction hypothesis there is a $a^* \in max(S, \succeq)$ from the previous step.

By definition I know that, $a^* \succeq b$ for all $b \in S$.

By totality or completeness it has to be that either $a \succeq a^*$ or $a^* \succeq a$.

In the former case, $a \in max(S', \succeq)$, in the latter case $a^* \in max(S', \succeq)$. Hence,

$max(S', \succeq) \neq \emptyset$.

$\square$

Chapter 2

Basics of Consumer Theory

2.1 Preliminaries

The heart of the classical Microeconomic Theory is the rational consumer. We are going to learn how to model decision making under the assumption of rationality. Of course, along the way we will try to understand also what economists mean when they say a consumer is rational. The rational decision maker theory that we are going to study is limited to a very specific environment that we are going to present next, but be wary that most of the concepts we are going to learn in this chapter can be extended to other situations.

2.2 Environment: Commodities and Budget Set

First we describe formally the goods and services that a consumer may consume. We call the object of consumption **commodity.** We assume that the **commodity space** is $\mathbb{R}^L$, where $L \geq 1$ is the number of commodities. We refer to any commodity by $l \in \{1, \cdots, L\}$ and to the quantity of the commodity by $\overline{x}_l$. The **consumption bundle or commodity bundle** is given by the vector $\overline{x} \in \mathbb{R}^L$ ($\overline{x} = (\overline{x}_1, \cdots, \overline{x}_L)'$).

Because the commodity space may be to 'generic' for some modelling purposes (e.g., the situation at hand may rule out extremely large quantities of some commodities, or negative quantities are impossible) we use from now on the **consumption set** $X \subseteq \mathbb{R}^L$**.** Remember that the consumption set is an abstract concept that can be defined according to each modelling situation, but for now on we will let $X = \mathbb{R}^L_+$.

In real life, the consumer cannot consume anything he likes from the consumption set X. He is constrained either by his available wealth or by other reasons. Here we focus in cases where the only constraint is given by his wealth. Let $w \in \mathbb{R}_+$ be the consumer's wealth (broadly defined). Let $p \in \mathbb{R}^L_+ \backslash \{0\}$ be the price vector associated with each commodity. The consumer is then limited to choose from his budget set. The budget set is a subset of the choice set $B(p,w) \subseteq X$ and it is defined by $B(p,w) = \{x \in X | p'x \leq w\}$.

2.3 Demand Functions

After studying the consumer environment, we need to establish what is the **primitive** or what is the data that we are going to try to model. The primitive is the demand function. The demand function is a mapping $x : \mathbb{R}^L_+ \backslash \{0\} \times \mathbb{R}_+ \mapsto X$ $((p,w) \mapsto \overline{x})$ that takes as inputs price-wealth pairs and produces a commodity bundle such that $x(p,w) \in B(p,w)$ (i.e., $p'x(p,w) \leq w$).

In fact $x_l(p,w)$ denotes the quantity of commodity l consumed at prices and wealth (p,w).

Observe that x is an empirical object, we have not explained yet how it came to be. Also note that we have made a very strong assumption when taking a function as our primitive. Observing a demand function means that we know what the consumer is going to choose at all possible price-wealth pairs which is clearly impossible to do. But do not fret, there are ways to deal with this in a reasonable way.

2.4 Consistency in Consumption

Now that we have established the ingredients, so to speak, (i.e., the environment and the primitive of our study), we are going to proceed to the modelling part of our study of consumption. The first approach to 'model' consumption is to talk about consistency in consumption choices. This is a somewhat different approach to modelling consumption to what you may have studied at the undergraduate level.

We will impose a set of restrictions or axioms over the Marshallian demand function that we expect to hold in all situations irrespective of the details. Here we understand modelling consumption as restricting the set of allowable behaviors.

You may not want to call this 'modelling', but in any case, economists like to think of this axioms a the minimal requirements that a demand function has. Sadly, we must underline that this restrictions are the product of the instrospection of theorist and not the result of careful empirical work that has established them as stylized facts. However, the theoretical principle of requiring minimal conditions or imposing minimal restrictions on the set of allowable behaviors is very important if we want to understand any behavioral phenomenon.

Axiom 1. *(Walras' law) A demand function x is said to satisfy Walras' law if and only if $p'x(p, w) = w$ for all price-wealth pairs.*

You may find little objection to Walras' law, but this simple restriction rules out a lot of consumption behaviors that we do not want to pay attention when thinking about the consumer problem.

Axiom 2. *(No money illusion/ Homegeneity of Degree Zero (HD0)) A demand function x is said to satisfy Homogeneity of Degree Zero if and only if for any $\alpha > 0$ $x(\alpha p, \alpha w) = x(p, w)$.*

The HD0 restriction is clearly stronger and here we are assuming that the consumer is intelligent enough so that he is not fooled by spurious changes in wealth that are countered by a change in prices of the same magnitude.

Axiom 3. *(Weak Axiom of Revealed Preference -WARP) A demand function x is said to satisfy the WARP if and only if for any two distinct price-wealth pairs (p, w) and $(p*, w*)$, and $x(p, w) \neq x(p*, w*)$ it follows that if $p'x(p*, w*) \leq w$ then $(p*)'x(p, w) > w*$.*

The WARP is the most interesting of the restrictions presented until this point. It captures the idea that there is stability in the choices that consumers make. Notice the 'revealed preference' in the name of the WARP, it comes from the idea that by observing demand behavior underlying consumer preferences are revealed to the observer.

Let's define a revealed preference relation R^D that stands for "directly revealed preferred to", this is a binary relation on the consumption set X (i.e., $R^D \subseteq X \times X$) defined as follows:

Let (p^1, x^1) and (p^2, x^2) be observed consumption bundles with its corresponding price, then

$$x^1 R^D x^2 \iff p^{1\prime} x^1 \geq p^{1\prime} x^2.$$

Notice that the hypothesis in the WARP says $p'x(p*, w*) \leq w$, let $y = x(p*, w*)$ and recall that by Walras' law $p'x(p, w) = w$, also let $z = x(p, w)$. This means that $w = p'z \geq p'y$ which implies that zR^Dy, or in words, z is directly revealed preferred to y. The WARP is then the requirement that if zR^Dy then it has to be the case that we do not allow $yR^D(P^D)z$. This means that we are not allowing that $(p')'z \leq (p')'y$ or equivalently it must be the case that $(p*)'x(p, w) > w*$. The binary relation R^D is declared to satisfy the weak axiom. Recall that this relation is not necessarily representing actual tastes/preferences, we do not need them to do that. We are only establishing a condition that for many economists (back in the time of Samuelson) was the minimal requirement that all consumers fulfilled.

2.5 Comparative Statics and Slutsky Matrix

After establishing the axioms of behavior, the nest step is to study its implications in terms of the parameters of the model. The parameters of the model in our study of consumer behavior are prices and wealth pairs.

- We need some preliminaries and notation. We denote $D_p x(p,w) = (\partial_{p_k} x_l(p,w))_{l,k\in\{1,\cdots,L\}} \in \mathbb{R}^{L\times L}$ the matrix of price effects of the demand function.
- We let $D_w x(p,w) = (\partial_w x_l(p,w))_{l\in\{1,\cdots,L\}}$ be the vector of wealth effects.
- We say that the **Slutsky Matrix** of a demand function x, is $S(p,w) = D_p x(p,w) + D_w x(p,w)x'$ or the total change of demand with respect to a "Slutsky compensated variation of prices". A Slutsky compensated variation of prices is a price change that leaves the previously commodity bundle (at the original price situation) affordable at the new price situation. $S_{lk} = \partial_{p_k} x_l(p,w) + \partial_w x_k(p,w)x_l$.

Think of the Slutsky compensated demand $x(p,p'q)$ for q a commodity bundle I want to make affordable then the derivative with respect to prices of this compensated demand is $D_p x(p,p'q) = D_p x(p,p'q) + D_w x(p,p'q)q'$, when we fix $q = x$ to the original commodity bundle we get the Slutsky matrix $S(p,w) = D_p x(p,w) + D_w x(p,w)x' = D_p x(p,p'q)|_{q=x}$.

Implications of Walras' law for the the Slutsky Matrix.

Fact 2. *Let x satisfy Walras' law then: (i) $p'D_w x(p,w) = 1$, (ii) $p'D_p x(p,w) = -x(p,w)'$, (iii) $p'S(p,w) = 0$.*

The previous fact part (i) implies that there exists at least one normal good, the reason is that there has to be at least one l such that $\partial_w x_l(p,w) > 0$ for $\sum_l p_l' \partial_w x_l(p,w) = 1$. In general, you can see that the Walras' law also restricts the behavior of the wealth effects. The previous fact part (iii) are restrictions on the behavior of the consumption changes to Slutsky compensated price changes.

Implications of Homogeneity of Degree Zero.

Claim 1. Let x satisfy Walras' law and homogeneity of degree zero then $S(p,w)p = 0$.

Proof. By HD0 we have that $x(\lambda p, \lambda w) = x(p,w)$ for all $\lambda > 0$, we derive by λ and obtain.

$D_p x(\lambda p, \lambda w)p + D_w x(\lambda p, \lambda w)w = 0$. We let $\lambda = 1$, and recall that $p'x(p,w) = w$ by Walras' law then:

$D_p x(p, w)p + D_w x(p, w)p'x = S(p, w)p.$

□

Implications of WARP.

The main implication of the WARP while assuming Walras' law is the Compensated Law of Demand (CLD).

Definition 14. (CLD) We say that x satisfies the Compensated Law of Demand if and only if $(p* - p)'(x(p*, w*) - x(p, w)) < 0$ when $x(p*, w*) \neq x(p, w)$ and $w* = (p*)'x(p, w)$.

We are going to establish that WARP implies the Law of Demand.

Proposition 2. *Let x satisfy the Walras' law and WARP then x satisfies the CLD.*

Proof. Let $z = x(p, w)$ and $y = x(p^*, w^*)$.

By the assumption that $p^*x(p, w) = w^*$ and by Walras' law we have that:$p'z = w$, and $(p*)'z = w*$. Also we have $(p*)'y = w*$.

Thus $(p*)'(y - z) = 0$. By WARP since $(p*)'z \leq w*$ it must be the case that $p'y > w$.

We have then $p'y - p'z > 0$ or $p'(y - z) > 0$ subtract this from that $(p^*)'(y - z) = 0$ to obtain:

$$(p^* - p)'(y - z) < 0,$$

$y \neq z.$

□

Now we are going to establish a corollary of this result.

Corollary 2. *Let x satisfies the Walras' law and WARP then x is such that $S(p, w)$ is negative semi-definite ($S(p, w) \leq 0$) for all price-wealth pairs.*

Proof. If x satisfies the Walras' law and WARP then,

$$(p^* - p)'[x(p^*, p^*x(p, w)) - x(p, w)] \leq 0,$$

I consider a case such that p^*, such that $dp = (p^* - p)$.

The differential version of the CLD is:

$$dp'dx \leq 0.$$

I compute $dx = D_p x(p,w)dp + D_w x(p,w)dw$, notice that by Slutsky compensation:

$$dw = x(p,w)'dp = dp'x(p,w).$$

$$dp'dx = dp'[D_p x(p,w) + D_w x(p,w)x(p,w)]dp,$$

$$dp'dx = dp'S(p,w)dp \leq 0.$$

□

Exercise 1. Write down a Giffen goods demand function. Compute the Slutsky and the substitution matrix compare. Write down a utility function that produces giffen goods, compute the demand, slutsky matrix and substition matrix. Show that the Slutsky matrix is NSD for the giffen good.

$x(p,w)' = (x_1^G(p,w), x_2(p,w))'$

$x_1^G(p,w) = p_1,$

$p_1x_1 + p_2x_2 = w$

$x_2(p,w) = \frac{w}{p_2} - p_1^2.$

$D_p x = \begin{pmatrix} 1 & 0 \\ -2p_1 & -\frac{w}{p_2^2} \end{pmatrix}$

$D_w x = \begin{bmatrix} 0 \\ \frac{1}{p_2} \end{bmatrix}$

$S(p,w) = \begin{pmatrix} 1 & 0 \\ -2p_1 & -\frac{w}{p_2^2} \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{p_2} \end{bmatrix} [p_1 \quad (\frac{w}{p_2} - p_1^2)] = \begin{pmatrix} 1 & 0 \\ -2p_1 & -\frac{w}{p_2^2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{p_1}{p_2} & \frac{w}{p_2^2} - \frac{p_1^2}{p_2} \end{pmatrix}$

Ramen noodles vs. meat.

Consider a consumer that has to decide how much to consume of two goods x_1 (Ramen noodles) and x_2 (meat). The consumer utility function for the two goods is given by:

$$u(x_1, x_2) := \alpha_1 ln(x_1 - \gamma_1) - \alpha_2 ln(\gamma_2 - x_2)$$

$$\alpha_1 ln(x_1 - \gamma_1) + \alpha_2 ln(\frac{1}{\gamma_2 - x_2})$$

where $\alpha_2 > \alpha_1 > 0$ are positive numbers, and γ_1, γ_2 are positive numbers that represent a "basic need" level of consumption of the goods.

We are going to assume that the solution to the problem has to satisfy that $x_1 > \gamma_1$, and $0 \leq x_2 < \gamma_2$. Variables x_1 and x_2 refer the the quantities of Ramen noodles and meat respectively.

The idea here is that Ramen noodles provide a subsistence level of categories and the consumer gets positive utility from being above this level. In contrast, x_2 is a luxury good, so the consumer will not go over the level γ_2, however she gets positive utility from consuming meat.

The marginal rate of substitution between good x_1 and x_2 is:

$$-\frac{dx_1}{dx_2} = \frac{\alpha_1(\gamma_2 - x_2)}{\alpha_2(x_1 - \gamma_1)} > 0.$$

The demand functions for this model are:

$$x_1 = \gamma_1 - \frac{\beta_1}{p_1}(w - p_1\gamma_1 - p_2\gamma_{x2})$$

$$x_2 = \gamma_2 + \frac{\beta_2}{p_2}(w - p_1\gamma_1 - p_2\gamma_{x2})$$

$\beta_1 = \frac{\alpha_1}{(\alpha_2-\alpha_1)}$, $\beta_2 = \frac{\alpha_2}{(\alpha_2-\alpha_1)}$,
for $w - p_2\gamma_{x2} < p_1\gamma_1 < w - p_2\gamma_{x2} + \frac{p_2\gamma_2}{\beta_2}$ that ensure that
$x_1 > \gamma_1$, and $0 \leq x_2 < \gamma_{x2}$.

Fix a point in this domain (in the interior of the domain) of prices and wealth, such that we have the solution above and obtain the Slutsky matrix.

Price Effects:

$$D_p x(p,w) = \begin{pmatrix} -\frac{\alpha 1(w - \gamma 2 \mathrm{p}2)}{\mathrm{p}1^2(\alpha 1 - \alpha 2)} & \frac{\alpha 1 \gamma 2}{\mathrm{p}1(\alpha 2 - \alpha 1)} \\ \frac{\alpha 2 \gamma 1}{\alpha 1 \mathrm{p}2 - \alpha 2 \mathrm{p}2} & \frac{\alpha 2(w - \gamma 1 \mathrm{p}1)}{\mathrm{p}2^2(\alpha 1 - \alpha 2)} \end{pmatrix}$$

The $\partial_{p_1} x_1 \geq 0$ when $w - p_2\gamma_2 \geq 0$. The slope of the demand for Ramen noodles has upward slope! Why? When the price of Ramen increases then the consumer now is poorer than before. This is the Giffen Paradox, in which, under subsistence conditions, a rise in the price of a cheap foodstuff (Ramen noodles) can force poor families to consume more, rather than less of it.

Wealth Effects:

$$D_w x(p,w) = \begin{pmatrix} \frac{\alpha 1}{\alpha 1 \mathrm{p}1 - \alpha 2 \mathrm{p}1} \\ \frac{\alpha 2}{\mathrm{p}2(\alpha 2 - \alpha 1)} \end{pmatrix}.$$

The consumption of Ramen noodles will decrease as wealth increases. The richer the families are the lowest the consumption of Ramen noodles.

Slutsky Matrix:

We have seen that the price effects are an imperfect measure of the effects of prices in consumption. Due to the fact that prices determine the choice set by their effect in the budget constraint they have two effects. The first effect is the direct price effect on consumption, the second through wealth or purchasing power. The Slutsky matrix corrects for the second effect and gives you a net effect:

$$S(p,w) = \begin{pmatrix} \frac{\alpha1\alpha2(-\gamma1\mathrm{p}1-\gamma2\mathrm{p}2+w)}{\mathrm{p}1^2(\alpha1-\alpha2)^2} & \frac{\alpha1\alpha2(\gamma1\mathrm{p}1+\gamma2\mathrm{p}2-w)}{\mathrm{p}1\mathrm{p}2(\alpha1-\alpha2)^2} \\ \frac{\alpha1\alpha2(\gamma1\mathrm{p}1+\gamma2\mathrm{p}2-w)}{\mathrm{p}1\mathrm{p}2(\alpha1-\alpha2)^2} & \frac{\alpha1\alpha2(-\gamma1\mathrm{p}1-\gamma2\mathrm{p}2+w)}{\mathrm{p}2^2(\alpha1-\alpha2)^2} \end{pmatrix}$$

$$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$$

By our conditions $w - p_2\gamma_{x2} < p_1\gamma_1$ it must be that the Slutsky compensated effect of the Giffen good (Ramen noodles) is negative $S(p,w)_{11} \leq 0$.

Chapter 3

Traditional Consumer Theory

3.1 Introduction

The traditional approach to consumer theory is to assume that the consumer has well-defined preferences over all alternative bundles and that the consumer attempts to select the most preferred bundle from among those available to him. In other words, we are trying to write a model of how consumer decide and choose. The main practical benefit of this type of approach is that we can write tracktable/computable economic models that include “intelligent” consumers.

3.2 Rational Preferences

The most powerful modelling technique to represent consumer tastes are binary relations. We say that a consumer is rational if his preferences can be represented by a binary relation $\succeq \subseteq X \times X$ that has two properties: (i) Completeness. (ii) Transitivity. Alternatively, this is a complete/total preorder (reflexive, transitive and complete). Observe that completeness implies reflexivity.

Definition 15. (Rational Preferences) A binary relation $\succeq$ on X is said to be a rational preference relation if and only if it is a complete/total preorder on X.

When we have $x \succeq y$ we say that $x \in X$ is “at least as good” as $y \in X$ or equivalently x is preferred to y. The antisymmetric part of $\succeq$ is $\succ$ (note that $\succ \cup D$ where $D = \{(a, a) \in X \times X\}$ is the diagonal ordering is a linear order). The symbol $\succ$ is used to represent strict preferences or $x \succ y$ means x is strictly preferred to y. The symmetric part of $\succeq$ represents indifference in preferences, when we have $x \sim y$ we say that x is indifferent to y.

Important Sets

We are going to study three very important sets in consumer theory.

Definition 16. (Upper contour set) Let $\succeq$ be a rational preference on X. The upper contour set of $\succeq$ and $x \in X$ is $B^{\succeq}(x) = \{y \in X | y \succeq x\}$.

This set captures the better than equal alternatives of any given commodity under the rational preferences.

Definition 17. (Lower contour set) Let $\succeq$ be a rational preference on X. The lower contour set of $\succeq$ and $x \in X$ is $L^{\succeq}(x) = \{y \in X | x \succeq y\}$.

And finally the well-known set (indifference curve/set).

Definition 18. (Indifference set) Let $\succeq$ be a rational preference on X. The indifference set of $\succeq$ and $x \in X$ is $I^{\succeq}(x) = \{y \in X | x \sim y\}$.

For now we will just state the definitions, in a moment, we will use them to prove important results of consumer theory.

Now we are going to state properties of the rational preferences, that we are going to impose across our study of the consumer theory.

Perhaps the most important one is **monotonicity.**

Definition 19. ($\succeq$ monotonicity) A preference relation $\succeq$ is monotone if $x \succ y$ for any $x, y \in X$ such that $x_l > y_l$ for all commodities l. It is strongly monotone if $x_l \geq y_l$ for all l and $x_j > y_j$ for some j implies $x \succ y$.

Monotonicity and strong monotonicity correspond to the behavior idea of "more is better".

Preferences that are strongly monotone are monotone but not the other way around. If preference are monotone or strongly monotone it follows immediately that the consumer will choose something on the frontier of his budget constraint. Clearly, this implies Walras' law.

An even weaker condition on preferences that implies the Walras' law and is very useful in our upcoming results is local non-satiation.

Definition 20. ($\succeq$ local non-satiation) A preference relation $\succeq$ satisfies local non-satiation if for every x and every $\epsilon > 0$ there is a point y such that $||x - y|| \leq \epsilon$ and $y \succ x$.

The local non-satiation assumption also rules "thick" indifference sets.

Example 9. If a demand function x is the result of maximizing a locally non-satiated preference relation $\succeq$ on $B(p,w)$, then $p'x(p,w) = w$ ($P \implies Q$) , $P = x$ is $\succeq$-maximizing and affordable such that $p'x \leq w$, and $\succeq$ is LNS. $Q = p'x(p,w) = w$ (Walras' Law). I negate the statement P and $\neg Q$, I will prove that this gives me a contradiction. (P and $\neg Q$) x is $\succeq$ maximizing, affordable and $\succeq$ are LNS and $p'x(p,w) < w$. By LNS I know that there exist and $\epsilon > 0$ small enough such that there exist a $y \in B(p,w)$ (affordable) such that $||y - x|| \leq \epsilon$, such that $y \succ x$. But that implies that x cannot be $\succeq$ maximizing because this implies that $x \succeq y$ for all $y \in B(p,w)$. This is a contradiction. I conclude that $P \implies Q$.

Another very important assumption that we are going to use in the upcoming chapters is convexity of preferences.

Definition 21. ($\succeq$ Convexity) Preferences are convex whenever $x \succeq y$ and $z \succeq y$ then $tx + (1 - t)z \succeq y$ for all $t \in [0,1]$. Preferences are strictly convex if whenever $x \succeq y$ and $z \succeq y$ and $x \neq z$ then for every $t \in [0,1]$ we have $tx + (1-t)z \succ y$.

Alternatively, we can say that consumers with convex preferences prefer averages to extremes. For example if $x \succeq y$ then for all $t \in [0,1]$ $tx + (1-t)y \succeq x$. The preferences are strictly convex if $x \sim y$ then $tx + (1-t)y \succ x$ and $tx + (1-t)y \succ y$.

3.3 Utility Representation Theorem

Preferences are the most general way to represent consumer tastes. However, they are not particularly easy to work with when modelling complex economic problems. For this reason, and also because economists are more used to deal with real analysis/calculus than with order theory we usually use a utility function to represent consumer preferences.

A utility function is a continuous mapping from the choice set to the reals, $u : X \mapsto \mathbb{R}$. A utility function is a representation of a preference relation $\succeq$ when $u(a) \geq u(b)$ if and only if $a \succeq b$.

Theorem 2. *If $\succeq$ are complete and transitive and X is finite then there exists a continuous mapping $u : X \mapsto \mathbb{R}$, that represents the preference $\succeq$:*

$$u(a) \geq u(b) \iff a \succeq b, \forall a, b \in X.$$

Proof. Necessity: (We want to show) If there is a utility function $u : X \to \mathbb{R}$ such that for any $a, b \in X$, if $u(a) \geq u(b)$ then $a \succeq b$, with $\succeq$ is rational (complete and transitive).

If there is a utility function u, such that $u(a) \geq u(b)$ then I declare a preference relation $a \succeq b$ whenever $u(a) \geq u(b)$.

I want to check that $\succeq$ is complete, this follows from the fact that either $u(a) \geq u(b)$ or $u(b) \geq u(a)$, $u(a), u(b) \in \mathbb{R}$.

I want to check that $\succeq$ is transitive, this follows from the fact that if $u(a) \geq u(b)$ and $u(b) \geq u(c)$ then it must be that $u(a) \geq u(c)$, this implies that $\succeq$ constructed above is transitive itself because it inherits the transitivity from the order $\geq$ on $\mathbb{R}$.

Sufficiency: If there is an ordering $\succeq$ on X that is rational (complete and transitive) then there exist a utility mapping u, such that if for any $a, b \in X$, if $a \succeq b$ then $u(a) \geq u(b)$.

I want to prove that if $a \succeq b$ for any $a, b \in X$ (X finite) then there is a utility function $u : X \to \mathbb{R}$ such that $u(a) \geq u(b)$.

First, I build the Lower Contour Set of $\succeq$ for any $x \in X$:

$$L^{\succeq}(x) = \{y \in X | x \succeq y\}.$$

Now, I propose the following utility function:

$$u(x) = |L^{\succeq}(x)|,$$

where $|\cdot|$ is the cardinality of a set.

The first step is to check that the candidate utility function is well-defined:

$X \backslash \sim$, $u = \pi \circ \hat{u}$, where π is taking the canonical projection to the Quotient set $X/\sim$.

$\hat{u} : X \backslash \sim \to \mathbb{R}$, is defined used the linear order induced by $\succeq$, as follows $[a]_\sim \succ^* [b]_\sim \iff a \succ b$ for any $a, b \in X$, the linear order induces an ennumeration of items from best to worst, in fact use the strict lower counter set so that $\hat{u}(x) = |L^{\succ^*}(x)|$ for $x \in X \backslash \sim$.

I want to check that for any $y \sim x$ such that $y \neq x$,

then $|L^{\succeq}(x)| = |L^{\succeq}(y)|$. In fact, we know that by definition $L^{\succeq}(x) = \{y \in X | x \succ y\} \cup I^{\succeq}(x)$.

I want to show that $I^{\succeq}(x) \equiv I^{\succeq}(y)$ when $x \sim y$, first if $z \in I^{\succeq}(x)$ it must be that $z \sim x$, by assumption $x \sim y$, by transitivity I conclude that $z \sim y$ therefore $z \in I^{\succeq}(y)$.

If $z' \in I^{\succeq}(y)$ that means that $z' \sim y$ and by assumption $y \sim x$, by transitivity I conclude that $z' \sim x$ therefore $z' \in I^{\succeq}(x)$.

I know that this is the same set for both $x \sim y$, then the cardinality has to be the same.

The second step is to prove that this particular construction represents the preference relation: $a \succeq b \implies |L^{\succeq}(a)| \geq |L^{\succeq}(b)|$.

It means that $a \succeq c$ for all $c \in L^{\succeq}(b)$ by transitivity, because $a \succeq b$ and $c \in L^{\succeq}(b)$ is such that $b \succeq c$, we conclude that $a \succeq c$. This implies that $L^{\succeq}(b) \subseteq L^{\succeq}(a)$.

We conclude that in fact, if $\succeq$ are such that $a \succeq b$ for any $a, b \in X$, then there is a mapping $u : X \to \mathbb{R}$ such that $u(a) \geq u(b)$.

□

We must first remark that not all rational preferences can be represented by a utility function. The assumption that we need on preferences to guarantee that they can be represented by a utility function is **continuity.**

Definition 22. ($\succeq$-Continuity) Let $x^n \succeq y^n$ for all $n \in N$ then if $x^n \to x$ and $y^n \to y$ it follows that $x \succeq y$.

Theorem 3. *If $\succeq$ are complete, transitive and continuous then there is a continuous mapping $u : X \mapsto \mathbb{R}$ such that $u(x) \geq u(y)$ if and only if $x \succeq y$ for all $x, y \in X$.*

We are going to prove this result formally in what follows, for now let's take it as given.

Ordinal Meaning of the Utility Function.

The utility representation of a rational preference relation has only ordinal meaning.

Theorem 4. *If $\succeq$ are represented by $u : X \mapsto \mathbb{R}$, then there exists a monotone mapping $f : \mathbb{R} \mapsto \mathbb{R}$ such that $v = f \circ u$ also represents $\succeq$.*

The proof is very simple and you should do it for practice.

3.4 Utility Maximization Problem (UMP)

We assume that preferences are well-behaved, in particular preferences are rational, continuous, strictly convex and locally nonsatiated. These assumptions imply that the consumer has a continuous utility function $u : X \mapsto \mathbb{R}$ and that the consumer choices will satisfy Walras' law. For now we will assume that the preferences are also differentiable.

The consumer's problem or utility maximization problem (UMP) is to choose the bundle that maximizes utility from among those available to him.

The set of available bundles is given by the Budget Constraints $B(p, w) = \{x \in X | p'x \leq w\}$. We will assume that prices are positive $p >> 0$ and that wealth is strictly positive as well. The consumer's problem can be written as

$$max_{x \geq 0} u(x)$$

s.t. $p'x \leq w$.

First we establish that the problem has a solution.

Lemma 1. *The UMP has a solution. Moreover, the solution is unique.*

Proof. Recall that u is continuous, and $B(p,w)$ is closed and bounded set (i.e., for $X \subseteq \mathbb{R}^L$ this implies that X is compact). By the Weiestrass theorem, we know that a continuous function on a compact set always achieves its maximum. Uniqueness follows from the strict convexity of preferences, this means that u is strictly quasi-concave and given that $B(p,w)$ is convex too then the solution to the maximization problem is also unique.

□

Now we are going to see how to find the solution under the differentiability assumption.

We write the KKT FOC necessary conditions.

$\mathcal{L} = u(x) + \lambda(w - p'x)$

$\partial_{x_l} u(x^*) - \lambda^* p_l \leq 0$ and $x_l[\partial_{x_l} u(x^*) - \lambda^* p_l] = 0$ for all $l \in \{1, \cdots, L\}$.

$w - p'x^* \geq 0$ and $\lambda^*(w - p'x^*) = 0$.

This is a system with $L + 1$ unknowns.

We are going to simplify a little bit the problem. By local non-satiation we know that the consumer is on the budget hyperplane (frontier). This means that

$$w - p'x^* = 0.$$

Also we are going to assume that the solution in interior, this means that the KKT conditions are reduced to

$$\frac{\partial_{x_l} u(x^*)}{\partial_{x_k} u(x^*)} = \frac{p_l}{p_k} \quad \forall l, k \in \{1, \cdots, L\}.$$

Note that for $L = 2 - \frac{p_1}{p_2}$ is the slope of the budget line and $-\frac{\partial_{x_1} u(x^*)}{\partial_{x_2} u(x^*)}$ is the slope of the indifference curve, usually this is also referred as the MRS.

3.5 Properties of the Marshallian Demand Function

The Marshallian demand function is the unique solution to the UMP problem above. $x(p, w) = argmax_{x \geq 0} u(x)$ s.t. $p'x \leq w$.

1. Walras' Law: $p'x(p, w) = w$ for all p, w. This follows from local non-satiation.
2. Homogeneity of degree zero in (p, w). The definition of homogeneity is the same as always. It is very easy to see that $B(p, w) \equiv B(\alpha p, \alpha w)$ for all $\alpha > 0$, thus changing α does not alter the constraint set. Check the FOC conditions also.
3. WARP. The Marshallian demand x satisfies WARP.
4. The Slutsky Matrix $S(p, w)$ is NSD, singular in prices and symmetric.

3.6 Implication of Rationality and Recoverability

Theorem 5. *A demand function x that satisfy Walras' law and is cont. differentiable can be generated by the UMP if and only if $S(p, w)$ is symmetric and NSD.*

3.7 Properties of the the Indirect Utility

The indirect utility is the maximal utility achieved given the budget constraint $v(p, w) = max_{x \geq 0} u(x)$ s.t. $p'x \leq w$.

Alternatively it is the utility evaluated in the Marshallian demand $v(p, w) = u(x(p, w))$.

1. Homogeneity of degree zero.
2. $v(p, w)$ is strictly increasing in w and non-increasing in p_l.

We have that $v(p, w)$ is increasing in w due to local non-satiation.

3. $v(p, w)$ is quasiconvex in (p, w). The set $\{(p, w) | v(p, w) \leq \overline{v}\}$ is convex.
4. $v(p, w)$ is continuous in p and w.

3.8 Roy's Identity

Recall the definition of indirect utility $v(p, w) = u(x(p, w))$.

We use the Envelope's theorem

$v(p, w) = max_{x \geq 0} u(x) + \lambda(w - p'x)$

$\partial_w v(p, w) = \lambda.$

$\nabla_p v(p,w) = -\lambda x$

This provides the Roy's identity

$$x(p,w) = -\frac{\nabla_p v(p,w)}{\partial_w v(p,w)}.$$

3.9 Expenditure Minimization Problem (EMP)

The Expenditure Minimization Problem (EMP) asks the question, if prices are p, what is the minimum amount the consumer would have to spend to achieve utility level $\overline{u}$. That is:

$$min_{x \geq 0} p'x$$

s.t.

$$u(x) \geq \overline{u}.$$

The Lagrangian for this problem is:

$$\mathcal{L}_{EMP} = p'x - \lambda(u(x) - \overline{u}).$$

Assuming an interior solution, the FOC are given by

$$p - \lambda \nabla_x u(x) = 0$$

$$\lambda(u(x) - \overline{u}) = 0$$

If u is quasiconcave, increasing in each argument then the constraint will bind and we can write the second constraint as $u(x) = \overline{u}$ and a unique solution for the problem will exists.

The solution to this problem $h(p,\overline{u}) = argmin_x p'x$ s.t $u(x) \geq \overline{u}$ is called the **Hicksian demand**.

The **expenditure function** is the minimum $e(p,\overline{u}) = p'h(p,\overline{u})$ value attained at the given prices and target utility.

There is a **duality relation** between the EMP and the UMP that we will not explore at this point. However, we will state the following relations.

$$h(p, v(p,w)) = x(p,w)$$

$$x(p, e(p, u)) = h(p, u).$$

Also,

$$\overline{u} = v(p, e(p, \overline{u}))$$

$$w = e(p, v(p, w)).$$

3.10 Properties of the Hicksian Demand and Expenditure Functions.

Properties of the Hicksian Demand Function.

1. Homogeneity of degree zero in p.

Why? Because $min_x \alpha p'x$ s.t. $u(x) \geq \overline{u}$ is the same as optimizing

$min_x p'x$ s.t. $u(x) \geq \overline{u}$, $\alpha > 0$ changes the minimum expenditure (objective function value) but not the minimizer, the hicksian demand.

2. No excess utility $u(h(p, \overline{u})) = \overline{u}$. This follows from continuity. Suppose not, $u(h(p, \overline{u})) > \overline{u}$ then we have h' that is smaller than $h(p, \overline{u})$ on all dimensions such that $p'h'(p, \overline{u}) < p'h(p, \overline{u})$, this is a contradiction.

3. Under strict convexity recall that $h(p, \overline{u})$ is unique. Without it, $h(p, \overline{u})$ is a set, and convexity makes it a convex set.

Properties of the Expenditure Function.

1. $e(p, \overline{u})$ is homogeneous of degree one in p.

2. $e(p, \overline{u})$ is strictly increasing in $\overline{u}$ and non-decreasing in p_l for any l.

3. $e(p, \overline{u})$ is concave in p. $e(\alpha p + (1-\alpha)p', \overline{u}) \geq \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$.

Let $p^\alpha = \alpha p + (1-\alpha)p'$

$e(p^\alpha, \overline{u}) = p^{\alpha\prime} h(p^\alpha, \overline{u})$

$= \alpha p' h(p^\alpha, \overline{u}) + (1-\alpha)(p')' h(p^\alpha, \overline{u})$

$\geq \alpha p' h(p, \overline{u}) + (1-\alpha)(p')' h(p', \overline{u})$.

Because $\alpha p' h(p^\alpha, \overline{u}) \geq \alpha p' h(p, \overline{u}) = e(p, \overline{u})$ by optimality of $h(p, \overline{u})$.

3.11 Sheppard's Lemma

Again using the Envelope's theorem

$$e(p,\overline{u}) = p'h + \lambda(u(h) - \overline{u})$$

We obtain:

$$\nabla_p e(p,\overline{u}) = h(p,\overline{u})$$

3.12 Additional Properties

1. $D_p h(p,u) = \mathcal{H}e(p,u)$ the derivative with respect to prices of the Hicksian demand is the Hessian of the expenditure function. $D_p h(p,u) = \nabla_{p'}\nabla_p e(p,u)$.

2. $D_p h(p,u)$ is a NSD matrix this follows from the fact that a Hessian of a concave function is NSD.

3. $D_p h(p,u)$ is symmetric. Young's theorem (continuity of e)

4. $D_p h(p,u)p = 0$ follows from HD0 of $h(p,\overline{u})$ in p and Euler's identity.

3.13 Slutsky Equation

We establish now a very useful equation.

$$h(p,u) = x(p, e(p,u))$$

Differentiating with respect ot p

$$D_p h(p,u) = D_p x(p,e(p,u)) + D_w x(p,e(p,u))\nabla_p e(p,u)'$$

By Sheppard's lemma and by the identities that relate the expenditure and wealth

$$D_p h(p,v(p,w)) = D_p x(p,w) + D_w x(p,w)x(p,w)'.$$

3.14 Implications of Rationality

Theorem 6. *A demand function x that satisfy Walras' law and is continuously differentiable can be generated by the UMP if and only if $S(p, w)$ is symmetric and NSD.*

The full proof is beyond the reach of the course but we have already proved the necessity part.

3.15 Welfare Evaluation

We will compare the wealth needed to achieve a certain level of utility at a given price-wealth pair.

$$u^1 = v(p^1, w)$$

$$u^0 = v(p^0, w).$$

We are interested in comparing the expenditure needed to achieve u^1 or u^0.

For a fixed reference price and a given utility level u, the needed expenditure to reach it is given by

$$e(p^{ref}, u).$$

Thus the comparison in terms of expenditure is given by

$$e(p^{ref}, u^1) - e(p^{ref}, u^0)$$

$$e(p^{ref}, v(p^1, w)) - e(p^{ref}, v(p^0, w)).$$

This expression will be positive when it takes more wealth to achieve u^1 at prices p^{ref} than to achieve u^0. This difference is measured in dollar terms. That is why $e(p^{ref}, v(p, w))$ is often called money metric indirect utility function.

We can construct many money metric indirect utility functions with different reference price.

When $p^{ref} = p^0$ is the original price situation then the change is expenditure is equal to the change in wealth such that the consumer would be indifferent between the new price with the old wealth and the old price and the new wealth. This asks what change is wealth would be equivalent to the change in price.

The **equivalent variation is**

$$EV(p^0, p^1, w) = e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = e(p^0, v(p^1, w)) - w,$$

since $e(p^0, v(p^0, w)) = w$.

The Equivalent variation is the amount of money the consumer is willing to pay to avoid a price change.

The other case considered is the one where the new price is taken as the reference price, when $p^{ref} = p^1$, the change is expenditure is equal to the change in wealth such that the consumer is indifferent between the original situation (p^0, w) and the new situation $(p^1, w + \Delta w)$.

The **compensating variation** is defined as:

$$CV(p^0, p^1, w) = e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = w - e(p^1, v(p^0, w)).$$

How much must I compensate you to make you as well off as you were before the price change?

Example:

When is one concept more useful than the other? – Los Angeles decides to build a new freeway which cuts through a neighborhood. How much would the city have to pay the residents of this neighborhood to keep them as well off as they were before? CV – What is the most the residents would pay not to have the freeway? EV

Some Additional Material

1. Microeconomic Theory MWG.
2. MWG has a Mathematical Appendix. Correspondences and Duality.
3. Matrix Cookbook Notes Online. Matrix/Vector Derivatives. Shortcut to understand derivation in Multivariate Cases.
4. Real Analysis for Economists Efe Ok. Order Theory. Efe Ok.
5. Notes Online David Levine in Standford General Equilibrium Notes. Nolan Miller Illinois he has notes on Consumer Theory plus Partial Equilibrium.

6. Micro Kreps Vol 1. Advanced Varian could be useful for the Revealed Preference Part.

7. Jehle & Reny Advanced Micro Theory.

Chapter 4

Preference and Choice (MGW Cp 1)

4.1 Introduction

Decision Theory uses three languages:

1. Choice rules or correspondences
2. Preferences relations.
3. Utility functions.

4.2 Choice structure/environment.

Let X be non-empty, choice set.

Definition 23. Choice correspondence: $c : 2^X \backslash \emptyset \mapsto 2^X \backslash \emptyset$, $c(A) \subseteq A$ for all $A \in 2^X$.

If X is infinite sometimes we don't want deal with the whole power set.

Let $\mathcal{B} \subseteq 2^X \backslash \emptyset$ this set of menus.

- Choice Structure is pair $(\mathcal{B}, c)$ where $c(A) \subseteq A$ and $A \in \mathcal{B}$.

-Rich dataset when X is finite: $\{a, b\} \in \mathcal{B}$, $\{a, b, c\} \in \mathcal{B}$.

Definition 24. Rationalizable choice correspondence.

c is rationalizable when $c(A) = c^*(\succeq, A) = \{b \in A | b \succeq a, \forall a \in A\}$, for any given $A \in \mathcal{B}$.

Assume that $\succeq$ is rational (i.e., complete and transitive).

4.3 Preference Relations

$\succeq$ a binary relation is a subset of the cartesian product $\succeq \subseteq X \times X$.

An asymmetric part $\succ$, $a \succ b$ if and only if $a \succeq b$ and $\neg b \succeq a$

A symmetric part $\sim$, $a \sim b$ if and only if $a \succeq b$ and $b \succeq a$.

Proposition 3. *(1.B.1 MWG) If $\succeq$ is rational then:*

(i) $\succ$ is irreflexive and transitive.

(ii) $\sim$ is reflexive, transitive and symmetric.

(iii) If $x \succ y \succeq z$ then $x \succ z$.

Proof. (i)

$\neg x \succ x$ irreflexive

$x \succ x$ it implies that $x \succeq x$ and $\neg x \succeq x$.

This is a contradiction.

Transitivity $a \succ b$, $b \succ c$ then $a \succ c$

(i) $a \succ b$ implies that $a \succeq b$ and $\neg b \succeq a$

(ii) $b \succ c$ implies that $b \succeq c$ and $\neg c \succeq b$

(i) and (ii) imply by trans. of $\succeq$ that $a \succeq c$ and $\neg c \succeq a$ (neg. transitivity).

This implies that $a \succ c$.

$x \sim x$ reflexive

$x \sim y$ this implies $y \sim x$ (symmetry)

(iii) $x \succ y \succeq z \implies x \succ z$

Case 1. $x \succ y \succ z$ this follows from (i). Transitivity of $\succ$.

Case 2. $x \succ y \sim z$ this follows from (ii).

$x \succeq y$ and $\neg y \succeq x$, and $y \succeq z$ and $z \succeq y$

ruling out $x \sim z$, by transitivity of $\succeq$ we have $x \succ z$.

□

Problems with transitivity:

- Just perceptible differences. Inattention or Missperception.
- Framing effects
- Condorcet Paradox.

  $O \succ_M R \succ_M I$, $I \succ_D O \succ_D R$ and $R \succ_C I \succ_C O$

Imagine pairs competing for a majority.

$OvsR$: O wins.

$RvsI$: R wins

$IvsO$: I wins

We conclude that the household preference relation is given by:

$O \succ R \succ I \succ O$

- Utility function is $u : X \mapsto \mathbb{R}$.
  - Utility represents $\succeq$, $u(a) \geq u(b)$ if and only if $a \succeq b$.

Proposition 4. *A preference relation $\succeq$ can be represented by a utility function only if it rational (X is finite).*

Proof. if $\succeq$ binary relation is represented by u, it means that $u(a) \geq u(b)$ iff $a \succeq b$ (by the definition of representability).

(i) $\succeq$ is complete.

$a \succeq b$ or $b \succeq a$

$u(a) \geq u(b)$ or $u(b) \geq u(a)$

(ii) $\succeq$ is transitive.

$a \succeq b$ and $b \succeq c$ this means $u(a) \geq u(b)$ and $u(b) \geq u(c)$

$u(a) \geq u(c)$ then $a \succeq c$.

□

4.4 Choice Rules

-Let the choice structure be $(\mathcal{B}, c)$

Definition 25. Weak Axiom of Revealed Preference (WARP). If for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in c(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in c(B')$ we must also have $x \in c(B')$.

If x is revealed at least preferred to y then y cannot be revealed strictly preferred to x.

$\succeq^*$: $a \succeq^* b \iff a \in c(B)$ when $b \in B$ for some $B \in \mathcal{B}$.

WARP. If $x \succeq^* y$ then $\neg y \succ^* x$.

Proposition 5. *Let $\succeq$ be a rational preference relation. Then $(\mathcal{B}, c^*(\succeq))$ satisfies the WARP.*

Proof. If $B, B' \in \mathcal{B}$, such that $x, y \in B \cap B'$: if $x \in c^*(\succeq, B)$ and $y \in c^*(\succeq, B')$, this means that $x \succeq y$ and $y \succeq x$, by the definition of $\succeq$ that means that $x \sim y$. This implies that $x \in c^*(\succeq, B')$ because $x \succeq b \forall b \in B'$ given that $y \succeq b \forall b \in B'$ and $x \sim y$, by transitivity it follows that $x \in c^*(\succeq, B')$.

□

The relation between preferences and choice correspondences.

Definition 26. We say that $\succeq$ rationalizes $c(\cdot)$ relative to $\mathcal{B}$ when $c(B) = C^*(B, \succeq)$.

Proposition 6. *If $(\mathcal{B}, c)$ is such that (i) WARP is satisfied and (ii) $\mathcal{B}$ includes all subsets of X up to three elements and X is finite. Then there is a rational $\succeq$ preference relation that rationalizes $c(\cdot)$ with respect to $\mathcal{B}$. Moreover, this is the only preference relation that does so.*

Proof. My objective is to build a model:

First we propose $\succeq \equiv \succeq^*$ where $\succeq^*$ is the revealed preference relation defined as $x \succeq^* y \iff x \in C(B)$ and $y \in B$.

(i) Check that $\succeq$ is complete. We want to check that $x \succeq y$ or $y \succeq x$ for all $x, y \in X$. **Because the data is rich** we have that all pairs $\{x, y\} \in \mathcal{B}$ are included in the dataset. This means that either $x \in c(\{x, y\})$ or $y \in c(\{y, x\})$ (because the choice correspondence is assumed to be non-empty).

(ii) Check that $\succeq$ is transitive. We have that $x \succeq y$, $y \succeq z$ when $x \in c(B \cup \{y\})$ and $y \in c(B' \cup \{z\})$, now say we have B'' such that $\{x, y, z\} \equiv B''$. We have the following cases, $z \in C(B'')$ in this case by WARP we have $y \in c(B'')$ and by WARP again $x \in c(B'')$, in which case $x \sim y \sim z$ (so $\succeq$ is transitive). The other case is $z \notin c(\{x, y, z\})$ because $c(\{x, y, z\}) \neq \emptyset$ is non-empty by assumption we know that either $x \in c(\{x, y, z\})$ in which case $x \succeq z$ and $\succeq$ is transitive, or $y \in c(\{x, y, z\})$ in which case by WARP $x \in c(\{x, y, z\})$ and $x \sim y \succeq z$ in which case $\succeq$ is transitive.

Second we check that $c^*(\succeq)$ rationalizes c.

If $x \in c(B)$ this means that $x \succeq^* y$ for all $y \in B$ then $x \in c^*(B, \succeq)$.

If $x \in c^*(B', \succeq)$ then this implies that $x \succeq y$ for all $y \in B'$ this implies that there exists some B_y for all $y \in B'$ such that $x \in c(B_y)$. Because $c(B') \neq \emptyset$, the WARP implies that $x \in c(B')$. Then

$c^*(B, \succeq) \equiv c(B)$.

Finally, for uniqueness, note that $\mathcal{B}$ being rich includes all pairs thus we determine completely the preferences over pairs.

□

Chapter 5

Consumer Choice (MGW Ch. 2)

5.1 Introduction.

We define a Walrasian demand function $x : P \times W \mapsto X$, where P is the price domain and W is the wealth domain and X is the choice set. We recall, $X \subseteq \mathbb{R}^L$, L was the number of commodities.

- Walras' Law $p'x(p,w) = w$.

- HD0 (No money illusion) $x(\alpha p, \alpha w) = x(p,w)$ $\alpha > 0$.

Proposition 7. *If the Walrasian demand function $x(p,w)$ satisfies HD0 then for all p, w.*

$$D_p x(p,w)p + D_w x(p,w)w = 0.$$

In elasticity form.

$$\epsilon_{lk}(p,w) = \partial_{p_k} x_l(p,w) \frac{p_k}{x_l(p,w)}$$

$$\epsilon_{lw}(p,w) = \partial_w x_l(p,w) \frac{w}{x_l(p,w)}.$$

We can express the previous condition as:

$$\sum_{k=1}^{L} \epsilon_{lk}(p,w) + \epsilon_{lw}(p,w) = 0,$$

for all $l = 1, \cdots, L$.

An equal percentage change in all prices and wealth implies no change in demand.

Proposition 8. *If the Walrasian demand $x(p,w)$ satisfies Walras' law then.*

- The total expenditure cannot change in response to a change in prices: $p'D_px(p,w) + x(p,w)' = 0$
- Total expenditure must change by an amount equal to any wealth change: $p'D_wx(p,w) = 1$.

In budget share form:

$$b_l(p,w) = p_l x_l(p,w)/w.$$

We have:

$$\sum_{l=1}^{L} b_l(p,w)\epsilon_{lk}(p,w) + b_k(p,w) = 0.$$

$$\sum_{l=1}^{L} b_l(p,w)\epsilon_{lw}(p,w) = 1.$$

5.2 The Weak Axiom of Revealed Preference and the Law of Demand.

Definition 27. The Walrasian demand function $x(p,w)$ satisfies the weak axiom of revealed preference if the following property holds for any two price wealth situations (p,w) and (p^*,w^*):

If $p'x(p^*,w^*) \leq w$ and $x(p,w) \neq x(p^*,w^*)$ then $p^{*\prime}x(p,w) > w^*$.

The implications of the WARP are:

Definition 28. (CLD) We say that x satisfies the Compensated Law of Demand if and only if $(p * -p)'(x(p^*,w^*) - x(p,w)) < 0$ when $x(p^*,w^*) \neq x(p,w)$ and $w^* = (p^*)'x(p,w)$.

We are going to establish that WARP implies the Law of Demand.

Proposition 9. *Let x satisfy the Walras' law and WARP then x satisfies the CLD.*

Also in differential terms. We say that the **Slutsky Matrix** of a demand function x, is $S(p,w) = D_px(p,w) + D_wx(p,w)x'$ or the total change of demand with respect to a "Slutsky compensated variation of prices".

Corollary 3. *Let x satisfy the Walras' law and WARP then x is such that $S(p,w)$ is negative semi-definite ($S(p,w) \leq 0$) for all price-wealth pairs.*

Proof. We have by Slutsky compensation that $dw = x(p,w)'dp$, also we have the CLD $dp'dx \leq 0$.

$dx = D_p x(p,w)dp + D_w x(p,w)dw$ by the total differential of x, now because the wealth compensation in the Slutsky sense $dw = x(p,w)'dp$.

$dx = D_p x(p,w)dp + D_w x(p,w)x(p,w)'dp$.

By the CLD, we have

$dp'dx = dp'D_p x(p,w)dp + dp'D_w x(p,w)x(p,w)'dp$

This means $dp'dx = dp'S(p,w)dp$, recall that any $v \in \mathbb{R}^L$ can be written as αdp for some scalar α thus, $S(p,w) \leq 0$ is NSD.

□

- Other results include $S(p,w)p = 0$ and $p'S(p,w) = 0$.

Definition 29. Rationalizable Demand. We say x (Walrasian Demand) is rationalizable if and only if there exists a preference relation $\succeq$ on X such that if $p^{t\prime}x^t = w^t \geq p^{t'}y$ this implies that $u(x^t) \geq u(y)$, where $x^t = x(p^t, w^t)$.

Chapter 6

Consumer Theory: Utility Maximization. (Ch. 3).

6.1 Demand Correspondences.

A Walrasian demand correspondence is a mapping $x : P \times W \mapsto 2^X \backslash \emptyset$.

Example 10. Observe that using Ch 4. notation a choice correspondence is defined as $c : \mathcal{B} \to 2^X \setminus \emptyset$, now we define the budget correspondence as $B : P \times W \to X$ $B(p, w) = \{x \in X | p'x \leq w\}$, now we define the demand correspondence as the composition of this two correspondences $x = c \circ B$, $(P \times W \to^B \to \mathcal{B} \to^c 2^X \setminus \emptyset)$.

We can illustrate this concept with the following example.

Example 11. Let $L = 2$, $x_1(p_1, p_2, w) = w/p_1$ if $p_1 < p_2$ and $x_2(p_1, p_2, w) = w/p_2$ when $p_2 < p_1$ and $x_i(p_1, p_2, w) \equiv [0, w/p_i]$ for $i \in \{1, 2\}$ for $p_1 = p_2$.

- Continuity (notion): Continuity for functions was defined as $(p^n, w^n) \to (p, w)$ then the condition was $x(p^n, w^n) \to x(p, w)$.
- Continuity for Correspondences: The map is closed, the graph of the correspondence is closed and Upper Hemi-continuity. Let x be a choice correspondence. If $(p^n, w^n) \to (p, w)$, and if $x^n \in x(p^n, w^n)$ such that $x^n \to x \in X$. Then it has to be the case that $x \in x(p, w)$.
- Properties of the image of the correspondence.
  - The correspondence x is called compact valued if $x(p, w)$ is a compact set for all (p, w) i.e., $x^n \in x(p, w)$ then $x^n \to x$.

- The correspondence x is convex valued iff $\alpha z + (1-\alpha)y \in x(p,w)$ when $z, y \in x(p,w)$, for $\alpha \in [0,1]$.

6.2 Utility Representation

Choice set $X = \mathbb{R}^L_+$.

Definition 30. Continuity $\succeq$. We say $\succeq$ are continuous if and only if $x^n \succeq y$ and $x^n \to x$ then $x \succeq y$.

Theorem 7. *If $\succeq$ is a continuous rational preference on $X = \mathbb{R}^L_+$ then there is a continuous utility function $u : X \mapsto \mathbb{R}$ that represents $\succeq$.*

First we provide the following intermediate result.

Lemma 2. *If $\succeq$ is a rational preference on $X = \mathbb{Q}^L$ (i.e. the rational numbers) then there is a function $u : X \mapsto \mathbb{R}$ that represents $\succeq$.*

Proof. Let $L^{\succeq}(x) = \{y \in \mathbb{Q}^L | x \succeq y\}$ be the lower contour set of x.

Now, notice that $L^{\succeq}(x)$ is a countable set, moreover it is countably infinite, thus we can define an indexed set $N^{\succeq}(x) \subseteq \mathbb{N}$ such that there is an isomorphism f (bijection) such that $f : L^{\succeq}(x) \twoheadrightarrow N^{\succeq}(x)$.

Now we define a candidate utility function $u : X \mapsto \mathbb{R}$, $u(x) = 0 + \sum_{i \in N^{\succeq}(x)} \frac{1}{2^i}$.

Now we prove that if $x \succeq y$ then $u(x) \geq u(y)$.

If $x \succeq y$, by transitivity, this implies that $L^{\succeq}(x) \supseteq L^{\succeq}(y)$. This means that $N^{\succeq}(x) \supseteq N^{\succeq}(y)$, then given that $\frac{1}{2^i} > 0$ for all $i \in \mathbb{N}$, this means that $u(x) \geq u(y)$.

Now we prove that if $u(x) \geq u(y)$ then $x \succeq y$.

If $u(x) \geq u(y)$ this means that $u(x) - u(y) \geq 0$, this in turn means $\sum_{i \in N^{\succeq}(x) \setminus N^{\succeq}(y)} \frac{1}{2^i} \geq 0$.

There are two cases:

(i) $\sum_{i \in N^{\succeq}(x) \setminus N^{\succeq}(y)} \frac{1}{2^i} = 0$ in which case $N^{\succeq}(x) = N^{\succeq}(y)$, this means that $L^{\succeq}(x) = L^{\succeq}(y)$, which implies $x \sim y$.

(ii) $\sum_{i \in N^{\succeq}(x) \setminus N^{\succeq}(y)} \frac{1}{2^i} > 0$ in which case $N^{\succeq}(x) \supset N^{\succeq}(y)$ this means that $x \succ y$.

We conclude that in general, it must be that if $u(x) \geq u(y)$ then $x \succeq y$.

□

E.g. $L^{\succeq}(x) = \{x, x_1, x_2, x_3, \cdots\}$ $N^{\succeq}(x) = \{1, 2, 3, 4, \cdots\}$.

Now we can prove the main result.

Proof. From the previous representation Theorem, I know that a rational preference $\succeq$ defined on $\mathbb{Q}^L$ have a utility representation $u : \mathbb{Q}^L \to \mathbb{R}$, in fact, I can also think of the same utility restricted to $u : \mathbb{Q}^L \to [0, 1]$.

Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, also $\mathbb{Q}^L$ is dense in $\mathbb{R}^L$. This means that any point in $x \in \mathbb{R}$ is either in $x \in \mathbb{Q}$, or there is a sequence (x^n) such that $x^n \in \mathbb{Q}$, it converges to x in $\mathbb{R}$, $x^n \to x$ then $x \in \mathbb{R}$.

A first try would be to define a utility function $v : \mathbb{R}^L \mapsto \mathbb{R}$, in the following way $v(x) = lim_{n\to\infty} u(x^n)$ such that $x^n \to x \in X$, and $x^n \succeq x^{n-1}$ for all n where u is defined as in the previous lemma.

This means that $u(x^n) \geq u(x^{n-1})$ and second $u(x^n) \in [0, 1]$ for all n (we have a monotone and bounded sequence). I define a new utility function as $v(x) = lim_{n\to\infty} u(x^n)$.

If we have $x \succeq y$ and pick a pair of sequences $x^n \succeq y^n$ with $x^n, y^n \in \mathbb{Q}^L$ such that $x^n \to x$ and $y^n \to y$. By continuity of $\succeq$ we are sure that such sequences exist, because the continuity assumption implies that $lim_{n\to\infty} x^n \succeq lim_{n\to\infty} y^n$. By the previous lemma, we have that $u(x^n) \geq u(y^n)$ for all n. By definition of v, we have that $v(x) = lim_{n\to\infty} u(x^n)$, thus $v(x) \geq v(y)$.

Rubinstein.

Another candidate for a utility function $w : X \mapsto \mathbb{R}$ such that w respresents $\succeq$, $X = \mathbb{R}^L_+$.

$w(x) = sup\{u(y)|x \succeq y|y \in \mathbb{Q}^L\}$.

Continuity follows from Debreu's "Gap Theorem".

□

6.3 The Utility Maximization Problem

We assume that u is continuous and $X = \mathbb{R}^L_+$.

The utility maximization problem (UMP) is:

$$max_{x\geq 0} u(x)$$

$$s.t. \quad p'x \leq w.$$

Alternatively we can define the restriction as $x \in B(p, w) = \{x \in \mathbb{R}^L : p'x \leq w\}$.

Proposition 10. *If $p >> 0$ and u is continuous, then the UMP has a solution.*

Proof. Check $B(p, w)$ is compact, in this case it is equivalent to proving that it is closed and bounded.

It is bounded because $x_l \leq w/p_l$ for all l. It is closed because x^n, such that $p'x^n \leq w$ and if $x^n \to x$ it must be the case that $lim_{n\to\infty} p'x^n \leq lim_{n\to\infty} w$ implies that $p'x \leq w$.

Then by the maximum theorem we know that the UMP has a solution.

□

We now prove further properties of the demand corespondence.

Proposition 11. *The Marshallian demand correspondence* $x(p, w) \equiv argmax_y\{u(y)|y \in B(p, w)\}$ *is non-empty, compact valued and upper hemi-continuous. Moreover,* $v(p, w) = u(x)$ *for* $x \in x(p, w)$ *(i.e., the indirect utility) is continuous.*

Proof. The results follows trivially from Berge's Theorem if we verify that the correspondence $B(\cdot)$ is a continuous correspondence. We already know that u is continuous and that $B(p, w)$ is a compact set, thus making the $B(\cdot)$ correspondence compact valued.

Take $z^n = (p^n, w^n)$ now assume that $z^n \to z$, now we want to prove that $B(z^n) \to B(z)$ where the convergence in in terms sets. Recall that, $B(p^n, w^n) = \{x \in X|p^{n\prime}x \leq w^n\}$. First assume that $x^n \in B(z^n)$ and $x^n \to x$ then we prove that $x \in B(z)$. If $x^n \in B(z^n)$ it must be the case that $p^{n\prime}x \leq w^n$ for all n, applying the limits over this inequality we conclude that $p'x \leq w$ where $x^n \to x$ and $(p^n, w^n) \to (p, w)$. Then $x \in B(z)$ with $z = (p, w)$.

Now we prove that if $x \in B(z)$ then there is a subsequence of the sequence z^n such that there exists $x^k \in B(z^{n,k})$ such that $x^k \to x$. The proof is more complicated and it can be seen in Border Example 30 p11.[1] With that in hand we apply Berge's theorem to obtain our conclusion.

□

We need some definitions.

Definition 31. Local non-satiation. We say $\succeq$ are locally non-satiated whenever for any $x \in X$, for any $\epsilon > 0$, there exists another commodity bundle $y \in X$ such that $||x - y|| \leq \epsilon$ and $y \succ x$.

Proposition 12. *Assume that the Marshallian demand correspondence is generated by maximizing a locally non-satiated utility function* u. *Then* $x(p, w)$ *has the following properties.*

(i) Homogeneous of degree zero in $(p, w) : x(\alpha p, \alpha w) = x(p, w)$ *for any scalar* $\alpha > 0$.

(ii) Walras' law $p'x = w$ *for all* $x \in x(p, w)$.

(iii) Convexity/uniqueness: If $\succeq$ *is convex (*u *is quasiconcave), then* $x(p, w)$ *is convex valued. Moreover, if* $\succeq$ *is strictly convex so that* u *is strictly quasiconcave then* $x(p, w)$ *is a single element.*

[1] http://people.hss.caltech.edu/~kcb/Notes/Correspondences.pdf

6.4 KKT Conditions for the UMP

Now we are going to see how to find the solution under the differentiability assumption.

We write the KKT FOC necessary conditions.

$\mathcal{L} = u(x) + \lambda(w - p'x)$

$\partial_{x_l} u(x_l^*) - \lambda^* p_l \leq 0$ and $x_l^*[\partial_{x_l} u(x_l^*) - \lambda^* p_l] = 0$ for all $l \in \{1, \cdots, L\}$.

$w - p'x^* \geq 0$ and $\lambda^*(w - p'x^*) = 0$.

This is a system with $L + 1$ unknowns.

We are going to simplify a little bit the problem. By local non-satiation we know that the consumer is on the budget hyperplane (frontier). This means that

$$w - p'x^* = 0.$$

Also we are going to assume that the solution in the interior (positive quantities), this means that the KKT conditions are reduced to

$$\frac{\partial_{x_l} u(x^*)}{\partial_{x_k} u(x^*)} = \frac{p_l}{p_k} \quad \forall l, k \in \{1, \cdots, L\}.$$

Note that for $L = 2, -\frac{p_1}{p_2}$ is the slope of the budget line and $-\frac{\partial_{x_1} u(x^*)}{\partial_{x_2} u(x^*)}$ is the slope of the indifference curve, usually this is also referred as the MRS.

6.5 Properties of the the Indirect Utility

The indirect utility is the maximal utility achieved given the budget constraint $v(p, w) = max_{x \geq 0} u(x)$ s.t. $p'x \leq w$.

Alternatively it is the utility evaluated in the Marshallian demand $v(p, w) = u(x(p, w))$.

Proposition 13. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}_+^L$. The indirect utility function $v(p, w)$ is:*

1. *Homogeneity of degree zero.*
2. *$v(p, w)$ is strictly increasing in w and non-increasing in p_l.*
3. *$v(p, w)$ is quasiconvex in (p, w). The set $\{(p, w) | v(p, w) \leq \overline{v}\}$ is convex.*
4. *$v(p, w)$ is continuous in p and w.*

Proof. 1. 2. and 4. are consequences of previous results presented here.

3. is the only non-trivial part of the proof.

Suppose that $v(p,w) \leq \overline{v}$ and $v(p',w') \leq \overline{v}$, for any $\alpha \in [0,1]$, consider then $(p'',w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$,

We want to show that $v(p'',w'') \leq \overline{v}$. Equivalently, for any x such that $(p'') \cdot x \leq w''$ it is the case that $u(x) \leq \overline{v}$.

If $(p'') \cdot x \leq w''$ then

$$(\alpha p + (1-\alpha)p') \cdot x \leq \alpha w + (1-\alpha)w'.$$

This means that either $p \cdot x \leq w$ or $(p') \cdot x \leq w'$ (it can be both). If the former inequality holds, then $u(x) \leq v(p,w) \leq \overline{v}$, and we have the result. If $(p') \cdot x \leq w'$ then $u(x) \leq v(p',w') \leq \overline{v}$ then the same conclusion follow.

□

6.6 Roy's Identity

Recall the definition of indirect utility $v(p,w) = u(x(p,w))$.

We use the Envelope's theorem

$v(p,w) = max_{x\geq 0} u(x) + \lambda(w - p'x)$

$\partial_w v(p,w) = \lambda.$

$\nabla_p v(p,w) = -\lambda x$

This provides the Roy's identity

$$x(p,w) = -\frac{\nabla_p v(p,w)}{\partial_w v(p,w)}.$$

Example 12. Imagine I want to model a data set $((p_l^t)_{l=1}^L, (x_l^t)_{l=1}^L)_{t\in T}$

$$\phi_l(p_1^t, \cdots, p_l^t) = x_l^t$$

for all $t \in T$.

Model 1:

$$x_l = \alpha_o^l + \sum_{k=1}^{L} \alpha_k^l p_k^{\beta_k^l}.$$

Model 2:

$$v(p_1^t, \cdots, p_L^t) = \sum_k \sum_l \alpha_{l,k} p_l^{1/2} p_k^{1/2}$$

$$x_l = \partial_{p_l} v(p_1^t, \cdots, p_L^t)$$

6.7 Expenditure Minimization Problem (EMP)

The Expenditure Minimization Problem (EMP) asks the question, if prices are p, what is the minimum amount the consumer would have to spend to achieve utility level $\overline{u}$. That is:

$$min_{x \geq 0} p'x$$

s.t.

$$u(x) \geq \overline{u}.$$

Proposition 14. *Suppose u is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}_+^L$ and the price vector is $p >> 0$. We have.*

(i) If x^* is optimal in the UMP when $w > 0$ then x^* is optimal in the EMP when the required level of utility is $u(x^*)$. Moreover, the minimized expenditure level $p'x^* = w$ is exacly the wealth w.

(ii) If x^* is optimal in the EMP when the required level of utility is $u > u(0)$ then x^* is optimal in the UMP when wealth is $p'x^*$. Moreover, the maximized utility level $u(x^*) = u$ is exactly u.

We omit the proof. For details go to Chapter of MGW.

Expenditure function is $e(p, u) = p'x^*$ where x^* is an element in the solution to the EMP.

Proposition 15. *Suppose that u is a continuous utility function representing a locally non-satiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}_+^L$. The expenditure $e(p, u)$ is*

(i) Homogeneous of degree one in p.

$min_{x \geq} \alpha p'x$ *s.t.* $u(x) \geq \overline{u}$.

(ii) Strictly increasing in u and non-decreasing in p_l for all l

(iii) Concave in p

(iv) Continuous in p, u.

Proof. (i),(ii) are omitted.

(iii) Concavity.

We want to show that $e(p,u)$ is concave in p for any u.

$e(p^\alpha,\overline{u}) \geq \alpha e(p,\overline{u}) + (1-\alpha)e(p',\overline{u})$,

when $p^\alpha = \alpha p + (1-\alpha)p'$ for some $\alpha \in [0,1]$.

Let $h(p,\overline{u})$ be the minimizer of the EMP, we abuse notation and let it denote any element of the set of optimizers at the given prices:

$e(p^\alpha,\overline{u}) = p^\alpha \cdot h(p^\alpha,\overline{u})$

$= \alpha p \cdot h(p^\alpha,\overline{u}) + (1-\alpha)(p') \cdot h(p^\alpha,\overline{u})$

Note that $e(p,\overline{u}) = p \cdot h(p,\overline{u}) \leq p \cdot h(p^\alpha,\overline{u})$, and we can conclude that:

$e(p^\alpha,\overline{u}) \geq \alpha p \cdot h(p,\overline{u}) + (1-\alpha)(p') \cdot h(p',\overline{u})$.

(iv) Continuity in p,u.

This follows from the Berge's Maximum Theorem.

Consider the problem

$$max_{x\geq 0} - p'x \quad s.t. \quad u(x) \geq \overline{u}.$$

First note that the budget constraint is not a compact-valued correspondence.

We observe that the above problem is equivalent to

$$max_{x\geq 0} - p'x \quad s.t. \quad u(x) = \overline{u}$$

since the problem is minimizing the expenditure we can create a non-binding upper bound, and by continuity we can make it equal to $\overline{u}$ (no excess utility).

Now $u(x) = \overline{u}$ is a continuous function in $\overline{u}$ (thus a continuous correspondence, that is singled valued and therefore compact valued). Given that $-p'x$ is also a continuous function in the parameter p, we conclude by the Berge's maximum theorem that $e(p,\overline{u}) = p'x^*$ for $x^* \in h(p,\overline{u})$ is continuous on $p,\overline{u}$.

□

6.8 Hicksian Compensated Demand

The Hicksian Compensated Demand or Hicksian demand is $h(p,u) = argmin_{x\geq 0, u(x)\geq \overline{u}} p'x$.

Proposition 16. *Suppose $u(\cdot)$ is continuous and it represents a locally non-satiated preference relation $\succeq$. Then the Hicksian demand correspondence $h(p, u)$ has the following properties:*

(i) HD0 in p

(ii) No excess utility: For $x^* \in h(p, u)$, $u(x^*) = \overline{u}$

(iii) Convexity/Uniqueness: If $\succeq$ is convex, then $h(p, u)$ is a convex set, and if $\succeq$ is strictly convex, then $h(p, u)$ is a unique element.

6.9 KKT Conditions for the EMP

Assume u is twice continously differentiable.

The Lagrangian for this problem is:

$$\mathcal{L}_{EMP} = p'x - \lambda(u(x) - \overline{u}).$$

The FOC are given by

$$p \geq \lambda \nabla_x u(x^*)$$

$$x_l^*[p_l - \lambda \nabla u(x^*)_l] = 0,$$

for some $\lambda \geq 0$.

6.10 Sheppard's Lemma

Proposition 17. $\nabla_p e(p, \overline{u}) = h(p, \overline{u})$.

Proof. We apply the envelope's theorem for constrained problem.

$$e(p, \overline{u}) = p'h + \lambda^*(u(h) - \overline{u})$$

We derive with respect to p and we obtain:

$$\nabla_p e(p, \overline{u}) = h(p, \overline{u})$$

□

6.11 Additional Properties

Let $D_p h(p,u)$ be the $L \times L$ matrix of derivatives of prices.

Proposition 18. *The Hicksian demand function has the following properties:*

1. $D_p h(p,u) = \mathcal{H}e(p,u)$.

The derivative with respect to prices of the Hicksian demand is the Hessian of the expenditure function.

2. $D_p h(p,u)$.

This is a NSD matrix. This follows from the fact that a Hessian of a concave function is NSD.

3. $D_p h(p,u)$ is symmetric.

Young's theorem (continuity of e).

4. $D_p h(p,u)p = 0$.

This follows from HD0 of $h(p,\overline{u})$ in p and Euler's identity.

6.12 Summary of Relationships between the EMP and UMP

The **expenditure function** is the minimum $e(p,\overline{u}) = p'h(p,\overline{u})$ value attained at the given prices and target utility.

There is a **duality relation** between the EMP and the UMP that we will not explore at this point. However, we will state the following relations.

$$h(p, v(p,w)) = x(p,w)$$

$$x(p, e(p,u)) = h(p,u).$$

Also,

$$\overline{u} = v(p, e(p,\overline{u}))$$

$$w = e(p, v(p,w)).$$

6.13 Slutsky Equation

We establish now a very useful equation.

$$h(p, u) = x(p, e(p, u))$$

Differentiating with respect ot p

$$D_p h(p, u) = D_p x(p, e(p, u)) + D_w x(p, e(p, u)) \nabla_p e(p, u)'$$

By Sheppard's lemma and by the identities that relate the expenditure and wealth

$$D_p h(p, v(p, w)) = D_p x(p, w) + D_w x(p, w) x(p, w)'.$$

6.14 Summary of the Implications of Utility Maximization.

Proposition 19. *If a demand function $x(p, w)$ is generated by maximizing rational preferences that are locally non-satiated subject to a linear budget constraint. Then: (i) $x(p, w)$ satisfies Walras' law, (ii) $x(p, w)$ is homogeneous of degree zero.*

Moreover, its Slutsky matrix function $S(p, w)$ is: (i) Negative semidefinite. (ii) Singular in prices $S(p, w)p = 0$, (iii) Symmetric $S(p, w) = S(p, w)'$.

This proposition summarizes our previous results.

6.15 Integrability

We are now interested in what are the sufficient conditions for integrability. That is given an observed demand function we want to be sure that there exists a utility function that generates the data. We saw in the first chapter that for rich data and finite choice sets. The WARP was necessary and sufficient for rationalizability of a choice correspondence. Here in the world of the consumer, where not all choice sets are allowed but only competitive budgets sets (linear budget constraints), we need further conditions. This conditions will be given in terms of the Slutsky matrix of the demand function.

Of course, we assume in this subsection that the demand function x is continously differentiable.

Recall that the Slutsky matrix function is $S(p, w) = D_p x(p, w) + D_w x(p, w)x(p, w)'$, or in the scalar version $s_{ij}(p, w) = \partial_{p_j} x_i(p, w) + \partial_w x_i(p, w)x_j(p, w)$ for all $i, j \in \{1, \cdots, L\}$.

Theorem 8. *(Hurwicz-Uzawa/Richter) Assume that (a) Walras' law holds for x. (b) The Slutsky matrix $S(p, w)$ is symmetric for all p, w, that is $S(p, w) = S(p, w)'$. (c) The Slutsky matrix is $S(p, w) \leq 0$ negative semi definite. Then there exists a continous utility function, differentiable that generates the data (i.e., the demand function $x = argmax_{q \geq 0} u(q)$ for some u subject to $p'q = w$).*

Proof. (Sketch)

1. Given condition (b) or the symmetry of the Slutsky matrix. We conclude that there is a solution to the differential equation

$$\frac{\partial \mu(p)}{\partial p_i} = x_i(p, \mu(p)) \forall i \in \{1, \cdots, L\}.$$

For initial conditions $\mu(p^0) = w^0$. The solution depends on the initial conditions so it is written $\mu(p; p^0, w^0)$.

2. Use the solution to 1, to define a candidate indirect utility (a solution exists if $S(p, w)$ is symmetric -Frobenious Theorem, this implies that $\mu(p^*; p, w)$ is continuous on the second argument and differentiable).

$$v(p, w) = \mu(p^*; p, w).$$

I use the NSD restriction to show that $v(p, w)$ is HD0, and quasi-convex.

3. Invert the demand function to obtain (p, w) as a function of $x^* = x(p, w)$.

4. Define the utility on the range of x by

$$u(x) = \mu(p^*, p(x), w(x))$$

where $x = x(p, w)$.

5. Use the (c) or NSD condition to conclude that u is a good faith utility function.

□

Remark. Some textbooks require a boundedness condition on the elasticity of wealth, this is in general not required.

6.16 Examples and Applications of the Integrability of Demand.

There are two commodities $L = 2$ and you estimate a demand function $x_1(p, w) = \frac{\alpha w}{p_1}$, and you know that the demand of good two is given as a residual. Furthermore, we have checked that this demand satisfies WARP and therefore it is HD0, we normalize $p_2 = 1$. Thus $x_2 = (1 - \alpha)w$.

The $L - 1$ differential equation is:

$$\mu'(p_1, 1) = \frac{\alpha\mu(p_1, 1)}{p_1} = x_1((p_1, 1), \mu((p_1, 1))),$$

for short, we can write:

$$\mu'(p_1) = \frac{\alpha\mu(p_1)}{p_1} = x_1(p_1, \mu(p_1)),$$

We can write more simply:

$$\frac{\mu'}{\mu} = \frac{\alpha}{p_1}.$$

Simple integration provides a family of solutions:

$$ln\mu = \alpha lnp_1 + C,$$

and in levels:

$$\mu = Kp_1^{\alpha},$$

with $K = exp(C)$ a constant of integration.

Given some initial conditions (p_1^0, w^0), we must have:

$$w^0 = K(p_1^0)^{\alpha}, \quad K = \frac{w^0}{(p_1^0)^{\alpha}},$$

or

$$\mu(p_1^*; p_1^0, w^0) = \frac{w^0}{(p_1^0)^{\alpha}}(p_1^*)^{\alpha}.$$

Now let's define the indirect utility, with $p_1^* = 1$:

$$v(p_1, w) = \mu(p^*, p_1, w) = \frac{w}{p_1^\alpha}.$$

To recover the utility u, we need to invert the demand, note that our primitive is indeed invertible.

We have to use the demand function and obtain the inverse demand:

$$x_1(p, w) = \frac{\alpha w}{p_1}; \quad x_2(p, w) = (1-\alpha)w$$

$$\frac{x_1}{x_2} = \frac{\alpha}{(1-\alpha)} \frac{1}{p_1}$$

$$p_1 = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}, \quad w = \frac{x_2}{1-\alpha}.$$

Replacing back in the indirect utility we have:

$$u(x_1, x_2) = v \circ (p_1, w)(x_1, x_2) = v(\frac{\alpha}{1-\alpha} \frac{x_2}{x_1}, \frac{x_2}{1-\alpha}).$$

$$u(x_1, x_2) = \frac{\frac{x_2}{1-\alpha}}{\left(\frac{\alpha}{1-\alpha} \frac{x_2}{x_1}\right)^\alpha} = \left(\frac{x_2}{1-\alpha}\right)^{1-\alpha} \left(\frac{x_1}{\alpha}\right)^\alpha$$

$$u(x_1, x_2) = \hat{c} x_1^\alpha x_2^{1-\alpha}, \quad \hat{c} = (1-\alpha)^\alpha \alpha^\alpha.$$

This is a a Cobb Douglas Utility function.

6.17 Revealed Preference Theory / Strong Axiom of Revealed Preference

Up to this point this chapter has had very little concern about the dataset that we have at hand. It is time to solve this. We assume that we observe a dataset $\mathcal{O} = \{p^k, x^k\}_{k=1}^K$.

Definition 32. We say that x^t is directly revealed preferred to x ($x^t R^D x$) if and only if $p^{t\prime} x^t \geq p^{t\prime} x$.

Furthermore, we define an (indirect) notion of revealed preference.

Definition 33. We say that x^t is (indirectly) revealed preferred to x (x^tRx) if and only if there is a sequence of commodities such that $x^tR^Dx^1R^Dx^2R^D\cdots x^nR^Dx$.

Using some ideas from order theory we can as well that R is the transitive closure of R^D.

Definition 34. (Strong Axiom of Revealed Preference -SARP-) The revealed preference relation R is acyclic, i.e., If x^tRx^s then it is not the case that $x^sR^Dx^t$ $(t \neq s)$.

This is a result by Afriat/Varian/Houtaker.

Proposition 20. *(Afriat's Theorem) Given $\mathcal{O} = \{p^k, x^k\}_{k=1}^K$, the following are equivalent:*

1. There exists a continuous, nonsatiated utility function u, strictly concave that rationalizes the data in the sense that for all k, $u(x^k) > u(x)$ for all $x \neq x^k$ such that $p^{k\prime}x^k \geq p^{k\prime}x$.

2. The dataset $\mathcal{O} = \{p^k, x^k\}_{k=1}^K$ satisfies SARP.

3. There exist a positive solution (u^t, λ^t) to the set of linear inequalites

$$u^t < u^s + \lambda^s p^{s\prime}(x^t - x^s) \forall s, t \in \{1, \cdots, K\}, s \neq t$$

(If am able to solve for: $\mathbf{u} = \{u^t\}_{t=1}^K$ and $\lambda = \{\lambda^t\}_{t=1}^K$ given O).

Proof. • We prove that 1. implies 3.

If u is strictly concave then $u(x^t) < u(x^s) + \nabla u(x^s)'(x^t - x^s)$. Given that u is assumed smooth ∇u exists (super-gradient for concave functions). Now, by the FOC of the UMP we know that $\nabla u(x^s) \leq \lambda^s p^s$. This implies that

$u(x^t) < u(x^s) + \lambda^s p^{s\prime}(x^t - x^s)$.

$u(x^t) = u^t$.

We prove that 3. implies 1.

Let $u(x) = min_{t\in\{1,\cdots,K\}}\{u^t + \lambda^t p^{t\prime}(x - x^t)\}$, this function is continuous in u and concave, it is also strictly increasing in x (given that $\lambda^t > 0$, and $p^t >> 0$).

I want to show that u generates the dataset $\mathcal{O}$.

First check that if $u(x^t) = u^t$.

Assume that $u(x^t) = u^m + \lambda^m p^{m\prime}(x^t - x^m) < u^t$, but this means that the Afriat's inequality for m, t failes. Then $u(x^t) = u^t$. In fact, $u^m - u^t < \lambda^m p^{m\prime}(x^m - x^t)$ this violates (3).

If x^sRx I want to show that $u(x) < u(x^s)$, for $p^{s\prime}x^s > p^{s\prime}x\cdot$

Now assume that $p^{s\prime}x^s > p^{s\prime}x$, or $p^{s\prime}(x^s - x) > 0$.

This means that $u(x) = min_t\{u^t + \lambda^t p^{t\prime}(x - x^t)\} \leq u^s + \lambda^s p^s(x - x^s) < u^s = u(x^s)$.

If $p^{s\prime}x^s = p^{s\prime}x$ then $x = x^s$ then this holds trivially, and when $x \neq x^s$ again by Afriat's inequalities:

$u^s + \lambda^s p^s(x^t - x^s) < u^t$.

Thus $u(x^s) > u(x)$ when $x^s R x$.

- Now we prove that 1. implies 2.

If $x^t R x^s$ it means that $u(x^t) > u(x^s)$ then this implies that $p^{s\prime} x^s < p^{s\prime} x^t$, because if $p^{s\prime} x^s \geq p^{s\prime} x^t$ this means that $u(x^s) \geq u(x^t)$, if that is the case, then the only solution is that $u(x^s) = u(x^t)$ and $p^{s\prime} x^s = p^{s\prime} x^t$ which means that the solution of the UMP is non unique which contradicts the assumption that u is strictly concave.

If $x^1 R x^2 \cdots x^n$ and $x^n R x^1$ (this is a violation of SARP) then there is a contradiction of (1) because this implies that $u(x^1) > u(x^2) > \cdots u(x^n)$ and $u(x^n) > u(x^1)$ which is impossible.

□

6.18 Generalized Axiom of Revealed Preference

In the previous treatment of rationality we did not allowed for correspondences, we only allowed for functions. But what happens for cases when we observe for the same prices $p^t = p^s$ different demand choices $x^t \neq x^s$ such that $p^{t\prime} x^t = p^{s\prime} x^s$, recall that this is a case where from the same budget set two different choice bundles are chosen (this can be seen as the case of a correspondence generating the data).

Definition 35. A dataset $O = \{p^k, x^k\}_{k=0}^K$ is said to satisfies GARP if an only if R admits Only Weak Cycles (OWC), i.e., if $x^1 R x^2 \cdots R x^n$ and $x^n R^D x^1$ then it must be the case that $x^1 I x^n$, where I is the symmetric part of R.

Proposition 21. *(Varian/Afriat's Theorem) Given $\mathcal{O} = \{p^k, x^k\}_{k=1}^K$, the following are equivalent:*

1. There exists a continuous, nonsatiated, concave utility function u that rationalizes the data in the sense that for all k, $u(x^k) \geq u(x)$ for all x such that $p^{k\prime} x^t \geq p^{k\prime} x$.

1.' There is exists a continuous, increasing (monotone) utility function u that rationalizes the data in the sense that all k, $u(x^k) \geq u(x)$ for all x such that $p^{k\prime} x^t \geq p^{k\prime} x$.

2. The dataset $\mathcal{O} = \{p^k, x^k\}_{k=1}^K$ satisfies GARP.

3. There exist a positive solution (u^t, λ^t) to the set of linear inequalites

$$u^t \leq u^s + \lambda^s p^{s\prime}(x^t - x^s) \forall s, t$$

6.19 Forecasting: Varian Support Set

Definition 36. Varian's Support Set. $S(p^{K+1}, w^{K+1}) = \{x \in X : O \cup \{p^{K+1}, x\}, \quad satisfy \quad GARP; \quad p^{K+1}x = w^{K+1}\}$.

The set of demand observations that given a new price level p^{K+1} is such that GARP still holds in the extended dataset.

$x \in S(p^{K+1}, w^{K+1})$

if:

1) $p^{K+1}x < p^{K+1}x^t$ for all cases such that $x^t P x$

2) $p^{K+1}x = w^{K+1}$.

6.20 Upper Bounds to Welfare Analysis.

Definition 37. A money metric utility function is defined as $m(p, x) = e(p, u(x))$ where e is the expenditure function that corresponds to u such that $e(p, u) = minp'x$ such that $u(x) = u$.

For any given p, $m(p, x)$ is a utility function that represents the same preferences as u but is expressed in money units.

Given a finite dataset O we Varian suggested an upper bound for the money metric utility using:

$$m^+(p, x) = min_t px^t, \quad s.t. x^t Rx.$$

6.21 Experiments about Testing Rationality

(This material is presented in Quah[2])

- Data from the portfolio choice experiment in Choi, Fisman, Gale, and Kariv (AER, 2007).
- 93 undergraduate subjects participated in the experiment at UC Berkeley, each completing 50 decision problems under risk.
- There were two states of the world, each occurring with a known probability, and two Arrow-Debreu securities, one for each state.

[2]http://www.johnquah.com/uploads/7/5/6/7/75676677/survey-qutweb.pdf

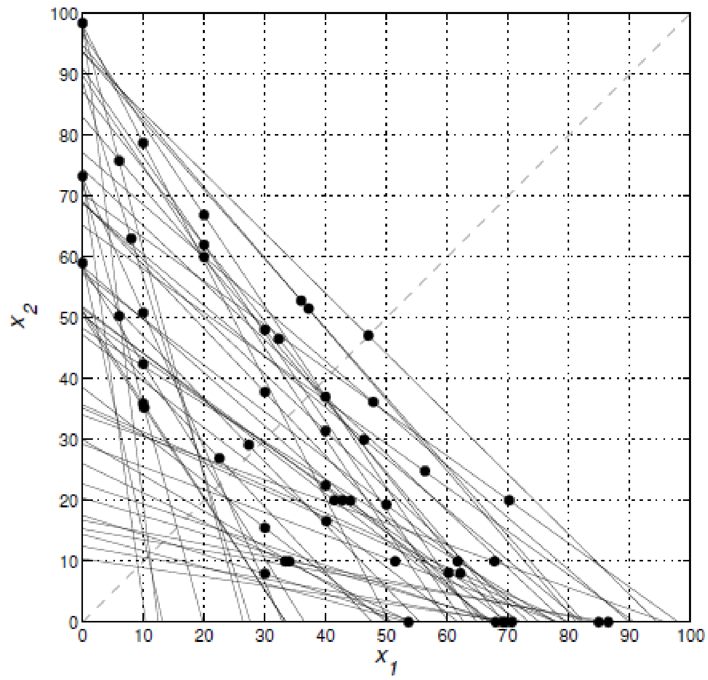


Figure: Subject 216 (symmetric treatment)

Figure 6.1: Subject 216 (symmetric treatment). Source: Quah.

- In each decision problem, every subject was given a budget; income was normalized to one and state prices were chosen at random. 47 subjects were subjected to the symmetric treatment, where $\pi_1 = \pi_2 = \frac{1}{2}$. The rest had the asymmetric treatment where $\pi_1 = \frac{1}{3}$, $\pi_2 = \frac{2}{3}$ (or vice versa).

6.22 Results

- In the symmetric treatment, 12/47 pass GARP.
- In the asymmetric treatment, 4/46 pass GARP.
- Roughly 17% pass GARP.

6.23 Afriat's Cost Efficiency Index: Rationality Measures of Performance.

Suppose a dataset O cannot be rationalized by any u.

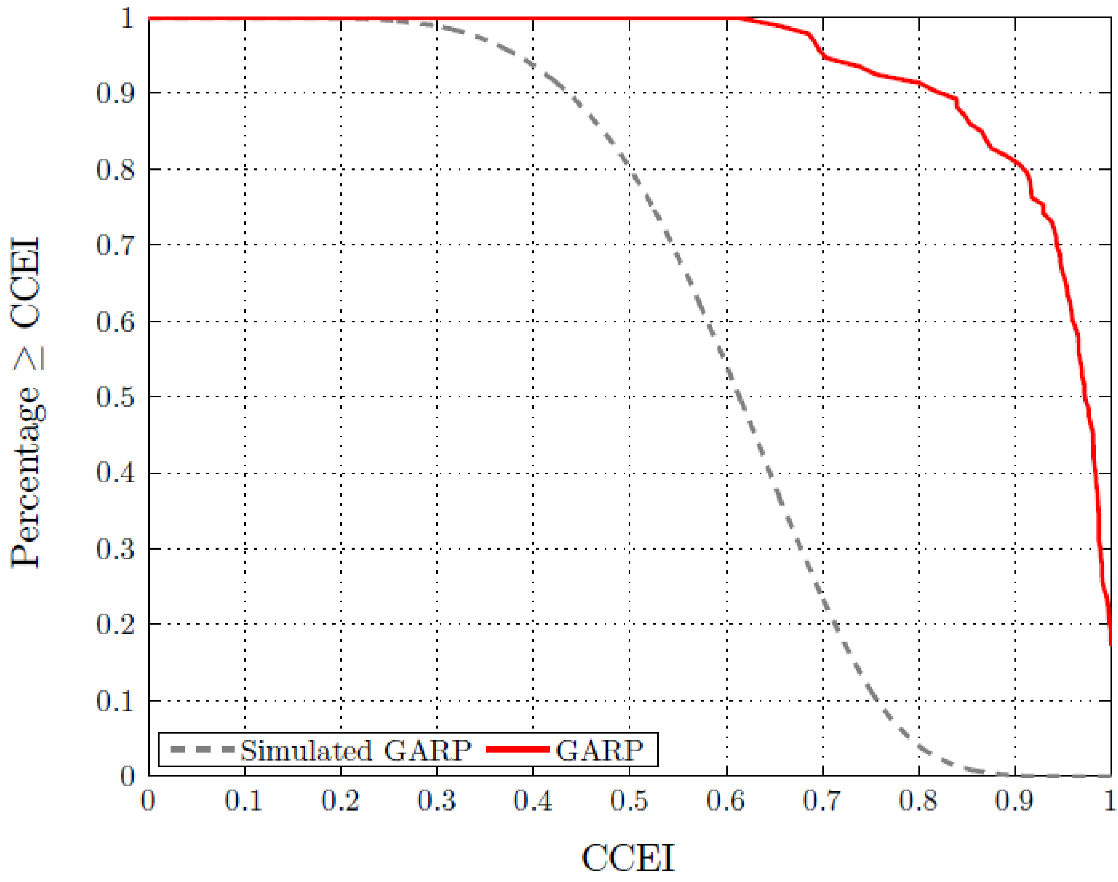


Figure 6.2: CCEI Source: Quah

Now consider the case where we shrink budget sets implied by O by an efficiency index $\alpha \in [0, 1)$:

Then every dataset O can be α-rationalized if we find $u(x^t) \geq u(x)$ for all $x \in B^t(\alpha)$, where $B^t(\alpha) = \{x \in \mathbb{R}^L_+ : p^{t\prime}x \leq \alpha p^{t\prime}x^t\}$.

Definition 38. The Critical Cost Efficiency Index (CCEI) of a data set O is $\alpha^* = sup\{\alpha : a \in A\}$ where $\alpha \in A$ if there is a u that rationalizes the data $O(\alpha) = \{B^t(\alpha), x^t\}_{t=0}^T$. It amounts to checking SARP/GARP for a grid of α values and checking whether it rationalizes the data. The supremum of those α numbers is the CCEI.

Now, what is the CCEI distribution in the case of the above experiment.

6.24 Welfare Evaluation

We will compare the wealth needed to achieve a certain level of utility at a given price-wealth pair.

$$u^1 = v(p^1, w)$$

$$u^0 = v(p^0, w).$$

We are interested in comparing the expenditure needed to achieve u^1 or u^0.

For a fixed reference price and a given utility level u, the needed expenditure to reach it is given by

$$e(p^{ref}, u).$$

Thus the comparison in terms of expenditure is given by

$$e(p^{ref}, u^1) - e(p^{ref}, u^0)$$

$$e(p^{ref}, v(p^1, w)) - e(p^{ref}, v(p^0, w)).$$

This expression will be positive when it takes more wealth to achieve u^1 at prices p^{ref} than to achieve u^0. This difference is measured in dollar terms. That is why $e(p^{ref}, v(p, w))$ is often called money metric indirect utility function.

We can construct many money metric indirect utility functions with different reference price.

When $p^{ref} = p^0$ is the original price situation then the change is expenditure is equal to the change in wealth such that the consumer would be indifferent between the new price with the old wealth and the old price and the new wealth. This asks what change is wealth would be equivalent to the change in price.

The **equivalent variation is**

$$EV(p^0, p^1, w) = e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = e(p^0, v(p^1, w)) - w,$$

since $e(p^0, v(p^0, w)) = w$.

The Equivalent variation is the amount of money the consumer **is willing to pay** to avoid a price change.

The other case considered is the one where the new price is taken as the reference price, when

$p^{ref} = p^1$, the change is expenditure is equal to the change in wealth such that the consumer is indifferent between the original situation (p^0, w) and the new situation $(p^1, w + \Delta w)$.

The **compensating variation** is defined as:

$$CV(p^0, p^1, w) = e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = w - e(p^1, v(p^0, w)).$$

How much must I compensate you to make you as well off as you were before the price change? Willingness to accept (compensation).

6.25 Homothetic and Quasilinear Utility Functions

Definition 39. Homothetic Prefereces. If $x \sim y$ then $\alpha x \sim \alpha y$ for $\alpha > 0$.

Note that if that is the case $u(\alpha x) = \alpha u(x)$.

Examples: Cobb-Douglas $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Definition 40. Quasilinear Preferences. If $x \sim y$, $x + e_L \alpha \succeq y$ where $e_L = (0, \cdots, 0, \cdots 1)'$ and $\alpha > 0$.

The utility function for quasilinear cases are $u(x) = V(x_1, \cdots, x_{L-1}) + x_L$,

Chapter 7

Aggregate Demand (Ch. 4)

7.1 Introduction

- When can aggregate demand be expressed as a function of prices and aggregate wealth? (Macro-Econometrician).
- When does aggregate demand satisfy the weak axiom? (Positive theorist)
- When does aggregate demand have welfare significance? (Welfarist Theorist)

Definition 41. (Aggregate Demand) Aggregate demand is a mapping from prices (common to all consumers) and the wealth distribution $x(p, \{w_i\}_{i=1}^{I}) = \sum_{i=1}^{I} x_i(p, w_i)$.

Now we tackle the first question.

Claim 2. We can write aggregate demand as a function of aggregate wealth $x(p, \{w_i\}_{i=1}^{I}) = x(p, \sum_{i=1}^{I} w_i)$ when $\sum_{i=1}^{I} \partial_{w_i} x_{l,i}(p, w_i) dw_i = 0$ for every l when $\sum_{i=1}^{I} dw_i = 0$, this in turn can happen if

$$\partial_{w_i} x_{l,i}(p, w_i) = \partial_{w_j} x_{l,j}(p, w_j)$$

for all $i, j \in \{1, \cdots, I\}$.

Now we need to define a class of preferences that admit this possibility.

Definition 42. (Gorman Form) A preference relation $\succeq_i$ satisfies the Gorman Form property if they allow an indirect utility of the form:

$$v_i(p, w_i) = a_i(p) + b(p) w_i.$$

Proposition 22. *A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths is* $\partial_{w_i} x_{l,i}(p, w_i) = \partial_{w_j} x_{l,j}(p, w_j)$ for all $i, j \in \{1, \cdots, I\}$, at any price vector p is that preferences admit indirect utility functions of the Gorman form.

Proof. Using Roy's identity we notice that:

$x_i(p, w_i) = -\frac{\nabla_p a_i(p) + \nabla_p b(p) w_i}{b(p)}$.

Now $\partial_{w_i} x_i(p, w_i) = -\frac{\nabla_p b(p)}{b(p)}$ that does not depend on $i \in \{1, \cdots, I\}$.

□

Examples of Gorman form are identical homothetic preferences or quasi-linear preferences with respect to the same good.

Theorem 9. *We can write the aggregate demand as a function of aggregate wealth* $x(p, \{w_i\}_{i=1}^I) = x(p, \sum_{i=1}^I w_i)$ *if* $x_i(p, w_i)$ *for all* i *are generated by a Gorman form utility.*

The proof follows from the previous results.

Corollary 4. *We can write aggregate demand as a function of aggregate wealth* $x(p, \{w_i\}_{i=1}^I) = x(p, \sum_{i=1}^I w_i)$ *when* w_i *is a wealth distribution rule based on prices or* $w_i = \omega_i(p, w)$ *where* w *is aggregate wealth.*

7.2 Aggregate Demand and the Weak Axiom

For illustration purposes, we can suppose that the wealth rule is simple and that there is a distribution rule vector $\{\alpha_i\}_{i=1}^I$.

Such that $\alpha_i \geq 0$ and $\sum_{i=1}^L \alpha_i = 1$. So that $w_i = \omega_i(p, w) = \alpha_i w$.

We have then:

$$x(p, w) = \sum_{i=1}^{I} x_i(p, \alpha_i w).$$

Definition 43. (WARP) The aggregate demand $x(p, w)$ satisfies the Weak Axiom (WA) if $p' x(p^*, w^*) \leq w$ and $x(p, w) \neq x(p^*, w^*)$ imply that $p^{*\prime} x(p, w) > w^*$ for any (p, w) and (p^*, w^*).

Claim 3. Aggregate demand $x(p, w)$ may not satisfy the WA even when $x_i(p, \alpha_i w)$ satisfy the WARP.

The main reason is that the WA requires that the wealth compensation is done at the aggregate level, this may not imply that individuals are compensated.

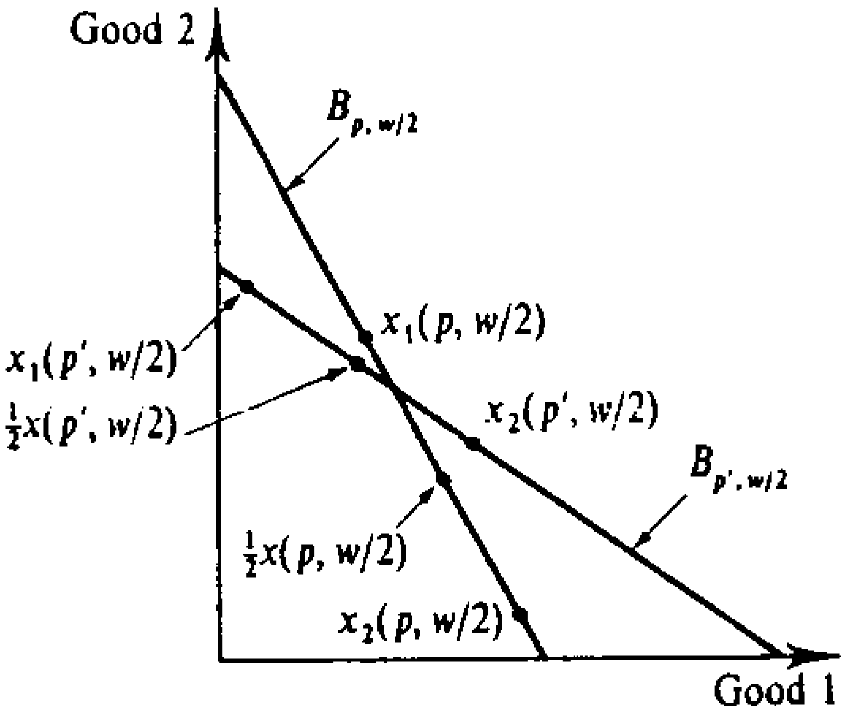


Figure 7.1: Violations of Weak Axiom by the Aggregate Demand

7.3 Restrictions on Behavior to Guarantee WA on the Aggregate Demand

Definition 44. (Uncompensated Law of Demand ULD) The individual demand function $x_i(p, w_i)$ satisfies the ULD property if

$$(p^* - p)'[x_i(p^*, w_i) - x_i(p, w_i)] \leq 0$$

for any p, p^* and w_i, with strict inequality if $x_i(p^*, w_i) \neq x_i(p, w_i)$.

Claim 4. The x_i satisfies the ULD if and only if $D_p x_i(p, w_i)$ is negative definite for all p.

Proof. In differential terms the ULD means that $dpdx_i < 0$, for dp big enough to make $x_i(p + dp, w_i) \neq x_i(p, w_i)$.

Because it is uncompensated then $dw_i = 0$.

It follows that: $dpD_p x_i(p, w_i)dp < 0$.

□

Proposition 23. *If for all $i \in \{1, \cdots, I\}$ $x_i(p, w_i)$ satisfies the ULD, so does the aggregate demand $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i)$. As a consequence the aggregate demand $x(p, w)$ satisfies the Weak Axiom.*

Proof. We have that for all i,

$(p^* - p)'[x_i(p^*, w_i) - x_i(p, w_i)] \leq 0$ (by ULD).

Then we sum over i:

$(p^* - p)'[\sum_{i=1}^{I} x_i(p^*, w_i) - \sum_{i=1}^{I} x_i(p, w_i)] \leq 0$

to obtain $(p^* - p)'[x(p^*, w) - x(p, w)] \leq 0$ (this is ULD for $x(p, w)$).

Because this works for any w and p, p^* it also works for compensated wealth levels.

Thus the WA also holds.

□

Proposition 24. *If $\succeq_i$ is homothetic, then $x_i(p, w_i)$ satisfies the ULD property.*

Proof. By $\succeq_i$ homothetic preferences the elasticity of quantity of wealth is 1 $D_{w_i} x_i(p, w_i) w_i = x_i(p, w_i)$.

Or $\partial_{w^i} x_i(p, w_i) \frac{w_i}{x_i(p, w_i)} = 1$, this implies that

$D_p x_i(p, w_i) = S_i(p, w_i) - \frac{1}{w_i} x_i(p, w_i) x_i(p, w_i)'$.

The definition of $S_i(p, w_i) = D_p x_i(p, w_i) + D_{w_i} x_i(p, w_i) x_i(p, w_i)'$, we replace,

$D_{w_i} x_i(p, w_i) w_i = x_i(p, w_i)$.

We notice that $-\frac{1}{w_i} x_i(p, w_i) x_i(p, w_i)'$ is ND and $S_i(p, w_i)$ is NSD.

This means that $dp D_p x_i(p, w_i) dp \leq 0$, because

$dp S_i(p, w_i) dp \leq 0$.

□

7.4 Conditions for UDL to Arise from Aggregation.

Proposition 25. *Suppose we have a continuum of consumers $i \in [0, 1]$, and preferences are identical and defined on $\mathbb{R}_+^L$ with individual demands $x(p, w)$ where individual wealth is uniformly distributed on $[0, \overline{w}]$. Then the average demand $\overline{x}(p, \overline{w}) = \int_0^{\overline{w}} x(p, w) dw$ satisfies ULD.*

Proof. Consider $v \in \mathbb{R}^L$ such that $v \neq 0$, under regularity conditions on $x(\cdot)$ we can interchange derivatives and integrals and we obtain:

$v' D_p \overline{x}(p, \overline{w}) v = \int_0^{\overline{w}} v' D_p x(p, w) v dw$

Note that:

$D_p x(p,w) = S(p,w) - D_w x(p,w)x(p,w)'$, where $S(p,w)$ is the individual Slutsky matrix then,

$v' D_p \overline{x}(p, \overline{w})v = \int_0^{\overline{w}} v' S(p,w) v dw - \int_0^{\overline{w}} (v' D_w x(p,w))(v' x(p,w)) dw.$

Note that, $\frac{d(v'x(p,w))^2}{dw} = 2\frac{(v'x(p,w))(v'D_w x(p,w))dw}{dw}$, then

$-\int_0^{\overline{w}} (v' D_w x(p,w))(v' x(p,w)) dw = -\frac{1}{2}\int_0^{\overline{w}} \frac{d(v'x(p,w))^2}{dw} dw = -\frac{1}{2}(v'x(p,w))^2 \leq 0.$

This means that,

$v' D_p \overline{x}(p, \overline{w})v \leq 0$, or the ULD property.

□

The previous result depends on the fact that the distribution of wealth is uniform and that preferences are identical.

Claim 5. The Slutsky matrix of the aggregate demand is $S(p,w) = \sum_{i=1}^{I} S_i(p,w) - C(p,w)$, where $C(p,w) = \sum_{i=1}^{I} \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p,w)][\frac{1}{\alpha_i} x_i(p, \alpha_i w) - x(p,w)]'$.

Finally, we can write a sufficient conditions for $S(p,w)$ to be NSD as $C(p,w)$ being positive semidefinite, roughly, on average we require there is a positive correlation across consumers between consumption in one commodity and the wealth effect for that commodity.

Theorem 10. *The aggregate demand $x(p,w)$ satisfies the WA if $C(p,w)$ is positive semidefinite.*

7.5 Welfare Content of Aggregate Demand

Definition 45. A positive representative consumer exists if there is a rational preference relation $\succeq$ on X such that the aggregate demand function $x(p,w)$ is precisely the Walrasian demand function generated by the preference relation. That is $x(p,w) \succ x$ when $x \neq x(p,w)$ and $p'x \leq w$.

The existence of a positive representative consumer is not enough to assign welfare significance at the aggregate demand.

We first need to define a Social Welfare Function.

Definition 46. A (Bergson-Samuelson) social welfare function is a function $W = \mathbb{R}^I \mapsto \mathbb{R}$ that assigns a utility value of each possible vector $(u_1, \cdots, u_I) \in \mathbb{R}^I$ of utility levels for the I consumers in the economy.

We are now going to pose a problem, from the point of view of fictional central planner. The central planner can redistribute wealth for any given price to maximize social welfare.

Problem 1. The fictional central planner solves:

$$Max_{\{w_i\}_{i=1}^I} W(\{v_i(p, w_i)\}_{i=1}^I)$$

$$s.t. \quad \sum_{i=1}^{I} w_i \leq w.$$

As usual the v_i functions represent the individual indirect utilities.

Proposition 26. *The function* $v(p, w) = Max_{\{w_i\}_{i=1}^I, \sum_{i=1}^I w_i \leq w} W(\{v_i(p, w_i)\}_{i=1}^I)$ *is an indirect utility function for a positive representative consumer for the aggregate demand* $x(p, w) = \sum_{i=1}^I x_i(p, w_i)$.

The proof is left for the reader.

Definition 47. The positive representative consumer $\succeq$ for the aggregate demand $x(p, w)$ is a normative representative consumer relative to the social welfare function $W(\cdot)$ if for every (p, w) the distribution of wealth $\{w_i(p, w)\}_{i=1}^I$ solves the fictional central planner problem.

If all $\succeq_i$ admit a Gorman form then the aggregate demand resulting of this admits a normative representative consumer.

Chapter 8

Choice under Uncertainty (Ch. 6)

8.1 Dataset or Primitives

We consider Z to be a set of finite alternatives. We say that $p : Z \mapsto \mathbb{R}$ is a probability distribution on Z if $p(z) \geq 0$ for all $z \in X$ and $\sum_{z \in Z} p(x) = 1$.

Let $\Delta(Z)$ be the set of probability distributions on Z.

For any $z \in Z$ let $\delta_z \in \Delta(Z)$ be the Dirach distribution $\delta_z(z) = 1$. This is also called degenerate lotteries.

For any two probabilities $p, q \in \Delta(Z)$ and a number $\alpha \in [0,1]$ we define a third probability distribution $\alpha p + (1-\alpha)q \in \Delta(Z)$ as follows: $[\alpha p + (1-\alpha)q](z) = \alpha p(z) + (1-\alpha)q(z)$.

We observe a choice structure $\mathcal{B}, C$ where $\mathcal{B} \subseteq \Delta(Z)$ and C is a choice correspondence.

8.2 Model

We consider a preference relation $\succeq$ on $\Delta(Z)$.

Definition 48. we say that $\succeq$ has a expected utility representation if there exists a (Bernoulli) utility function $u : Z \mapsto \mathbb{R}$ such that the function $U(p) := \sum_{z \in Z} u(z)p(z)$ represents $\succeq$.

The Bernoulli utility is bounded and continuous and is defined over prices Z.

8.3 Independence Axiom

We are assuming $\succeq$ are rational.

Axiom 4. *(Independence Axiom) For all $p, q, r \in \Delta(Z)$ and $\alpha \in [0,1]$, $p \succeq q$ implies $\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$.*

The other axiom that we need is more technical

Axiom 5. *(Mixture continuity) For all $p, q, r \in \Delta(Z)$, if $p \succ q \succ r$, then the set $\{\alpha \in [0,1] | \alpha p + (1-\alpha)r \succ q\}$ and $\{\alpha \in [0,1] | q \succ \alpha p + (1-\alpha)r\}$ are open in $[0,1]$.*

8.4 Representation Theorem

Theorem 11. *The preference relation $\succeq$ satisfies the Independence Axiom and Mixture continuity if and only if there exists a utility function $u : Z \mapsto \mathbb{R}$ such that $p \succeq q$ and $\sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z)$.*

First we prove that there is a linear representation of $\succeq$.

Lemma 3. *If the preference relation $\succeq$ satisfies the Independence Axiom and Mixture continuity then there is a linear function $U : \Delta(Z) \mapsto \mathbb{R}$ that represents $\succeq$.*

Proof. First find $r_1, r_2 \in \Delta(Z)$ such that $r_1 \succeq p \succeq r_2$ for a fixed $p \in \Delta(Z)$. To ensure that this r_1, r_2 exists we use continuity and finiteness of Z. The second step is to define $U(p) = \alpha_p$ for $p \sim \alpha_p r_1 + (1-\alpha_p) r_2$.

Step 1. If $p \succ q$ and $\alpha \in (0,1)$ then $p \succ \alpha p + (1-\alpha)q \succ q$.

This follows from the independence axiom, note that if we have $p \succ q$ then $\alpha p + (1-\alpha)q \succ \alpha q + (1-\alpha)q$, finally notice that $p \succ \alpha p + (1-\alpha)q$ by definition. Now by transitivity $p \succ \alpha p + (1-\alpha)q \succ q$.

Step 2. Define the best lottery p^+, such that $p^+ \succeq p$ for all p and the worst lottery p^-. Now $\alpha, \beta \in [0,1]$. We want to show that $\beta p^+ + (1-\beta)p^- \succ \alpha p^+ + (1-\alpha)p^-$ if and only if $\beta > \alpha$.

Step 3. For any $p \in \Delta(Z)$ there is a unique α_p such that $\alpha_p p^+ + (1-\alpha_p)p^- \sim p$.

Existence is a consequence of Mixture continuity. Note that $p^+ \succeq p \succeq p^-$ then the set $\{\alpha \in [0,1] | \alpha p^+ + (1-\alpha)p^- \succeq p\}$ and $\{\alpha \in [0,1] | p \succeq \alpha p^+ + (1-\alpha)p^-\}$ are closed thus there is a sequence of $\alpha^n \in [0,1]$ in the intersection of this sets that converges to α_p such that $\alpha_p p^+ + (1-\alpha_p)p^- \sim p$. We know that a bounded sequence in a closed interval converges, so α_p exists and is between $[0,1]$. Uniqueness comes due to Step 2.

Step 4. $U(p) = \alpha_p$ represents the preferences $\succeq$.

By step 3, we have that for any $p, q \in \Delta(Z)$ such that $p \succeq q$, $\alpha_p p^+ + (1-\alpha_p)p^- \succeq \alpha_q p^+ + (1-\alpha_q)p^-$ if and only if $\alpha_p \geq \alpha_q$ by step 2.

Step 5. The $U(p) = \alpha_p$ is linear.

We want to to show that $U(\beta p + (1-\beta)q) = \beta U(p) + (1-\beta)U(q)$.

Note that $p \sim U(p)p^+ + (1-U(p))p^-$ and $q \sim U(q)p^+ + (1-U(q))p^-$.

By the independence axiom:

$\beta p + (1-\beta)q \sim \beta[U(p)p^+ + (1-U(p))p^-] + (1-\beta)q$

$\sim \beta[U(p)p^+ + (1-U(p))p^-] + (1-\beta)[U(q)p^+ + (1-U(q))p^-]$.

Rearranging:

$$\beta p + (1-\beta)q \sim [\beta U(p) + (1-\beta)U(q)]p^+ + [1 - [\beta U(p) + (1-\beta)U(q)]]p^-,$$

thus $U(\beta p + (1-\beta)q) = \beta U(p) + (1-\beta)U(q)$.

□

The second step of the proof is to show that a linear representation leads to an expected utility representation.

Lemma 4. *Let $U : \Delta(Z) \mapsto \mathbb{R}$. Then U is linear if and only if there exists a $u : Z \mapsto \mathbb{R}$ such that $U(p) = \sum_{z \in Z} u(z)p(z)$.*

Note that U is linear on $\Delta(Z)$ and continuous, then it is a continuous and linear operator. The Riesz representation theorem guarantees that we can always represent it as an inner product of the dual of the linear operator and the object itself, $U(p) = u'p$ where u is the dual. We let $u : Z \mapsto \mathbb{R}$ be a Bernoulli utility function.

8.5 Recoverability

Theorem 12. *Let $U : \Delta(Z) \mapsto \mathbb{R}$ and $U' : \Delta(Z) \mapsto \mathbb{R}$ be linear maps that represent $\succeq$ on $\Delta(Z)$, then there exist real numbers $\alpha > 0$ and b such that $U'(p) = \alpha U(p) + b$ for all $p \in \Delta(Z)$. If u, v are Bernoulli utilities such that $u'p$ and $v'p$ represent $\succeq$ then there are real numbers $\alpha > 0$ and b such that $v'p = \alpha u'p + b$.*

Think about this. How it differs from the recoverability result of the common rational preference $\succeq$.

8.6 Attitudes towards Risk

Definition 49. Expected value certainty equivalent. We say that $\delta_{\overline{z}}$ degenerate lottery is an expected value certainty equivalent of p if $\overline{z} = \sum_{z \in Z} zp(z)$.

Definition 50. A decision maker is risk averter if for any expected utility certainty equivalent of p, $\delta_{\overline{z}}$ with $\overline{z} = \sum_{z \in Z} zp(z)$, if $\delta_{\overline{z}} \succeq p$.

Under the Expected Utility framework.

Claim 6. The expected utility consumer is risk averter if u is concave,

$$\sum_{x \in Z} u(x)p(x) \leq u(\sum xp(x))$$

for all p.

The proof is simply an application of Jensen's inequality.

Definition 51. Given a Bernoulli utility function u we define the following concepts.

(i) Certainty equivalent. $CE(p, u) = u(\delta_z) = \sum_{z \in Z} u(z)p(z)$ for some δ_z.

(ii) Probability premium. For a fixed prize x, and a positive number ϵ (small) $\pi(x, \epsilon, u)$ is the probability premium if

$$u(x) = (\frac{1}{2} + \pi(x, \epsilon, u))u(x + \epsilon) + (\frac{1}{2} - \pi(x, \epsilon, u))u(x - \epsilon).$$

This is the excess probability over fair-odds that make the individual indifferent between the certain outcome x and a lottery over $x + \epsilon$ and $x - \epsilon$. It can be negative.

Proposition 27. *Suppose a consumer is an expected utility maximizer, with a Bernoulli utility u. Then the following are equivalent.*

(i) The decision maker is risk averse.

(ii) u is concave.

(iii) The expected certain equivalent of p is $CE(p, u) \leq \sum xp(x)$ for all p.

(iv) $\pi(x, \epsilon, u) \geq 0$ for all x, ϵ.

8.7 Expected Utility for Infinite Sets

Let Z be a set of prizes, we say that the collection of subsets $\Sigma \subseteq 2^Z$ is a σ-algebra when

- $\emptyset \in \Sigma$

- if $E \in \Sigma$ then $E^c \in \Sigma$
- If $\{A_n\}$ is a countable collection of sets such that $A_n \in \Sigma$ then $\cup_n A_n \in \Sigma$.

We say that (Z, Σ) is a measurable space whenever Σ is a $\sigma-$algebra.

Definition 52. We say that $p : \Sigma \mapsto [0, 1]$ is a probability measure if:

- $p(Z) = 1$
- if $\{A_n\}$ is a countable collection of pairwise disjoint sets such that $A_n \in \Sigma$, then $p(\cup_n A_n) = \sum_n p(A_n)$.

We call the collection of probabililities $\Delta(Z, \Sigma)$ for short $\Delta(Z)$ when Σ is fixed.

$\alpha p + (1 - \alpha)q \in \Delta(Z)$ is defined as $(\alpha p + (1 - \alpha)q)(E) = \alpha p(E) + (1 - \alpha)q(E)$.

We fix a topology on this pace $p^n \rightarrow^{w*} p$ under the weak$*$ topology if $\int f dp^n \rightarrow \int f dp$ for continuous and bounded functions f.

Axiom 6. *(w*-Continuity) For any $p \in \Delta(Z)$ the sets $\{q \in \Delta(Z) : q \succ p\}$ and $\{q \in \Delta(Z) : p \succ q\}$ are open in the weak* topology.*

Theorem 13. *A preference relation $\succeq$ on $\Delta(Z)$ that satisfies the Independence Axiom and w*-Continuity if and only if there exists a $u : Z \mapsto \mathbb{R}$ bounded and continuous such that $U(p) = \int u(z)dp(z)$ represents $\succeq$.*

8.8 Application: Demand for a risky asset.

An asset is a divisible claim to a financial return in the future.

Let α, β denote the amounts of money invested in a risky and safe asset respectively. The risky asset pays a random return of Z dollars per dollar invested and the safe asset pays 1 dollar with certainty per dollar invested. The random return has a distribution $F(z)$ (CDF). We assume that $\int z dF(z) > 1$, or that the risky asset expected return exceeds the return of the safe asset. Then the total return is $\alpha Z + \beta$. Of course, we must have that $\alpha + \beta = w$.

$$max_{\alpha,\beta \geq 0} \int u(\alpha z + \beta) dF(z)$$

$$s.t. \quad \alpha + \beta = w.$$

Equivalently we want to solve.

$$max_{0\leq\alpha\leq w}\int u(w+\alpha(z-1))dF(z).$$

The first order conditions are:

$$\phi(\alpha^*) = \int u'(w+\alpha^*[z-1])[z-1]dF(z)\begin{cases}\leq 0 & if \quad \alpha^* < w\\ \geq 0 & if \quad \alpha^* > 0.\end{cases}$$

Note that $\int zdF(z) > 1$ implies that $\phi(0) > 0$.

This follows from $\underline{u'(w) > 0.}$

And $\phi(0) = \int u'(w)[z-1]dF(z)$.

So $\alpha^* = 0$ cannot satisfiy the first order conditions.

If a risk is actuarially favorable then a risk averter will always accept at least a small amount of it.

8.9 Dominance

Theorem 14. *If $F \succeq^{FOD} G$ then $U(F) \geq U(G)$ when U is expected utility (u is strictly increasing $\succ$ and increasing $\succeq$).*

$F \succeq^{FOD} G$ FOSD means that $F(x) \leq G(x)$ for all $x \in Z$ with strict inequality for some $x \in Z$.

8.10 The Allais Paradox

Z is finite in fact $N = 3$,

First Prize	Second Price	Third Price
2.5 millions	0.5 millions	0

Two choice sets:

The first,

$L_1 = (0, 1, 0)$ and $L_1' = (0.10, 0.89, 0.01)$.

The second,

$L_2 = (0, 0.11, 0.89)$ and $L_2' = (0.10, 0, 0.90)$.

Some individuals choose $L_1 \succ L_1'$ and $L_2' \succ L_2$.

This is inconsistent with expected utility.

If $L_1 \succ L_1'$ then

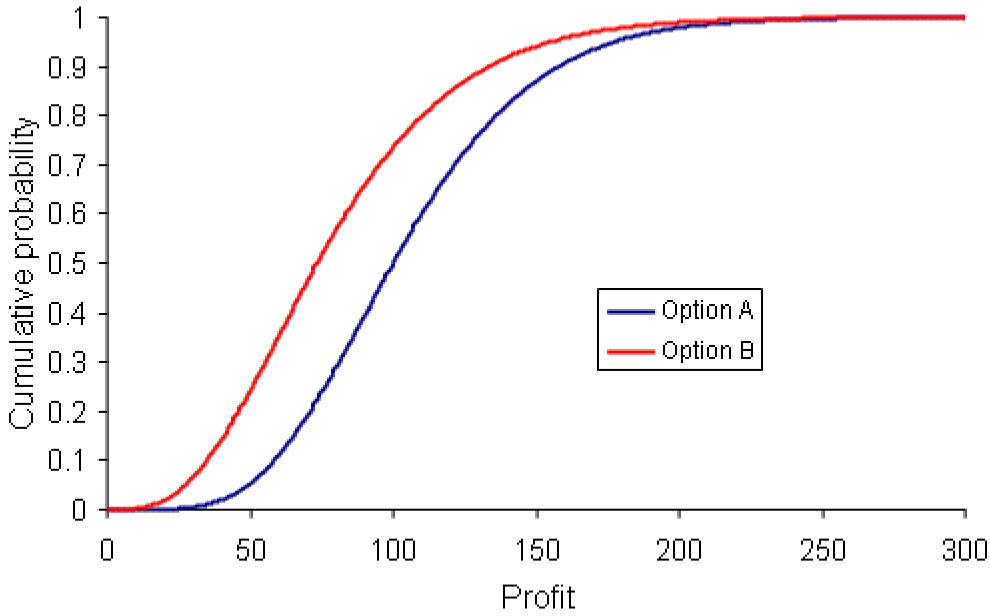


Figure 8.1: First order stochastic dominance. Option A dominates Option B.

$$u_{0.5} > (0.10)u_{2.5} + (0.89)u_{.5} + (0.01)u_0.$$

Adding, $(0.89)u_0 - (0.89)u_{0.5}$ to both sides we get,

$$(0.11)u_{0.5} + (0.89)u_0 > (0.1)u_{2.5} + (0.9)u_0,$$

and thus $L_2 \succ L_2'$.

8.11 Revealed Preference of Expected Utility under Concavity

Consider now the problem of maximizing expected utility subject to a budget. The consumption space is state-contingent commodities paying $x_s \in \mathbb{R}$ in state $s \in S$. There is a lottery defined over the states of the world $\pi \in \Delta(S)$, where $\pi_s \geq 0$ is the probability that state s happens and the consumer gets x_s units of consumption. The problem is:

$$max_x \sum_{s \in S} \pi_s u(x_s)$$

s.t.

$$\sum_{s\in S} p_s x_s = w.$$

We are going to devise a nonparametric test of expected utility on the basis of the first order conditions. The Bernoulli utility $u : X \to \mathbb{R}$, is assumed to be for this subsection continuous, increasing, and **concave.** We do not need it to be differentiable but for simplicity assume it is. Then, let's obtain the FOC of this problem:

$$\pi_s u'(x_s) = \lambda p_s,$$

for all $s \in S$.

Note that a consumption vector $x = (x_1 \cdots x_S)'$ is expected utility maximizing if and only if it satisfies the FOC.

Now, consider a finite data set of prices and consumption $O^T = \{p^t, x^t\}_{t=1}^T$ and a known probability over states π, where each observation is a different trial. Note that x_s^t, p_s^t, π_s denote the state contingent quantities, prices and probabilities respectively. Then:

Definition 53. We say a data set O^T is rationalized by a risk averse expected utility function if there is a continuous, concave, increasing utility function u such that:

$$\sum_{s\in S} \pi_s^t u(x_s^t) \geq \sum_{s\in S} \pi_s^t u(y_s)$$

for all $y \in X$ such that $\sum_{s\in S} p_s^t x_s^t \geq \sum_{s\in S} p_s^t y_s$.

Theorem 15. *The following are equivalent:*

(i) O^T is rationalized by a risk averse expected utility function.

(ii) If there are numbers $\beta_s^t > 0$ for all $t \in \{1, \cdots, T\}$ and $s \in S$ such that:

(1) if $x_s^t > x_{s'}^{t'}$ then $\beta_s^t \leq \beta_{s'}^{t'}$,

(2) for every $t \in \{1, \cdots, T\}$,

$$\frac{\pi_s \beta_s^t}{p_s^t} = \frac{\pi_{s'} \beta_{s'}^t}{p_{s'}^t}.$$

Proof:

Necessity:

By the FOC, we have that

$$\pi^s u'(x_s^t) = \lambda^t p_s^t, \forall s \in S, \forall t.$$

	GARP	RDEU	EU	EU*
Drop 0	12/47 (26%)	2/47 (4%)	2/47 (4%)	0/47 (0%)
Drop 1	14/47 (30%)	7/47 (15%)	5/47 (11%)	0/47 (0%)
Drop 2	27/47 (57%)	9/47 (19%)	7/47 (15%)	2/47 (4%)
Drop 3	32/47 (68%)	12/47 (26%)	10/47 (21%)	2/47 (4%)

Table 1: Results

Figure 8.2: Passing rates of Expected Utility.

Then:

$$\lambda^t = \frac{\pi_s u'(x_s^t)}{p_s^t}.$$

This implies:

$$\frac{\pi_s \beta_s^t}{p_s^t} = \frac{\pi_{s'} \beta_{s'}^t}{p_{s'}^t},$$

for $\beta_s^t = u'(x_s^t)$.

Conditions in (1), are implied by the fact that u is concave and increasing.

8.12 Experimental Evidence

Polisson and Quah develop a revealed preferences test for expected utility. They apply their framework to the famous Choi, Fisman, Gale and Kariv (2007) dataset and find the following passing rates.

-GARP is already known to us.

-RDEU is an alternative model to expected utility called rank dependent expected utility.

-EU is the expected utility model.

-EU* is the expected utility with the restriction that the Bernoulli utility is concave.

The “drop” column corresponds to how much observations I dropped from the sample.

Table 1: Holt and Laury (2002)

Row No.	Option A Outcome A 1 = \$2.00	Outcome A 2 = \$1.60	Option B Outcome B 1 = \$3.85	Outcome B 2 = \$0.10	RRA if row was last choice of A and below all B
1	Prob. 1/10	Prob. 9/10	Prob. 1/10	Prob. 9/10	$[-1,71;-0.95]$
2	Prob. 2/10	Prob. 8/10	Prob. 2/10	Prob. 8/10	$[-0.95;-0.49]$
3	Prob. 3/10	Prob. 7/10	Prob. 3/10	Prob. 7/10	$[-0.49;-0.14]$
4	Prob. 4/10	Prob. 6/10	Prob. 4/10	Prob. 6/10	$[-0.14;0.15]$
5	Prob. 5/10	Prob. 5/10	Prob. 5/10	Prob. 5/10	$[0.15;0.41]$
6	Prob. 6/10	Prob. 4/10	Prob. 6/10	Prob. 4/10	$[0.41;0.68]$
7	Prob. 7/10	Prob. 3/10	Prob. 7/10	Prob. 3/10	$[0.68;0.97]$
8	Prob. 8/10	Prob. 2/10	Prob. 8/10	Prob. 2/10	$[0.97;1.37]$
9	Prob. 9/10	Prob. 1/10	Prob. 9/10	Prob. 1/10	$[1.37;\infty)$
10	Prob. 10/10	Prob. 0/10	Prob. 10/10	Prob. 0/10	non-monotone

Figure 8.3: Price list method for eliciting risk aversion

8.13 Measuring Risk Aversion in the Lab

The Holt and Laury price list method.

Individuals choose between A and B in each of the ten rows.

There are two outcomes for each option A_1, A_2 and B_1, B_2 with changing probabilities for each row.

The expected outcome of A is higher for the firt 4 rows, and it switches to lower for the last six rows.

-A risk neutral subject will choose A from $t = 1, \cdots, 4$ switch at 5 and remain there.

- Option B has higher variance:

-An individual who switches to B between row 6 and row 10 is classified as being risk-averse and the more risk-averse individual the switch will be later.

At random you pay one of the rows. To incentivize decision making.

Chapter 9

Theory of the Firm

9.1 Introduction

A **firm** represents the production possibilities of household, individuals, corporations or other businesses. The firm may represent actual production or potential of production. The theory of the firm is concerned with the question of: What can a firm do? The firm for us, is a "black box" that is able to transform inputs into outputs.

9.2 Production Sets

We fix $L \geq 1$ commodities, a **production vector or plan** $y = (y_1, \cdots, y_L) \in \mathbb{R}^L$ describes the (net) outputs of the L commodities. If $y_l \geq 0$ denotes output and $y_l < 0$ denotes input.

The set of all production plans that are technologically feasible is called the **production set,** $Y \subset \mathbb{R}^L$**.**

Any $y \in Y$ is feasible, any $y \notin Y$ is not.

The production set is the data or the primitive of the theory.

Sometimes it is useful to describe the production set, using a function F called the **transformation function.**

This is $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$ and $F(y) = 0$ if and only if y is an element of the boundary of Y.

The set of $\{y \in \mathbb{R}^L | F(y) = 0\}$ is called the **transformation frontier.**

Definition 54. The Marginal Rate of Transformation MRT of any two goods l, k is the ratio:

$$MRT_{lk}(\overline{y}) = \frac{\partial F(y)/\partial y_l}{\partial F(y)/\partial y_k}, \quad F(\overline{y}) = 0.$$

The MRT measures how much the net output of k can increase if the firm decreases the net output of l by one marginal unit.

9.3 Technologies with Different Inputs and Outputs

We let M denote the number of outputs and $L - M$ the number of inputs, usually the vector of outputs is denotes as $q = (q_1, \cdots, q_M) \geq 0$ and the vector of used inputs is $z = (z_1, \cdots, z_{L-M}) \geq 0$. Here, we eliminate the negative sign for inputs as they stand separate from outputs.

A particular case of interest is when $M = 1$, in that case we can write the production set using a **production function,** that gives the maximum amount of output one can reach with a level of input z, $q \leq f(z)$.

The production set is then:

$$Y = \{(-z', q)' \in \mathbb{R}^L | q - f(z) \leq 0 \quad z \geq 0, z \in \mathbb{R}^{L-1}\}.$$

Definition 55. Marginal rate of technical substitution -MRTS-. The MRTS of goods l, k is the ratio:

$$MRTS_{lk}(\overline{z}) = \frac{\partial f(\overline{z})/\partial z_l}{\partial f(\overline{z})/\partial z_k}.$$

For a fixed level $\overline{q} = f(\overline{z})$, the MRTS of l, k measures by how much the input of k must be increased for a reduction of a marginal unit of l to keep the production at the maximum level $\overline{q}$.

This is analogous to the Marginal Rate of Substitution in consumer theory, where we asked by how much the demand of one good changed with respect to other to keep the utility constant.

9.4 Properties of the Production Set (a list)

(i) (Always assumed) Y is non-empty.

(ii) (Always assumed) Y is closed. The limit of feasible production plans is also feasible $y^n \to y$ and $y^n \in Y$ implies that $y \in Y$.

(iii) No free lunch. If $y \geq 0$ and $y \in Y$, then $y = 0$. $(Y \cap \mathbb{R}^L_+ \subset \{0\})$.

(iv) Possibility of inaction. $0 \in Y$.

(v) Free disposal. If $y \in Y$ and $y' \leq y$ (so that y' produces the same amount of outputs using at least the same amount of inputs), then $y' \in Y$. $(Y \backslash \mathbb{R}^L_+ \subset Y)$. Additional amounts of input (that produces at least the same output) can be disposed at no cost.

(vi) Irreversibility. Suppose that $y \in Y$ and $y \neq 0$. Then $-y \notin Y$. It is technologically impossible to transform an amount of output into the same required amount of input.

9.4.1 Returns to Scale

(viii) Nonincreasing returns to scale. Y exhibits nonincreasing returns to scale if $y \in Y$ implies that $\alpha y \in Y$ for all scalars $\alpha \in [0, 1]$.

(viii) Nondecreasing returns to scale. Y If $y \in Y$ then $\alpha y \in Y$ for any scale $\alpha \geq 1$.

(ix) Constant returns to scale. If $y \in Y$ this implies that $\alpha y \in Y$ for any $\alpha \geq 0$ (in this case Y is a cone).

(x) Additivity (free entry). Suppose $y \in Y$ and $y' \in Y$ then $y + y' \in Y$.

(xi) Convexity. Y is convex.

Proposition 28. *The production set Y is additive and satisfies the nonincreasing returns conditions if and only if it is a convex cone.*

Proof. The convex cone property of Y implies Y is convex and that for all $\alpha \geq 0$, if $y \in Y$ then $\alpha y \in Y$. Convexity and the cone property imply noincreasing returns and additivity properties.

Conversely, we want to show that if nonincreasing returns hold and additivity then the convex cone property follows.

We want to show that $\alpha, \beta > 0$ and $y, y' \in Y$, then $\alpha y + \beta y' \in Y$. Let k an integer be such that $k > max\{\alpha, \beta\}$. By additivity $ky \in Y$ and $ky' \in Y$. Since $\frac{\alpha}{k} < 1$ and $\alpha y = (\frac{\alpha}{k})ky$ the nonincreasing returns condition implies that $\alpha y \in Y$. Similarly, $\beta y' \in Y$. Finally, by additivity $\alpha y + \beta y' \in Y$.

□

This means that if we want to assume non-increasing returns and free entry we need the convexity assumption.

Proposition 29. *For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns convex production $Y' \subset \mathbb{R}^{L+1}$ such that $Y' = \{y' \in \mathbb{R}^L : y' = (y, -1), y \in Y\}$.*

Let $Y' = \{y' \in \mathbb{R}^{L+1} : y' = \alpha(y, -1)$ for some $y \in Y$ and $\alpha \geq 0\}$.

Example 13. One output. q is the output, one input z, $f(z) = z^\alpha$ for $0 < \alpha < 1$,

ω efficiency, $\hat{f}(z,\omega) = z^{\alpha}\omega^{1-\alpha}$, $\lambda(z \quad \omega)$, λq, for any $\lambda > 0$, $\hat{f}(\lambda z, \lambda\omega) = \lambda\hat{f}(z,\omega)$, $\lambda q = \lambda\hat{f}(z,\omega)$.

9.5 Profit Maximization and Cost Minimization

The **profit maximization problem** PMP, is

$$max_y \quad p'y$$

$$s.t. \quad y \in Y.$$

Using the transformation function we have:

$$Max_y p'y$$

$$s.t. \quad F(y) \leq 0.$$

The **profit function** is the value function or $\pi(p) = max\{p'y : y \in Y\}$.

The maximizer is the **supply correspondence** $y(p) = argmax\{p'y : y \in Y\}$ given prices.

The profit function may be infinite $\pi(p) = +\infty$.

FOC:

If $y^* \in y(p)$ then for some $\lambda \geq 0$, y^* must satisfy the FOC

$$p_l = \lambda \frac{\partial F(y^*)}{\partial y_l} \quad for \quad l = 1, \cdots, L$$

or equivalently in matrix notation,

$$p = \lambda \nabla F(y^*).$$

For single output technologies we have:

$$max_{z \geq 0} pf(z) - w'z.$$

For f a production function and z a vector of inputs.

For $l \in \{1, \cdots, L-1\}$ we have:

$$p_l \frac{\partial f(z^*)}{\partial z_l} \leq w_l, \quad z_l^* > 0 \quad equality \quad holds.$$

Or in matrix notation:

$$p' \nabla f(z^*) \leq w, \quad [p \nabla f(z^*) - w]' z^* = 0.$$

If Y is convex then the FOC are necessary and sufficient for optimality.

Proposition 30. *(Properties of the profit function) Suppose* $\pi(p) = max_y\{p'y|y \in Y\}$ *and* $y(p) = argmax_y\{p'y|y \in Y\}$

Assume also Y *is closed, and satisfies the free disposal. Then,*

(i) $\pi(p)$ *is homogeneous of degree one.*

(ii) $\pi(p)$ *is convex*

(iii) If Y *is convex then* $Y = \{y \in \mathbb{R}^L : p'y \leq \pi(p) \forall p >> 0\}$.

(iv) $y(p)$ *is homogeneous of degree zero.*

(v) If Y *is convex then* $y(p)$ *is convex set for all* p. *Moreover, if* Y *is strictly convex, then* $y(p)$ *is singled-valued. (if non-empty).*

(vi) Hotelling's lemma. If $y(p)$ *consists of a single point then* π *is differentiable at* p *and* $\nabla \pi(p) = y(p)$.

(vii) If $y(p)$ *is a differentiable function at* p *then* $Dy(p) = D^2\pi(p)$ *is symmetric and positive semidefinite matrix with* $Dy(p)p = 0$.

Proof. (ii) $\pi(\cdot)$ is a convex function.

$y \in y(\alpha p + (1-\alpha)p^*)$ with $\alpha \in (0,1)$ then

Convexity means $\pi(\alpha p + (1-\alpha)p^*) \leq \alpha \pi(p) + (1-\alpha)\pi(p^*)$

$\pi(\alpha p + (1-\alpha)p^*) = \alpha p \cdot y(\alpha p + (1-\alpha)p^*) + (1-\alpha)(p^*) \cdot y(\alpha p + (1-\alpha)p^*) \leq$

$$\alpha \pi(p) + (1-\alpha)\pi(p^*).$$

$\pi(p) = p \cdot y(p) = p \cdot y' \; p \cdot y' \geq p \cdot y(\alpha p + (1-\alpha)p^*)$.

Because $y \in y(\alpha p + (1-\alpha)p^*)$ may not be optimal at p and at p^*.

(iii) If Y is convex then $p'y \leq p'y^* \; y^* \in y(p)$ for all $y \in Y$ by definition.

(vi) Follows from the Envelope's theorem, $\mathcal{L} = p'y + \lambda(F(y))$

$\partial_p \mathcal{L} = y(p) = \nabla_p \pi(p)$.

(viii) Given (ii) we know that π is convex, and by (vi) $Dy(p) = D^2\pi(p)$. Thus $Dy(p)$ is PSD and symmetric. The singularity property $Dy(p)p = 0$ follows from (i).

□

Property (viii) can also be stated in non-differentiable terms as the **law of supply.**

$$(p - p^*)'(y - y^*) \geq 0.$$

9.6 Cost Minimization

A property of profit-maximizing firms is that there is no way to produce the same amount at a lower cost. A necessary condition for profit-maximization is cost minimization.

Sometimes firms are not price takers, or there are increasing returns to scale, in that cases the use of the profit functions is limited and we can use instead the cost function. The **cost minimization problem:**

$$Min_{z \geq 0} w'z$$

$$s.t \quad f(z) \geq q.$$

The optimized value is the cost function $c(w, q)$. The optimizing set is $z(w, q)$ is known as the conditional factor demand correspondences. The produced level $q \in \mathbb{R}_{++}$ is fixed.

The FOC,

$$w_l \geq \lambda \frac{\partial f(z^*)}{\partial z_l}, \quad z_l > 0.$$

In matrix notation,

$$w \geq \lambda \nabla f(z^*) \quad [w - \lambda \nabla f(z^*)]'z = 0.$$

The λ Lagrange multiplier can be interpreted as the marginal value of relaxing the constraint. That is $\frac{\partial c(w,q)}{\partial q} = \lambda$ the marginal cost of production.

Proposition 31. *The properties of the $c(w, q)$ and the $z(w, q)$ are,* *(i) c is HD1 in w and nondecreasing in q.*

(ii) c is a concave function on w.

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ *are convex for every* q *then* $Y = \{(-z, q) : w'z \geq c(w, q)$ *for all* $w \geq 0\}$.

(iv) z (intermediate supply) is HD0 in w

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ *is convex then* $z(w, q)$ *is a convex set. Moreover if* $\{z \geq 0 : f(z) \geq q\}$ *is strictly convex then* $z(p, q)$ *is singled valued.*

(vi) Sheppard's lemma. If $z(\overline{w}, q)$ *consists of a single point then* $c(\cdot)$ *is differentiable with respect to* w *at* $\overline{w}$ *and* $\nabla_w c(\overline{w}, q) = z(\overline{w}, q)$.

(vii) If z *is differentiable at* $\overline{w}$, *then* $D_w z(\overline{w}, q) = D_w^2 c(\overline{w}, q)$ *is symmetric and NSD with* $D_w z(\overline{w}, q)\overline{w} = 0$.

(ix) If f *is concave then* c *is a convex function of* q *(in particular, marginal costs are nondecreasing in* q*).*

The proof of this properties are very similar the expenditure case in the consumer part of this course.

We can state the firm's problem with the cost function as follows:

$$Max_{q \geq 0} pq - c(w, q).$$

The necessary and first-order conditions for q^* to be profit maximizing is

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0$$

with equality if $q^* > 0$.

9.7 Aggregation

Suppose there are J production units (firms) and each production set is denoted as Y_j (non-empty and closed also satisfies the free-disposal). The individual profit function and supply functions are $\pi_j(p)$ and $y_j(p)$ respectively.

The aggregate supply function is

$$y(p) = \sum_{j=1}^{J} y_j(p).$$

Note that $Dy(p)$ is symmetric and PSD because $Dy_j(p)$ are also symmetric and PSD for all j.

The aggregate production set is

$$Y = \sum_{j=1}^{J} Y_j = \{y \in \mathbb{R}^L | y = \sum_j y_j \quad for \quad some \quad y_i \in Y_j, j = 1, \cdots, J\}.$$

Let $\pi^*(p)$ and $y^*(p)$ be the profit function and supply function of the aggregate production set.

Proposition 32. *For all* $p >> 0$ *we have*

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$.

Proof. For the first equality note that if we take any feasible plan $y_j \in Y_j$ and note that $\pi^*(p) \geq p'(\sum_j y_j)$ because $\pi^*(p)$ is maximum then this means that $\pi^*(p) \geq \sum_j p' y_j(p)$, but this means that $\pi^*(p) \geq \sum_j \pi_j(p)$.

Consider any $y \in Y$ feasible in the aggregate, then $p'y = p'(\sum_j y_j) = \sum_j p' y_j \leq \sum_j \pi_j(p)$, but this implies that $\pi^*(p) \leq \sum_j \pi_j(p)$.

□

This result implies that we can decentralize the aggregate firm profit maximization problem into maximizing the profit of each firm separately.

9.8 Efficiency

We have a definition of efficiency.

Definition 56. A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

An efficient plan has to be on the boundary but not all boundary points are efficient.

Proposition 33. *If* $y \in Y$ *is profit maximizing for some* $p >> 0$ *then* y *is efficient.*

Proof. Suppose otherwise, i.e., there is a $y' \in Y$ such that $y \neq y'$ and $y' \geq y$. By $p >> 0$, this means that $p \cdot y' > p \cdot y$, but that means that y is not profit maximizing.

□

Corollary 5. *If a collection of firms* J *independently maximizes profits for a given price* $p >> 0$*, then the aggregate production is socially efficient.*

Figure 1. Strong separation.

Figure 2. These sets cannot be separated by a hyperplane.

Figure 9.1: Source:Border

This is a consequence of the aggregation result and the previous result.

Proposition 34. *Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.*

Proof. Suppose $y \in Y$ is efficient, and define the set $P_y = \{y' \in \mathbb{R}^L : y' >> y\}$. The set P_y is convex, and given that y is efficient $P_y \cap Y = \emptyset$. We can invoke the separating hyperplane theorem to establish that there is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$. In particular this implies that $p \cdot y' \geq p \cdot y$ for every $y' >> y$. Thus, we must have $p \geq 0$ because if $p_l < 0$ for some l, then we should have $p \cdot y' < p \cdot y$ for some $y' >> y$ with $y'_l - y_l$ sufficiently large.

Finally, since $p \cdot y' \geq p \cdot y''$ for all $y' \in P_y$ and $y'' \in Y$ and I can choose $y' >> y$ to be as close as possible to y then $p \cdot y \geq p \cdot y''$, thus y is profit maximizing.

□

Chapter 10

Competitive Markets and Partial Equilibrium

10.1 Competitive Equilibria

We consider a market.

Consider I consumers and J firms. There are L goods. Consumer's i preferences over $x_i = (x_{i,l})_{l=1}^{L}$ are given by $u_i(\cdot)$. The initial amount of the goods that are in the economy or **endowment** is given by $\omega_i = (\omega_{i.l})_{l=1}^{L}$ with the restriction that $\omega_{i,l} \geq 0$. The firms may transform some of the endowment of a good into additional amounts of other goods.

Each firm j has available to it the production possibilities summarized by the production set $Y_j \subset \mathbb{R}^L$. An element $y_j \in Y_j$ is a production plan $y_j = (y_{j,l})_{l=1}^{L}$. The net amount of good on the economy is $\omega_l + \sum_{j=1}^{J} y_{l,j}$.

Definition 57. (Economic allocation) An economic allocation $(x_1, \cdots, x_I; y_1, \cdots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \cdots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \cdots, J$.

Now, we define the main property of any allocation.

Definition 58. (Feasibility) The allocation $(x_1, \cdots, x_I; y_1, \cdots, y_J)$ is feasible if

$$\sum_{i=1}^{I} x_{l,i} \leq \omega_l + \sum_{j=1}^{J} y_{l,j} \quad \forall l = 1, \cdots, L.$$

Now we are ready to define a competitive equilibrium.

Definition 59. The allocation $(x_1, \cdots, x_I; y_1, \cdots, y_J)$ and price vector $p \in \mathbb{R}^L$ constitute a competitive (or Walrasian) equilibrium if the following conditions are satisfies:

(i) Profit maximization: For each firm j, y_j solves

$$max_{y_j \in Y_j} p \cdot y_j.$$

(ii) Utility maximization: For each consumer i, x_i solves

$$max_{x \in X_i} u_i(x_i)$$

$$s.t. \quad p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j),$$

for $\theta_{ij} \in [0,1]$ a share of firm j owned by i such that $\sum_{i=1}^{I} \theta_{ij} = 1$.

(iii) Market clearing: For each good $l = 1, \cdots, L$

$$\sum_{i=1}^{I} x_{l,i} = \sum_{i=1}^{I} \omega_{l,i} + \sum_{j=1}^{J} y_{l,j}.$$

10.2 Some Preliminary Results

Claim 7. If the allocation $(x; y)$ and $p \in \mathbb{R}^L_{++}$ constitute a competitive equilibrium then so does $(x; y)$ and $\alpha p \in \mathbb{R}^L_{++}$ for $\alpha > 0$.

This means that we can always normalize prices and fix $p_l = 1$ for a fixed commodity l.

Lemma 5. *If the allocation $(x; y)$ and the price vector $p >> 0$ satisfies the market clearing condition for all goods $l \neq k$ for a fixed k, and if the consumers' budget constraints so that $p \cdot x_i = p \cdot \omega_i + \sum_{i=1}^{I} \theta_{ij} p \cdot y_j$ for all i. Then the market for good k also clears.*

10.3 Partial Equilibrium Competitive Analysis

We study an economy with two goods, good l x_i, and a numeraire m_i each consumer has a quasilinear utility

$$u_i(m_i, x_i) = m_i + \phi_i(x_i).$$

The consumption set is $X = \mathbb{R} \times \mathbb{R}^L_+$ so that the numeraire may be negative (no boundary problems).

The function ϕ_i is bounded above and twice differentiable, with $\phi_i'(x_i) > 0$ and $\phi_i''(x_i) < 0$ at all $x_i \geq 0$. We normalize $\phi_i(0) = 0$.

The numeraire m_i is a composite good of all other goods. The key idea is that x_i is small relatively to the whole expenditure and thus the wealth effects are close to null.

We normalize the price of the numeraire to 1 and p is the price of good l.

Each firm $j = 1, \cdots, J$ produces good l from m. The cost function $c_j(q_j)$ captures how much input of m do the firm j needs to produce $q_j \geq 0$ of good l (efficiently) recall that the price of the numeraire is 1. The intermediate or m demand of firm j is given by z_j. Its production set is

$$Y_j = \{(-z_j, q_j) : q_j \geq 0 \quad and \quad z_j \geq c_j(q_j)\}.$$

The cost function is twice continuously differentiable $c_j'(q_j) > 0$ and $c_j''(q_j) > 0$ for all $q_j \geq 0$ (strictly convex technology).

We assume that there is no initial endowment of good l. Consumers' initial endowment of the numeraire is $\omega_{mi} > 0$ and the total endowment is $\omega_m = \sum_{i=1}^I \omega_{mi}$.

Definition 60. (Partial Competitive Equilibrium) We say that $(x_1, \cdots, x_I; q_1, \cdots, q_J)$ and the price p constitute a (partial) competitive equilibrium if and only if

$$p \leq c_j'(q_j), \quad \text{with equality if} \quad q_j > 0 \quad \forall j \in \{1, \cdots, J\}$$

$$\phi_i'(x_i) \leq p \quad \text{with equality if} \quad x_i > 0 \quad \forall i \in \{1, \cdots, I\}$$

$$\sum_{i=1}^I x_i = \sum_{j=1}^J q_j.$$

This is a system of equations of $I + J + 1$ that determines I consumption levels, J production levels and one price.

We use here the fact that we can check that the market clears for $L - 1$ goods.

It should be understood that the consumer's i equilibrium consumption of the numeraire is

$$m_i = \omega_{mi} + \sum_{j=1}^J \theta_{ij}(pq_j - c_j(q_j)) - px_i.$$

And the firm $j's$ equilibrium usage of the input is

$$z_j = c_j(q_j).$$

As long as $Max_i\phi_i'(0) > Min_jc_j^{'}(0)$ the aggregate consumption and production of good l must be strictly positive in a competitive equilibrium. We will assume this is the case in what follows.

Observe that the (partial) competitive equilibrium definition does not involve the endowments or the ownership shares of the consumers. This is a consequence of the use of quasi-linear preferences and the defining feature of partial equilibrium analysis.

10.4 Existence of the Partial Competitive Equilibrium

Claim 8. Assume that $Max_i\phi_i'(0) > Min_jc_j^{'}(0)$, then there is always a p^* and $(x;q)$ that satisfies the condition of the (partial) competitive equilibrium.

Proof. First we notice that to find prices p^*, we only need to find p at which aggregate supply equals aggregate demand.

$x^*(p) = q^*(p)$. Under the assumption $Max_i\phi_i'(0) > Min_jc_j^{'}(0)$ at any $p \geq Max_i\phi_i'(0)$ we have $x(p) = 0$ and $q(p) > 0$. Similarly at any $p \leq Min_jc_j'(0)$ we have $x(p) > 0$ and $q(p) = 0$. The existence of an equilibrium price $p \in (Min_jc_j'(0), Max_i\phi_i'(0))$ follows from the continuity of the aggregate demand and the aggregate supply. Note that $x^*(\cdot)$ is strictly decreasing at all $p < Max_i\phi_i'(0)$ and $q^*(\cdot)$ is strictly increasing at all p. After finding p^*, the individual level of consumption $x_i^* = x_i(p^*)$ and the individual level of production is given likewise $q_j^* = q_j(p^*)$.

□

10.5 Pareto Optimality

An "optimal" economic outcome for most economists is the Pareto Optimality or Pareto Efficiency.

Definition 61. (Pareto Optimal) A **feasible** allocation $(x_1, \cdots, x_I; y_1, \cdots, y_J)$ is Pareto optimal (or Pareto efficient) if there is no other **feasible** allocation $(x_1', \cdots, x_I'; y_1', \cdots, y_J')$ such that $u_i(x_i') \geq u_i(x_i)$ for all $i = 1, \cdots, I$ and $u_i(x_i') > u_i(x_i)$ for some i.

10.6 The Fundamental Welfare Theorems in Partial Equilibrium Contexts

Proposition 35. *(The First Fundamental Theorem of Welfare Economics) If the price p^* and allocation $(x_1^*, \cdots, x_I^*, q_1^*, \cdots, q_J^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal.*

To prove this we need some preliminaries.

We imagine that there is a central planner that wants to maximize aggregate surplus.

$$\max_{\substack{(x_1,\cdots,x_I)\geq 0\\(q_1,\cdots,q_J)\geq 0}} \sum_{i=1}^{I}\phi_i(x_i)-\sum_{j=1}^{J}c_j(q_j)+\omega_m$$

$$s.t. \quad \sum_{i=1}^{I}x_i-\sum_{j=1}^{J}q_j=0.$$

The term $\sum_{i=1}^{I}\phi_i(x_i)-\sum_{j=1}^{J}c_j(q_j)$ is the Mashallian aggregate surplus. It can be though as the total utility generated by the consumption of good l less the cost of production in terms of the numeraire. It is clear that an allocation that solves the central planner problem is Pareto Optimal.

Now notice the first order conditions, for μ a Lagrange multiplier associated with the only constraint.

$$\mu \leq c_j'(q_j), \quad \text{with equality if} \quad q_j>0 \quad \forall j\in\{1,\cdots,J\}$$

$$\phi_i'(x_i)\leq \mu \quad \text{with equality if} \quad x_i>0 \quad \forall i\in\{1,\cdots,I\}$$

$$\sum_{i=1}^{I}x_i=\sum_{j=1}^{J}q_j.$$

This means that if the price p^* and allocation $(x_1^*, \cdots, x_I^*, q_1^*, \cdots, q_J^*)$ constitute a partial competitive equilibrium then $(x_1^*, \cdots, x_I^*, q_1^*, \cdots, q_J^*)$ and $\mu = p^*$ satisfy the FOC of the central planner problem. Moreover given the convexity assumptions we have made the FOC are also sufficient. We conclude that the partial equilibrium allocation is Pareto optimal.

Proposition 36. *(The Second Fundamental Theorem of Welfare Economics) For any Pareto* *optimal levels of utility* $(u_1, \cdots, u_I)$ *there are transfers of the numeraire commodity* $(T_1, \cdots, T_I)$ *satisfying* $\sum_{i=1}^{I} T_i = 0$*, such that a competitive equilibrium reached from the endowments* $(\omega_{m1} + T_1, \cdots, \omega_{nI} + T_I)$ *yields precisely the utilities* $(u_1, \cdots, u_I)$.

Chapter 11

General Equilibrium Theory

11.1 Pure Exchange Economies: The Edgeworth Box

Assume that there are two consumers $i = 1, 2$ and two commodities $l = 1, 2$. Consumer $i's$ consumption vector is $x_i = (x_{1i}, x_{2i})$ that is x_{li} denotes the consumption of good l and individual i.

The consumption set $X_i = \mathbb{R}^2_+$ and i has preferences $\succeq_i$ defined over X_i. Each consumer has an endowment of $\omega_{li} \geq 0$ of good l. Consumer's i endowment vector is $\omega_i = (\omega_{1i}, \omega_{2i})$. The total endowment of good l is the endowment vector $\omega_l = \omega_{l1} + \omega_{l2}$. This quantity is strictly positive.

An allocation is $x \in \mathbb{R}^4$, $x = (x_1, x_2) = ((x_{11}, x_{21}), (x_{12}, x_{22}))$.

We say that an allocation is feasible if

$$x_{l1} + x_{l2} \leq \omega_l \quad \forall 1, 2.$$

The main feature of general equilibrium is that wealth is given endogenously.

We have for any given vector of prices $p = (p_1, p_2)$, consumer's i wealth equals the market endowment of commodities:

$$p \cdot \omega_i = p_1\omega_{1i} + p_2\omega_{2i}.$$

The consumer's i budget constraint is:

$$B_i(p) = \{x_i \in X : p \cdot x_i \leq p \cdot \omega_i\}.$$

In an Edgeworth box, the budget constraint of both consumers can be captured by the

budget line, that is the line that goes through the market endowment and has slope $-(p_1/p_2)$.

Definition 62. (Walrasian Equilibrium) A Walrasian (or competitive) equilibrium for an Edgeworth box economy is a price vector p^* an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$

$$x_i^* \succeq_i x_i \quad \forall x_i \in B_i(p^*).$$

Example 14. (Cobb-Douglas Exchange Economy) Suppose that each consumer i has the Cobb-Douglas utility function $u_i(x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$. In addition, endowments are $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$. At prices $p = (p_1, p_2)$, consumer's 1 wealth is $(p_1 + 2p_2)$ and his demand lie on the offer curve:

$$OC_1(p) = \left(\frac{\alpha(p_1 + 2p_2)}{p_1}, \frac{(1-\alpha)(p_1 + 2p_2)}{p_2} \right).$$

The consumer's 2 OC is:

$$OC_2(p) = \left(\frac{\alpha(2p_1 + p_2)}{p_1}, \frac{(1-\alpha)(2p_1 + p_2)}{p_2} \right).$$

To determine the Walrasian equilibrium prices, note that at these prices the total amount of good 1 consumed by the two consumers must equal 3 ($\omega_{11} + \omega_{12} = 3$). Thus:

$$\frac{\alpha(p_1^* + 2p_2^*)}{p_1^*} + \frac{\alpha(2p_1^* + p_2^*)}{p_1^*} = 3.$$

Solving this:

$$\frac{p_1^*}{p_2^*} = \frac{\alpha}{1-\alpha}.$$

Definition 63. (Pareto Optimality/Edgeworth Box) An allocation x in the Edgeworth box is Pareto optimal (or Pareto efficient) if there is no other allocation x' in the Edgeworth box with $x_i' \succeq x_i$ for $i = 1, 2$ and $x_i' \succ x_i$ for some i.

- Pareto Set: In the Edgeworth Box a Pareto Set is the set of allocations that are Pareto Optimal for all possible endowments (that add up to the fixed market endowment).
- The Pareto set has the property that any element has to be the in the tangency of the preferences curves.
- Any Walrasian Equilibrim allocation x^* necessarily belongs to the Pareto Set.

- Under convexity assumptions a planner can achieve any desired Pareto optimal allocation by appropriately redistributing wealth in a lump-sum fashion.

Definition 64. (Equilibrium with Transfers) An allocation x^* in the Edgeworth box is supportable as an equilibrium with transfers if there is a price system p^* and a wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$, such that for each consumer i we have

$$x_i^* \succeq x_i' \quad \forall x_i' \in \mathbb{R}_+^2 \quad s.t. \quad p^* \cdot x_i' \leq p^* \cdot \omega_i + T_i.$$

Claim 9. If the preferences of the consumers in the Edgeworth box are continuous, convex, and strongly monotone, then any Pareto optimal allocation is supportable as an equilibrium with transfers.

11.2 The One-Consumer, One-Producer Economy

There is a consumer and a technology. The consumer has continuous, convex, and strongly monotone preferences $\succeq$ defined over his consumption of leisure x_1 and the consumption of good x_2. The endowment is time $\overline{L}$ (leisure) (e.g. 24 hours) and no endowment of the consumption good.

The consumer can produce the consumption good by means of labor according to the production function $f(z)$, where z is the firm's labor input. Thus to produce output, the firm must hire the consumer, effectively purchasing some of the consumer's leisure from him. We assume that the firm seeks to maximize profits taking prices as given.

Let p be the price of its output and w be the price of labor, the firm solves

$$max_{z \geq 0} pf(z) - wz.$$

Given prices (p, w), the firm's optimal labor demand is $z(p, w)$, its output is $q(p, w)$, and its profits are $\pi(p, w)$.

The consumer is the sole owner of the firm and receives all profits earned by the firm $\pi(p, w)$.

Let $u(x_1, x_2)$ be a utility function $\succeq$, the consumer's problem given prices (p, w) is

$$max_{\{x_1, x_2\} \in \mathbb{R}^2} u(x_1, x_2)$$

$$s.t. \quad px_2 \leq w(\overline{L} - x_1) + \pi(p, w).$$

The budget constraint of the consumer is $w(\overline{L} - x_1) + \pi(p, w)$.

The consumer optimal demand at prices (p, w) is

$$(x_1(p, w), x_2(p, w)).$$

A Walrasian equilibrium in this economy involves a price vector (p^*, w^*) at which the consumption and labor markets clear, that is, at which

$$x_2(p^*, w^*) = q(p^*, w^*)$$

$$z(p^*, w^*) = \overline{L} - x_1(p^*, w^*).$$

Claim 10. A particular consumption-leisure combination can arise in a competitive equilibrium if and only if it maximizes the consumer's utility subject to the economy's technological and endowment constraints.

11.3 Equilibrium and Its Basic Welfare Properties

Consider $I > 0$ consumers, $J > 0$ firms and L commodities.

Each consumer $i = 1, \cdots, I$ is characterized by a consumption set $X_i \subset \mathbb{R}^L$ and a preference relation $\succeq_i$ defined on X_i. The preferences are rational.

Each firm $j = 1, \cdots, J$ is characterized by a technology or production set, $Y_j \subset \mathbb{R}^L$.

We assume that Y_j is non-empty and closed.

The initial resources in the economy are given by $\omega = (\omega_1, \cdots, \omega_L) \in \mathbb{R}^L$.

The data is:

$$(\{X_i, \succeq_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega).$$

An economy is pure exchange if the only technology disposable is free disposal, that is,

$$Y_j = -\mathbb{R}^L_+,$$

for all $j = 1, \cdots, J$.

Definition 65. An allocation $(x, y) = (x_1, \cdots, x_I, y_1, \cdots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \cdots, I$ and a production vector $y_j \in Y_j$ for each

$j = 1, \cdots, J$.

Definition 66. A feasible allocation (x, y) is Pareto optimal if there is no other feasible allocation (x', y') that Pareto dominates it, that is, if there is no feasible allocation $(x'y')$ such that $x'_i \succeq_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

11.4 Private Ownership Economies

Consumers have ownership claims (shares) to the profit of firms.

Formally consumer i has initial endowment vector $\omega_i \in \mathbb{R}^L$ and a claim to a share $\theta_{ij} \in [0, 1]$ of the firm j, where $\omega = \sum_i \omega_i$ and $\sum_j \theta_{ij} = 1$ for every j).

Definition 67. Given a private ownership economy specified by $(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{\omega_i, \theta_{i1}, \cdots, \theta_{iJ}\}_{i=1}^I)$ and allocation (x^*, y^*) and a price vector $p = (p_1, \cdots, p_L)$ constitutes a Walrasian equilibrium if:

(i)

$$p \cdot y_j \leq p \cdot y_j^* \forall j, \forall y_j \in Y_j$$

(ii)

$$x_i^* \succeq_i x_i$$

$$\forall i, \forall x_i \in \{x_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii)

$$\sum_i x_i^* = \omega + \sum_j y_j^*.$$

11.5 Price Equilibria with Transfers

Definition 68. Given an economy specified by $(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \omega)$ and allocation (x^*, y^*) an a price vector $p = (p_1, \cdots, p_L)$ constitute a price equilibrium with transfers if there is an assignment of wealth levels $(w_1, \cdots, w_I)$ with $\sum_i w_i = p \cdot \omega + \sum_j p \cdot y_j^*$ such that

(i) For every j, y_j^* maximizes profits in Y_j that is

$$p \cdot y_j \leq p \cdot y_j^* \forall y_j \in Y_j$$

(ii)

$$x_i^* \succeq_i x_i$$

$$\forall i, \forall x_i \in \{x_i : p \cdot x_i \leq w_i\}.$$

(iii)

$$\sum_i x_i^* = \omega + \sum_j y_j^*.$$

11.6 The First Fundamental Theorem of Welfare Economics

Definition 69. The preference relation $\succeq_i$ on the consumption set X_i is locally nonsatiated if for every $x_i \in X_i$ and every $\epsilon > 0$, there is an $x_i' \in X_i$ such that $||x_i' - x_i|| \leq \epsilon$ and $x_i' \succ_i x_i$.

Proposition 37. *(First Fundamental Theorem of Welfare Economics) If preferences are locally nonsatiated and if (x^*, y^*, p) is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian Equilibrium allocation is Pareto Optimal.*

Proof. Suppose that (x^*, y^*, p) is a price equilibrium with transfers and that the associated wealth levels are $(w_1, \cdots, w_I)$. Recall that $\sum_i w_i = p \cdot \omega + \sum_j p \cdot y_j^*$.

The preference maximization part of the notion of price equilibrium with transfers requires that if

$$x_i \succ x_i^* \quad then \quad p \cdot x_i > w_i = p \cdot x_i^*.$$

Under local non-satiation we also have:

$$If \quad x_i \succeq x_i^* \quad then \quad p \cdot x_i \geq w_i.$$

We verify this by contradiction, consider that $x_i \succeq x_i^*$ and $p \cdot x_i < w_i$ then there is $x_i' \succ x_i$ that is affordable thus making x_i^* non-optimal.

Now consider an allocation $(x, y) \succ_{PD} (x^*, y^*)$ that dominates the given allocation in the Pareto sense.

This means that $x_i \succeq_i x_i^*$ for all i and $x_i \succ_i x_i^*$ for some i.

By the previous arguments we have that

$$p \cdot x_i \geq w_i \forall i \quad and \quad p \cdot x_i > w_i \quad \exists i$$

Hence,

$$\sum_i p \cdot x_i > \sum_i w_i = p \cdot \omega + \sum_j p \cdot y_j^*.$$

By profit maximization we know that

$$\sum_j p \cdot y_j^* \geq \sum_j p \cdot y_j.$$

Thus,

$$\sum_i p \cdot x_i > p \cdot \omega + \sum_j p \cdot y_j.$$

This means that (x, y) is not feasible. There is a contradiction.

Thus (x^*, y^*) must be Pareto optimal.

□

11.7 Second Fundamental Theorem of Welfare Economics.

The second welfare theorem gives conditions under which a Pareto Optimum can be supported as a price equilibrium with transfers.

We need some preliminaries.

Definition 70. (Price quasi-equilibrium with transfers) Given an economy specified by $(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \cdots, p_L) \neq 0$ constitutes a price quasi-equilibrium with transfers if there is an assignment of wealth levels $(w_1, \cdots, w_I)$ with $\sum_i w_i = p \cdot \omega + \sum_j p \cdot y_j^*$ such that

(i) For every j, $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.

(ii) For every i, if $x_i \succ_i x_i^*$ then $p \cdot x_i \geq w_i$.

(iii) $\sum_i x_i^* = \omega + \sum_j y_j^*$.

Note that the price quasi-equilibrium differs from the price equilibrium notion with the part (ii) of the definition.

Preferences maximization imply (ii) but (ii) does not implies preference maximization. Hence, any price equilibrium with transfers is a price quasi-equilibrium with transfers but the converse is not true.

Under local non-satiation we can write (ii) as:

(ii) For every i, if $x_i \succ_i x_i^*$ then $p \cdot x_i \geq p \cdot x_i^*$.

Lemma 6. *(Second Fundamental Theorem of Welfare Economics) Consider an economy specified by* $(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \omega)$ *and suppose that every* Y_j *is convex and every preference relation* $\succeq_i$ *is convex and locally non-satiated. Then for every Pareto optimal allocation* (x^*, y^*) *there is a price vector* $p = (p_1, \cdots, p_L) \neq 0$ *such that* (x^*, y^*, p) *is a price quasiequilibrium with transfers.*

Proof. Define for every i, the set V_i, $V_i = \{x_i \in X_i : x_i \succ_i x_i^*\} \subset \mathbb{R}^L$.

Then define:

$$V = \sum_i V_i = \left\{ \sum_i x_i \in \mathbb{R}^L : x_1 \in V_1, \cdots, x_I \in V_I \right\}$$

and

$$Y = \sum_j Y_j = \left\{ \sum_j y_j \in \mathbb{R}^L : y_1 \in Y_1, \cdots, y_J \in Y_J \right\}.$$

The set $Y + \{\omega\}$ is the shifted production function by the endowments.

Step 1.WTS that every set V_i is convex. This follows from the preferences being rational and convex. Suppose that $x_i \succ x_i^*$ and $x_i' \succ_i x_i^*$. Take a mixing parameter $\alpha \in [0, 1]$. We want to show that $\alpha x_i + (1 - \alpha) x_i' \succ_i x_i^*$.

Given completeness of preferences, without loss of generality we can assume $x_i \succeq_i x_i'$. By convexity we have, $\alpha x_i + (1 - \alpha) x_i' \succeq_i x_i'$, which by transitivity yields the desired conclusion $\alpha x_i + (1 - \alpha) x_i' \succ_i x_i^*$.

Step 2. The sets V and $Y + \{\omega\}$ are convex. The sum of convex sets is also a convex set.

Step 3. $V \cap (Y + \{\omega\}) = \emptyset$. This is a consequence of Pareto Optimality of (x^*, y^*).

Step 4. There is $p = (p_1, \cdots, p_L) \neq 0$ and a number r such that $p \cdot z \geq r$ for every $z \in V$ and $p \cdot z \leq r$ for every $z \in Y + \{\omega\}$. This follows directly from the separating hyperplane theorem.

Step 5. WTS If $x_i \succeq_i x_i^*$ for every i then $p \cdot (\sum_i x_i) \geq r$.

Suppose that $x_i \succeq_i x_i^*$for every i. By local non-satiation for each i there is a $\hat{x}_i$ close to x_i such that $\hat{x}_i \succ_i x_i$ and thus $\hat{x}_i \in V_i$.

Hence, $\sum_i \hat{x}_i \in V$ and $p \cdot (\sum_i \hat{x}_i) \geq r$, taking limits $\hat{x}_i \rightarrow x_i$we have $p \cdot (\sum_i x_i) \geq r$.

Step 6. WTS $p \cdot (\sum_i x_i^*) = p \cdot (\omega + \sum_j y_j^*) = r$.

Given Step 5 we have $p \cdot (\sum_i x_i^*) \geq r$. Also $\sum_i x_i^* = \sum_j y_j^* + \omega \in Y + \{\omega\}$, thus $p \cdot (\sum_i x_i^*) \leq r$.

Thus $p \cdot (\sum_i x_i^*) = r$. Since, $\sum_i x_i^* = \omega + \sum_j y_j^*$, we also have $p \cdot (\omega + \sum_j y_j^*) = r$.

Step 7. WTS For every j, we have $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.

For any firm j and $y_j \in Y_j$ we have $y_j + \sum_{h \neq j} y_h^* \in Y$. Therefore,

$$p \cdot (\omega + y_j + \sum_{h \neq j} y_h^*) \leq r$$

Given Step 6:

$$r = p \cdot (\omega + y_j^* + \sum_{h \neq j} y_h^*).$$

Thus, $p \cdot y_j \leq p \cdot y_j^*$.

Step 8. WTS For every i, if $x_i \succ_i x_i^*$, then $p \cdot x_i \geq p \cdot x_i^*$.

Consider any $x_i \succ_i x_i^*$. Because of steps 5 and 6 we have.

$$p \cdot (x_i + \sum_{h \neq i} x_h^*) \geq r = p \cdot (x_i^* + \sum_{h \neq i} x_h^*).$$

Hence, $p \cdot x_i \geq p \cdot x_i^*$.

□

Step 9. WTS The wealth levels $w_i = p \cdot x_i^*$ for $i = 1, \cdots, I$ support (x^*, y^*, p) as a price quasi-equilibrium with transfers.

Lemma 7. *Suppose that for every i, X_i is convex, $0 \in X_i$ and $\succeq_i$ is continuous. Then any price quasiequilibrium with transfers that has $(w_1, \cdots, w_I) >> 0$ is a price equilibrium with transfers.*

The proof requires the following result.

Lemma 8. *Assume that X_i is convex and $\succeq_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p and the wealth level w_i are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq w_i$. Then if there is a consumption vector $x_i' \in X_i$ such that $p \cdot x_i' < w_i$ (a cheaper consumption for (p, w_i)), it follows that $x_i \succ_i x_i^*$ implies $p \cdot x_i > w_i$.*

Proof. We consider only the case $p \cdot x_i = w_i$.

Suppose to the contrary that $x_i \succ_i x_i^*$ such that $p \cdot x_i = w_i$.

By the cheaper consumption assumption, there is an $x_i' \in X_i$ such tht $p \cdot x_i' < w_i$.

Then for all $\alpha \in [0, 1)$ we have $\alpha x_i + (1 - \alpha) x_i' \in X_i$ and $p \cdot (\alpha x_i + (1 - \alpha) x_i') < w_i$.

But for $\alpha \to 1$ the continuity of $\succeq_i$ imply that $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$. This is a contradiction because we have found a consumption bundle that is preferred to x_i^* and cost less than w_i.

$\square$

Chapter 12

Positive Theory of Equilibrium

12.1 Introduction

In this chapter we study the Walrasian Equilibrium.

Definition 71. (Walrasian Equilibrium) Given a private ownership economy specified by $(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{\omega_i, \theta_{i1}, \cdots, \theta_{iJ}\}_{i=1}^I)$ and allocation (x^*, y^*) and a price vector $p = (p_1, \cdots, p_L)$ constitutes a Walrasian equilibrium if:

(i)

$$p \cdot y_j \leq p \cdot y_j^* \forall j, \forall y_j \in Y_j$$

(ii)

$$x_i^* \succeq_i x_i$$

$$\forall i, \forall x_i \in \{x_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii)

$$\sum_i x_i^* = \omega + \sum_j y_j^*.$$

12.2 Pure Exchange Economies and Excess Demand Function

We recall pure exchange is the situation where the only technology available is free disposal, we let $J = 1$ and $Y_1 = -\mathbb{R}^L_+$. We take $X_i = \mathbb{R}^L_+$ and we assume that the consumers' preferences are continuous, strictly convex and locally nonsatiated.

We assume that $\sum_i \omega_i >> 0$.

The Walrasian Equilibrium can be redefined here as follows:

Definition 72. (Pure Exchange Walrasian Equilibrium)

(i)' $y_1^* \leq 0$, $p \cdot y^* = 0$ and $p \geq 0$.

(ii)' $x_i^* = x_i(p, p \cdot \omega_i)$ for all i ($x_i(\cdot)$ is the Walrasian demand function).

(iii') $\sum_i x_i^* - \sum_i \omega_i^* = y_1^*$.

We can state the first result:

Proposition 38. *In a pure exchange economy in which consumer preferences are continuous, strictly convex and locally non-satiated, $p \geq 0$ is a Walrasian equilibrium price vector if and only if*

$$\sum_i (x_i(p, p \cdot \omega_i) - \omega_i) \leq 0.$$

Proof. If $p \geq 0$ is a Walrasian equilibrum price vector then (i)'-(iii)' imply

$$\sum_i (x_i(p, p \cdot \omega_i) - \omega_i) \leq 0.$$

If $\sum_i (x_i(p, p \cdot \omega_i) - \omega_i) \leq 0$ holds then we let $y_1^* = \sum_i (x_i(p, p \cdot \omega_i) - \omega_i)$, we check that $p \cdot \sum_i (x_i(p, p \cdot \omega_i) - \omega_i) = 0$, this follows from local non-satiation and we let $x_i^* = x_i(p, p \cdot \omega_i)$ so (i)'-(iii)' holds.

□

We note that the difference (vector) $x_i(p, p \cdot \omega_i) - \omega_i$ is the consumer's i net or excess demand for each good according to what he had in terms of endowments of the given good.

Definition 73. (Excess demand function of consumer i) The excess demand function of consumer i is

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$$

where $x_i(p, p \cdot \omega_i)$ is consumer's i's Walrasian demand function.

With this in hand we define the aggregate excess demand:

Definition 74. The aggregate excess demand function of the economy is

$$z(p) = \sum_i z_i(p).$$

$z : \mathbb{R}^L_+ \mapsto \mathbb{R}^L$, is a mapping from non-negative prices to the euclidean space.

We can then write the following corollary.

Corollary 6. *$p \in \mathbb{R}^L_+$ is an equilibrium price vector if and only if $z(p) \leq 0$.*

Moreover, $z_l(p) = 0$ if $p_l > 0$, thus at equilibrium a good l can be in excess supply $z_l(p) < 0$ only if it is free $p_l = 0$.

We can add additional assumptions in terms of preferences to strengthen the previous definition.

We require all preferences to be strongly monotone.

Corollary 7. *If consumers preferences are strongly monotone (continuous, strictly convex), a price vector $p >> 0$ is a Walrasian equilibrium price vector if and only if it clear markets, that is, if and only if $z_l(p) = 0$ for all l.*

We can now study the properties of the excess demand.

Proposition 39. *Suppose that for all i, $X_i = \mathbb{R}^L_+$ and $\succeq_i$ is continuous, strictly convex and strongly monotone. Suppose also that $\sum_i \omega_i >> 0$. Then the aggregate excess demand function $z(p)$, defined for all price vectors $p >> 0$ satisfies the properties:*

(i) $z(\cdot)$ is continuous.

(ii) $z(\cdot)$ is homogeneous of degree zero.

(iii) $p \cdot z(p) = 0$ for all p (Walras' law).

(iv) (Bounded below) There is an $s > 0$ such that $z_l(p) > -s$ for every commodity l and all p.

(v) If $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l then

$$max_l\{z_l(p^n)\} \to \infty.$$

All the properties follow from the demand properties. The last item follows from $p_l = 0$ for some l and strong monotonic preferences.

Claim 11. Consider the system $\hat{z}(p) = \{z_l(p)\}_{l=1}^{L-1}$, $p >> 0$ is such that $z(p) = 0$ if and only if $\hat{z}(p) = 0$.

This follows from Walras' law and Homogeneity of degree zero.

12.3 Existence of Walrasian Equilibrium

Proposition 40. *Suppose that $z(p)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^L_{++}$ and satisfying conditions (i)-(v) in the previous proposition. Then the system of equations $z(p) = 0$ has a solution. Hence a Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i >> 0$ and every consumer has continuous, strictly convex, and strongly monotone preferences.*

Proof. We normalize prices:

$$\Delta = \{p \in \mathbb{R}^L_+ : \sum_l p_l = 1\}$$

the unit simplex in $\mathbb{R}^L$. Because the function $z(\cdot)$ is homogeneous of degree zero, we can restrict ourselves, in our search for an equilibrium, to price vectors in Δ.

Observe $z(\cdot)$ is well-defined for the interior of Δ,

$$interior\Delta = \Delta^I = \{p \in \Delta : p_l > 0 \forall l\}.$$

$$boundary\Delta = \Delta^B = \Delta \setminus \Delta^I$$

Proof's plan:

First, we construct a mapping $f : \Delta \mapsto 2^{\Delta}$. Second, we show that if $p^* \in f(p^*)$ then $z(p^*) = 0$. Third, we show that f is convex valued and UHC (closed graph). Finally, we apply Kakutani's fixed point theorem to show that at p^* with $p^* \in f(p^*)$ necessarily exists.

Step 1:

For $p >> 0$:

We let $f(p) = \{q \in \Delta : z(p) \cdot q \geq z(p) \cdot q' \forall q' \in \Delta\}$.

Note that we have:

$$f(p) = \{q \in \Delta : q_l = 0 \quad if \quad z_l(p) < Max_l\{z_l(p)\}\}.$$

Also notice that if $z(p) = 0$ then $f(p) = \Delta$.

Step 2:

For $p \in Boundary(\Delta)$ we have

$$f(p) = \{q \in \Delta : p \cdot q = 0\} = \{q \in \Delta : q_l = 0 \quad if \quad p_l > 0\}.$$

With this construction no price at the boundary can be a fixed point, there is no $p \in Boundary(\Delta)$ such that $p \in f(p)$.

Step 3: A fixed point of $f(\cdot)$ is a Walrasian equilibrium.

Suppose that $p^* \in f(p^*)$. It is the case that $p^* \in \Delta^I$ (interior) $p^* >> 0$. If $z(p^*) \neq 0$, then according to the definition of f, $f(p^*) \subset Boundary(\Delta)$.

Observe that if $z(p) \neq 0$ for $p >> 0$ because of Walras' law, we have $z_l(p) < 0$ for some l and $z_l(p) > 0$ for some $l' \neq l$. Then for such p^* any $q \in f(p^*)$ has $q_l = 0$ for some l.

Thus $p \in f(p)$ cannot occur because $p \cdot p > 0$ while $p \cdot q = 0$ is the requirement for $p \in f(p)$. Thus it must be the case that if $p \in f(p)$, $z(p) = 0$.

Step 4. The fixed point correspondence is convex-valued and upper hemi-continuous.

Case I: If $p \in \Delta^I$ and $q \in \Delta^B$, then $f(p)$ is a level set of linear functions defined on the convex set Δ $\{q \in \Delta : \lambda \cdot q = k\}$ for some scalar k and vector $\lambda \in \mathbb{R}^L$, and thus it is convex.

The rest of cases are easy to verify.

To establish upper hemi continuity consider sequences $p^n \to p$ and $q^n \to q$ such that $q^n \in f(p^n)$ for all n. We have to show that $q \in f(p)$.

Case I: $p \in \Delta^I$ and $q \in \Delta^B$. If that is the case, then $p^n >> 0$ for n large, using the definition $q^n \cdot z(p^n) \geq q' \cdot z(p^n)$ using the continuity of z and limits we get $q \cdot z(p) \geq q' \cdot z(p)$ thus $q \in f(p)$.

Case II: $p \in \Delta^B$, take any l with $p_l > 0$. For n large enough $q_l^n = 0$ and thus $q_l = 0$ follows, thus $q \in f(p)$.

Because $p_l > 0$ there is a $\epsilon > 0$ such that $p_l^n > \epsilon$ for sufficiently large n.

If in addition $p^n \in \Delta^B$ then $q_l^n = 0$ by definition of $f(p^n)$.

If instead $p^n >> 0$ then the properties of excess demand (iv) and (v) are used:

For sufficiently large n, we have

$$z_l(p^n) < max_l\{z_l(p^n)\}$$

and then again we have $q_l^n = 0$.

The RHS goes to $+\infty$ as $p \in \Delta^B$ but the LHS is bounded above because if it is positive:

$$z_l(p^n) \leq \frac{1}{\epsilon} p_l^n z_l(p^n) = \frac{1}{\epsilon} \sum_{l' \neq l} p_l^n z_l(p^n) < \frac{s}{\epsilon} \sum_{l' \neq l} p_l^n < \frac{s}{\epsilon},$$

where s is the bound in excess supply given in condition (iv).

In summary for p^n close enough to the Boundary Δ, the maximal demand corresponds to some of the commodities whose price is close to zero.

For large n, any $q^n \in f(p^n)$ will put nonzero weight only on commodities whose price approach zero But this guarantees $p \cdot q = 0$ so $q \in f(p)$.

Step 5: Existence of Fixed Point.

Kakutani's fixed point theorem says that a convex-valued, upper hemicontinuous correspondence from a non-empty, compact, convex set into itself has a fixed point. Since Δ (is a simplex) thus nonempty, convex and compact set, and since $f(\cdot)$ is convex-valued upper hemi-continuous correspondence from Δ to Δ then we conclude that there is a $p^* \in \Delta$ with $p^* \in f(p^*)$.

□

12.4 Local Uniqueness and Regularity

We know that under certain conditions on preferences there exists a Walrasian equilibrium. The next question is whether the solution is unique. The quick answer is that in general the equilibrium is not unique.

The next best thing is to have local uniqueness. This is important because we would like to make stable predictions. An equilibrium is expected to be "stable".

Definition 75. A Walrasian equilibrium price vector $p^* \in \Delta$ is locally unique if there is an $\epsilon > 0$ such that any other $p \in \Delta$ such that $z(p) \leq 0$ is $||p - p^*|| > \epsilon$.

A sufficient condition for locally uniqueness is that the economy is regular.

Definition 76. (Regular Economy) An equilibrium price vector $p = (p_1, \cdots, p_{L-1})$ is regular if the matrix of price effects $D\hat{z}(p) \in \mathbb{R}^{L-1 \times L-1}$ is non-singular, that is, has rank $L - 1$. If every normalized equilibrium price vector is regular, we say that the economy is regular.

We can state the first result.

Proposition 41. *Any regular (normalized) equilibrium price vector is locally isolated (or locally unique). That is, there is an $\epsilon > 0$ such that if $p' \neq p$, $p'_L = p_L = 1$ and $||p' - p|| < \epsilon$, then $z(p') \neq 0$. Moreover, if the economy is regular, then the number of normalized equilibrium price vectors is finite.*

Proof. The local uniqueness of a regular solution is a direct consequence of the inverse function theorem. For any infinitesimal change in normalized prices, $dp \neq 0$, the nonsingularity of $D\hat{z}(p)$ implies that $D\hat{z}(p)dp \neq 0$, thus we cannot remain in equilibrium.

Continuity of the excess demand and the properties that it blows up when prices are close to zero and the incompatibility of zero prices with equilibrium following form strongly montone preference, imply that the set of equilibrium prices is a closed set in $\mathbb{R}^{L-1}$. Any set that is closed and bounded and discrete (by the regularity part) is necessarily finite.

□

In fact, we can say a bit more, the number of equilibria in regular economies is **odd.**

Formally, we need some preliminaries.

Definition 77. (Index of an Economy) Suppose that $p = (p_1, \cdots, p_{L-1}, 1)$ is a regular equilibrium of an economy. Then we denote

$$index(p) = (-1)^{L-1} sign|D\hat{z}(p)|,$$

where $|D\hat{z}(p)|$ is determinant of the $(L-1) \times (L-1)$ matrix of $D\hat{z}(p)$.

Now we can state the following result.

Proposition 42. *(The Index Theorem) For any regular economy, we have*

$$\sum_{\{p:z(p)=p,p_L=1\}} index(p) = +1.$$

This basically means that the number of equilibria is odd. In particular, it cannot be zero, so existence is implied by this theorem. The equilibrium with positive sign are more fundamental because all economies have at least one.

Finally, we conclude this section with a genericity analysis, most of the formalities of this part are going to be omitted, but the main point is that non-regular economies are very unlikely to exists because we can always perturb a bit the endowments and get a regular economy, thus empirically non-regular economies are of "little" interest.

We state (without proof) the Theorem of Debreu (1970):

Proposition 43. *For almost every vector of initial endowments* $(\omega_1, \cdots, \omega_L) \in \mathbb{R}^L_{++}$ *the economy defined by* $\{(\succeq_i, \omega_i)\}_{i=1}^I$ *is regular.*

12.5 Everything Goes

First we provide a local negative result.

Notice before that thanks to Walras' law and HD0 we have that $D_p z(p)p = 0$ and $p \cdot Dz(p) = -z(p)$. These are the only implications we have.

Proposition 44. *Given a price vector p. Let $z \in \mathbb{R}^L$ be an arbitrary vector, A is an arbitrary $L \times L$ matrix satisfying $p \cdot z = 0$, $Ap = 0$ and $p \cdot A = -z$. Then there is a collection of L consumers generating an aggregate excess demand function $z(\cdot)$ such that $z(p) = z$ and $Dz(p) = A$.*

The nature of the impossibility result is stronger, in fact, it is of a global nature.

Proposition 45. *Suppose that $z(\cdot)$ is a continuous function defined on*

$$P_\iota = \{p \in \mathbb{R}^L_+ : p_l / p_{l'} \geq \epsilon \quad \forall l, l'\}$$

and with values in $\mathbb{R}^L$ ($z : P_\iota \mapsto \mathbb{R}^L$). Assume that in addition, $z(\cdot)$ is HD0 and satisfies Walras' law. Then there is an economy of L consumers whose aggregate excess demand function coincides with $z(p)$ in the domain P_ι.

Proof. We start with an arbitrary $z(\cdot)$ satisfying the assumptions that we are given. We fix $L = 2$. We pick an $\epsilon > 0$ that also satisfies the assumptions above.

The continuity and homogeneity of degree zero of $z(\cdot)$ imply the existence of a number $r > 0$ such that $|z_1(p)| < r$ for every $p \in P_\iota$.

Now, we specify two functions $z^1(\cdot)$ and $z^2(\cdot)$ with domain P_ι, and values in $\mathbb{R}^2$, which are also continuous and homogeneous of degree zero, and satisfy Walras' law. In particular, we let

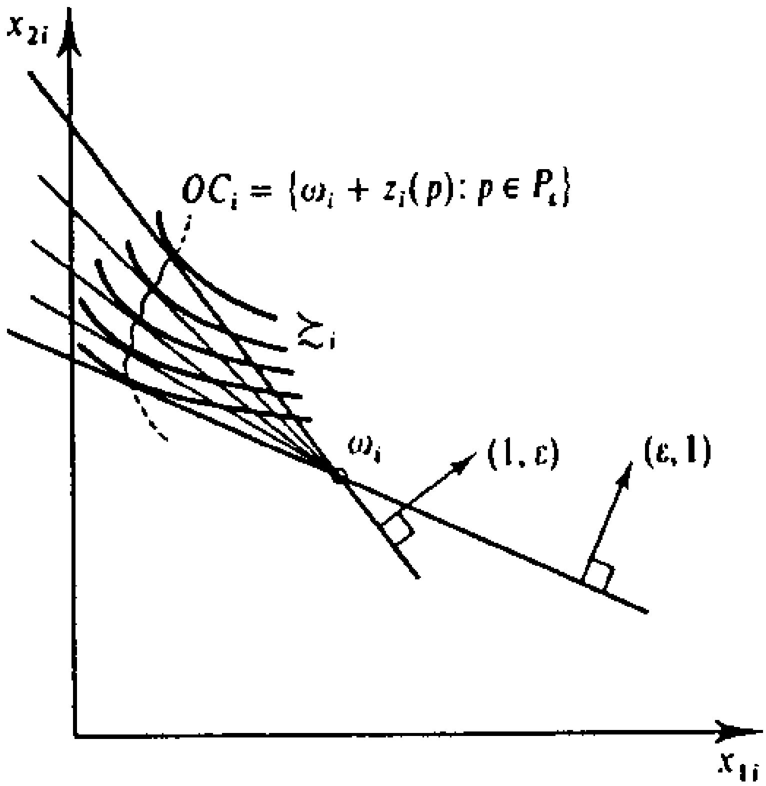
$$z_1^1(p) = \frac{1}{2} z_1(p) + r; \quad z_2^1(p) = -(p_1/p_2) z_1^1(p).$$

$$z_1^2(p) = \frac{1}{2} z_1(p) - r; \quad z_2^2(p) = -(p_1/p_2) z_1^2(p).$$

By construction $z(p) = z^1(p) + z^2(p)$ for every $p \in P_\iota$. We want to show that for $i = 1, 2$ the function $z^i(\cdot)$ coincides in the domain P_ι with the excess demand function of a consumer.

We use the properties of $z^i(\cdot)$, continuity homogeneous of degree zero, Walras' law and the fact that $z^i(p) \neq 0$ for all $p \in P_\iota$.

Choose $\omega_i >> 0$ such that $\omega_i + z^i(p) >> 0$ for every $p \in P_\iota$. We notice that $\omega_i + z^i(p)$ is the intersection point of the offer curve with the budget line perpendicular to p. Because any preference $\succeq_i$ indifference curve is tangent to the budget line perpendicular to p for some p.



□

12.6 Uniqueness of Equilibria

In applications we usually want to impose uniqueness of equilibria.

Proposition 46. *Given an economy specified by the constant returns to technology Y and the aggregate excess demand function of the consumers $z(\cdot)$, a price vector p is a Walrasian equilibrium price vector if and only if*

(i) $p \cdot y \leq 0$ for every $y \in Y$, and

(ii) $z(p)$ is a feasible production, that is, $z(p) \in Y$.

Proof. If p is a Walrasian equilibrium price vector, then (ii) follows from market clearing and (i) is a necessary condition for profit maximization with a constant returns technology.

If (i) and (ii) hold, then consumptions $x_i^* = x_i(p, p \cdot \omega_i)$ for all $i = 1, \cdots, I$, the production vector $y^* = z(p) \in Y$ is profit maximizing, and price vector p constitute a Walrasian equilibrium. To verify this, the only thing that is not automatic is profit maximizing, because $p^* \cdot y \leq 0$ for all $y \in Y$ and $p \cdot y^* = p \cdot z(p) = 0$ from Walras' law).

□

We define the weak axiom for excess demand functions.

Definition 78. (The Weak Axiom for Excess Demand Functions) The excess demand function $z(\cdot)$ satisfies the weak axiom of revealed preference (WA) if for any pair or price vectors p and p' we have

$$z(p) \neq z(p') \quad and \quad p \cdot z(p') \leq 0 \implies p' \cdot z(p) > 0.$$

Proposition 47. *Suppose that the excess demand function $z(\cdot)$ is such that, for any constant returns convex technology Y, the economy formed by $z(\cdot)$ and Y has a unique (normalized) equilibrium price vector. Then $z(\cdot)$ satisfies the weak axiom. Conversely, if $z(\cdot)$ satisfies the weak axiom then, for any constant returns convex technology Y, the set of equilibrium price vectors is convex (and so, if the set of normalized price equilibria is finite, there can be at most one normalized price equilibirum).*

Proof. Suppose that the WA was violated, that is suppose for some p, p' we have $z(p) \neq z(p')$, $p \cdot z(p') \leq 0$, and $p' \cdot z(p) \leq 0$. then we claim that both p and p' are equilibrium prices for the convex, constant returns production set given by

$$Y^* = \{y \in \mathbb{R}^L : p \cdot y \leq 0 \quad and \quad p' \cdot y \leq 0\}.$$

Note that we have $z(p) \in Y^*$ and $p \cdot y \leq 0$ for every $y \in Y^*$. Thus p is an equilibrium, the same is true for p'. Since $z(p) \neq z(p')$, we conclude that the equilibrium is not unique for the economy formed by $z(\cdot)$ and the production set Y^*.

□

12.7 Gross Substitutes

A sufficient condition different from the WA that is also used for ensuring uniqueness is the gross substitutes property.

Definition 79. The function $z(\cdot)$ has the gross substitutes (GS) property if whenever p' and p are such that, for some l, $p'_l > p_l$ and $p'_k = p_k$ for $k \neq l$, we have $z_k(p') > z_k(p)$ for $k \neq l$. Moreover if $z(\cdot)$ is HD0 we also have $z_l(p') < z_l(p)$.

Proposition 48. *An aggregate excess demand function $z(\cdot)$ that satisfies the gross substitute property has at most one exchange equilibrium, that is, $z(p) = 0$ has at most one normalized solution.*

Proof. We have to check that $z(p) = z(p')$ cannot occur whenever p and p' are price vectors that are not collinear. By homogeneity of degree zero, we can assume that $p' \geq p$ and $p_l = p'_l$ for some l. Now, consider altering the price vector p' to obtain the price vector p in $L - 1$ steps, lowering the price of every commodity $k \neq l$ one at a time. By gross substitution, the excess demand of good l cannot decrease in any step, and, because $p \neq p'$, it will actually increase in at least one step. Hence, $z_l(p) > z_l(p')$.

□

12.8 The Core

Formally consider an economy wth I consumers. Every i consumer has consumption set $\mathbb{R}^L_+$, and endowment vector $\omega_i \geq 0$, and a continuous, strictly convex, strongly monotone preference relation $\succeq_i$. There is also a publicly available constant returns convex technology $Y \subset \mathbb{R}^L$. For example, we could have $Y = -\mathbb{R}^L_+$, that is a pure exchange economy.

Recall that an allocation $x \in \mathbb{R}^{LI}_+$ is feasible if $\sum_i x_i = y + \sum_i \omega_i$ for some $y \in Y$.

Definition 80. Coalition. A coalition $S \subseteq I$ (nonempty) improves upon, or blocks, the feasible allocation x^* if for every $i \in S$ we can find a consumption $x_i \geq 0$ with properties: (i) $x_i \succ_i x_i^*$ for every $i \in S$, (ii) $\sum_{i \in S} x_i \in Y + \{\sum_{i \in S} \omega_i\}$.

This definition means that a coalition S can improve upon a feasible allocation x^* if there is some way that, by using only their endowments $\sum_{i \in S} \omega_i$ and the publicly available technology.

Definition 81. (The core property) We say that the set of feasible allocations x^* has the core property if there is no coalition of consumers $S \subseteq I$ that can improve upon x^*.

Now we are able to define the core.

Definition 82. (The core) The core is the set of allocations that have the core property.

- For $I = 2$, the Core is equivalent to the Contract Curve of the Edgeworth box.

Proposition 49. *Any Walrasian equilibrium allocation has the core property.*

Proof. Let x^* be a Walrasian allocation with corresponding equilibrium price $p \geq 0$. Consider an arbitrary coalition $S \subset I$ and suppose that the consumption $\{x_i\}_{i \in S}$ are such that $x_i \succ_i x_i^*$

for every $i \in S$. then $p \cdot x_i > p \cdot \omega_i$ for every $i \in S$ and therefore $p \cdot (\sum_{i \in S} x_i) > p \cdot (\sum_{i \in S} \omega_i)$. But then $\{x_i\}_{i \in S}$ is not feasible. So coalition S cannot block x^*.

$\square$

- However, not all allocations in the Core are Walrasian allocations.
- When $I \to \infty$ (achieved by replica economies) then the Core converges to the Walrasian allocations.

Chapter 13

Testable restrictions on the equilibrium manifold.

13.1 Contribution.

Brown Matzkin (BM) present a finite system of polynomial inequalities in unobservable variables and market data that observations on market prices, individual incomes, and aggregate endowments must satisfy to be consistent with the equilibrium behavior of some pure trade economy.

Quantifier elimination is used to derive testable restrictions on finite data sets for the pure trade model.

A characterization for aggregate endowment economy Robinson Crusoe's economy is also provided.

The problem is that due to Sonnenschein-Debreu-Mantel theorem SDM, the Slutsky restrictions on individual excess demand functions do not extend to market excess demand functions.

Mas-Collel (1977) shows that utility maximization subject to a budget constraint imposes no testable restrictions on the set of equilibrium prices.

13.2 The setting.

- Market excess demand function $F_{\hat{w}}(p)$ where the profile of individual endowments $\hat{w}$ is fixed but market prices p may vary.

- Equilibrium manifold, we denote the market excess demand as $F(\hat{w}, p)$ where both $\hat{w}$ and

p may vary.

Definition 83. The equilibrium manifold is defined as the set $\{(\hat{w}, p) | F(\hat{w}, p) = 0\}$.

- Contrary to the result of Mas-Colell, BM show that utility maximization subject to a budget constraint does impose testable restrictions on the "equilibrium manifold".

- Afriat's theorem (1967) is the key for the result.

13.2.1 Afriat's theorem.

Afriat use the Theorem of Alternatives, proved the equivalence of a finite family of linear inequalities that contains unobservable utility levels with "observable" cyclical consistency or GARP. And the existence of a concave, continuous monotonic utility function rationalizing the observed data.

- Afriat's theorem is an instance of a deep theorem in model theory, Tarski-Seidenberg theorem or quantifier elimination.

- Tarski-Seidenberg theorem proves that any finite system of polynomial inequalities can be reduced to an equivalent finite family of polynomial inequalities in the coefficients of the given system.

- The Tarski-Seidenberg theorem provides and algorithm which in principle can be used to carry out the elimination of the unobservable in a finite number of steps.

- Each time a variable is eliminated an equivalent system of polynomial inequalities is obtained, which contains all the variables except those that have been eliminated up to that point.

ALGORITHM.

- Algorithm ends if:

(i) 1=0 the original system of polynomial inequalities is never satisfied.

(ii) 1=1 the original system is always satisfied.

(iii) an equivalent finite family of polynomials inequalities in the coefficients of the original system which is satisfied only by some parameters values of the coefficients.

13.2.2 The solution.

The plan is to apply the Tarski-Seidenberg theorem.

- Step 1: Express the structural conditions of the pure trade model as a finite family of polynomial inequalities.

- Step 2: To derive equivalent conditions on the data, the coefficients in this family of polynomial must be market observables - individual endowments and market prices-. The unknowns must be the unobservables in the theory. (Utility level of individuals, marginal utilities of income, and consumption bundles).

DATA:

Prices and endowments.

Unknowns:

Utility level. Consumption can be derived using Roy and marginal utility of income is derived form the primitive

INEQUALITIES.

- For every agent the Afriat's inequalities.

- The budget constraint of each agent.

- Market clearing equations for each observation.

TARSKI-SEIDENBERG ALGORITHM.

(i) 1=0 the given equilibrium conditions are inconsistent.

(ii) 1=1 there is no finite data set that refutes the model.

(iii) the equilibrium conditions are testable.

- THE ALGORITHM IS NON POLYNOMIAL AND DOUBLE EXPONENTIAL.

HOWEVER, for the case of pure trade model it suffices to show that the algorithm cannot terminate in 1=0 and 1=1, to show that it must end in (iii) and therefore it has testable implications.

This does not solve the problem of actually testing the model.

PROOF:

From Arrow-Debreu existence Tarski-Seidenberg algorithm will not terminate in 1=0 (i).

They construct an example of Pure trade model where no values of the unobservables are consistent with the values of the observables. Hence the algorithm will not stop 1=1 (ii).

CONCLUSION.

Therefore the Tarski-Seidenberg theorem implies for any finite family of profiles of individual endowments $\hat{w}$ and market prices p that these observations lie on the equilibrium manifold of a pure trade economy, for some family of concave, continuous, an monotonic utility functions, if an only if they satisfy the derived family of polynomial inequalities in $\hat{w}$ and p.

- This family of polynomial inequalities constitute the testable restrictions of the Walrasian model of pure trade.

- They derive restrictions for two observations. For more than two the conditions are necessary not sufficient.

- WARE weak axiom of revealed equilibrium. Even if pairwise it is fulfilled, a third observation may violate the conditions then this conditions are not sufficient, but certainly necessary.

- Key assumption: The authors do not assume that individual consumptions are observed as did Afriat. As a consequence the Afriat inequalities in our model are nonlinear in the unknowns.

13.2.3 Restriction in the pure trade model.

- K commodities

- T traders, under pure trade model.

- Commodity space $X = R^K$

- Each agent has R^K_+ as the consumption set.

- Traders characterized by $w_t \in R^K_{++}$ and a utility function $V_t : R^K_+ \to R$.

- V_t are assumed to be continuous, monotone and concave.

- Allocation: consumption vector $x_t \in R^K_+$ for each t and $\sum_{t=1}^T x_t = \sum_{t=1}^T w_t$.

- The price simplex $\Delta = \{p \in R^K_+ | \sum_{i=1}^K p_i = 1\}$.

- Only positive prices $S = \{p \in \Delta | p_i > 0 \quad \forall i\}$.

Definition 84. Competitive equilibrium.

An allocation $\{x_t\}_{t=1}^T$ and prices p

- Each x_t is utility maximizing for t s.t. to budget constraint.

The prices p are called "equilibrium prices".

DATA:

Suppose we observe $\{w^r\}_{r=1}^N$ and prices p^r for $r = 1, \cdots, N$.

We observe the Walrasian correspondence.

Map from endowment profiles to equilibrium prices.

UNKNOWNS:

- $\{V_t\}_{t=1}^T$ and consumption vectors.

Testability.

Definition 85. Testability.

A pure trade model is testable, if for every N there exists a finite family of polynomials inequalities in w^r and p^r for $t = 1, \cdots, T$ and $r = 1, \cdots, N$ such that observed pairs of profiles of individual endowments and market prices satisfy the given system of polynomial inequalities if an only if they lie on some equilibrium manifold.

Before proving testability of the pure trade model recall Afriat's theorem.

Theorem 16. *The following conditions are equivalent.*

(A.1) There exists a nonsatiated utility function that rationalizes the data $(p^i, x^i)_{i=1,\cdots,N}$ *i.e. there exists a non satiated utility function* $u(x)$ *such that for all* $i = 1, \cdots, N$ *and all* x *such that* $p^i \cdot x^i \geq p^i \cdot x$, $u(x^i) \geq u(x)$.

(A.2.) The data satisfies "Cyclical Consistency CC" i.e. for all $\{r, s, t, \cdots, q\}$ $p^r \cdot x^r \geq p^r \cdot x^s$, $p^s \cdot x^s \geq p^s \cdot x^t, \cdots, p^q \cdot x^q \geq p^q \cdot x^r$ *implies* $p^r \cdot x^r = p^r \cdot x^s$, $p^s \cdot x^s = p^s \cdot x^t, \cdots, p^q \cdot x^q = p^q \cdot x^r$.

(A.3) There exists numbers $U^i, \lambda^i > 0, i = 1, \cdots, n$ *such that* $U^i \leq U^j + \lambda^j p^j \cdot (x^i - x^j)$ *for* $i, j = 1, \cdots, N$.

(A.4) There exists a non satiated, continuous, concave, monotonic utility function that rationalizes the data.

Versions of Afriat's theorem for the SARP and SSARP due to Chiappori Rochet can be found in Matzking and Ritcher and Chiappori and Rochet respectively.

Proof. EQUILIBRIUM INEQUALITIES.

$\exists\{\overline{V}^r_t\}_{r=1,\cdots,N;t=1,\cdots,T}; \{\lambda^r_t\}_{r=1,\cdots,N;t=1,\cdots,T}; \{x^r_t\}_{r=1,\cdots,N;t=1,\cdots,T}$

such that

(1.1) $\overline{V}^t_r - \overline{V}^t_s - \lambda^t_r p^t_r \cdot (x^t_r - x^t_s) \leq 0$

$(r, s = 1, \cdots, N; t = 1, \cdots, T)$

(1.2) $\lambda^r_t > 0$, $x^r_t \geq 0$

$(r = 1, \cdots, N; t = 1, \cdots, T)$

(1.3) $p^r \cdot x^r_t = p^r \cdot w^r_t$

$(r = 1, \cdots, N; t = 1, \cdots, T)$

(1.4) $\sum_{t=1}^T x^r_t = \sum_{t=1}^T w^r_t$

$(r = 1, \cdots, N.)$

□

For observables $\{w^r_t\}_{t=1}^T$ and market prices p^r for $r = 1, \cdots, N$.

Arrow-Debreu proves that the algorithm will not stop at 1=0 (i.e. no inconsistencies of the inequalities).

COUNTEREXAMPLE. Proof 1=1.

- Two agents.

- Two boxes. w^1, p^1 and w^2, p^2, $r = 1, 2$

- Agent 1 lives always in A for box 1 and 2.

- Agent 2 lives in C in box 1 and in F in box 2.

- Equilibrium allocation Intersection of box r with budget sets r.

Agent 1 breaks WARE, that is WARP since the allocations of box 1 are still affordable for her under box 2 but she chooses below the budget set.

13.2.4 The pure trade model is testable.

Theorem 17. *(1) The pure trade model is testable.*

Proof. The system of equilibrium inequalities is a finite family of polynomial inequalities. Then this is a consequence of Tarski-Seidenberg algorithm. BM have shown cannot terminate in 1=0 or 1=1.

□

When income is observable. Only aggregate endowment is observed.

I_t^r is the income of consumer t a in observation r and w^r the aggregate endowment in observation r.

Theorem 18. *(2) Let* $< p^r, \{I_t^r\}_{t=1}^T, w^r >$ *for* $r = 1, \cdots, N$ *be given.*

Then there exists a set of continuous, concave, and monotone utility functions $\{V_t\}_{t=1}^T$ *such that for each* $r = 1, \cdots, N$*:* p^r *is an equilibrium price vector for the exchange economy*

$< \{V_t\}_{t=1}^T, \{I_t^r\}_{t=1}^T, w^r >$ *if and only if there exists numbers* $\{\overline{V}_r^r\}_{t=1,\cdots,T;r=1,\cdots,N}$ *and* $\{\lambda_t^r\}_{t=1,\cdots,T;r=1,\cdots,N}$ *and vectors* $\{x_t^r\}_{t=1,\cdots,T;r=1,\cdots,N.}$ *satisfying*

(2.1) $\overline{V}_r^t - \overline{V}_s^t - \lambda_r^t p_r^t \cdot (x_r^t - x_s^t) \leq 0$ *for all* $r, s \in \{1, \cdots, N\}; t \in \{1, \cdots, T\}$

(2.2) $\lambda_t^r > 0, x_t^r \geq 0$ *for all* $r \in \{1, \cdots, N\}; t \in \{1, \cdots, T\}$

(2.3) $p^r \cdot x_t^r = I_t^r$

(2.4) $\sum_{t=1}^T x_t^r = w^r$ *for all* $r = 1, \cdots, N$.

Proof. (2.1)-(2.3) by Afriat's theorem imply that for each t, there exists a continuous concave, monotone utility function $V_t : R_+^K \to R$ such that for each r, x_t^r is one of the maximizers of V_t subject to $p^r \cdot y \leq I_t^r$.

(2.4.) Guarantees that this maximization is a feasible allocation and therefore p^r is an equilibrium.

The converse is immediate, by Arrow Debreu and Kuhn-Tucker theorem that assure the existence of $\{\lambda^r\}_{t=1}^T$ such that (2.1) and (2.2) hold where $V_t^r = V_t(x_t^r)$.

□

13.2.5 Revealed Equilibrium.

For two observation $r = 1, 2$ and Chiappori-Rochet version of the Afriat's theorem we use in the proof of Theorem 3 below, quantifier elimination to derive testable restrictions for a pure trade model with two consumers $t = a, b$ from the equilibrium inequalities.

WEAK AXIOMS OF REVEALED EQUILIBRIUM. WARE.

Definition 86. Most expensive bundle.

$\overline{z}_t^r$ for $r = 1, 2; t = a, b$, is

$$\overline{z}_t^r \in argmax_x\{p^s \cdot x | p^r \cdot x = I_t^r, 0 \le x \le w^r\}$$

where $r \neq s$. Hence among all the bundles that are feasible in observation r and are on the budget hyperplane of consumer t in observation r,

$\overline{z}_t^r$ is any of the bundles that cost the most under prices $p^s(s \neq r)$.

Definition 87. WARE. We say that observations $\{p^r\}_{r=1,2}; \{I_t^r\}_{r=1,2;t=a,b}; \{w^r\}_{r=1,2}$ satisfy WARE if

(I) $\forall r = 1, 2,\ I_a^r + I_b^r = p^r \cdot w^r$.

(II) $\forall r, s = 1, 2\ (r \neq s), \forall t = a, b,\ [(p^s \cdot \overline{z}_t^r \le I_t^s) \implies (p^r \cdot \overline{z}_t^s > I_t^r)]$,

(III) $\forall r, s = 1, 2\ (r \neq s),\ [(p^s \cdot \overline{z}_a^r \le I_a^s) \wedge (p^s \cdot \overline{z}_b^r \le I_b^s)] \implies (p^r \cdot w^s > p^r \cdot w^r)$.

(I) Condition for income and aggregate endowment.

(II) WARP in this setting. This applies when all bundles in the budget hyperplane of consumer t in observation r that are feasible in observation r can be purchased with the income and prices faced by consumer t in observation s $(s \neq r)$.

Then it must be the case that some of the bundles that are feasible in observation s and are in the budget hyperplane of consumer t in observation s cannot be purchased with the income and prices faced by consumer t in observation r $(p^r \cdot \overline{z}_t^s > I_t^r)$. The consumer A does not satisfy this conditions.

(III) This guarantees that at least one of the pairs of consumption bundles in observation s that contain for each agent feasible and affordable bundles that could not be purchased with the income and price of observation r are such that they add up to the aggregate endowment.

Theorem 19. *(3) Let $\{p^r\}_{r=1,2}; \{I_t^r\}_{r=1,2;t=a,b}; \{w^r\}_{r=1,2}$ be given such that p^1 is not a scalar multiple of p^2. Then the equilibrium inequalities for strictly monotone, strictly concave, C^∞ utility functions have a solution, i.e. the data lies on the equilibrium manifold of some economy whose consumers have strictly monotone, strictly concave C^∞ utility functions, if an only if the data satisfy WARE.*

Proof. Using the Tarski-Seidenberg theorem, we need to show WARE can be derived by quantifier elimination from the equilibrium inequalities for strictly monotone, strictly concave C^∞ utility functions.

Making use of Chiappori and Rochet (1987) these inequalities are

$\exists\{\overline{V}^r_t\}_{r=1,2;t=a,b}, \{\lambda^r_t\}_{r=1,2;t=a,b}, \{x^r_t\}_{r=1,2;t=a,b}$ such that.

(C.1) $\overline{V}^2_1 - \overline{V}^1_t - \lambda^1_t p^1 \cdot (x^2_t - x^1_t) < 0,\ t = a, b$

(C.2) $\overline{V}^1_t - \overline{V}^2_t - \lambda^2 p^2 \cdot (x^1_t - x^2_t) < 0,\ t = a, b$

(C.3) $\lambda^r_t > 0,\ r = 1, 2; t = a, b$

We eliminate $\{\lambda^r_t\}_r$ and

$\exists\{\overline{V}^r_t\}_{r=1,2;t=a,b}, \{x^r_t\}_{r=1,2;t=a,b}$

(C.1)' $p^1 \cdot (x^2_t - x^1_t) \leq 0 \implies \overline{V}^2_t < \overline{V}^1_t,\ t = a, b$

(C.2)' $p^2 \cdot (x^1_t - x^2_t) \leq 0 \implies \overline{V}^1_t < \overline{V}^2_t,\ t = a, b$

(C.4) $p^r \cdot x^r_t = I^r_t,\ \forall r, t$

(C.5) $p^1 \neq p^2 \implies x^1_t \neq x^2_t, t = a, b;$

(C.6) $x^r_t \geq 0,\ r = 1, 2; t = a, b;$

(C.7) $x^r_a + x^r_b = w^r,\ r = 1, 2$

Necessity is trivial. Sufficiency follows by noticing that $(C,1)'$ and $(C.2)'$ imply $\exists\{\lambda^1_t\}_{t=a,b}$ satisfying C.1 and C.3 and $\exists\{\lambda^2_t\}$ satisfying C.2 and C.3.

Elimination of $\{\overline{V}^r_t\}_{r=1,2;t=a,b}$ yields the equivalent expression.

$\exists\{x^r_t\}_{r=1,2;t=a,b}$ such that

(C.1") $p^1 \cdot (x^2_t - x^1_t) \leq 0 \implies p^2 \cdot (x^1_t - x^2_t) > 0,\ t = a, b$

(C.4) $p^r \cdot x^r = I^r_t,\ r = 1, 2, t = a, b$

(C.5) $p^1 \neq p^2 \implies x^1_t \neq x^2_t\ t = a, b$

(C.6) $x^r_t \geq 0,\ r = 1, 2; t = a, b;$

(C.7) $x^r_a + x^r_b = w^r,\ r = 1, 2$

This is Afriat's theorem for two observations. Strong SARP.

Next elimination is $\{x^r_b\}_{r=1,2}$, using (C.7) yield the equivalent expression $\exists x^1_a, x^2_a$ s.t. .

Next reduce x^1_a and x^2_a and let everything in terms of $\overline{z}^r_t$.

□

Chapter 14

Sequential Trade

We consider I consumers, J firms, S states of the world each of them with probability $\pi_{is} \in [0, 1]$ of being realized according to i, LS state-contingent commodities.

Definition 88. For every physical commodity $l = 1, \cdots, L$ and states $s = 1, \cdots, S$, a unit of (state-)contingent commodity ls is a title to receive a unit of the physical commodity l if and only if s occurs. Accordingly, a (state-)contingent commodity vector is specified by

$$x = (x_{11}, \cdots, x_{L1}, \cdots, x_{1S}, \cdots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive commodity vector $(x_{1s}, \cdots, x_{Ls})$ if state s occurs.

The state-contingent endowments for consumers $i = 1, \cdots, I$ be a contingent commodity vector:

$$\omega_i = (\omega_{11i}, \cdots, \omega_{L1i}, \cdots, \omega_{1Si}, \cdots, \omega_{LSi}) \in \mathbb{R}^{LS}.$$

Preferences are defined over $X_i \subset \mathbb{R}^{LS}$, and denoted by $\succeq_i$ and are of the Expected Utility Form.

$$x_i \succeq_i x_i' \iff \sum_s \pi_{si} u_{si}(x_{1si}, \cdots, x_{Lsi}) \geq \sum_s \pi_{si} u_{si}(x_{1si}', \cdots, x_{Lsi}').$$

Similarly technological possibilities are represented by $Y_j \subset \mathbb{R}^{LS}$, or state-contingent production plan $y_j \in \mathbb{R}^{LS}$ is a member of Y_j if for every s the plan y_s of physical commodities is feasible for firm j when state s occurs.

Definition 89. An allocation

$$(x_1^*, \cdots, x_I^*, y_1^*, \cdots, y_J^*) \in \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j \subset \mathbb{R}^{LS(I+J)}$$

and a system of prices for the contigent commodities $p = (p_{11}, \cdots, p_{LS}) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium if:

(i) For every j, y_j^* satisfies $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.

(ii) For every i, x_i^* is maximal for $\succeq_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii) $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$.

Claim 12. At $t = 1$, if the economy is in Arrow-Debreu Eq. when uncertainty is realized there is no incentive to trade further.

The reason is that due to the first welfare theorem the allocations are efficient and thus there is no incentive to trade.

14.1 Sequential Trade

Consider a case where there are not S markets, but only one good can be traded at $t = 0$, at $t = 0$ consumers make predictions about spot prices for each $s \in S$, $p = (p_1, \cdots p_S) \in \mathbb{R}^{LS}$ at $t = 1$. Then the consumer trades the only state contingent commodity at prices $q \in \mathbb{R}^S$. The problem of consumer i is:

$$\underset{\substack{(x_{1i}, \cdots, x_{Si}) \in \mathbb{R}^L \forall s \in S \\ (z_{1i}, \cdots, z_{Si}) \in \mathbb{R}^S}}{Max} U_i(x_{1i}, \cdots, x_{Si})$$

$$s.t. \quad (i) \sum_s q_s z_{si} \leq 0.$$

$$(ii) p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + p_{1s} z_{si} \forall s \in S$$

Restriction (i) corresponds to trade in $t = 0$, and $t = 1$.

Definition 90. (Radner Equilibrium) A collection formed by a price vector $q \in \mathbb{R}^S$ for state-contingent first good commodity at $t = 0$ a spot price vector

$$p = (p_1, \cdots, p_S) \in \mathbb{R}^{SL}$$

for every s and for every consumer$i$, consumption plans $z_i^* = (z_{1i}^*, \cdots, z_{Si}^*) \in \mathbb{R}^S$ at $t = 0$ and $x_i^* = (x_{1i}^*, \cdots x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitutes a Rander equilibrium if

(i) For every i, the consumption plans z_j^*, x_i^* solve

$$\underset{\substack{(x_{1i}, \cdots, x_{Si}) \in \mathbb{R}^L \forall s \in S \\ (z_{1i}, \cdots, z_{Si}) \in \mathbb{R}^S}}{Max} \quad U_i(x_{1i}, \cdots, x_{Si})$$

$$s.t. \quad (i) \sum_s q_s z_{si} \leq 0.$$

$$(ii) p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + p_{1s} z_{si} \forall s \in S$$

(ii) $\sum_i z_{si}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every s.

Remark 1. We make two normalizations (i) $p_{1s} = 1$ and (ii) $q_1 = 1$.

Proposition 50. *(i) If the allocation $x^* \in \mathbb{R}^{LSI}$ and the contingent commodities price vector $p \in (p_1, \cdots, p_S) \in \mathbb{R}^{LS}_{++}$ constitue an Arrow-Debreu equilibrium, then there are prices $q \in \mathbb{R}^S_{++}$for contingent first good commodities and consumption plans for these commodities $z^* \in \mathbb{R}^{SI}$ such that x^*, z^*, q, p constitute a Radner Equilibrium.*

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$ and $z^* \in \mathbb{R}^{SI}$ and the price $q \in \mathbb{R}^S_{++}$ and $(p_1, \cdots, p_S) \in \mathbb{R}^{LS}_{++}$ constitute a Rander equilibrium, then there are multipliers $(\mu_1, \cdots, \mu_S) \in \mathbb{R}^S_{++}$ such that the allocation x^* and the contingent commodities price vector $(\mu_1 p_1, \cdots, \mu_S p_S) \in \mathbb{R}^{LS}_{++}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value at $t = 0$ of a dollar at $t = 1$ and state s).

Proof. (i) If we have an Arrow-Debreu equilibrium, we can let $q_s = p_{1s}$ for every s. With this we claim that for all i, the budget set for the Arrow-Debreu problem is:

$$B_i^{AD} = \{x \in \mathbb{R}^{LS}_+ : \sum_s p_s \cdot (x_{si} - \omega_{si}) \leq 0\},$$

is identical to the budget set of Radner's.

$$B_i^R = \{x \in \mathbb{R}_+^{LS} : \exists z_i \in \mathbb{R}^S, s.t. \sum_s q_s z_{si} \leq 0 \quad and \quad p_s \cdot (x_{si} - \omega_{si}) \leq p_{1s} z_{si} \forall s\}.$$

To see this suppose that $x_i \in B_i^{AD}$. For every s, denote $z_{si} = (1/p_{1s})p_s \cdot (x_{si} - \omega_{si})$. Then $\sum_s q_s z_{si} = \sum_s p_{1s} z_{si} = \sum_s p_s \cdot (x_{si} - \omega_{si}) \leq 0$ and $p_s \cdot (x_{si} - \omega_{si}) = p_{1s} z_{si}$ fo all s. Hence $x_i \in B_i^R$.

The reverse is also true, if $x_i \in B_i^R$, that is for some z we have $\sum_s q_s z_{si} \leq 0$ and $p_s \cdot (x_{si} - \omega_{si}) \leq p_{1s} z_{si}$ for all $s \in S$. Summing over s, $\sum_s p_s \cdot (x_{si} - \omega_{si}) \leq \sum_s p_{1s} z_{si} = \sum_s q_s z_{si} \leq 0$. Hence, $x_i \in B_i^{AD}$.

We have established that the following sets are equal/equivalent: $B_i^{AD} \equiv B_i^R$.

Thus AD is Radner with $q = (p_{11}, \cdots, p_{1S}) \in \mathbb{R}^S$, the spot prices $p_1, \cdots, p_S$ and the contingent trades $(z_{1i}^*, \cdots, z_{Si}^*) \in \mathbb{R}^S$ defined by $z_{si}^* = (1/p_{1s})p_s \cdot (x_{si}^* - \omega_{si})$. Note that the contingent market clears since for all s

$$\sum_i z_{si}^* = (1/p_{1s})p_s \cdot (\sum_i (x_{si}^* - \omega_{si})) \leq 0.$$

(ii) Choose μ_s, such that $\mu_s p_{1s} = q_s$. Then we can write Radners budget set of every consumer i as

$$B_i^R = \{(x_{i1}, \cdots, x_{si}) \in \mathbb{R}^{LS} : \exists z_i \in \mathbb{R}^S, s.t., \sum_s q_s z_{si} \leq 0 \quad and \quad \mu_s p_s \cdot (x_{si} - \omega_{si}) \leq q_s z_{si} \forall s\}$$

We can rewrite this as the AD budget

$$B_i^R = B_i^{AD} = \{x \in \mathbb{R}^{LS} : \sum_s \mu_s p_s \cdot (x_{si} - \omega_{si}) \leq 0\}.$$

Hence the consumption plan x_i^* is also preference maximizing in the budget set B_i^{AD}. Since this is true for all i, we conclude that the price vector $(\mu_1 p_1, \cdots, \mu_S p_S) \in \mathbb{R}^{LS}$ clears the markets for the LS contingent commodities.

$\square$

Chapter 15

Behavioral Economics: The Sparse-Max Consumer

15.1 Introduction

We consider a generalization of the rational consumer model and most of the theory we have developed so far. Why we need this? There is evidence that the rationality assumption is not the best assumption to make. Thus it is an important question how relaxing this property affects our main results.

The main ingredient is a perceived price p^s that may be different from actual prices p.

Definition 91. Sparse max consumer. $smax_{x|p^s} u(x)$ subject to $p'x \leq w$.

The sparse max operator is the given by the following process:

1. $max_x \mathcal{L}(x, p^s, w, \lambda^s)$ where $\mathcal{L}(x, p^s, w, \lambda^s) = u(x) + \lambda^s(w - p^{s\prime}x)$

Obtain $x(p^s, \lambda^s) = argmax_x \mathcal{L}(x, p^s, w, \lambda^s)$ for unknown λ^s.

2. Find λ^s, such that $p'x(p^s, \lambda^s) = w$.

Formally, this implies the following:

$$\nabla_x u(x) = \lambda^s p^s$$

$$p'x(p^s, \lambda^s) = w.$$

This implies that there exists a $w' > 0$ such that the solution to the smax $x^s(p, p^s, w) =$

$arg(smax)_{x|p^s} u(x)$ subject to $p'x \leq w$:

$$x^s(p, p^s, w) = x^r(p^s, w'),$$

where $p'x^r(p^s, w') = w$ and $x^r(\cdot) = argmax_x u(x)$ s.t. $p'x \leq w$, this is the solution to the UMP.

Proposition 51. *If x^r is linear in wealth then* $x^s(p, p^s, w) = \frac{x^r(p^s,w)}{p'x^r(p^s,1)}$.

Proof. $x^r(p, w)$ is linear in wealth means that $x^r(p, w) = x^r(p, 1)w$, then we have $p \cdot x^r(p^s, w') = p \cdot x^r(p^s, 1)w' = w$, this implies that $w' = \frac{w}{p \cdot x^r(p^s,1)}$. Now, by our previous result we have that $x(p, p^s, w) = x^r(p^s, w')$, by linearity in wealth $x(p, p^s, w) = x^r(p^s, 1)w' = \frac{x^r(p^s,1)w}{p \cdot x^r(p^s,1)}$, this implies that $x(p, p^s, w) = \frac{x^r(p^s,w)}{p \cdot x^r(p^s,1)}$.

□

We have an important example:

Example 15. Cobb-Douglas Sparse Max.

$x^s_l(p, p^s, w) = \frac{\alpha_l w}{p^s_l} \frac{1}{(\sum_l (\alpha_l p_l)/p^s_l)}$.

If $p^s = p$ then we are back to the rational case.

15.2 The Perceived Price and Limited Attention

The perceived price is going to be modelled as an average of actual prices p and default prices p^d. Default prices, are what the consumer has in mind, maybe a forecast about prices $p^d = E[\mathbf{p}]$ if $\mathbf{p}$ is the distribution of prices in the market. More formally $\mathbf{p}$ can be the history of prices or the memory of the consumer about prices.

The linear model of perceived prices is as follows:

$$p^s_l(m, p, p^d) = m_l p_l + (1 - m_l)p^d_l,$$

for $m_l \in [0, 1]$ an attention parameter.

The consumer is rational when $m_l = 1$ for all l and he becomes more behavioral when $m < 1$.

We let $M = diag(\{m_l\}^L_{l=1})$, the diagonal matrix of all attention parameters.

We have the following result. If we have any demand $x(p, w)$ then we can find a p^s and a w' such that:

$$x^r(p^s, w') = x(p, w).$$

This means that p^s without restrictions is too general.

Proposition 52. *The Slutsky matrix of the sparse max is such that* $S^s(p, w; p^d)|_{p=p^d} = S^r(p, w)M$.

Proof. We have $x^s(p, w) = x^r(\{p_l^d, +m_i(p_l - p_l^d)\}_{l=1}^L, w'(p, p^d))$

Then we derive with respect to p_l and evaluate everything to $p = p^d$:

$\partial_{p_l} x^s(p, w) = \partial_{p_l} x^r(p, w)m_l + \partial_w x^r(p, w)\frac{\partial w'(p,p^d)}{\partial p_l}$

Now, $p \cdot x^r(p^s, w'(p)) = w$, we differentiate with respect to p_l:

$0 = x_l^r + p \cdot \partial_{p_l} x^r m_l + p \cdot \partial_w x^r \partial_{p_l} w'$. As, $p \cdot x^r(p, w) = w$, we have $x_l^r + p \cdot \partial_{p_l} x^r = 0$, and $p \cdot \partial_w x^r = 1$, so $0 = x_l^r - x_l^r m_l + \partial_{p_l} w'$ thus

$\partial_{p_l} x' = (m_l - 1)x_l^r$.

Finally,

$$\partial_{p_l} x^s(p, w) = \partial_{p_l} x^r m_l + \partial_w x^r(m_l - 1)x_l^r.$$

Notice that $\partial_w x^s(p, w') = \partial_w x^r(p, w)$ when $p = p^d$ and also we have $x^s(p^d, w'(p^d, p^d)) = x^r(p^d, w)$.

This implies that at $p = p^d$ and thus $w'(p^d, p^d) = w$ we have:

$\partial_{p_l} x_k^s(p, w) + \partial_w x_l^s(p, w)x_k^s(p, w) = [\partial_{p_l} x_k^r(p, w) + \partial_w x_l^r(p, w)x_k^r(p, w)]m_l$

$S^s(p, w)|_{p=p^d} = S^r(p, w)|_{p=p^d} M$.

□

It turns out that

$S^s(p, w)$ is not singular in prices that is $S^s(p, w)p \neq 0$ for some (p, w). This is a consequence of failures of homogeneity of degree zero.

$S^s(p, w)$ is also not necessarily NSD, in fact the model $x^s(p, w)$ fails WARP.

15.3 General Equilibrium (Pure Exchange)

We have as before an endowment ω and $z^{i,s}(p, M^i) = x_{M^i}^{i,s}(p, p \cdot \omega^i) - \omega^i$. The aggregate excess demand is $z^s(p) = \sum_i z^{i,s}(p, M^i)$.

The equilibrium prices are such that $z^s(p) = 0$.

Proposition 53. *Assume Pareto Efficient Interior Allocations. Consumers are such that* $M^i = M^j$ *for all* $i, j \in \{1, \cdots, I\}$ *if and only if the allocation* x^* *and* p *such that* $z(p) = \sum_i (x^{i,*}(p, p \cdot \omega) - \omega^i) = 0$ *is Pareto Efficient.*

Proof. $\nabla_x u^i(x) = \lambda^i p^s(m_i)$, in a PE interior equilibrium we have for any two goods $\frac{u^i_l(x)}{u^i_k(x)} = \frac{u^j_l(x)}{u^j_k(x)}$, and we have also $\frac{u^i_l(x)}{u^i_k(x)} = \frac{p_l}{p_k}$ due to utility maximization, however say that for any two goods $m^i_l \neq m^j_l$, for some two goods and then $\frac{p^s_l(m^i)}{p^s_k(m^i)} < \frac{p^s_l(m^j)}{p^s_k(m^j)}$ which implies that $\frac{u^i_l(x)}{u^i_k(x)} < \frac{u^j_l(x)}{u^j_k(x)}$. Now, we have that this is not a PE (interior) allocation. □

Also the Second Welfare Theorem does not hold either, because WARP does not hold.

Offer curves are not longer lines.

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Recall that the offer curve is $OC^i(p_1, p_2) = \{p : x^i(p_1, p_2, p \cdot \omega^i)\}$ because HD0 fails then we cannot normalize, this means that $OC^i(p_1, p_2)$ has two parameters not only one as in the rational case, this means that it is a “ribbon”.

Chapter 16

Discrete Choice and Logit

There are J alternatives in the market, indexed by $j = 1, \cdots, J$. Each purchase occasion, each consumer i divides her income y_i on at most one alternative and on an outside good.

$$max_{j,z} u_i(x_j, z) \quad s.t. \quad p_j + p_z z = y_i,$$

where x_j are characteristics of the brand j and p_j the price, z is the quantity of the outside good and p_z its price. The outside good, denoted by $j = 0$ is the non-purchase of any alternative.

Substitute $z = y - p_j/p_z$ to derive the conditional indirect utility function for each brand:

$$U^*_{ij}(x_j, p_j, p_z, y_i) = U_i(x_j, y_i - p_j/p_z).$$

If the outside good is bought then:

$$U^*_{i0}(0, y_i/p_z).$$

The consumer chooses the brand yielding the highest conditional indirect utility:

$$max_j U^*_{ij}(x_j, p_j, p_z, y_i).$$

Usually we let the utility to be random:

$$U^*_{ij}(x_j, p_j, p_z, y_i) = V_{ij}(x_j, p_j, p_z, y_i) + \epsilon_{ij},$$

where ϵ_{ij} is a random shock observed by i but not to the researcher (unobservable attributes, preference shock, preference heterogeneity).

We have now the probability of choice:

$$P_{ij}(p_1, \cdots, p_J, p_z, y_i) = Prob\{\epsilon_{i0}, \cdots, \epsilon_{i,J} : U^*_{ij} > U^*_{ij'} \forall j' \neq j\}.$$

16.1 Multinomial Logit

Assume $\epsilon_{ij}, j = 0, \cdots, J$ is distributed *i.i.d.* type I extreme value across i, with CDF:

$$F_\epsilon(x) = exp(-exp(-\frac{x-\eta}{\mu})) = Pr(\epsilon \leq x)$$

with $\eta = 0.577$ (Euler's constant) and the scale parameter $\mu = 1$, then we have:

$$P_{ij} = \frac{exp(V_{ij})}{\sum_{j'=0,\cdots,J} exp(V_{ij'})}.$$

16.2 Independence of Irrelevant Alternatives

You may be asking what are the empirical implications of the Mutinomial Utility Model. The most important implication is the Independence of Irrelevant Alternatives (IIA). Before defining it we allow J to change, $P_{ij,J} = \frac{exp(V_{ij})}{\sum_{j'=0,\cdots,J} exp(V_{ij'})}$ so that the probability of choice depends on the number of alternatives.

Axiom 7. *(IIA)* $\frac{P_{ij,J}}{P_{ik,J}} = \frac{P_{ij,J'}}{P_{ik,J'}}$ *where J' possibly is different from J, for all j, k.*

It is very easy to check that Multinomial Logit satisfy the IIA condition.
$\frac{P_{ij,J}}{P_{ik,J}} = \frac{exp(V_{ij})}{exp(V_{ik})}$ for all J.

Example 16. Red bus/blue bus problem

Say you have two options W for walking and RB for a red bus. Now, assume that $p_{W,2} = 0.5$ and $p_{RB,2} = 0.5$. The odds ratio is $\frac{p_{W,2}}{p_{RB,2}} = 1$.

Now the city transportation corporation introduced a Blue Bus BB, that is identical to the red bus in all attributes, (except for color), in fact, it is a (nearly) perfect substitute while not being a substitute for walking. Given this, we should expect that the probability under this situation with $J = 3$, we have probabilities such as $p_{W,3} = 0.5$ and $p_{RB,3} = 0.25$, $p_{BB,3} = 0.25$. The odds ratio here is $\frac{p_{W,3}}{p_{RB,3}} = 2$, but the IIA requires that $\frac{p_{W,3}}{p_{RB,3}} = 1$ is the same independently of the other alternatives. This limits a lot the capabilites of the multinomial logit to capture substitution patterns.

16.3 The Supply Side of the Discrete Products Economy

We assume that we have one firm that produces all J products. The size of the market is the constant M, the profit function of the firm is given by

$$\pi = \sum_{j=1,\cdots,J} (p_j - mc_j) M \cdot s_j(p, x)$$

Where $ln(mc_j) = x_j\gamma + \omega_j$ is the marginal cost of the production of one unit of good j where γ is a unitary cost for observable attributes x_j and ω_j is a productivity shock.

The product share is the proportion of the total size of the economy M, that chooses product j.

$s_j(p, x) = \int \frac{exp(V_j(x_j, p_j, p_z, y))}{\sum_{j'} exp(V_{j'}(x_j, p_j, p_z, y))} dF(y)$.

The problem for the firm is that they can price a good according to its attributes but they do not observe the wealth distribution.

The problem of the firm is to maximize profits by choosing price. This is a monopolist, it is not a price taker.

$$max_p \pi$$

The FOC are:

$$s_j(p, x) + \sum_{r=1,\cdots,J} (p_r - mc_r)\partial_{p_j} s_r(p, x) = 0.$$

16.4 Partial Equilibrium

The pair $(p, s(p, x))$ of observed prices for J commodities and observed market shares $s(p, x)$ is a partial equilibrium if and only if:

(i) Consumer maximize their random utility given prices and their wealth.

$$s_j(p, x) = \int \frac{exp(V_j(x_j, p_j, p_z, y))}{\sum_{j'} (V_{j'}(x_j, p_j, p_z, y))} dF(y)$$

(ii) The firm maximizes its profit with respect to prices and observed market shares:

$$s_j(p, x) + \sum_{r=1,\cdots,J} (p_r - mc_r)\partial_{p_j} s_r(p, x) = 0.$$

16.5 General Random Utility

Consider X a finite choice set. Consider the set of linear orders on X , $R \subset X \times X$. There is a distribuion over linear orders $\pi \in \Delta(R)$ that captures preference variability, heterogeneity or preference shocks. Given a menu $A \in 2^X \setminus \emptyset$, the probability of choosing $a \in A$ is:

$$p(a, A) = \sum_{r \in R} \pi(r) \mathbb{I}(arb \forall b \in A \backslash \{a\}).$$

Where $\mathbb{I}(arb \forall b \in A \backslash \{a\}) = 1$ when the argument is true, else $\mathbb{I}(arb \forall b \in A \backslash \{a\}) = 0$.

Definition 92. (ASRP) Axiom of Stochastic Revealed Preference. The choice probability p satisfy the ASRP if and only for every sequence of $(a_k, D_k)_{k=1}^n$, such that a_k is in D_k and $D_k \subseteq X$ and $D_k \neq \emptyset$ we have $\sum_{k=1}^n p(a_k, D_k) \leq max_{r \in R} \sum_{k=1}^n \mathbb{I}(a_k r b \forall b \in D_k \backslash \{a_k\})$.

Lemma 9. *The following are equivalent:*

(i) p satisfy the ASRP

(ii) The following system of linear equations has a solution:

$$M\pi = p,$$

where, $M = \mathbb{I}(a_k r^s b \forall b \in D_k \backslash \{a_k\})_{k=1,s=1}^{k=n,s=S}$, $\pi = (\pi(r^s))_{s=1}^S$ and $p = (p(a_k, D_k))_{k=1}^K$, and the unknown is a vector $\pi \in \Delta(R)$ is in the simplex of the linear orderings over X.

(iii) There is random utility that is $\pi \in \Delta(R)$ such that $p(a, A) = \sum_{r \in R} \pi(r) \mathbb{I}(arb \forall b \in A \backslash \{a\})$.

(iv) The mapping $f_C(a) = \sum_{D \in \mathbf{B}(C)} (-1)^{|D \backslash C|} p(a, D)$ where $\mathbf{B}(C) = \{D \in 2^X | C \subseteq D\}$, is $f_C(a) \geq 0$ for all a and all $C \subseteq X$ (this is called total monotonicity).

Proof. We prove that (iii) implies ASRP.

Note that if (iii) holds then for any sequence $(a_k, D_k)_{k=1}^n$, such that a_k is in D_k and $D_k \subseteq X$ and $D_k \neq \emptyset$ we have $\sum_{k=1}^n p(a_k, D_k) = \sum_{k=1}^n \sum_{r \in R} \pi(r) \mathbb{I}(a_k r b \forall b \in D_k \backslash \{a_k\})$for some $\pi \in \Delta(R)$.

Notice that $\sum_{k=1}^n \sum_{r \in R} \pi(r) \mathbb{I}(a_k r b \forall b \in D_k \backslash \{a_k\}) \leq max_{\pi \in \Delta(R)} \sum_{r \in R} \pi(r) \mathbb{I}(a_k r b \forall b \in D_k \backslash \{a_k\}) = max_{r \in R} \sum_{r \in R} \mathbb{I}(a_k r b \forall b \in D_k \backslash \{a_k\})$. Thus,

$\sum_{k=1}^n p(a_k, D_k) \leq max_{r \in R} \sum_{k=1}^n \mathbb{I}(a_k r b \forall b \in D_k \backslash \{a_k\})$.

□

There is a surprising result about random utility.

Consider the following definition.

Definition 93. The aggregate probability of choice of $a \in A$ for a population of agents with different random utilities supported in Π such that $\mu \in \Delta(\Pi)$, is $p^M(a, A) = \sum_{\pi \in \Pi} \mu(\pi) p_\pi(a, A)$.

Proposition 54. *There exist a representative random consumer, i.e., there exist a random utility* $\pi^{rep} \in \Delta(R)$ *such that* $p^M(a, A) = \sum_{r \in R} \pi^{rep}(r) \mathbb{I}(arb \forall b \in A \backslash \{b\})$.

Proof. $p^M(a, A) = \sum_{\pi \in \Pi} \mu(\pi) p_\pi(a, A)$, then

$f_{C,\pi}(a) = \sum_{D \in \mathbf{B}(C)} (-1)^{|D \backslash C|} p_\pi(a, D)$ where $\mathbf{B}(C) = \{D \in 2^X | C \subseteq D\}$, is $f_{C,\pi}(a) \geq 0$ for all a and all $C \subseteq X$ by assumption.

Define

$f_C(a) = \sum_{D \in \mathbf{B}(C)} (-1)^{|D \backslash C|} p^M(a, D)$ where $\mathbf{B}(C) = \{D \in 2^X | C \subseteq D\}$, I want to show that $f_C(a) \geq 0$ for all a and all $C \subseteq X$.

Replace:

$f_C(a) = \sum_{D \in \mathbf{B}(C)} (-1)^{|D \backslash C|} \sum_{\pi \in \Pi} \mu(\pi) p_\pi(a, A) = \sum_{\pi \in \Pi} \mu(\pi) \sum_{D \in \mathbf{B}(C)} (-1)^{|D \backslash C|} p_\pi(a, A)$

$f_C(a) = \sum_{\pi \in \Pi} f_{C,\pi}(a) \geq 0.$

By the previous theorem this implies that there is a $\pi^{rep} \in \Delta(R)$ that generates the aggregate probability.

$p^M(a, A) = \sum_{r \in R} \pi^{rep}(r) \mathbb{I}(arb \forall b \in A \backslash \{b\}).$

□

Chapter 17

Matching

17.1 One-to-One Matching: Marriage Problems

A marriage problem (Gale and Shapley 1962) is a triple $< M, W, \succeq >$ where M is a finite set of men and W is a finite set of women, $\succeq = (\succeq_i)_{i \in M \cup W}$ is a list of preferences. Here, $\succeq_m$ denotes the preference relation of man m over $W \cup \{m\}$, $\succeq_w$ denotes the preference of woman w over $M \cup \{w\}$, and $\succ_i$ denotes the strict preference derived for agent $i \in M \cup W$.

Consider man m:

- $w \succ_m w'$ means that man m prefers woman w to w'
- $w \succ_m m$ means that man m prefers w to remain single, and
- $m \succ_m w$ means that woman w is unacceptable to man m

Assumption 1. *All preference are strict.*

The outcome of a marriage problem is a matching. Formally, a matching is a function $\mu : M \cup W \mapsto M \cup W$ such that

1. $\mu(m) \notin W \implies \mu(m) = m \quad \forall m \in M,$
2. $\mu(w) \notin M \implies \mu(w) = w \quad \forall w \in W$
3. $\mu(m) = w \iff \mu(w) = m \quad \forall m \in M, w \in M$

Here $\mu(i) = i$ represents that the agent i remains single.

A matching μ is Pareto efficient if there is no other matching ν such that $v(i) \succeq_i \mu(i)$ for all $i \in M \cup W$ and at least some $i \in M \cup W$ such that $v(i) \succ_i \mu(i)$.

A matching is blocked by an individual if $i \succ_i \mu(i)$. A matching is individually rational if it is not blocked by any individual.

A matching μ is blocked by a pair $(m, w) \in M \times W$ if they both prefer each other to their partners under μ, i.e.,

$$w \succ_m \mu(m) \quad and \quad m \succ_w \mu(w).$$

A matching is stable if it is not blocked by any individual pair. The next result follows inmediately.

Proposition 55. *Stability implies Pareto Efficiency.*

The following is an algorithm and its versions played a central role for almost 50 years in ot only matching theory but also its application in real-life matching markets.

17.2 Men-Proposing Deferred Acceptance Algorithm

Step 1. Each man m proposes to his first choice (if any). Each woman rejects any offer except the best acceptable proposal and "holds" the most-preferred acceptable proposal (if any).

In general, at

Step k. Any man who was rejected at step $k-1$ makes a new proposal to his most-preferred acceptable potential mate who has not yet rejected him (if no acceptable choice remains, then he makes no proposal). Each woman "holds" her most-preferred acceptable proposal to date and rejects the rest.

The algorithm terminates when there are not more rejections. Each woman is matched with a man she has been holding in the last step. Any woman who has not been holding an offer or any man who was rejected by all acceptable woman remains single.

Theorem 20. *(Thm. 1,2 in Gale and Shapley, 1962) The men proposing deferred acceptance algorithm gives a stable matching for each marriage problem. Moreover, every man weakly prefers this matching to any other stable matching.*

Proof. Improvement Lemma. If m proposes to w on the kth day, then on $k+i$ for all $i \geq 1$, w has someone m' that has proposed to w such that $m' \succ_w m$.

Suppose towards contradiction that on the jth day $j > k$ we have the first counter-example, where w has either nobody or some m^* such that $m \succ_w m^*$. On $j-1$, w has m' that proposed to her and likes $m' \succ_w m$. According to the algorithm, m' still proposes to w on jth, since she hold his proposal the previous day. So w has the choice of at least one man on jth, moreover the best choice has to be at least as good as m', there is an $m'' \succeq_w m'$ with $\sim_w$ if $m' = m''$, according to the algorithm she will choose it over m^*. This contradicts the initial statement.

Lemma. The algorithm terminates with a pairing. For $|M| = n$ and $|W| = n$

Suppose for contradiciton that there is a man m who is left unpaired, at the end of the algorithm. He must have proposed to every single woman on his list. By the Improvement Lemma, each of these women thereafter has someone that has proposed to them. Thus when the algorithm terminates, n women have n men proposing to them non including m. So there must be at least $n + 1$ men. Contradiction.

Lemma. The matching is stable.

We want to prove that no man can be in a rouge couple, or in a blocking pair. Consider $\mu(m) = w$ and suppose that $w^* \succ_m w$. We will prove that $\mu(w^*) \succ_{w^*} m$. So that (m, w^*) cannot be a rouge couple. Since $w^* \succ_m w$ then m proposed to her first and w^* rejected him for some other m'. By the improvement lemma and transitivity we have that $\mu(w^*) \succeq_{w^*} m' \succ_{w^*} m$. Thus no man m can be involved in a rouge couple, and the pairing is stable.

□

17.3 One-Sided Matching: House Allocation and Housing Markets

A house allocation problem (Hylland and Zeckhauser 1979) is a triple $< I, H, \succ >$ where I is a set of agents, H is a set of of indivisible objects (henceforth houses). Preferences are $\succ = (\succ_i)_{i \in I}$. Throughout this section we assume $|H| = |I|$ and the preference are strict.

The outcome of a house allocation problem is an assignment of houses to agents such that each agent receives a house.

Definition 94. A house matching is a mapping $\mu : I \mapsto H$ such that μ is one-to-one and onto from I to H (isomorphism).

A matching μ Pareto dominates another matching if $\mu(i) \succeq_i \nu(i)$ for all $i \in I$ and $\mu(i) \succ_i v(i)$ for some $i \in I$. A matching is Pareto efficient if it is not Pareto dominated by any other matching.

A house allocation problem is simply a collective ownership economy where a number of houses shall be assigned to a number of agents. It is an economy where the grand coalition I owns the set of houses H, but no subset of I has any say over a house or set of houses.

In constrast, the following economy is a private ownership economy, where each agent holds the property rights of a specific house.

A housing market (Shapley and Scarf 1974) is a four-tuple $< I, H, \succ, \mu >$ where μ is an initial endowment of houses. Formally a housing market is a set of houses. Let $h_i = \mu(i)$ denote the initial endowment of agent $i \in I$.

17.4 Core of a Housing Market

A matching η is individually rational if $\eta(i) \succeq_i h_i$ for all $i \in I$. A matching η is in the core of the housing market $< I, H, \succ, \mu >$ if there is no coalition $T \subseteq I$ and matching ν such that

1. $v(i) \in \{h_j\}_{j \in T} \forall i \in T$
2. $v(i) \succeq_i \eta(i) \forall i \in T$
3. $v(i) \succ_i \eta(i) \exists i \in T$.

The following algorithm, along with the deferred acceptance algorithms, plays a key role in the matching literature.

17.5 Gale's Top Trading Cycles (TTC) Algorithm

Step 1. Each agent "points to" the owner of his favorite house. Since there is a finite number of agents, there is at least one cycle of agents pointing to one another. Each agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment. If there is at least one remaining agent, proceed with the next step.

In general, at

Step k. Each remaining agent points to the owner of his favorite house among the remaining houses. Every agent in a cycle is assigned the house of the agent he points to and removed form the market with his assignment. If there is at least one remaining agent, proceed with the next step.

Theorem 21. *(Thm. 2 in Roth and Postlewaite 1977): The outcome of Gale's TTC algorithm is the unique matching in the core of each housing market. Moreover, this matching is the unique competitive allocation.*

Proof. There exists a core matching for any housing market.

Let μ be the resulting matching from the TTC. Suppose there is a coalition T who deviates profitably from μ, inducing v, then each $i \in T$ prefers $v(i) \succ_i \mu(i)$. Let a be an agent that is matched first among this subset in the TTC algorithm. Then $v(a)$ is owned by an agent $b \in T$ who is removed by the TTC algorithm in a strictly early step (say cycle C_m).

Then b obtains a house $b' \in T \cap C_m$ both in v and in μ, $b^k \in T \cap C_m$ obtains $v(a)$ both at v and μ. This is a contradiction.

The matching produced by the TTC algorithm is unique.

Consider an arbitrary matching $v \neq \mu$, and fix a to be one of the first agents with $v(a) \neq \mu(a)$ according to the order of being matched in TTC.

Let C_m be the set of agents that form a cycle that includes a.

Then, any agents b who are matched before C_m satisfy $v(b) = \mu(b)$.

By construction of TTC, $\mu(b) \succeq_b v(b)$ for all $b \in C_m$ (because, in v, all preferred goods are allocated to those who are matched before C_m).

Furthermore, since $v(a) \neq \mu(a)$ and $a \in C_m$, we have $\mu(a) \succ_a v(a)$.

Since, for any $b \in C_m$, $\mu(b)$ is an initial house owned by some other agent in C_m, these facts imply that C_m can profitable deviate from v to μ.

$\square$

Chapter 18

Computable General Equilibrium (CGE)

18.1 Introduction

The Computable General Equilibrium (CGE) analysis is an ex-ante policy evaluation tool that uses a mix of micro and macro data, calibration and estimation, to generate consistent escenarios for counterfactual policies.

In sum, we write a parametric general equilibrium model that reproduces the sectorial macro-accounts of a country. Then we use the model to ask what would happen under a change of a policy, such that the macro accountability identities are respected and such that individual households and firms optimization principles are taken into account.

This tool is widely used by policy making institutions around the world.

I am going to use Erinc Yeldan's notes on a simple CGE for an open 2x2 Ricardo-Viner General Equilibrium model with sector specific capital.

18.2 A 2×2 Small Open Economy Model

By small open economy we mean that the economy is price-taker for internationally traded goods. There are two goods and two factors of production, labor and capital (L and K).

Labor is mobile across sectors, but capital is sector-specific. There are two households, workers and capitalists. Preferences are Cobb-Douglas.

Sets:	
Goods	I, A agricultural and N industral goods
Households	HH
Time Periods	TP
Parameters:	
Labor supply	$\overline{L}^S$
Total capital	$\overline{K}_i$
Production CET	ρ_i
Production Labor Subs.	σ_i
World Prices	PW_i
Shift Parameter in Prod. Function	A_i
CES Function share param.	α_i
Exchange rate	ER
Prices:	
	$P_i = ER \cdot PW_i$
Factor Markets	
Production	$Q_i^s = A_i[\alpha_i K_i^{-\rho_i} + (1-\alpha_i)L_i^{-\rho_i}]^{-1/\rho_i}$
Labor demand	$L_i^D = [(P_i/W)(1-\alpha_i)A_i^{-P}]^{\sigma_i} Q_i^S$
Labor market Equilibrium	$\sum_{i \in I} L_i^D = \overline{L}^S$
Capital is sector specific	$K_i = \overline{K}_i$
Wage income	$Y^W = \sum_{i \in I} W \cdot L_i^D$
Capitalist income	$Y^K = \sum_{i \in I}(P_i Q_i^S - W \cdot L_i^D)$
Commodity Markets:	
Consumer demand:	$Q_{h,i}^D = \beta_{h,i}\left[\frac{Y_h}{P_i}\right]$
Import (excess demand):	$Q^M = \sum_{h \in HH} Q_{h,A} - Q_A^S$
Export (excess supply)	$Q^E = Q_N^S - \sum_{h \in HH} Q_{h,N}^D$
Trade Balance	$P_A^W Q^M - P_N^W Q^E = 0$

18.3 Social Accounting Matrix (SAM)

A Social Accounting Matrix (SAM) is a comprehensive, economy-wide data framework, typically representing the economy of a country. A SAM is a square matrix S in which each account is represented by a row and a column. Each cell S_{ij} represents the payment from account j to account i.

- Example: Consumption is given by the cells $S_{3,7}$ for expenditure of households of type Worker to commodities of the rural type.
- Wages are given by cells $S_{5,1}$ the payment of industry 1 Agriculture to labor factor 5.
- Imports are in cell $S_{9,3}$(only rural good) and Exports are in cell $S_{4,9}$ (only industrial good).

The SAM is based on the double-entry accounting principle, that requires that for each column total (revenue) we have the same corresponding amount for expenditures of the same account.

SAM									
		Activities		Commodities		Factors		Househo	
		1. Agr	2. Ind	3.Rural	4.Urban	5. Labor	6. Capital	7.Workers	8.
Activities	1. Agriculture			2.206	0				
	2. Industry			0	5.721				
Commodities	3. Rural							2.167	
	4. Urban							0.722	
Factors	5. Labor	0.986	1.903						
	6. Capital	1.220	3.818						
Households	7. Workers					2.889	0		
	8. Capitalists					0	5.038		
ROW	9.Row			2.480	0				
Totals	Totals	2.206	5.721	4.686	5.721	2.889	5.038	2.889	