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Imperial College London

Department of Mathematics

MSc Research Project

Higher Order Approximation of Nonlinear SPDEs with Additive Space-Time White Noise

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Submitted in partial fulfillment of the requirements for the MSc Pure Mathematics at Imperial College London

September 5, 2025

Abstract

This project serves as a literature review on convergence rates of higher order numerical approximations for nonlinear reaction–diffusion SPDEs with additive space–time white noise on the one-dimensional torus, and mainly follows the work in [1]. The main focus is to prove functional bounds of numerical approximations' convergence rate for two classes of nonlinearities: (i) globally bounded f with bounded derivatives, and (ii) polynomially growing f under a one-sided Lipschitz condition. In both cases, convergence rates of $M^{-1+\epsilon}$ in time and $N^{-1/2+\epsilon}$ in space are proved. The paper also focuses on two theoretical frameworks that are crucial in the derivations of said results, namely the Stochastic Sewing Lemma and the implementation of Littlewood-Paley Theory/Besov spaces in analysis.

Plagiarism statement

The work contained in this thesis is my own work unless otherwise stated.

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Date: September 5, 2025

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Chapter 1

Introduction

This thesis serves as a mathematically rigorous literature review on higher order numerical approximations of nonlinear SPDEs driven by additive space-time white noise on the one-dimensional torus $\mathbb{T}$, and largely follows the results and proofs obtained in [1]. In particular, in this paper, the numerical discretisations of the stochastic reaction diffusion equation is considered, which is given by

$$\partial_t u = \Delta u + f(u) + \xi, \tag{1.1}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{T}$, ξ[1] is a space-time white noise, $f : \mathbb{R} \to \mathbb{R}$ is a non linear function and $u(0, x) = u_0$ is the initial condition. The existence and uniqueness of the solution (1.1) are guaranteed under certain regularity conditions on f (see Chapter 2).

The first attempt at full discretizations was presented in [2], where the authors have used a finite difference method in space and Euler (both explicit and implicit) scheme in time, to achieve the L^p rate of convergence of order $M^{-1/4} + N^{-1/2}$ [2], via sampling rectangular increments of W. In further work, such as in [3] a progress was made by considering sampling from the stochastic convolution in the mild solution directly, generated by Δ, which was possible since it follows a Gaussian distribution with a known covariance matrix (so sampling is straightforward). However, in [3] assumptions that were considered were too restrictive(in particular, f was assumed to be Frechet differentiable), so the result could not be generalised for truly nonlinear functions f. In some other works, distributional convergence rates or strong coloring conditions on the noise were considered. Thus, the problem, in its full generality, remained open.

The goal of this paper is to thoroughly follow the work in [1], where a general result[3], namely the strong rate of convergence in functional space, is shown to be $M^{-1+\epsilon}$ for the temporal variable, while the space convergence remains at $N^{-1/2+\epsilon}$ as in other literature. The analysis is carried out in two scenarios with different assumptions

[1] $\xi(t, x) = \partial_t \partial_x W(t, x)$

[2] Here grid with meshsize M^{-1} in time and N^{-1} in space is used

[3] generality is in terms of assumption on f and the employed scheme is proved

on f. In the first case, a globally bounded f is considered with bounded derivatives up to order two. More formally,

Assumption 1.0.1. *(a) There exists constant K such that $\forall i = 0, 1, 2$ and $\forall x \in \mathbb{R}$ one has*

$$|\partial^i f(x)| \leq K \quad \textit{(bounded derrivatives)}$$

(b) u_0 is $\mathcal{F}_0$ measurable with values in $\mathscr{C}^{1/2}(\mathbb{T})$ and for any $p \in [1, \infty)$ $\exists \mathcal{M}(p)$ such that $\mathbb{E}\|u_0\|^p_{\mathscr{C}^{1/2}}$

In this case, the spectral Galerkin scheme in space, and the accelerated exponential explicit Euler scheme are used. A priori bounds are easily constructed for such f, and the Girsanov theorem is used to equate the drift of the mild flow with that of the Orenstein-Uhlenbeck process (OU) under a new measure[4], which allows for transferring the obtained Hölder-Besov regularity bounds of OU to the discretised solution.

In the second scenario, a maximal polynomial growth with one–sided Lipschitz condition is assumed for f. In particular,

Assumption 1.0.2. *(a) There exists a constant $K \geq 1$ and $m \geq 0$ such that for any $i = 0, 1, 2, 3$ and all $x \in \mathbb{R}$ one has*

$$|\partial^i f(x)| \leq K(1 + |x|^{(2m+1-i)\vee 0}), \tag{1.2}$$

and furthermore for all $x \in \mathbb{R}$ one has

$$\partial f(x) \leq K. \tag{1.3}$$

(b) The initial condition u_0 is an $\mathcal{F}_0$-measurable random variable with values in $\mathscr{C}^2(\mathbb{T})$ and for any $p \in [1, \infty)$ there exists a constant $\mathscr{M}(p)$ such that $\mathbb{E}\|u_0\|^p_{\mathscr{C}^2} \leq \mathscr{M}(p)$.

A prime example of such an SDE is the famous Allen-Cahn equation. In this case, while the proof strategy remains the same, but the construction of a priori bounds on discretisations is more complex, and requires the introduction of auxiliary processes. Additionally that, for this class of equations, simple schemes like explicit Euler blow up due to non-linearity, and as such are not suitable. Thus, a splitting scheme for the time variable is used, for which the rate of $M^{-1+\epsilon}$ is proved uniform in time and L^2 in space. Lastly, application of the Girsanov theorem is not possible, since f doesn't satisfy the Novikov condition, so a more involved argument is used.

To quantify the regularity of functions/distributions appearing in the SPDE, as a main analytical tool, the paper relies on the framework of Littlewood-Paley decomposition and Besov spaces. In particular, one uses the

[4]OU pricess is the solution to (1.1) with $f = 0, u_0 = 0$

Fourier transform of a function and splits its Fourier coefficients into groups according to their size. This allows one to establish estimates on each such group of coefficients individually and only then recombine them, which greatly simplifies the analysis. Concretely, one starts by introducing dyadic partition of unity (see (2.0.5)), which is a set of functions ρ_j defined on a disjoint sets of annuli $\{k \in \mathbb{Z}^d : 2^{j-1} < |k| < 2^{j+1}\}$ in the space of Fourier frequencies[5], and as such when multiplied by other functions can act as localising or "bump" functions that limit the said function to a given Fourier frequency range. Littlewood-Paley decomposition arises naturally in this context and is simply a decomposition of a function into dyadic frequency blocks via

$$u = \sum_{j \geq -1} \Delta_j u \quad \text{with} \quad \Delta_j u = \mathscr{F}^{-1}(k \to \rho_j \mathscr{F}(u))(k), \quad \forall u \in \mathcal{S}'(\mathbb{T})$$

To actually measure the regularity at a given frequency range, one has to introduce Besov spaces, denoted $B^{\theta}_{p,q}$ (See Chapter 2). Formally given as a space of distributions, for which the norm $\|u\|_{B^{\theta}_{p,q}} := \|(2^{j\theta}\|\Delta_j u\|_{L^p})_{j\geq -1}\|_{\ell^q}$ is finite. Intuitively, the norm 1) decomposes u into dyadic pieces $\Delta_j u$, then 2) measures each piece in the spatial L^p norm, and finally 3) combines the localised measurements via ℓ^q, after weighting by $2^{j\theta}$. The parameter θ dictates how strongly high frequencies must decay[6]. In this analysis, proving regularity in Besov norm is particularly useful, due its embedding properties that allow to transfer the bounds obtained in the distributional sense to the functional equivalent (for instance, Besov $B^{\theta}_{\infty,\infty}$ coincides with the Hölder $\mathscr{C}^{\theta}$, which naturally embeds into L^{∞} for $\theta > 0$).

Using this analytic framework, two actual main strategies allow the construction of better bounds compared to previous literature. The first such idea is to note the fact that error bounds are highly dependent on the topology in which they are measured, i.e., the choice of norm. Therefore, since regularity properties are linked to improved rates of convergence, one can consider weaker topologies for error measurement, in order to obtain better error bounds, leading to a faster convergence rate. For instance, it is known that the Orenstein - Uhlenbeck process is 1/4-Hölder continuous process in time, however, the OU process enjoys higher regularity properties in a weaker norm $\mathscr{C}^{-1/2}(\mathbb{T})$[7], providing 1/2 temporal error rate. The key idea in [1] is to use a similar idea to obtain first bounds in distributional norms. However, since it's desirable to obtain functional bounds in the case of 1 1-dimensional torus, one can apply appropriate embedding results for the Besov space, as discussed earlier.

The second main idea is to construct more efficient bounds on the following integral

$$E_M := \Big| \int_0^T P_{T-s}\big(f(O_s) - f(O_{k_M(s)})\big)\, ds \Big|,$$

where P is the heat kernel, M is the number of timesteps in an equidistant partition of the time horizon, and

[5] Here "frequency" refers to the magnitude $|k|$ of the Fourier mode $e^{ik\cdot x}$ the Littlewood–Paley block Δ_j retains precisely those coefficients $\hat{u}(k)$ with $|k|$ lying in the dyadic annuli $(2^{j-1}, 2^{j+1})$

[6] Here high frequency refers to positive j in the dyadic partition, and low frequencies correpsond to $j = -1$

[7] Negative regularity Besov-Hölder space (See (2.0.7))

$k_M(s)$ is the last gridpoint before s. This integral controls the rate of temporal convergence of the discretisation, however, the estimation of the integral is non-trivial. Under the Lipschitz bound on f, using the triangle inequality is sufficient to obtain the bound $M^{-1/4+\epsilon}$, but for the non-linear case (which is not Lipschitz w.r.t. $\mathscr{C}^{-1/2}(\mathbb{T})$) the bound would be $M^{-1/2+\epsilon}$. The point is to notice that oscillatory, mean-reverting processes, like OU, enjoy a lot of cancellations, which are not captured when the absolute value is brought inside the integral. So the alternative route to approximate the integral is Stochastic Sewing lemma, first introduced in [4]. Instead of dealing with E_M directly, the SSL requires two bounds[8] , the two–parameter family of short increments $A_{s,t}$ that represent the contribution of the integrand on $[s,t]$. If the bounds are satisfied, the lemma provides an adapted process $\mathcal{A}_t$ such that $\mathcal{A}_t - \mathcal{A}_s$ approximates $A_{s,t}$ to order $(t-s)^\beta$ and coincides with the target integral E_M in the limit as the partition gets smaller. In the context of this paper, one constructs

$$A_{s,t} \;=\; \int_s^t \mathbb{E}_s\big[\Delta_j P^N_{R-r}\big(f(O_s) - f(O_{k_M(s)})\big]\,(x)\,dr,$$

Sewing then returns the desired process $\mathcal{A}_t = \int_0^t \Delta_j P^N_{R-r}\big(f(O_s) - f(O_{k_M(s)})\,dr$[9] with explicit L^p bounds that sum over j to the required Besov/Hölder bound and, importantly, the time–rate $M^{-1+\varepsilon}$.

Conceptually, the SSL uses oscillatory cancellations on smaller intervals; however, patching together the increments is not trivial as they are are non-additive, i.e. $A_{s,t} - (A_{s,u} + A_{u,t}) \neq 0$. To handle that, it is useful to condition on s on each interval $(s,t]$ as above, since when conditioned on s any stochastic integral over $(s,t]$ has conditional mean zero, i.e., is a martingale increment. This gives the edge over classical Grönwall inequality (see Chapter 2) in the mild formulation (which fails under superlinear assumption on f). In short, sewing lets us prove the discretization rate by verifying small increment inequalities, to get a global bound

In short, neither of these instruments is individually sufficient to achieve the desired bound of $1-\epsilon$; however, together, by bounding $f(O_s) - f(O_{k_M(s)})$ in distributional topology, and approximating via SSL, one can achieve the temporal rate of $M^{-1+\epsilon}$.

[8] See Chaper 2

[9] This statement is equivalent to E_M, see Chapter 3 and 4

Chapter 2

Background

2.0.1 PDE theory and Set-up

Consider equation (1.1). Before stating the main result, one needs to establish the background for the problem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space. Define a one-dimensional Torus to be $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $T > 0$ be the finite time horizon. The white noise ξ is defined as a map $\xi : \mathscr{B}([0,T] \times \mathbb{T}) \to L^2(\Omega)$, such that for all $A_1, \ldots, A_k \in \mathscr{B}([0,T] \times \mathbb{T})$, one has $(\xi(A_1), \ldots, \xi(A_k)) \sim \mathcal{N}(0, \mathbb{E}[\xi(A_i)\xi(A_j)] = \lambda(A_i \cap A_j))$, where λ is the Lebesgue measure. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is complete and for any $t \in [0,T]$, $A \in \mathscr{B}([0,t] \times \mathbb{T})$, $B \in \mathscr{B}([t,T] \times \mathbb{T})$, $\xi(A)$ is $\mathcal{F}_t$-measurable and $\xi(B)$ is independent of $\mathcal{F}_t$.Then, the stochastic integral given by $\int_0^T \int_{\mathbb{T}} g(s,y)\xi(ds,dy)$ can be defined for all $\mathscr{P} \times \mathscr{B}(\mathbb{T})$-measurable integrands[1] $g : \Omega \times [0,T] \times \mathbb{T} \to \mathbb{R}$ with $g \in L^2(\Omega \times [0,T] \times \mathbb{T})$.

Recall that the set $e_k(x) := e^{-2\pi ikx}$, $k \in \mathbb{Z}$ forms an orthonormal basis in $L^2(\mathbb{T}, \mathbb{C})$. For $f \in \mathcal{S}'(\mathbb{T})$[2] set

$$\widehat{f}(k) := \int_{\mathbb{T}} f(x)\, e^{-2\pi ikx}\, dx, \qquad f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\, e^{2\pi ikx}$$

to be the Fourier transform of f. It's useful to note for further analysis that the Parseval identity holds, i.e. $\|f\|^2_{L^2(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2$, and hence for the heat semigroup one has

$$\widehat{P_t f}(k) = e^{-4\pi^2 k^2 t}\, \widehat{f}(k), \qquad p_t(x) = \sum_{k\in\mathbb{Z}} e^{-4\pi^2 k^2 t}\, e^{2\pi ikx}, \qquad P_t f = p_t * f.$$

In particular, $\|p_t\|^2_{L^2(\mathbb{T})} = \sum_{k\in\mathbb{Z}} e^{-8\pi^2 k^2 t}$. One is now in the position to discuss the solutions of (1.1). To do so, some basic definitions and results need to be established. Consider a general Banach space X and the general form of the stochastic Cauchy problem

$$\partial_t u(t,x) = Au(t,x) + f(u) + \xi, \quad u(0,x) = u_0 \tag{2.1}$$

[1] The predictable σ-algebra on $\Omega \times [0,T]$ will be denoted by $\mathscr{P}$

[2] Space of tempered distributions

where $A : \mathcal{D}(A) \to X$, and $\mathcal{D}(A)$ is the domain of the operator A. Consider the following definitions due to [5]

Definition 2.0.1. *(Semigroup) A family of bounded linear operators $\{S(t)\}_{t\geq 0}$ mapping X into X is called a semigroup if the following conditions hold:*

1. $S(0)u = u$
2. $S(t+s)u = S(t)S(s)u = S(u)S(t)u$ *for all* $t, s \geq 0$ *(semigroup property);*
3. *the map* $t \mapsto S(t)u$ *is continuous from* $[0, \infty)$ *into* X.

Definition 2.0.2. *(Infitesimal generator) Let*

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in \mathcal{D}(A) := \left\{ u \in X \,\middle|\, \lim_{t\to 0^+} \frac{S(t)u - u}{t} \text{ exists in } X \right\}$$

then A is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

To analyse the solutions, one naturally requires that (i) A generates a strongly continuous semigroup $S(\cdot)$ in X, (ii) f is a predictable process with Bochner integrable[3] trajectories on finite interval $[0, T]$, and u_0 is $\mathcal{F}_0$ measurable[4]. The notion of the semigroup helps to formulate the existence and uniqueness of equation (1.1), via the following definition

Definition 2.0.3. *(Stochastic Convolution) For a semigroup $(S_t)_{t\geq 0}$ stochastic process given by*

$$O_t := \int_0^t S(t-s)\, dW(s)$$

is called the Stochastic Convolution

The stochastic convolution is in fact a Gaussian process with a known mean and covariance matrix [6, Theorem 5.2]. Then, using this definition, as stated in [6, Theorem 5.4] , the **weak (mild) solution to the equation (1.1) exists, is unique and is of the form**

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(u(s))\, ds + \int_0^t S(t-s)\, dW(s), \tag{2.2}$$

For $A = \Delta$, it has been shown in [7] that the semigroup that generates the mild solution is given by

$$S(t)[f(x)] = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{T}} \exp\left(-\frac{|x-y|^2}{4t} \right) f(y)\, dy,$$

[3]A function $f : X \to B$ (B is Banach space) is **Bochner integrable** if there exists a sequence $\{f_n\}$ of simple maps such that $f_n \to f$ a.e. (hence $x \mapsto \|f_n - f\|_B$ is measurable) and $\int \|f_n - f\|_{\mathbb{B}}\, d\mu \to 0$ .Then **Bochner integral** of f is $\int_E f\, d\mu = \lim_n \int_E f_n\, d\mu$, for any measurable set $E \subset X$.

[4]See [5] for details

called the **Heat Semigroup**, denoted $P_t := S(t)$. The following definition is useful when working with the heat semigroup

Definition 2.0.4. *(Heat Kernel)*

$$p_t = \frac{1}{\sqrt{4\pi t}} \sum_{m \in \mathbb{Z}} \exp\left(-\frac{|x-m|^2}{4t}\right)$$

Then one can write $P_t f = p_t * f$

Then, the mild solution can be written as:

$$\begin{aligned} u(t) &= P_t u_0 + \int_0^t P_{t-s} f(u_s)\, ds + \int_0^t P_{t-s}\, dW(s) \\ &= P_t u_0 + \int_0^t P_{t-s} f(u_s)\, ds + \int_0^t \int_{\mathbb{T}} p_{t-s}(x-y)\, \xi(s,y)\, dy\, ds. \end{aligned} \tag{2.3}$$

The stochastic convolution enjoys certain regularity properties, which, in combination with Littlewood-Paley decomposition (see next section), will further help with establishing the regularity of the weak solution. To state and prove these properties, one requires first to establish the following analytic framework.

2.0.2 Littlewood - Paley Theory

As mentioned previously, to understand the weak solution of the SPDE and analyze the convergence rate of the numerical approximation, one needs to carefully establish the regularity properties of the solution u.

On the torus $\mathbb{T}$, the Fourier series of any periodic distribution u converges to u in the sense of distributions (i.e. in $\mathcal{S}'(\mathbb{T})$). It is well known that regularity properties of u in $\mathcal{S}'(\mathbb{T})$ are influenced by the rate at which $\hat{u}_k \to 0$ as $k \to \infty$, however, direct estimation is computationally difficult, so one can look into the decomposition of the function in the Fourier space, and conduct analysis in "in blocks", via the use of dyadic partition of unity, as mentioned in the introduction.

Definition 2.0.5. *(Dyadic Partition of Unity) A collection of functions* $(\rho_i)_{i \geq -1}$ *with* $\rho_i \in C_c^\infty(\mathbb{R})$*is called a smooth dyadic partition of unity if*

- ρ_{-1} *and* ρ_0 *are even, greater than zero and have* $supp(\rho_{-1}) \in B_{\frac{1}{2}}(0)$[a] *and* $supp(\rho_0) \in B_1(0)/B_{1/4}(0)$
- $\rho_j(x) = \rho_0(2^{-j}x) \quad \forall x \in \mathbb{R}, j \geq 0$
- $\sum_{j=-1}^{\infty} \rho_j(x) = 1, \quad \forall x \in \mathbb{R}$
- $supp(\rho_j) \cap supp(\rho_i) = \emptyset, \quad \forall i,j$ ***s.t.*** $|i-j| > 1$

[a]In (X,d), for any $x \in X$, an open ball is $B_r(x) := \{y \in X : d(y,x) < r\}$. Hence, for $r_1 > r_2$, $B_{r_1}(x)/B_{r_2}(x)$ defines an annulus

Intuitively, the functions are defined on disjoint sets of annuli, in other words, $\rho(2^{-j}x)$ being supported on $\{x \in \mathbb{Z}^d : 2^{j-1} < |x| < 2^{j+1}\}$ and hence represent localising "bump" functions that act to limit Fourier frequencies of a distribution to a certain range[5]. Consider the following definition.

Definition 2.0.6. *(Littlewood - Paley Projectors (Blocks)) The operator $\Delta_j : \mathcal{S}'(\mathbb{T}) \to C^\infty(\mathbb{T})$ is defined by:*

$$\Delta_j u = \mathscr{F}^{-1}(k \to \rho_j \mathscr{F}(u))(k), \quad \forall u \in \mathcal{S}'(\mathbb{T})$$

Hence, any distribution $u \in \mathcal{S}'(\mathbb{T})$, can be written as a decomposition into dyadic frequency blocks

$$u = \sum_{j \geq -1} \Delta_j u,$$

which is very convenient for analysis, since bounds could be made at individual frequencies first, and recombined later using Besov Spaces. In particular, to formally analyse the regularity of the distributions under this framework, one introduces Besov spaces on the torus for $p, q \in [1, \infty]$, $\theta \in \mathbb{R}$ as

Definition 2.0.7. *(Besov Space) A Besov space $B^\theta_{p,q}$ is defined as*

$$B^\theta_{p,q} := \{u \in \mathcal{S}'(\mathbb{T}) \mid \|u\|_{B^\theta_{p,q}} := \|(2^{j\theta}\|\Delta_j u\|_{L^p})_{j \geq 1}\|_{\ell^q}\}$$

Besov spaces, in contrast to standard function spaces like Sobolev or L^p, contain a wider range of functions, like certain nontrivial homogeneous functions, that could be non integrable or singular[6]. As mentioned before in the introduction, intuitively, the Besov norm decomposes u into dyadic pieces $\Delta_j u$, then measures each piece in the spatial L^p norm, and finally combines the localised measurements via ℓ^q, after weighting by $2^{j\theta}$. The parameter θ controls how strongly high-frequencies must decay, which in this context refers to positive j in the dyadic partition(low frequencies correspond to $j = -1$). In this analysis, proving regularity in Besov norm is particularly useful, due it's embedding properties that allow to transfer the bounds obtained in distributional sense to the functional equivalent (for instance, Besov $B^\theta_{\infty,\infty}$ coincides with θ-Hölder continuous functions $\mathscr{C}^\theta$, which naturally embeds into L^∞ for $\theta > 0$). They are also quite convenient to work with, as they enjoy many other properties.

[5]Again, "frequency" refers to the magnitude $|k|$ of the Fourier mode $e^{ik\cdot x}$ the Littlewood–Paley block Δ_j retains precisely those coefficients $\hat{u}(k)$ with $|k|$ lying in the dyadic annuli $(2^{j-1}, 2^{j+1})$

[6]An example of such function is $|x|^{-\sigma} \in \dot{B}^{\frac{d}{p}-\sigma}_{p,\infty}$, see [8] for details

Proposition 2.0.0.1. *(Embedding) Let $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$ and $\theta \in \mathbb{R}$, then $B^{\theta}_{p_1,q_1}$ is continuously embedded in $B^{\theta-d(1/p_1-1/p_2)}_{p_2,q_2}$*

Proof. Following the proof of [8, Proposition 2.71], Consider the Bernstein type lemma given as [8, Lemma 2.1]. It yields that

$$\|S_0 u\|_{L_{p_2}} \leq C\|S_0 u\|_{L_{p_1}} \text{ and } \|\Delta_j u\|_{L_{p_2}} \leq C^{2jd}(\frac{1}{p_1} - \frac{1}{p_2})\|\Delta_j u\|_{L_{p_1}} \quad \text{for all} \quad j \in \mathbb{N}.$$

Since $\ell_{r_1}(\mathbb{Z})$ is continuously embedded in $\ell_{r_2}(\mathbb{Z})$, one has the required embedding. ☐

Proposition 2.0.0.2. *(Weak Derivatives) If $u \in B^{\theta}_{p,q}$ and $n \in \mathbb{N}$, $\|\partial^n u\|_{B^{\theta-n}_{p,q}} \lesssim \|u\|_{B^{\theta}_{p,q}}$.*

Proof. Recall first, that $\|u\|_{B^{\theta}_{p,q}} = \|(2^{j\theta}\|\Delta_j u\|_{L^p})_{j\geq -1}\|_{\ell^q}$. Then, expanding the LHS, one has

$$\begin{aligned}
\|\partial^n u\|_{B^{\theta-n}_{p,q}} &= \Big\|\big(2^{j(\theta-n)}\|\Delta_j(\partial^n u)\|_{L^p}\big)_j\Big\|_{\ell^q} \\
&= \Big\|\big(2^{j(\theta-n)}\|\mathcal{F}^{-1}\big[(i\lambda)^n\rho_j(\lambda)\hat{u}(\lambda)\big]\|_{L^p}\big)_j\Big\|_{\ell^q} \\
&= \Big\|\big(2^{j(\theta-n)}\|\partial^n(\Delta_j(u))\|_{L^p}\big)_j\Big\|_{\ell^q} \\
&\leq C\Big\|\big(2^{j\theta}\|\Delta_j u\|_{L^p}\big)_j\Big\|_{\ell^q} = C\,\|u\|_{B^{\theta}_{p,q}},
\end{aligned}$$

where in the last line one uses Bernstein Lemma [8, Lemma 2.1/Lemma 2.2], which gives $\|\Delta_j(\partial^\alpha u)\|_{L^p} \leq C\,2^{j|\alpha|}\|\Delta_j u\|_{L^p}$ ☐

Proposition 2.0.0.3. *(Products) If θ, β are such that $\theta + \beta > 0$, then for any two distributions $u \in \mathscr{C}^{\theta}$ and $v \in \mathscr{C}^{\beta}$ their product uv is well-defined and there exists a constant $C = C(\theta, \beta)$, such that the bound*

$$\|uv\|_{\mathcal{C}^{\min(\theta,\beta)}} \leq C\|u\|_{\mathcal{C}^{\theta}}\|v\|_{\mathcal{C}^{\beta}}$$

holds

Proof. Refer to [1, equation 3.3] ☐

Proposition 2.0.0.4. *(Heat kernel bounds) If $\theta \geq 0$, $\tilde{\theta} \in [0, 2]$, then there exist constants $C_1(\theta)$ and $C_2(\tilde{\theta})$ such that for any $\beta \in \mathbb{R}$, $u \in \mathcal{C}^{\beta}$, and $t \in (0, 1]$, the bounds*

$$\|P_t u\|_{\mathscr{C}^{\beta+\theta}} \leq C_1 t^{-\theta/2}\|u\|_{\mathscr{C}^{\beta}}, \quad \|(\mathrm{Id} - P_t)u\|_{\mathscr{C}^{\beta-\tilde{\theta}}} \leq C_2 t^{\tilde{\theta}/2}\|u\|_{\mathscr{C}^{\beta}} \tag{2.4}$$

hold

Proof. To prove the two statements, it is first crucial to prove a more general result given in [9, Lemma A.5]. Using it, for $P_t = \varphi(t^{1/(2\sigma)})$ with $\varphi(z) = e^{-|z|^{2\sigma}}$, since heat semigroup is inifinitely differentiable, and φ and its derivatives decay faster than any rational function at ∞, it is sufficient to show that $\hat{\phi} \in L_1$.

For $\sigma \leqslant 1$, φ constitutes a symmetric 2σ-stable density, and therefore in L^1. For $\sigma > 1$, all derivatives up to order $d+1$ are in L^1, which yields $(1+|x|^{d+1})\varphi \in L^\infty$, implying that $\varphi \in L^1$. Applying the lemma, the first bound is complete.

To show the estimate for $(P_t - Id)u$, view $P_t - Id$ as convolution operator; let $K(x) = \mathcal{F}^{-1}\varphi$, then

$$\begin{aligned}|(P_t - Id)u(x)| &= \left| t^{-d/(2\sigma)} \int K\left(\frac{x-y}{t^{1/(2\sigma)}}\right)(u(y)-u(x))y \right| \\ &\lesssim t^{-d/(2\sigma)} \int K\left(\frac{x-y}{t^{1/(2\sigma)}}\right)|y-x|^\beta \|u\|_{\mathscr{C}^\beta} y \lesssim t^{\beta/(2\sigma)}\|u\|_{\mathscr{C}^\beta},\end{aligned}$$

hence the bound holds ☐

To quote certain results, the introduction of function spaces on $\mathbb{R}$ is needed. In particular, consider a canonical space of bounded measurable functions C_b^k for $k = 0, 1, \ldots$, whose distributional derivative up to order k is essentially bounded[7], equipped with the canonical Sobolev norm[8]. Using this framework, one can also define $(P_t^{\mathbb{R}})_{t\geq 0}$ as the heat semigroup on $\mathbb{R}$, that is

$$P_t^{\mathbb{R}} f = p_t^{\mathbb{R}} * f, \qquad p_t^{\mathbb{R}}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}. \tag{2.5}$$

with the direct estimate given by $0 \leq s \leq t \leq 1$ one has

$$\|(P_t^{\mathbb{R}} - P_s^{\mathbb{R}})u\|_{C_b^0} \leq |t-s| \|u\|_{C_b^2}. \tag{2.6}$$

Since the mild solution is valued in a spatial function space on the torus (e.g., a Besov or Hölder space), to quantify its time regularity, let the path space be given by the following definition. For a given Banach space X (like Besov Space), $C_T X$ denotes the space of continuous functions in time with values in X equipped with the supremum norm. For $\gamma \in (0,1]$, define

$$C_T^\gamma X := \big\{u \in C_T X : \|u\|_{C_T^\gamma X} := \sup_{t\in[0,T]} \|u_t\|_X + \sup_{0\leq s<t\leq T} \frac{\|u_t - u_s\|_X}{(t-s)^\gamma} < \infty\big\}.$$

Note, that in the analysis, the notation $a \lesssim b$ is used if there exists a constant $C > 0$, such that $a \leq Cb$[9].

2.0.3 Useful Properties and Results

One is now in a position to establish useful properties of the stochastic convolution and additional results that would help prove the main result. Consider first the regularity properties of stochastic convolution.

[7] Essentially Sobolev space $W^{k,\infty}$

[8] Elements of C_b^0 are not assumed to be continuous.

[9] To stress dependence of C on a parameter τ, $a \lesssim_\tau b$ is used.

Proposition 2.0.0.5. *Let $(O_t)_{t\geq 0}$ be as before, $p\in[1,\infty)$, $\lambda\in(0,1)$ and $\epsilon\in(0,1/2)$. Then $\exists$ a constant $C=C(p,T,\lambda,\epsilon)$, such that for all $0\leq s\leq t\leq T$ one has*

$$\mathbb{E}\|O_t-O_s\|^p_{\mathscr{C}^{1/2-\lambda-\epsilon}}\leq C(t-s)^{p\lambda/2}. \tag{2.7}$$

and

$$\mathbb{E}\|O\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}}\leq C. \tag{2.8}$$

If $\theta\in[0,1]$, then there exists a constant $C=C(p,T,\theta,\epsilon)$ such that for all $s,t\in[0,T]$ one has

$$\left(\mathbb{E}\sup_{r\in[0,T]}\|(P_{t+s}-P_t)O_r\|^p_{\mathscr{C}^{-1/2+\epsilon}}\right)^{1/p}\leq Ct^{-\theta/2-\epsilon}s^{\frac{1+\theta}{2}}. \tag{2.9}$$

Proof. First, note that to prove (2.7) it is sufficient to show that

$$(\mathbb{E}|\Delta_jO_t(x)-\Delta_jO_s(x)|^p)^{1/p}\lesssim 2^{j(-1/2+\lambda)}|t-s|^{\lambda/2}. \tag{2.10}$$

Indeed, via the definition of the Besov norm, it is easily shown that for any ϵ (in this case $1/2-\lambda$)

$$\left(\mathbb{E}\|O_t-O_s\|^p_{\mathscr{C}^{\frac{1}{2}-\lambda-\epsilon}}\right)^{1/p}\lesssim\sup_{j\geq -1}2^{(\frac{1}{2}-\lambda-\epsilon)j}\sup_{x\in\mathbb{T}}\left(\mathbb{E}|\Delta_j(O_t-O_s)(x)|^p\right)^{1/p}$$

To prove the statement, consider, first via Gaussian Hypercontractivity[10]

$$\begin{aligned}(\mathbb{E}|\Delta_jO_t(x)-\Delta_jO_s(x)|^p)^{2/p}&\lesssim\mathbb{E}|\Delta_jO_t(x)-\Delta_jO_s(x)|^2\\&\lesssim\mathbb{E}\left|\int_s^t\int_{\mathbb{T}}\Delta_jp_{t-r}(x-y)\xi(dr,dy)\right|^2+\mathbb{E}\left|\int_0^s\int_{\mathbb{T}}\Delta_j(p_{t-r}-p_{s-r})(x-y)\xi(dr,dy)\right|^2\\&=\int_s^t\|\Delta_jp_{t-r}(x-\cdot)\|^2_{L^2(\mathbb{T})}\,dr+\int_0^s\|\Delta_j(p_{t-r}-p_{s-r})(x-\cdot)\|^2_{L^2(\mathbb{T})}\,dr\\&=\int_s^t\sum_{k\in\mathbb{Z}}\rho_j(k)^2e^{-8\pi^2k^2(t-r)}dr+\int_0^s\sum_{k\in\mathbb{Z}}\rho_j(k)^2e^{-8\pi^2k^2(s-r)}(1-e^{-4\pi^2k^2(t-s)})^2dr.\end{aligned}$$

where in the last step one uses Parseval's identity, since $p_{t-s}(x)$ has Fourier coefficients $e^{-8\pi^2k^2(t-s)}$. Finally, integrating and using that $e^{-x}\leq\min(x^{-1},1)$ and $1-e^{-x}\leq\min(x,x^{1/2},1)$ for $x\geq 0$, we obtain

$$\begin{aligned}(\mathbb{E}|\Delta_jO_t(x)-\Delta_jO_s(x)|^p)^{2/p}&\lesssim\sum_{k\in\mathbb{Z}}\rho_j(k)^2\min(|t-s|,|k|^{-2})+\sum_{k\in\mathbb{Z}}\rho_j(k)^2\min(|k|^{-2},|k|^{-2}|k|^2|t-s|)\\&\lesssim 2^j\min(|t-s|,2^{-2j})\lesssim 2^{j(2\lambda-1)}|t-s|^{\lambda}\end{aligned}$$

for any $\lambda\in[0,1]$. To finish the last two statements, note that since $\|O\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}}=\sup_{t\in[0,T]}\|O_t\|_{\mathscr{C}^{1/2-\lambda-\epsilon}}+\sup_{0\leq s<t\leq T}\frac{\|O_t-O_s\|_{\mathscr{C}^{1/2-\lambda-\epsilon}}}{(t-s)^{\gamma}}$, the (2.8) follows directly from applying Kolmogorov continuity theorem (see Appendix A.0.1) with (2.7). To prove the (2.9), one can use the fact that $\|(P_{t+s}-P_t)O_r\|=\|(\mathrm{Id}-P_s)(P_tO_r)\|$ and apply both bounds in (2.0.0.4). □

[10] Via (A.0.2), for I_1, $\|\Delta_j(O_t-O_s)(x)\|_{L^p(\Omega)}\leq\sqrt{p-1}\,\|\Delta_j(O_t-O_s)(x)\|_{L^2(\Omega)}$

One is now in a position to establish the result that ensures the well-posedness of the stochastic reaction diffusion equation (**??**).

Proposition 2.0.0.6. *Let (1.0.1) or (1.0.2) hold. Then there exists a unique mild solution u to (1.1). Moreover, for any $\lambda \in (0,1)$, $\epsilon \in (0,1/2)$, $p \geq 1$ there exists a constant $C = C(T,p,\lambda,\epsilon,m,K)$ such that the solution satisfies the bound*

$$\mathbb{E}\|u\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}(\mathbb{T})} \leq C(1+\mathbb{E}\|u_0\|^{(2m+1)p}_{\mathscr{C}^{1/2}}). \tag{2.11}$$

Proof. Assume that f satisfies both conditions in assumption (1.0.2). Both of these conditions are implied by (1.0.1). Then, in particular, the function f satisfies Hypothesis 6.2 in [10]. Additionally, the conditions in Hypothesis 6.1 in [10] are satisfied by the $A = \Delta$ and $Q = Id$ operators, with the associated spaces $H = L^2(\mathbb{T}), E = C(\mathbb{T})$. Then, by Proposition 6.2.2, the solution to (1.1) exists, is unique, and the condition

$$\mathbb{E}\|u\|^q_{C_TC(\mathbb{T})} \lesssim (1+\mathbb{E}\|u_0\|^q_{C(\mathbb{T})}), \tag{2.12}$$

holds. Then since by assumption (1.0.2), one has that $|f(u)| \leq K(1+|u|^{2m+1})$, taking supremum in space and time, and using (2.12), one obtains that

$$\mathbb{E}\|f(u)\|^q_{C_TC(\mathbb{T})} \lesssim (1+\mathbb{E}\|u_0\|^{(2m+1)q}_{C(\mathbb{T})}) \tag{2.13}$$

It remains to prove the bound (2.11), which has been previously shown for O in (2.8). Note that, by applying (2.0.0.4), one has that

$$\begin{aligned}\|P_tu_0 - P_su_0\|_{\mathscr{C}^{1/2-\lambda-\epsilon}} &= \|(\mathrm{Id} - P_{t-s})(P_su_0)\|_{\mathscr{C}^{1/2-\lambda-\epsilon}} \\ &\lesssim |t-s|^{(\lambda+\epsilon)/2}\|u_0\|_{\mathscr{C}^{1/2}},\end{aligned}$$

Thus, if heuristically $u = v + O + Pu_o$, then with (2.8) and the bound above, it suffices to prove (2.11) for $v = u - O - P.u_0$. Then, the increments of v are

$$\begin{aligned}v_t - v_s &= \int_0^t P_{t-r}f(u_r)dr - \int_0^s P_{s-r}f(u_r)dr \\ &= \int_s^t P_{t-r}f(u_r)dr - \int_0^s P_{s-r}(\mathrm{Id} - P_{t-s})f(u_r)dr.\end{aligned}$$

Lastly, again, using the estimates (2.4), one has that for $\lambda \in (0,1)$, $\epsilon \in (0,1/2)$

$$\begin{aligned}\|v_t - v_s\|_{\mathscr{C}^{1/2-\lambda-\epsilon}} &\lesssim \int_s^t \|P_{t-r}f(u_r)\|_{\mathscr{C}^{1/2-\lambda-\epsilon}}dr + \int_0^s \|P_{s-r}(\mathrm{Id} - P_{t-s})f(u_r)\|_{\mathscr{C}^{1/2-\lambda-\epsilon}}dr \\ &\lesssim \Big(\int_s^t (t-r)^{(\lambda/2+\epsilon/2-1/4)\wedge 0}\,dr + \int_0^s (t-s)^{\lambda/2+\epsilon/2}(s-r)^{-1/4}\,dr\Big) \\ &\quad \times \|f(u)\|_{C_TC(\mathbb{T})} \\ &\lesssim (t-s)^{\lambda/2+\epsilon/2}\|f(u)\|_{C_TC(\mathbb{T})}.\end{aligned}$$

Hence, using the argument similar to (2.0.0.5), via the Kolmogorov inequality, one has

$$\mathbb{E}\|v\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} \lesssim \mathbb{E}\|f(u)\|^p_{C_T C(\mathbb{T})} \lesssim 1 + \mathbb{E}\|u_0\|^{(2m+1)p}_{\mathscr{C}^{1/2}},$$

where one uses (2.13) in the last inequality. Finally, via the Minkowski inequality, taking expectation and applying all three bounds

$$\begin{aligned}\mathbb{E}\|u\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} &\lesssim \mathbb{E}\|O\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} + \mathbb{E}\|P.u_0\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} + \mathbb{E}\|v\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} \\ &\lesssim C_1 + C_2\,\mathbb{E}\|u_0\|^p_{\mathscr{C}^{1/2}} + 1 + \mathbb{E}\|u_0\|^{(2m+1)p}_{\mathscr{C}^{1/2}}\end{aligned}$$

Since $(2m+1)p \geq p$, we have the elementary inequality

$$\mathbb{E}\|u_0\|^p_{\mathscr{C}^{1/2}} \leq 1 + \mathbb{E}\|u_0\|^{(2m+1)p}_{\mathscr{C}^{1/2}},$$

So the middle term is controlled by the last one up to constants. Therefore

$$\mathbb{E}\|u\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} \leq C\big(1 + \mathbb{E}\|u_0\|^{(2m+1)p}_{\mathscr{C}^{1/2}}\big)$$

for some $C = C(T, p, \lambda, \epsilon, m, K)$, which is exactly (2.11). □

Another useful result, which is often used to construct different bounds for differences of two processes, which is especially useful in this analysis, is the Grönwall inequality, a version of which is provided along with the proof as

Proposition 2.0.0.7. *Let V be a Banach space, $p \geq 1$, and take three processes X, Y, Z belonging to $L^p(\Omega; C([0,T]; V))$. Assume there exists a Lipschitz continuous function F on V with Lipschitz constant L_1, a family $(S(s,t))_{0\leq s\leq t\leq T}$ of uniformly bounded linear operators on V with uniform bound L_2 and such that $(s,t) \mapsto S(s,t)v$ is measurable for any $v \in V$, and a measurable mapping $\tau : [0,T] \to [0,T]$ such that $\tau(s) \leq s$ and that the following equality holds for all $0 \leq t \leq T$:*

$$X_t - Y_t = Z_t + \int_0^t S(s,t)\big(F(X_{\tau(s)}) - F(Y_{\tau(s)})\big)\,ds. \tag{2.14}$$

Then there exists a constant $C = C(p, L_1, L_2, T)$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t - Y_t\|^p \leq C\mathbb{E}\sup_{t\in[0,T]}\|Z_t\|^p. \tag{2.15}$$

Proof. Using the triangle inequality, the Lipschitz condition, and boundedness of stochastic convolution, one has trivially that

$$\sup_{s\in[0,t]}\|X_s - Y_s\| \;\leq\; \sup_{s\in[0,t]}\|Z_s\| + \sup_{s\in[0,t]}\int_0^s \|S(r,s)\|\,\|F(X_{\tau(r)}) - F(Y_{\tau(r)})\|\,dr \tag{2.16}$$

$$\leq \sup_{s\leq t}\|Z_s\| + L_1L_2\int_0^t \sup_{r\leq s}\|X_{\tau(r)} - Y_{\tau(r)}\|\,dr \tag{2.17}$$

Recall that the general Grönwall inequality guarantees that if one has that if $L_1 L_2$ and $\|X_t - Y_t\|$ are continuous and the negative part of $\sup_{t\in[0,T]} \|Z_t\|$ is integrable on every closed and bounded subinterval of $[0,T]$ and is non-decreasing, then if one has that:

$$\|X_t - Y_t\| \leq \sup_{t\in[0,T]} \|Z_t\| + \int_0^t L_1 L_2 \|X_s - Y_s\| \, ds$$

It implies that

$$\|X_t - Y_t\| \leq \sup_{t\in[0,T]} \|Z_t\| \exp\Big(\int_0^t L_1 L_2 \, ds\Big).$$

Then, by raising to the p^{th} power and taking expectation, one has the required result. □

2.0.4 Stochastic Sewing Lemma

As mentioned previously, a key instrument in achieving higher approximation rates is the Stochastic Sewing Lemma, presented in [4]. In general, the lemma generalizes how one constructs stochastic integrals from increments, when the integral is non-trivial to estimate traditionally. To understand better, recall that in error analysis, the temporal error is determined by the time integral

$$E_M = \left| \int_0^T P^N_{T-s}(f(O_s) - O_{k_M(s)}) \, ds \right|, \tag{2.18}$$

where $k_M(s)$ is the last gridpoint before s.The estimates of this integral are highly non-trivial even for $f \in C_c^\infty$, and the regular triangle inequality bound would result in a convergence rate of $M^{-1/2+\epsilon}$. Then, using the Stochastic Sewing would allow one to improve on this approximation of the integral, which comes from the fact that when estimating oscillatory processes, they have a lot of cancellations, which are not captured when bringing the absolute value inside the integral (i.e. via triangle inequality). Consider the setting of the lemma.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, so that any modification of the process remains adapted to the same filtration.

The stochastic sewing lemma provides conditions under which a discrete sum of adapted random variables $S = \sum_{i=1}^n Z_i$ approximates a stochastic integral, under some control on the approximation error in $L^p(\Omega)$-norm[11]. That is, it allows one to estimate the term in (2.18) via approximations constructed from its increments. To do that, one defines a two-parameter family of increments

$$A_{s,t} := \mathbb{E}_s \int_s^t P^N_{t-s}(f(O_r) - f(O_{k_M(r)}) \, dr,$$

The lemma then yields a unique adapted process $\mathcal{A}_t$ approximating the integral $\int_0^t P^N_{t-s}(f(O_r) - O_{k_M(r)}) \, dr$ as a limit of such irregular increments. However, as mentioned in the introduction, although the SSL uses oscillatory cancellations on smaller intervals, patching together the increments is not trivial, since they are non-additive,

[11]Here, the p^{th} moment refers to the L^p norm $\|S\|_{L^p(\Omega)} := (\mathbb{E}[|S|^p])^{1/p}$

i.e. $A_{s,t} - (A_{s,u} + A_{u,t}) \neq 0$. To handle that, it is useful to condition on s on each interval $(s,t]$ as above, since when conditioned on s, any stochastic integral over $(s,t]$ has conditional mean zero, i.e. is a martingale increment. This gives the edge over classical Grönwall inequality (see Chapter 2) in the mild formulation (which fails under superlinear assumption on f). To state the lemma formally, one first defines the following notions

$$[S,T]^2_{\leq} := \{(s,t) \in [S,T]^2 : S \leq s \leq t \leq T\} \quad [S,T]^*_{\leq} := \{(s,t) \in [S,T]^2_{\leq} : t-s \leq T-t\}$$

where S,T are fixed non-negative numbers such that $S < T$. To quantify the integral error, one defines

$$\delta A_{sut} = A_{st} - A_{su} - A_{ut}.$$

Then, A is said to be adapted to $(\mathcal{F}_t)_{t\geq 0}$ if A_{st} is $\mathcal{F}_t$-measurable for every $(s,t) \in [S,T]^2_{\leq}$; A is $L^p(\Omega)$-integrable if $A_{st} \in L^p(\Omega)$ for every $(s,t) \in [S,T]^2_{\leq}$. The same definitions hold for a single parameter $(\phi_t)_{S\leq t\leq T}$. The following lemma provides a regularity condition on the error δA_{st}, for the existence of a modified approximation to the integral.

Lemma 2.0.0.1. *(Stochastic Sewing Lemma) Let $p \geq 2$ and $0 \leq S < T \leq 1$. Let $(A_{s,t})_{S\leq s\leq t\leq T}$: $[S,T]^*_{\leq} \to L^p(\Omega)$ be a $(\mathcal{F}_t)_{t\in[S,T]}$ adapted stochastic process in $\mathbb{R}^d$ which is $L_p(\Omega)$-integrable . Assume there exists $C_1, C_2 \geq 0$ and $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$ such that:*

$$\|A_{st}\|_{L^p(\Omega)} \leq C_1(T-t)^{-\delta_1}|t-s|^{1/2+\epsilon_1}, \tag{2.19}$$

$$\|\mathbb{E}_s[\delta A_{sut}]\|_{L^p(\Omega)} \leq C_2(T-t)^{-\delta_2}|t-s|^{1+\epsilon_2} \tag{2.20}$$

Then there exists a unique $(\mathcal{F}_t)_{t\in[S,T]}$-adapted process $\mathcal{A} : [S,T] \to L^p(\Omega)$ and constants K_1, K_2 such that for all $(s,t) \in [S,T]^*_{\leq}$

$$\mathcal{A}_S = 0 \tag{2.21}$$

$$\|\mathcal{A}_{st} - A_{st}\|_{L^p(\Omega)} \leq K_1(T-t)^{-\delta_1}|t-s|^{1/2+\epsilon_1} + K_2(T-t)^{-\delta_2}|t-s|^{1+\epsilon_2}, \tag{2.22}$$

$$\|\mathbb{E}_s[\mathcal{A}_{st} - A_{st}]\|_{L^p(\Omega)} \leq K_2(T-t)^{-\delta_2}|t-s|^{1+\epsilon_2}. \tag{2.23}$$

Further, and equally importantly, for every fixed $t \in [S,T]$ and any partition $\pi = S = t_0 < t_1 < ... < t_N = t$of $[S,t]$, define the Riemann sum

$$A^\pi_t := \sum_{i=0}^{N-1} A_{t_i,t_{i+1}}$$

Then $(A^\pi_t)_\pi$ converges to $\mathcal{A}_t$ in $L^p(\Omega)$ as the mesh size $|\pi| := \max_i |t_{i+1} - t_i|$ goes to 0[12]. As such, the lemma provides a tool for defining and controlling integrals involving nonlinearities evaluated at discrete (numerical) approximations. Specifically, Lemma 2.0.0.1 will be used to control the approximation errors arising from replacing the integral term $\int_s^t P^N_{t-r} f(O^N_r)\,dr$ with its discretised counterpart $\int_s^t P^N_{t-r} f(O^N_{k_M(r)})\,dr$, in later sections, explicitly quantifying the convergence rate of the numerical approximation in terms of the chosen discretisation.

[12]In the context of the paper, the meshsize M in temporal variable determines the partition π

2.0.5 Numerical Methods

To handle the two cases outlined in the introduction, namely (1) f bounded with bounded derivatives and (2) Maximum polynomially growing f , that obeys one-sided Lipschitz conditon, different discretisation techniques are implemented.

Consider the equation 1.1, then, to construct numerical discretisation, one needs to consider a finite-dimensional projection of H, $L^2(\mathbb{T})$ in this case, which is given by the following definition

Definition 2.0.8. *(Finite dimensional projection) Let $(e_i)_{i\geq 1}$ be an eigenbasis for the Hilbert Space H with inner product $\langle\cdot,\cdot\rangle$, the projection $\Pi_N : H \to H_N :=$ span$\{e_1, e_2, ..., e_N\}$ onto of $u \in H$ a finite-dimensional eigenspace is defined as:*

$$\Pi_N(u) = \sum_{n=1}^{N} \langle u, e_n\rangle e_n,$$

where for the case of $H = L^2(\mathbb{T})$ and $D(\Delta) = \{u \in H : \sum_{n=1}^{\infty} |\langle e_n, u\rangle|^2 n^4 < \infty\}$ the inner product is defined by $\langle u, e_n\rangle = \int_{\mathbb{T}} u(x)e_n(x)dx$. This definition immediately allows us to truncate the variables of the equation, by using $(P_t^N := \Pi_N P_t)_{t\geq 0}$ as the corresponding finite dimensional semigroup. Then again, one can write $P_t^N f = p_t^N * f$, where

$$p_t^N(x) = \sum_{|k|\leq N} e^{-4\pi^2 k^2 t} e^{2\pi i k x}.$$

and thus define the **Galerkin projection**

Definition 2.0.9. *(Galerkin Projection)*

$$U_t^N = P_t^N u_0 + \int_0^t P_{t-s}^N f(U_s^N) ds + O_t^N, \quad t \in [0, T], \tag{2.24}$$

where O_t^N is given by

$$O_t^N = \int_0^t \int_{\mathbb{T}} p_{t-s}^N (x-y)\xi(ds, dy). \tag{2.25}$$

Uniqueness of the discrete scheme is by induction from its recursive definition. Further, to discretise U_t^N, $t \in [0, T]$ in time, one recursively defines **Explonential Euler approximation scheme**. For $M \in \mathbb{N}$, let $h = T/M$ and consider the temporal gridpoints $t_k = kh$ for $k = 0, \dots, M$. Let for $s \in [0, 1]$, $k_M(s) := \lfloor h^{-1}s\rfloor h$ be the last gridpoint before (or equal to) s. Set $V_0^{M,N} = \Pi_N u_0 =: u_0^N$ and then recursively

$$V_{t_{k+1}}^{M,N} = P_h^N V_{t_k}^{M,N} + \Delta_N^{-1}(P_h^N - \mathrm{Id})\Pi_N(f(V_{t_k}^{M,N})) + O_{t_{k+1}}^N - P_h^N O_{t_k}^N$$

for $k = 0, \dots, M-1$. Here Id denotes the identity operator on L^2. One can also re-express $V^{M,N}$ in a "mild" form

$$
\begin{aligned}
V^{M,N}_{t_{k+1}} &= P^N_{t_{k+1}} u_0 + \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} P^N_{t_{k+1}-s} f(V^{M,N}_{t_l}) ds + O^N_{t_{k+1}} \\
&= P^N_{t_{k+1}} u_0 + \int_0^{t_{k+1}} P^N_{t_{k+1}-s} f(V^{M,N}_{k_M(s)}) ds + O^N_{t_{k+1}}, \quad k = 0, \dots, M-1.
\end{aligned} \tag{2.26}
$$

An advantage of the form (2.26) is that one can easily extend it to arbitrary (i.e. not grid-points $t \in [0,T]$, by replacing each instance of t_{k+1} on the right-hand side by t.

Note, that $V^{N,M}_{k+1} \in H$, one has $V^{N,M}_{k+1} = \sum_{i \geq 1} V^{N,M}_{k+1,i} e_i$, where $V^{N,M}_{t_{k+1,1}} = \langle e_i, V^{N,M}_{k+1} \rangle$, which effectively transforms the original infinite-dimensional stochastic PDE into a finite-dimensional stochastic differential equation system.

When it comes to the polynomially growing non-linearirty with one sided Lipschitz condition, it is known that both the standard and the accelerated Euler discretizations fail in this setting, since the discretisation grow non linearly. The nonlinearity of f makes the solution explode, if u_t explodes, so to state and prove convergence results for the discretizations, one needs taming schemes. In particular, to evolve through f without introducing instability, but still keep the scheme explicit and high-order, one splits the SPDE into two sub-problems (nonlinear + linear with noise) over each step $[t_k, t_{k+1}]$. Consider the auxiliary function $\Phi_t(z)$ that solves the non-linear equation[13]

$$
\partial_t \Phi_t(z) = f(\Phi_t(z)), \quad \Phi_0(z) = z. \tag{2.27}
$$

Then one can construct the following scheme $X^{M,N}_0 = \Pi_N u_0 = u^N_0$ and

$$
\begin{aligned}
Y^{M,N}_{t_k} &= \Phi_h(X^{M,N}_{t_k}) \\
X^{M,N}_{t_{k+1}} &= P^N_h Y^{M,N}_{t_k} + O^N_{t_{k+1}} - P^N_h O^N_{t_k}
\end{aligned} \tag{2.28}
$$

for $k = 0, \dots, M-1$ and $h = T/M$ and O^N is the truncated Ornstein-Uhlenbeck process defined in (2.25). One can rewrite the scheme as a classical Euler scheme for an auxiliary function. Consider the following auxiliary definition

$$
g_t(z) = \frac{\Phi_t(z) - z}{t}, \quad t > 0, \quad g_0(z) := f(z). \tag{2.29}
$$

Then, in particular, by FTC

$$
\Phi_h(z) - z = \int_0^h f(\Phi_s(z))\, ds \quad \Rightarrow \quad g_h(z) = \frac{1}{h} \int_0^h f(\Phi_s(z))\, ds.
$$

So $g_t(z)$ is the time average of the true drift along the nonlinear flow. Using the definition of g_h, $X^{M,N}$ can equivalently be written in the form

$$
X^{M,N}_{t_{k+1}} = P^N_h X^{M,N}_{t_k} + h P^N_h g_h(X^{M,N}_{t_k}) + O^N_{t_{k+1}} - P^N_h O^N_{t_k}, \tag{2.30}
$$

[13] Note, that in some cases, the equation (2.27) has an explicit solution, like Allen - Cahn equation, for which $\Phi_t(z) = \operatorname{sgn}(z) \frac{e^t}{\sqrt{z^{-2} - 1 + e^{2t}}}$.

This is the exponential Euler step for the auxiliary SPDE. Finally, it is possible extend this formulation to arbitrary points $t \in [0, T]$, by giving a "mild" version of the approximation $X^{M,N}$ as

$$X_t^{M,N} = P_t^N u_0 + \int_0^t P_{t-k_M(s)}^N g_h(X_{k_M(s)}^{M,N}) ds + O_t^N, \tag{2.31}$$

Chapter 3

Globally bounded f with bounded derivatives

In this section, the main result for f bounded with bounded derivatives up to order 2 is derived. In particular, under the assumption (1.0.1), the following result holds

Theorem 3.0.1. *Let assumption 1.0.1 hold. Let u be the unique mild solution of* (1.1) *and for any $M, N \in \mathbb{N}$, let $V^{M,N}$ be as before. Then for any $\epsilon > 0$ and $p \in [1, \infty)$ there exists a constant $C = C(T, \epsilon, p, K, \mathcal{M})$ such that*

$$\big(\mathbb{E} \sup_{t\in[0,T]} \|u_t - V_t^{M,N}\|^p_{L^\infty(\mathbb{T})}\big)^{1/p} \leq C\big(N^{-1/2+\epsilon} + M^{-1+\epsilon}\big). \tag{3.1}$$

To prove the result, one first constructs the following bridging argument. Namely, one decomposes the approximation error $u - V^{M,N}$ into $u - U^N, U_N - \tilde{U}^N, V^{M,N} - \tilde{V}^{M,N}$ and $\tilde{U}^N - \tilde{V}^{M,N}$, where the processes $\tilde{U}^N, \tilde{V}^{M,N}$ are defined as follows:

$$\tilde{U}_t^N = P_t^N u_0^N + \int_0^t P_{t-s}^N f(\tilde{U}_s^N) ds + O_t, \quad t \in [0,T] \tag{3.2}$$

$$\tilde{V}_{t_{k+1}}^{M,N} = P_h^N \tilde{V}_{t_k}^{M,N} + \Delta_N^{-1}(P_h^N - \mathrm{Id})\Pi_N(f(\tilde{V}_{t_k}^{M,N})) + O_{t_{k+1}} - P_h^N O_{t_k}$$

for $k = 0, \ldots, M-1$, and for $t \in [0,T]$ via

$$\tilde{V}_t^{M,N} = P_t^N u_0^N + \int_0^t P_{t-s}^N f(\tilde{V}_{k_M(s)}^{M,N}) ds + O_t. \tag{3.3}$$

That is, the processes for which the convolution part is not discretised. First, to demonstrate the well-posedness of the solutions to (3.2) well-posedness and (3.3), via the following proposition.

Proposition 3.0.1.1. *Let f satisfy the assumption (1.0.1), then $\exists!$ solutions U^N and $\tilde{U}^N$ to equations (3.2) and (3.3). Furthermore, $\forall \lambda \in (0,1), \epsilon \in (0, 1/2)$, $p \geq 1$, $\exists$ constant $C = C(T, p, \epsilon, \lambda)$ such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|U^N\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}(\mathbb{T})} + \sup_{N\in\mathbb{N}} \mathbb{E}\|\tilde{U}^N\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}(\mathbb{T})} \leq C(1 + \mathbb{E}\|u_0\|^p_{\mathscr{C}^{1/2}}).$$

Proof. One starts by defining an operator $\Phi : C_T(L^\infty) \to C_T(L^\infty)$, by $\Phi(u) := P_t^N u_0 + \int_0^t P_{t-s}^N f(u_s)ds + O_t$, $t \in [0, T]$. Since $C_T(L^\infty)$ is complete, it is then sufficient to show that the operator is contractive, to apply the Banach Fixed Point Theorem and hence, prove existence and uniqueness.

In particular, for $u, v \in C_T(L^\infty)$

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{L^\infty} &= \left\|\int_0^t P_{t-s}^N(f(u_s) - f(v_s))ds\right\|_{L^\infty} \leq \int_0^t \left\|P_{t-s}^N(f(u_s) - f(v_s))ds\right\|_{L^\infty} \\ &\leq L\int_0^t \|u_s - u_s\|_{L^\infty} \leq Lt \sup_{s\in[0,t]} \|u_s - v_s\|_{L^\infty} = Lt\|u - v\|_{C_T(L^\infty)}\end{aligned}$$

where one uses that $\sup_{N\in\mathbb{N}} \|P_t^N u\|_{L^\infty} \leq \|u\|_{L^\infty}$, which gives Lipschitz condition $\|f(u_s) - f(v_s)\|_{L^\infty} \leq L\|u_s - v_s\|_{L^\infty}$, for some constant L. Taking supremum over $t \in [0, T]$, one has that

$$\|\Phi(u) - \Phi(v)\|_{C_T(L^\infty)} \leq LT\|u - v\|_{C_T(L^\infty)}.$$

So for $T < 1/L$, the operator Φ is a contraction on $C_T(L^\infty)$, and ny Banach's fixed point theorem existence and uniqueness of a mild solution in $C_T(L^\infty)$ is given. Combining solutions on subintervals gives a solution for an arbitrary time $T > 0$. A similar argument holds for U^N. The actual bound follows directly from applying the argument used in proving (2.11) for both U^N and $\tilde{U}^N$, where the argument for P is used for P^N □

Since the solutions do, in fact, exist and are unique, it remains to separately bound each term in the decomposition. One starts by estimating bounds on the $\tilde{U^N}$ and $\tilde{V}^{M,N}$, with the following lemma

Lemma 3.0.1.1. *Assume the setting of Theorem 3.0.1. Let $N \in \mathbb{N}$ and let $U^N, V^{M,N}$ be as before and $\tilde{U}^N, \tilde{V}^{M,N}$ be as in (3.2). Let $p \in [1, \infty)$ and $\epsilon > 0$. Then there exists a constant $C = C(T, p, \epsilon, K)$ such that the following bound holds*

$$\big(\mathbb{E} \sup_{t\in[0,T]} \|\tilde{U}_t^N - U_t^N\|^p_{L^\infty}\big)^{1/p} + \big(\mathbb{E} \sup_{t\in[0,T]} \|\tilde{V}_t^{M,N} - V_t^{M,N}\|^p_{L^\infty}\big)^{1/p} \leq CN^{-1/2+\epsilon}.$$

Proof. One can bound each term individually, starting with $\tilde{U}_t^N - U_t^N$.Using proposition 2.0.0.7, it is sufficient to show that $Z = O - O^N$ belongs to $L^p(\Omega; C([0, T]); V)$. First note that,

$$(O_t - O_t^N)(x) = \int_0^t \int_{\mathbb{T}} (p_{t-s} - p_{t-s}^N)(x - y)\xi(ds, dy), \quad \text{with} \quad (p_{t-s} - p_{t-s}^N)(x) = \sum_{|k|>N} e^{-4\pi^2 k^2(t-s)} e^{2\pi i k x}.$$

Then define $O_{r,t} = O_t - O_r$, $r \le t$, and let $O^N_{r,t}$ be defined in an analogous way. One has

$$\begin{aligned}(O_{r,t} - O^N_{r,t})(x) &= \int_r^t \int_{\mathbb{T}} (p_{t-s} - p^N_{t-s})(x-y)\xi(ds,dy) \\ &\quad - \int_0^r \sum_{|k|>N} e^{-4\pi^2 k^2 (r-s)}(1 - e^{-4\pi^2 k^2 (t-r)})e^{2\pi i k(x-y)}\xi(ds,dy) \\ &=: I_1 - I_2\end{aligned}$$

Then, similar to proof of (2.0.0.5), it sufficient to show the following bound

$$\begin{aligned}(\mathbb{E}|\Delta_j(O_{r,t} - O^N_{r,t})(x)|^p)^{1/p} &\lesssim (\mathbb{E}|\Delta_j(O_{r,t} - O^N_{r,t})(x)|^2)^{1/2} \quad \text{Gaussian Hypercontractivity} \\ &= (\mathbb{E}|\Delta_j I_1 - \Delta_j I_2|^2)^{1/2} \\ &= (\mathbb{E}|\Delta_j I_1|^2 + \mathbb{E}|\Delta_j I_2|^2)^{1/2} \quad \text{Product of Ito's integrals over disjoint regions}\end{aligned}$$

Now, again, by Parseval's identity, since $p_{t-s}(x)$ has Fourier coefficients $e^{-8\pi^2 k^2 (t-s)}$, one has that

$$\mathbb{E}|\Delta_j I_1|^2 = \int_r^t \sum_{|k|>N} \rho_j^2(k) e^{-8\pi^2 k^2 (t-s)} ds.$$

Similar argument holds for I_2. Thus, by bringing the integral inside the sum and evaluating

$$\begin{aligned}(\mathbb{E}|\Delta_j(O_{r,t} - O^N_{r,t})(x)|^p)^{1/p} &\lesssim \left(\int_r^t \sum_{|k|>N} \rho_j(k)^2 e^{-8\pi^2 k^2 (t-s)} ds + \int_0^r \sum_{|k|>N} \rho_j(k)^2 e^{-8\pi^2 k^2 (r-s)}(1 - e^{-4\pi^2 k^2 (t-r)})^2 ds\right)^{1/2} \\ &\le \left(\sum_{|k|>N} \frac{1 - e^{-8\pi^2 k^2 (t-r)}}{8\pi^2 k^2} + \sum_{|k|>N} \frac{(1 - e^{-8\pi^2 k^2 r})(1 - e^{-4\pi^2 k^2 (t-r)})^2}{8\pi^2 k^2}\right)^{1/2} \\ &\lesssim \left((t-r)^\epsilon \sum_{|k|>N} \frac{1}{k^{2-2\epsilon}}\right)^{1/2} \quad \text{Since } \forall \epsilon' \in [0,1],\ 1 - e^{-x} \le x^{\epsilon'},\ x \ge 0 \\ &\lesssim N^{-1/2+\epsilon'}(t-r)^{\epsilon'/2},\end{aligned}$$

Once the space bound was obtained, one can construct bound in frequencies, uniform in N.

$$\begin{aligned}(\mathbb{E}|\Delta_j(O_{r,t} - O^N_{r,t})(x)|^p)^{1/p} &\lesssim \left(\int_r^t \sum_{|k|>N} \rho_j(k)^2 e^{-8\pi^2 k^2 (t-s)}\, ds + \int_0^r \sum_{|k|>N} \rho_j(k)^2 e^{-8\pi^2 k^2 (r-s)}(1 - e^{-4\pi^2 k^2 (t-r)})^2\, ds\right)^{1/2} \\ &\lesssim \left(\int_r^t 2^j e^{-8\pi^2 2^{2j}(t-s)}\, ds + \int_0^r 2^j e^{-8\pi^2 2^{2j}(r-s)}(1 - e^{-4\pi^2 2^{2j}(t-r)})^2\, ds\right)^{1/2} \\ &= \left(2^j \frac{1 - e^{-8\pi^2 2^{2j}(t-r)}}{8\pi^2 2^{2j}} + 2^j \frac{(1 - e^{-8\pi^2 2^{2j} r})(1 - e^{-4\pi^2 2^{2j}(t-r)})^2}{8\pi^2 2^{2j}}\right)^{1/2} \\ &\lesssim (2^{1-j})^{1/2} = 2^{-j/2}.\end{aligned}$$

Hence, via trivial inequality[1] for any $\epsilon'' \in (0,1]$, one obtains:

$$\left(\mathbb{E}|\Delta_j(O_{r,t} - O^N_{r,t})(x)|^p\right)^{1/p} \lesssim 2^{-j\epsilon''/2} N^{(-1/2+\epsilon')(1-\epsilon'')}(t-r)^{(1-\epsilon'')\epsilon'/2},$$

[1] For any pair of nonnegative bounds A, B and for any $\epsilon'' \in (0,1]$: $\min\{A,B\} \le A^{1-\epsilon''} B^{\epsilon''}$.

Repeating the argument from (2.10) to (2.8), via the Besov norm embedding, the bound gives

$$\left(\mathbb{E}\,\|O_{r,t}-O_{r,t}^N\|_{C^{\epsilon'''}}^p\right)^{1/p} \lesssim N^{(-\frac{1}{2}+\epsilon')(1-\epsilon'')}\,(t-r)^{(1-\epsilon'')\epsilon'/2}. \tag{3.4}$$

Using Kolmogorov continuity theorem, choosing $\epsilon', \epsilon'' > 0$ sufficiently small and p sufficiently large,

$$\left(\mathbb{E}\,\|O-O^N\|_{C_T^{\tilde{\alpha}}C^{\epsilon'''}}^p\right)^{1/p} \lesssim N^{(-\frac{1}{2}+\epsilon')(1-\epsilon'')}.$$

Hence the Grönwall inequality applies. The bound for $\tilde{V}^{M,N}-V^{M,N}$ is done analogously: since

$$\tilde{V}_t^{M,N}-V_t^{M,N} = \int_0^t P_{t-s}^N(f(\tilde{V}_{k_M(s)}^{M,N})-f(V_{k_M(s)}^{M,N}))ds + O_t - O_t^N \tag{3.5}$$

we can use (2.0.0.7) almost exactly as before, with the only difference being that $\tau(s)=k_M(s)$. □

Further, one bounds the convergence error of Galerkin space approximation via the following result

Lemma 3.0.1.2. *Assume the setting of Theorem 3.0.1. Let $N\in\mathbb{N}$ and let u, U^N be as before. Let $p\geq 1$ and $\epsilon\in(0,1/2)$. Then there exists a constant $C=C(p,\epsilon,K,\mathscr{M})$ such that*

$$(\mathbb{E}\sup_{t\in[0,T]}\|u_t-U_t^N\|_{L^\infty}^p)^{1/p}\leq CN^{-1/2+\epsilon}.$$

Proof. Firstly, by adding and subtracting $\int_0^t P_{t-s}^N f(u_s)ds$ one has that

$$u_t-U_t^n = (P_t-P_t^N)u_0 + \int_0^t (P_{t-s}-P_{t-s}^N)f(u_s)ds + \int_0^t P_{t-s}^N(f(u_s)-f(U_s^N))ds + O_t - O_t^N$$

So it remains to prove the bound for each term individually. Starting with

$$\|(P_t-P_t^N)u_0\|_{L^\infty(\mathbb{T})} = \|P_t(u_0-\Pi_N u_0)\|_{L^\infty(\mathbb{T})} \leq \|(u_0-\Pi_N u_0)\|_{L^\infty(\mathbb{T})}$$

Furthermore one has:

$$\begin{aligned}
\int_0^t \left\|(P_{t-s}-P_{t-s}^N)f(u_s)\right\|_{L^\infty(\mathbb{T})}ds &= \int_0^t \left\|(p_{t-s}-p_{t-s}^N)*f(u_s)\right\|_{L^\infty(\mathbb{T})}ds \\
&\leq \int_0^t \left\|(p_{t-s}-p_{t-s}^N)\right\|_{L^2(\mathbb{T})}\|f(u_s)\|_{L^2(\mathbb{T})}ds \quad \text{by Youngs Convolution Inequality} \\
&\lesssim \|f\|_{L^\infty(\mathbb{R})}\int_0^t \left\|(p_{t-s}-p_{t-s}^N)\right\|_{L^2(\mathbb{T})}ds \\
&\lesssim \|f\|_{L^\infty(\mathbb{R})}\int_0^t \left(\sum_{|k|>N} e^{-8\pi^2k^2(t-s)}\right)^{1/2} ds \quad \text{by Parseval Identity} \\
&\lesssim \|f\|_{L^\infty(\mathbb{R})}\cdot N^{-3/2+\epsilon}\int_0^t (t-s)^{-1+\epsilon/2}ds
\end{aligned}$$

where $\|f(u_s)\|_{L^2(\mathbb{T})}^2 \leq \|f\|_{L^\infty(\mathbb{R})}^2$ is used in the third line. Finally, using the proposition (2.0.0.7) gives the desired claim by setting $Z_t = (P_t-P_t^N)u_0 + \int_0^t (P_{t-s}-P_{t-s}^N)f(u_s)ds + O_t - O_t^N$ (the arguments above with (2.0.0.5) ensure the regulation of Z), $\tau(s)=s$, $S(s,t)=P_{t-s}^N$ one gets the result. □

One finally can bound the difference between $\tilde{U}_t^N$ and $\tilde{V}_t^{N,M}$, via the following result

Lemma 3.0.1.3. *For any $\epsilon > 0$ and $p \geq 1$ there exists a constant $C = C(\epsilon, p, K, \mathscr{M})$ such that*

$$\big(\mathbb{E} \sup_{t\in[0,T]} \|\tilde{U}_t^N - \tilde{V}_t^{M,N}\|_{L^\infty(\mathbb{T})}^p\big)^{1/p} \leq CM^{-1+\epsilon}.$$

To prove the lemma, we first deconstruct the error as

$$\tilde{U}_t^N - \tilde{V}_t^{M,N} = \int_0^t P_{t-s}^N[f(\tilde{U}_s^N) - f(\tilde{U}_{k_M(s)}^N)]ds + \int_0^t P_{t-s}^N[f(\tilde{U}_{k_M(s)}^N) - f(\tilde{V}_{k_M(s)}^{M,N})]ds, \tag{3.6}$$

which allows one to treat the integrals separately. The second term in the expression can again be treated via the Grönwall inequality (2.0.0.7). To treat the first term, first consider a proof for a simpler process, given in the following proposition.

Proposition 3.0.1.2. *Let (1.0.1) hold. Let $p \geq 1$ and $\epsilon \in (0, 1/2)$. Then there exists a constant $C = C(T, p, \epsilon, K, \mathscr{M})$ such that for all $0 \leq s \leq t \leq R \leq T$ it holds that*

$$\left(\mathbb{E}\left\|\int_s^t P_{R-s}^N[f(O_s + P_s^N u_0) - f(O_{k_M(s)} + P_{k_M(s)}^N u_0)]ds\right\|_{L^\infty}^p\right)^{1/p} \leq CM^{-1+2\epsilon}|t-s|^{1/4+\epsilon/2}. \tag{3.7}$$

Proof. For notation convenience, consider the shifted OU process $\tilde{O}_t := O_t + P_t^N u_0, \quad t \in [0, T]$. Note, to prove the proposition, it suffices to show that for any $j \geq -1$, $x \in \mathbb{T}$, $0 \leq s \leq t \leq R \leq T$, one has

$$\left(\mathbb{E}\left|\int_s^t \Delta_j P_{R-r}^N[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr\right|^p\right)^{1/p} \lesssim M^{-1+2\epsilon}2^{-j\epsilon}(t-s)^{1/4+\epsilon/2}, \tag{3.8}$$

Indeed, similar to previous results, since one has

$$\Big(\mathbb{E}\| \textstyle\int_s^t u_r\, dr\|_{B_{p,p}^\epsilon}^p\Big)^{1/p} = \Big(\mathbb{E} \displaystyle\sum_{j\geq -1} 2^{j\epsilon p} \| \textstyle\int_s^t \Delta_j u_r\, dr\|_{L^p}^p\Big)^{1/p} \tag{3.9}$$

$$= \Big(\sum_{j\geq -1} 2^{j\epsilon p}\, \mathbb{E}\| \textstyle\int_s^t \Delta_j u_r\, dr\|_{L^p}^p\Big)^{1/p} \tag{3.10}$$

$$\leq \Big(\sum_{j\geq -1} 2^{j\epsilon p} \Big[\textstyle\int_s^t (\mathbb{E}\|\Delta_j u_r\|_{L^p}^p)^{1/p}\, dr\Big]^p\Big)^{1/p} \quad \text{(Minkowski in time)} \tag{3.11}$$

$$\leq \textstyle\int_s^t \Big(\sum_{j\geq -1} 2^{j\epsilon p}\, \mathbb{E}\|\Delta_j u_r\|_{L^p}^p\Big)^{1/p} dr \quad (\text{Minkowski in } \ell^p) \tag{3.12}$$

Therefore (3.8) implies that

$$\left(\mathbb{E}\left\|\int_s^t P_{R-r}^N[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr\right\|_{B_{pp}^\epsilon}^p\right)^{1/p} \lesssim M^{-1+2\epsilon}2^{-j\epsilon}(t-s)^{1/4+\epsilon/2},$$

then, finally using the embedding property (2.0.0.1), one has that $B^{\epsilon}_{pp} \hookrightarrow B^{\epsilon-d/p}_{\infty\infty}$, i.e. $B^{\epsilon}_{pp} \hookrightarrow \mathscr{C}^{\epsilon-1/p} \hookrightarrow L^{\infty}$ for p large enough (refer to [8, Proposition 2.39] for more details). To prove (3.8), one applies the Stochastic Sewing Lemma. Consider $j \geq -1$, $x \in \mathbb{T}$, $R \leq T$, then define for $0 \leq s \leq t \leq R$ the increments as

$$A_{st} = \mathbb{E}_s \int_s^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr, \tag{3.13}$$

so the error term is defined for $s < u < t$ as

$$\begin{aligned}\delta A_{sut} = \mathbb{E}_s \int_u^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr \\ - \mathbb{E}_u \int_u^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr.\end{aligned} \tag{3.14}$$

Then, the second condition in (2.0.0.1) is immediately satisfied for $C_2 = 0$, as by tower property $\mathbb{E}_s[\delta A_{sut}] = 0$. It is therefore suffices to show that the increment satisfies the following claim

Claim 1. *For all $0 \leq u < t < R$ with $|t-u| \leq |R-t|$ one has*

$$\|A_{ut}\|_{L^p(\Omega)} \lesssim 2^{-j\epsilon} M^{-1+2\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)^{1/2+\epsilon}. \tag{3.15}$$

uniformly in x, j.

Proof. To prove claim (1) one consideres two cases, depending on the size of the increment. In the **first case**, let $|t-u| \leq 3M^{-1}$, then one has the following properties

$$\|P_{t+s}u_0 - P_t u_0\|_{\mathscr{C}^{-1/2+\epsilon}} \lesssim_\theta s^{(1+\theta)/2} t^{-\theta/2-\epsilon/2} \|u_0\|_{\mathscr{C}^{1/2}}$$

for any $\theta \in [0,1]$ and

$$\|P_{t+s}u_0 - P_t u_0\|_{\mathscr{C}^{-1/2+\theta}} \lesssim_\theta s^{(1-\theta)/2} \|u_0\|_{\mathscr{C}^{1/2}}$$

for $\theta \in [0,1]$, both of which directly derived from the heat kernel bounds (2.0.0.4), as in previous results. Additionally, for $\theta \in (0, 1/2)$, $q \geq 1$, one has[2]

$$\mathbb{E} \sup_{r\in[0,R]} \|f'(\lambda\tilde{O}_r + (1-\lambda)\tilde{O}_{k_M(r)})\|^q_{\mathscr{C}^\theta} \lesssim \mathbb{E}\|f'\|^q_{C^1_b} \|\tilde{O}\|^q_{C_T\mathscr{C}^\theta} \lesssim_{q,\theta} 1. \tag{3.16}$$

Using these bounds, one has that:

$$\begin{aligned}\left(\mathbb{E}\left|\int_u^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr\right|^p\right)^{1/p} &\lesssim 2^{-j\epsilon}\left(\mathbb{E}\left\|\int_u^t P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})]dr\right\|^p_{\mathscr{C}^\epsilon}\right)^{1/p} \\ &\lesssim 2^{-j\epsilon}\int_u^t (R-r)^{-1/4-\epsilon/2} \\ &\quad \times \left(\mathbb{E}\left\|\left(\int_0^1 f'(\lambda\tilde{O}_r + (1-\lambda)\tilde{O}_{k_M(r)})d\lambda\right)(\tilde{O}_r - \tilde{O}_{k_M(r)})\right\|^p_{\mathscr{C}^{-1/2}}\right)^{1/p} dr\end{aligned}$$

[2] see Appendix A.0.1 for derivation

where the first line comes from the fact that since one has $2^{j\epsilon}|\Delta_j u(x)| \lesssim \|u\|_{\mathscr{C}^\epsilon}$, and in the second line, using the heat kernel bound (2.0.0.4), and the fact that $f(\tilde{O}_r)-f(\tilde{O}_{k_M(r)}) = \Big(\int_0^1 f'(\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)})\,d\lambda\Big)(\tilde{O}_r-\tilde{O}_{k_M(r)})$ (easy to check via FTC and substitution).Then, one has that

$$\begin{aligned}
2^{-j\epsilon}\bigg(\mathbb{E}\bigg\|\int_u^t P^N_{R-r}[f(\tilde{O}_r)-f(\tilde{O}_{k_M(r)})]dr\bigg\|^p_{\mathscr{C}^\epsilon}\bigg)^{1/p} &\lesssim 2^{-j\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)\\
&\quad\times\bigg(\mathbb{E}\sup_{r\in[0,T]}\|\int_0^1 f'(\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)})\,d\lambda\|^{2p}_{\mathscr{C}^{1/2-\epsilon/2}}\bigg)^{\frac{1}{2p}}\\
&\quad\times\big(\mathbb{E}\sup_{r\in[0,T]}\|\tilde{O}_r-\tilde{O}_{k_M(r)}\|^{2p}_{\mathscr{C}^{-1/2+\epsilon}}\big)^{\frac{1}{2p}},
\end{aligned}$$

via Hölder inequality. Finally, using the (3.16) and the Proposition (2.0.0.5) one gets

$$\begin{aligned}
\bigg(\mathbb{E}\bigg|\int_u^t \Delta_j P^N_{R-r}[f(\tilde{O}_r)-f(\tilde{O}_{k_M(r)})](x)dr\bigg|^p\bigg)^{1/p} &\lesssim 2^{-j\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)(r-k_M(r))^{1/2-\epsilon}\\
&\lesssim 2^{-j\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)M^{-1/2+\epsilon}\\
&\lesssim 2^{-j\epsilon}(R-t)^{-1/4-\epsilon/2}(t-s)^{1/2+\epsilon}M^{-1+2\epsilon},
\end{aligned}\tag{3.17}$$

using $|t-u|\leq 3M^{-1}$ in the last inequality.This shows that (3.15) holds for $|t-u|\leq 3M^{-1}$. For the second case, $|t-u|>3M^{-1}$, consider t' to be the second smallest value after u, with following decomposition for $\phi = \Delta_j P^N_{R-r}[f(\tilde{O}_r)-f(\tilde{O}_{k_M(r)})]$

$$\int_u^t \phi(x)dr = \int_u^{t'}\phi(x)dr + \int_{t'}^t \phi(x)dr,$$

Bounding the integrals separately, one can immediately notice that $|t'-u|\leq 2M^{-1}$(since the increments are of M^{-1} size), and thus one can use the previous argument to bound the first term. It remains to prove the second integral. First, notice that

$$\tilde{O}_r = O_r + P^N_r u_0 = P_{r-u}O_u + \int_u^r P_{r-v}\xi(dv) + P_{r-u}(P^N_u u_0) = P_{r-u}\tilde{O}_u + \int_u^r P_{r-v}\xi(dv).$$

Thus, in kernel notation, one has

$$\tilde{O}_r = P_{r-u}\tilde{O}_u + \int_u^r\int_{\mathbb{T}} p_{r-v}(\cdot-y)\xi(dv,dy),$$

where $P_{r-u}\tilde{O}_u$ is $\mathcal{F}_u-$ measurable and $\int_u^r\int_{\mathbb{T}} p_{r-v}(\cdot-y)\xi(dv,dy)$ is a centered Gaussian increment(Lemma 5.3 [6]). One can therefore apply the general result, that states if X is $\mathcal{F}_u-$ measurable and Y, centered Gaussian with variance σ, independent of $\mathcal{F}_u$, one has that $\mathbb{E}_u[f(X+Y)] = (P^{\mathbb{R}}_\sigma f)(X)$, where $P^{\mathbb{R}}$ denotes the heat-semigroup on $\mathbb{R}$. Using these results, one obtains

$$\begin{aligned}
\mathbb{E}_u\Big[\int_{t'}^t \Delta_j P_{R-r}^N[f(\tilde{O}_r)-f(\tilde{O}_{k_M(r)})](x)dr\Big] &= \mathbb{E}_u\Big[\int_{t'}^t \Delta_j P_{R-r}^N f(\tilde{O}_r)\Big] - \mathbb{E}_u\Big[\int_{t'}^t \Delta_j P_{R-r}^N[f(\tilde{O}_{k_M(r)})](x)dr\Big] \\
&= \int_{t'}^t \Delta_j P_{R-r}^N[(P_{Q(r-u)}^{\mathbb{R}} f)(P_{r-u}\tilde{O}_u) - (P_{Q(k_M(r)-u)}^{\mathbb{R}} f)(P_{k_M(r)-u}\tilde{O}_u)](x)dr, \qquad (3.18) \\
&\quad + \int_{t'}^t \Delta_j P_{R-r}^N[[(P_{Q(r-u)}^{\mathbb{R}} f) - (P_{Q(k_M(r)-u)}^{\mathbb{R}} f)](P_{k_M(r)-u}\tilde{O}_u)](x)dr. \qquad (3.19)
\end{aligned}$$

where $Q(r-u) = \mathbb{E}[(\int_u^r \int_{\mathbb{T}} p_{r-v}(x-y)\xi(dv,dy))^2]$ is a time only dependent variance. Dealing with (3.18) first, similar to (3.16), one has

$$\mathbb{E}\sup_{r\in[0,R]} \|(P_{Q(r-u)}^{\mathbb{R}} f)'(\lambda P_{r-u}\tilde{O}_u + (1-\lambda)P_{k_M(r)-u}\tilde{O}_u)\|_{\mathscr{C}^\theta}^q \lesssim_{q,\theta} 1. \qquad (3.20)$$

Using similar approach as for the smaller increment bound in the first case, one obtains the following bound

$$\begin{aligned}
&\Big(\mathbb{E}\Big|\int_{t'}^t \Delta_j P_{R-r}^N[(P_{Q(r-u)}^{\mathbb{R}} f)(P_{r-u}\tilde{O}_u) - (P_{Q(r-u)}^{\mathbb{R}} f)(P_{k_M(r)-u}\tilde{O}_u)](x)dr\Big|^p\Big)^{1/p} \\
&\lesssim 2^{-j\epsilon}\Big(\mathbb{E}\Big\|\int_{t'}^t P_{R-r}^N\Big[\Big(\int_0^t (P_{Q(r-u)}^{\mathbb{R}} f)'(\lambda P_{r-u}\tilde{O}_u + (1-\lambda)P_{k_M(r)-u}\tilde{O}_u)d\lambda\Big) \\
&\qquad\qquad \times (P_{r-u}\tilde{O}_u - P_{k_M(r)-u}\tilde{O}_u)\Big]dr\Big\|_{\mathscr{C}^\epsilon}^p\Big)^{1/p} \\
&\lesssim 2^{-j\epsilon}\int_{t'}^t \Big((R-r)^{-1/4-\epsilon/2} \\
&\quad \times \Big(\mathbb{E}\Big\|\int_0^t (P_{Q(r-u)}^{\mathbb{R}} f)'(\lambda P_{r-u}\tilde{O}_u + (1-\lambda)P_{k_M(r)-u}\tilde{O}_u)d\lambda\Big\|_{\mathscr{C}^{1/2-\epsilon/2}}^{2p}\Big)^{1/2p}\Big) \\
&\quad \times \big((\mathbb{E}\|P_{r-u}\tilde{O}_u - P_{k_M(r)-u}\tilde{O}_u\|_{\mathscr{C}^{-1/2+\epsilon}}^{2p})^{1/2p}\big)dr \\
&\lesssim 2^{-j\epsilon}\int_{t'}^t (R-r)^{-1/4-\epsilon/2}(r-k_M(r))^{1-2\epsilon}(k_M(r)-u)^{-1/2+\epsilon}dr \\
&\lesssim 2^{-j\epsilon}M^{-1+2\epsilon}\int_{t'}^t (R-r)^{-1/4-\epsilon/2}(r-u)^{-1/2+\epsilon}dr \\
&\lesssim 2^{-j\epsilon}M^{-1+2\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)^{1/2+\epsilon}, \qquad (3.21)
\end{aligned}$$

where one uses by (2.0.0.4), $\|P_{r-u}\tilde{O}_u - P_{k_M(r)-u}\tilde{O}_u\|_{\mathscr{C}^{-1/2+\epsilon}} \lesssim (r-k_M(r))^{1-2\epsilon}(k_M(r)-u)^{-1/2+\epsilon}$, and the fact that $k_M(r)-u \geq 1/2(r-u)$. For the second summand, namely (3.19), consider the following estimate for the variance Q, for $u \leq l \leq r$

$$\begin{aligned}
Q(r-u) - Q(l-u) &= \mathbb{E}\Big[\Big(\int_u^r\int p_{r-t}(\cdot-y)\xi(dt,dy)\Big)^2\Big] - \mathbb{E}\Big[\Big(\int_u^l\int p_{l-t}(\cdot-y)\xi(dt,dy)\Big)^2\Big] \\
&= \int_u^r \|p_{r-t}\|_{L^2(\mathbb{T})}^2 dt - \int_u^l \|p_{l-t}\|_{L^2(\mathbb{T})}^2 dt = \int_{l-u}^{r-u} \|p_s\|_{L^2(\mathbb{T})}^2\, ds \\
&\leq \int_{l-u}^{r-u} \|p_s\|_{L^1(\mathbb{T})}\|p_s\|_{L^\infty(\mathbb{T})}\, ds \lesssim \int_{l-u}^{r-u} s^{-1/2}\, ds \\
&\lesssim (r-l)^{1-\epsilon}(l-u)^{-1/2+\epsilon}. \qquad (3.22)
\end{aligned}$$

$|p_s|_{L^2}^2 \lesssim s^{-1/2}$ and the mean-value estimate. The latter bound together with the heat kernel estimate (2.6) yields

$$\|(P^{\mathbb{R}}_{Q(r-u)}f) - (P^{\mathbb{R}}_{Q(k_M(r)-u)}f)\|_{L^\infty(\mathbb{R})} \lesssim (Q(r-u) - Q(k_M(r)-u))$$
$$\lesssim (k_M(r)-u)^{-1/2+\epsilon}(r-k_M(r))^{1-\epsilon}.$$

Hence we obtain

$$\begin{aligned}
&\left(\mathbb{E}\left|\int_{t'}^t \Delta_j P^N_{R-r}[[(P^{\mathbb{R}}_{Q(r-u)}f) - (P^{\mathbb{R}}_{Q(k_M(r)-u)}f)](P_{k_M(r)-u}O_u + P^N_{k_M(r)}u_0)](x)dr\right|^p\right)^{1/p} \\
&\lesssim 2^{-j\epsilon}\int_{t'}^t (R-r)^{-\epsilon/2}\|(P^{\mathbb{R}}_{Q(r-u)}f) - (P^{\mathbb{R}}_{Q(k_M(r)-u)}f)\|_{L^\infty(\mathbb{R})}dr \\
&\lesssim 2^{-j\epsilon}(R-t)^{-\epsilon/2}\int_{t'}^t (k_M(r)-u)^{-1/2+\epsilon}(r-k_M(r))^{1-\epsilon}dr \\
&\lesssim 2^{-j\epsilon}M^{-1+\epsilon}(R-t)^{-\epsilon/2}(t-s)^{1/2+\epsilon} \qquad (3.23)
\end{aligned}$$

Using again that $k_M(r) - u \geq (r-u)/2$ for $r \in [t', t]$, thus concluding the proof of (1). □

To finish the proof of (3.8) and thus (3.0.1.2) is remains to justify that

$$\mathcal{A}_t = \int_0^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr.$$

For that, one needs to verify the two conditions in the stochastic sewing lemma, namely, (2.22) and (2.23). It is clear, that $\mathbb{E}_s[\mathcal{A}_{st} - A_{st}] = 0$, automatically satisfying (2.23). To verify (2.22), note that

$$\begin{aligned}
|\mathcal{A}_{st} - A_{st}| &= \left|\int_0^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr - \mathbb{E}_s\Big[\int_0^t \Delta_j P^N_{R-r}[f(\tilde{O}_r) - f(\tilde{O}_{k_M(r)})](x)dr\Big]\right| \\
&\leq (t-s)\|f\|_{L^\infty}
\end{aligned}$$

Since the f is bounded by the assumption (1.0.1) one gets the desired bound. □

Now, to complete the proof of the lemma, one simply requires stitching the results over the whole interval in time, and then uses a change of measure via the Girsanov theorem, to make the results hold for the original process $\tilde{U}$. Consider the following corollaries

Corollary 3.0.1.1. *Let (1.0.1) hold. Let $p \geq 1$ and $\epsilon \in (0, 1/4)$. Then there exists a constant $C = C(T, p, \epsilon, K, \mathscr{M})$ such that*

$$\left(\mathbb{E}\sup_{R\in[0,T]}\left\|\int_0^R P^N_{R-s}[f(O_s + P^N_s u_0) - f(O_{k_M(s)} + P^N_{k_M(s)}u_0)]ds\right\|^p_{L^\infty}\right)^{1/p} \leq CM^{-1+\epsilon}. \qquad (3.24)$$

Proof. Consider a variation of Kolmogorov Continuity Theorem for semigroups given in [1]. Let $(X_t)_{t\in[0,T]}$ be a continuous stochastic process starting from 0 with values in a Banach space E and let $(S_t)_{t\geq 0}$ be a continuous semigroup of bounded linear operators on V. Then, if for some $p > 0$, $\alpha > 0$, $C' < \infty$ it holds for all $0 \le s \le t \le T$ that

$$\mathbb{E}\|X_t - S_{t-s}X_s\|^p \le C'|t-s|^{1+\alpha}, \tag{3.25}$$

one has

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|^p \le C''C', \tag{3.26}$$

where C'' depends only on p, α, T. For a process

$$X_t = \int_0^t P^N_{t-s}[f(O_s + P^N_s u_0) - f(O_{k_M(s)} + P^N_{k_M(s)} u_0)]ds, \tag{3.27}$$

one has that

$$\mathbb{E}\|X_t - S_{t-s}X_s\|^p = \mathbb{E}\left\| \int_s^t P^N_{t-r}\left[f(\widetilde{O}_r) - f(\widetilde{O}_{k_M(r)})\right] dr \right\|^p \tag{3.28}$$

so via the (3.0.1.2), the term satisfies the above conditions with the semigroup $S = P^N$, $V = L^\infty$, any $p \ge 4$, $\alpha = 2\epsilon$, and $C' = (CM^{-1+\epsilon})^p$. $\square$

Corollary 3.0.1.2. *Let (1.0.1) hold. Let $p \ge 1$ and $\epsilon \in (0, 1/2)$. Let $\tilde{U}^N$ be the solution of (??). Then there exists a constant $C = C(T, p, \epsilon, K, \mathscr{M})$ such that*

$$\left(\mathbb{E} \sup_{R\in[0,T]} \left\| \int_0^R P^N_{R-s}[f(\tilde{U}^N_s) - f(\tilde{U}^N_{k_M(s)})]ds \right\|^p_{L^\infty}\right)^{1/p} \le CM^{-1+\epsilon}. \tag{3.29}$$

Proof. Following the proof due to [11], the goal is to change a probability measure under which the drifted noise becomes white (BM) again, so that under new $\mathbb{Q}$ the law of $\widetilde{U}^N$ coincides with the law of $\widetilde{O}$. Define the probability measure $\mathbb{Q}$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \rho = \exp\Big(-\int_0^T \int f(\widetilde{U}^N_s(y))\xi(dy, ds) - \frac{1}{2}\int_0^T \int |f(\widetilde{U}^N_s(y))|^2 dyds \Big).$$

By Girsanov's theorem ([6, Theorem 10.14]), since $f \in L^\infty$, the Novikov condition holds, and the random distribution $\xi(dy, ds) + f(\widetilde{U}^N_s(y))\, dy\, ds$ is a space–time white noise under $\mathbb{Q}$, independent of $\mathcal{F}_0$. Hence, under $\mathbb{Q}$, the process $(\widetilde{U}^N_t)_{t\in[0,T]}$ has law that equals that of $(\widetilde{O}_t)_{t\in[0,T]}$ under $\mathbb{P}$. To prove the statement, define first the functional

$$g(Z) := \sup_{R\in[0,T]} \left\| \int_0^R P^N_{R-s}[f(Z_s) - f(Z_{k_M(s)})]ds \right\|, \tag{3.30}$$

for $Z \in C([0,T]; L^\infty)$. Then it is sufficient to show that $(\mathbb{E}|g(\widetilde{U}^N)|^p)^{1/p}$. Using Cauchy Schwartz

$$\begin{aligned}\mathbb{E}[|g(\tilde{U}^N)|^p] = \mathbb{E}[\rho\rho^{-1}|g(\tilde{U}^N)|^p] &= \mathbb{E}_{\mathbb{Q}}[\rho^{-1}|g(\tilde{U}^N)|^p] \\ &\leq \mathbb{E}_{\mathbb{Q}}[\rho^{-2}]^{1/2}\mathbb{E}_{\mathbb{Q}}[|g(\tilde{U}^N)|^{2p}]^{1/2} \\ &= \mathbb{E}[\rho^{-1}]^{1/2}\mathbb{E}[|g(O + P.u_0^N)|^{2p}]^{1/2} \\ &\lesssim \mathbb{E}[|g(O + P.u_0^N)|^{2p}]^{1/2}.\end{aligned}$$

Note that here one uses the shorthand notation $\mathbb{E} := \mathbb{E}_{\mathbb{P}}$ and the fact that $\mathbb{E}[\rho^{-1}] \lesssim C(\|f\|_{L^\infty})$.The application of Corollary (3.0.1.1) with $2p$ instead of p follows directly. □

Using these results, the proof (3.0.1.3) follows directly

Proof of (3.0.1.3). By (3.6), since both bounds were constructed, the proof follows from application of (2.0.0.7) with $X = \tilde{U}^N$, $Y = \tilde{V}^{M,N}$,

$$Z_t = \int_0^t P_{t-s}^N[f(\tilde{U}_s^N) - f(\tilde{U}_{k_M(s)}^N)]ds, \tag{3.31}$$

$S(s,t) = P_{t-s}^N$, and $\tau(s) = k_M(s)$. Using (3.0.1.2) to bound Z, the claim is proved. □

Hence, the proof of (3.0.1) is complete.

Chapter 4

Maximally polynomial growing non-linearity

As mentioned in the introduction, in the second part of the investigation, the superlinearily growing nonlinearities f, that satisfy a one-sided Lipschitz condition are considered. A prime example of such nonlinearity is the Allen-Cahn equation, $f(x) = x - x^3$.

The assumption (1.0.2) immediately gives the local Lipschitz bound with polynomial growth and a global one-sided Lipschitz bound. Namely, by the mean value theorem, there exists c such that $\partial f(c) = \frac{f(x)-f(y)}{x-y}$, thus by applying the assumption and using $|c|^{2m} \lesssim |x|^{2m} + |y|^{2m}$ one has

$$|f(x) - f(y)| \leq K(1 + |x|^{2m} + |y|^{2m})|x - y|, \tag{4.1}$$

$$(x - y)(f(x) - f(y)) \leq K|x - y|^2. \tag{4.2}$$

Then the main result states

Theorem 4.0.1. *Let (1.0.2) hold. Let u be the unique mild solution of (1.1) and for any $M, N \in \mathbb{N}$, let $X^{M,N}$ be as above. Then for any $\epsilon > 0$ and $p \in [1, \infty)$ there exists a constant $C = C(T, \epsilon, p, K, \mathscr{M})$ such that*

$$\big(\mathbb{E} \sup_{t\in[0,T]} \|u_t - X_t^{M,N}\|^p_{L^2(\mathbb{T})}\big)^{1/p} \leq C\big(N^{-1/2+\epsilon} + M^{-1+\epsilon}\big). \tag{4.3}$$

4.0.1 Constructing a priori bounds

Before proving the theorem, some important bounds on the regularity of the auxiliary functions need to be established. Namely, recall numerical discretisation that the mild solution of the SPDE with nonlinearity g_h is given by

$$X_t^h = P_t u_0 + \int_0^t P_{t-s} g_h(X_s^h) ds + O_t. \tag{4.4}$$

with its Galerkin approximation, $X^{h,N}$, given by

$$X_t^{h,N} = P_t^N u_0 + \int_0^t P_{t-s}^N g_h(X_s^{h,N}) ds + O_t^N. \tag{4.5}$$

Like f, g_h also has a local Lipschitz condition, a polynomial growth and a one-sided Lipschitz condition, but Φ_h is globally Lipschitz continuous. Consider, the following lemma

Lemma 4.0.1.1. *Let f satisfy assumption (1.0.2) (a). Let Φ_h, g_h be given as in (??), (2.29) Then there exist constants $C, \tilde{K} > 0$ and $\tilde{m} \geq m$, depending only on K and m, such that for $i = 0, 1, 2, 3$ and for all $h \in [0, 1]$, the functions Φ_h and g_h satisfy*

$$\begin{aligned}
|\Phi_h(x) - \Phi_h(y)| &\leq e^{Kh/2}|x - y|, \quad \forall x, y \in \mathbb{R} \\
|\partial^i g_h(x)| &\leq \tilde{K}(1 + |x|^{2\tilde{m}+1-i}), \quad \forall x \in \mathbb{R}, \quad i = 0, 1, 2, 3 \\
\partial g_h(x) &\leq K, \quad \forall x \in \mathbb{R} \\
|g_h(x) - g_0(x)| &\leq Ch(1 + |x|^{4m+2}), \quad \forall x \in \mathbb{R}.
\end{aligned}$$

In particular, g_h satisfies

$$(g_h(x) - g_h(y))(x - y) \leq K(x - y)^2, \quad \forall x, y \in \mathbb{R} \tag{4.6}$$

$$|g_h(x) - g_h(y)| \leq \tilde{K}(1 + |x|^{2\tilde{m}} + |y|^{2\tilde{m}})|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{4.7}$$

Proof. See Appendix (A) □

The lemma allows us to obtain the bounds on the mild solutions X^h, $X^{h,N}$ via

Corollary 4.0.1.1. *Let (1.0.2) hold. Then there exist unique mild solutions X^h and $X^{h,N}$ to the equations (4.4) and (4.5), respectively. Moreover, for any $p \geq 1$, $\epsilon \in (0, 1/2)$, $\lambda \in (0, 1)$, there exists a constant $C = C(T, m, K, \epsilon, \lambda, p, \mathscr{M})$ such that the following bound holds*

$$\sup_{M \in \mathbb{N}} \mathbb{E}\|X^h\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} + \sup_{M,N \in \mathbb{N}} \mathbb{E}\|X^{h,N}\|^p_{C_T^{\lambda/2}\mathscr{C}^{1/2-\lambda-\epsilon}} \leq C.$$

Proof. One can prove the statement for two solutions separately. For X^h, one can note that the Proposition (2.0.0.6) holds under the Assumption (1.0.2), and since, by Lemma (4.0.1.1), g_h satisfies the Assumption (1.0.2), the statement follows directly.

For the second solution, one first notices that the operators $A = \Pi_N \Delta$ and $Q = \Pi_N$, with the spaces $H = L^2(\mathbb{T}), E = C(\mathbb{T})$ satisfy the conditions in Hypothesis 6.1 in [10]. The proof from Proposition (2.0.0.6) follows directly. □

Proposition 4.0.1.1. *Let Assumption (1.0.2) hold. Let $p \geq 1$ and $X^{M,N}$ be as in (2.31). Then there exists a constant $C = C(p, K, m, T, \mathscr{M})$ such that the following a priori bound holds*

$$\sup_{M,N\in\mathbb{N}} \mathbb{E} \sup_{k=0,\dots,M} \|X^{M,N}_{t_k}\|^p_{L^\infty} \leq C.$$

Proof. Consider $R^{M,N}_{t_k} = X^{M,N}_{t_k} - O^N_{t_k}$. Using the bounds given in Lemma (4.0.1.1), namely Lipschitz condition on Φ_h and the growth bound on g_h, via $\Phi_h(z) - z = hg_h(z)$, as well as the fact that P_h is bounded on L^∞ with norm 1, one achieves the following

$$\begin{aligned}
\|R^{M,N}_{t_{k+1}}\|_{L^\infty} = \|X^{M,N}_{t_k} - O^N_{t_k}\| &= \|P^N_h(\Phi_h(R^{M,N}_{t_k} + O^N_{t_k}) - \Phi_h(O^N_{t_k})) + P^N_h(\Phi_h(O^N_{t_k}) - O^N_{t_k})\|_{L^\infty} \\
&\leq \|P^N_h(\Phi_h(R^{M,N}_{t_k} + O^N_{t_k}) - \Phi_h(O^N_{t_k}))\|_\infty + \|P^N_h(\Phi_h(O^N_{t_k}) - O^N_{t_k})\|_{L^\infty} \\
&\leq e^{Kh/2}\|R^{M,N}_{t_k}\|_{L^\infty} + h\tilde{K}^p(1 + \|O^N\|^{(2\tilde{m}+1)}_{C_T L^\infty}) \\
&\vdots \\
&\leq e^{(k+1)Kh/2}\|\Pi_N u_0\|_{L^\infty} + \tilde{K}\sum_{j=0}^{k} e^{jKh/2}h(1 + \|O^N\|^{(2\tilde{m}+1)}_{C_T L^\infty}),
\end{aligned}$$

Finally, one has that since $(k+1)h \leq Mh = T$ and $\sum_{j=0}^k e^{jKh/2} \leq (k+1)e^{kKh/2}$, gives that

$$\sup_{k=0,\dots,M} \|R^{M,N}_{t_k}\|_{L^\infty} \leq e^{KT/2}\|u_0\|_{L^\infty} + \tilde{K}Te^{KT/2}\left(1 + \|O^N\|^{2\tilde{m}+1}_{C_T L^\infty}\right).$$

Using $\|O^N\|^p_{C_T L^\infty} = \|\Pi_N O\|^p_{C_T L^\infty} \leq \|O\|^p_{C_T L^\infty}$ to drop the N-dependence:

$$\sup_k \|R^{M,N}_{t_k}\|_{L^\infty} \leq C_1\left(1 + \|u_0\|_{L^\infty} + \|O\|^{2\tilde{m}+1}_{C_T L^\infty}\right),$$

□

Corollary 4.0.1.2. *Let (1.0.2) hold. Let $X^{M,N}$ be as in (2.31). Then for any $p \geq 1$, $\lambda \in (0,1)$, $\epsilon \in (0, 1/2)$, there exists a constant $C = C(p, T, \lambda, \epsilon, K, m, \mathscr{M})$ such that*

$$\sup_{M,N\in\mathbb{N}} \mathbb{E}\|X^{M,N}\|^p_{C^{\lambda/2}_T \mathscr{C}^{1/2-\lambda-\epsilon}} \leq C. \tag{4.8}$$

Let $R^{M,N} := X^{M,N} - O^N$. Then for any $\alpha \in (0,2)$, $\epsilon \in (0, 1-\alpha/2)$, there exists a constant $C = C(p, T, \epsilon, \alpha, K, m, \mathscr{M})$ such that,

$$\sup_{M,N\in\mathbb{N}} \mathbb{E}\|R^{M,N}\|^p_{C^{1-\alpha/2-\epsilon}_T \mathscr{C}^\alpha} \leq C.$$

Proof. To estimate the first bound, we use Proposition (4.0.1.1) it is completely analogous to how in Proposition (2.0.0.6) bound (2.12) is used to prove (2.11).

For the second bound, however,first write $R_t^{M,N}$ in its mild form as

$$R_t^{M,N} = P_t^N u_0 + \int_0^t P_{t-k_M(r)}^N g_h\big(X_{k_M(r)}^{M,N}\big)\,dr$$

which gives directly

$$R_t^{M,N} - R_s^{M,N} = (P_t^N - P_s^N)u_0 + \int_s^t P_{t-k_M(r)}^N g_h\big(X_{k_M(r)}^{M,N}\big)\,dr + \int_0^s P_{s-k_M(r)}^N (P_{t-s}^N - \mathrm{Id})\, g_h\big(X_{k_M(r)}^{M,N}\big)\,dr, \tag{4.9}$$

so each term is bounded separately in $\mathscr{C}^\alpha$, as in previous results. The first term is bounded using the (2.0.0.4) as in previous results, giving

$$\|(P_t^N - P_s^N)u_0\|_{\mathscr{C}^\alpha} \lesssim |t-s|^{1-\alpha/2}\,\|u_0\|_{\mathscr{C}^2}.$$

For the second term, consider the following expression

$$\begin{aligned}\Big\|\int_s^t P_{t-k_M(r)}^N g_h(\cdot)\,dr\Big\|_{\mathscr{C}^\alpha} &\lesssim \int_s^t (t-r)^{-\alpha/2}\,\|g_h(X_{k_M(r)}^{M,N})\|_{L^\infty}\,dr \quad \text{using (2.0.0.4) and that } \mathscr{C}^0 = L^\infty\\ &\lesssim (t-s)^{1-\alpha/2}\Big(1+\sup_k \|X_{t_k}^{M,N}\|_{L^\infty}^{2\tilde{m}+1}\Big) \quad \text{using } t-k_M(r) \geq t-r\end{aligned}$$

Finally, Fix $\epsilon \in (0, 1-\alpha/2)$. Then, same as with the second term

$$\Big\|\int_0^s P_{s-k_M(r)}^N (P_{t-s}^N - \mathrm{Id}) g_h(X_{k_M(r)}^{M,N})dr\Big\|_{\mathscr{C}^\alpha} \lesssim \int_0^s (s-k_M(r))^{-1+\epsilon}\|(P_{t-s}^N - \mathrm{Id}) g_h(X_{k_M(r)}^{M,N})\|_{\mathscr{C}^{-(2-\alpha-2\epsilon)}}dr$$

Therefore, using (2.0.0.4),

$$\|(P_{t-s}^N - \mathrm{Id})g_h(\cdot)\|_{\mathscr{C}^{\alpha-2\epsilon}} \lesssim (t-s)^{1-\alpha/2-\epsilon}\,\|g_h(\cdot)\|_{\mathscr{C}^0} \lesssim (t-s)^{1-\alpha/2-\epsilon}\,\|g_h(\cdot)\|_{L^\infty}.$$

Using $s - k_M(r) \geq s-r$ so that $\int_0^s (s-k_M(r))^{-1+\epsilon}dr \lesssim 1$, and the growth bound for g_h.

$$\begin{aligned}\Big\|\int_0^s P_{s-k_M(r)}^N (P_{t-s}^N - \mathrm{Id})\, g_h(X_{k_M(r)}^{M,N})\,dr\Big\|_{\mathscr{C}^\alpha} &\lesssim (t-s)^{1-\alpha/2-\epsilon}\int_0^s (s-k_M(r))^{-1+\epsilon}\,\|g_h(X_{k_M(r)}^{M,N})\|_{L^\infty}\,dr\\ &\lesssim (t-s)^{1-\alpha/2-\epsilon}\Big(1+\sup_k \|X_{t_k}^{M,N}\|_{L^\infty}^{2\tilde{m}+1}\Big),\end{aligned}$$

Hence, combining the estimates on achieves

$$\mathbb{E}\|R_t^{M,N} - R_s^{M,N}\|_{\mathscr{C}^\alpha}^p \lesssim |t-s|^{p(1-\alpha/2-\epsilon)}.$$

since by Proposition (4.0.1.1) $\sup_{M,N}\mathbb{E}\Big[\sup_k \|X_{t_k}^{M,N}\|_{L^\infty}^{p(2\tilde{m}+1)}\Big] < C.$ □

4.0.2 Proof of the main result

One is now in a position to begin the proof of Theorem (4.0.1), i.e. error bound of $M^{-1+\epsilon} + N^{-1/2+\epsilon}$ is achieved in the superlinearly growing case. The strategy is significantly more involved, since the Grönwall inequality cant always be applied to due to non-regularity and the Girsanov theory doesn't apply, since the f no longer satisfies the Novikov condition. The first correction that one needs to consider is that due to the lack of a global Lipschitz condition, one has to look at variational forms of certain equations. Consider the following definition

Definition 4.0.1. *Let $F, G \in C([0,T]\times\mathbb{T}), H \in C(\mathbb{T})$, then the mild solution of the equation*

$$\partial_t F = \Delta F + G, \qquad F_0 = H,$$

is given by

$$F(t,x) = \int_{\mathbb{T}} p_t(x-y)H(y)dy + \int_0^t \int_{\mathbb{T}} p_{t-s}(x-y)G(s,y)dsdy$$

satisfying

$$\partial_t \|F(t,\cdot)\|_{L^2}^2 \le 2\langle F(t,\cdot), G(t,\cdot)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the $L^2(\mathbb{T})$ inner product.

Note, that 1) multiplying by $\|F(t,\cdot)\|_{L^2}^{p-2}$ one immediately gets that

$$\partial_t \|F(t,\cdot)\|_{L^2}^p \le p\|F(t,\cdot)\|_{L^2}^{p-2}\langle F(t,\cdot), G(t,\cdot)\rangle,$$

and 2) the same conclusion holds for the projected P^N. Additionally, to constrain growing nonlinearities, i.e. to cancel out the polynomially growing f, some weighted spaces are to be introduced.

Definition 4.0.2. *For $k \in \mathbb{N}_0$, the weighted Hölder space is defined as*

$$C_\omega^k = \{f \in \mathcal{S}' \mid \omega f \in C_b^k\}, \quad \|f\|_{C_\omega^k} := \|\omega f\|_{C_b^k},$$

where $\omega(x) = (1+|x|^2)^{-\beta/2}$, $\beta \ge 1$, are polynomial weights.

For the weighted space the following semigroup estimate holds

$$\|(P_t^{\mathbb{R}} - P_s^{\mathbb{R}})u\|_{C_\omega^0} \lesssim |t-s|\|u\|_{C_\omega^2}. \tag{4.10}$$

Indeed, the equation is a direct consequence of (2.6). Note, additionally, that by the conditions in (4.0.1.1), that the derivatives of g_h up to order 3 satisfy polynomial growth bound, there exists ω with $g_h \in C_\omega^3$t. **Fix such weight**[1]. Similar to the previous chapter, one starts by decomposing the error, and then proving the results for each part of such decomposition

$$\mathbb{E}[\sup_{t\in[0,T]} \|u_t - X_t^{M,N}\|_{L^2}^p] \lesssim \mathbb{E}[\sup_{t\in[0,T]} \|u_t - X_t^h\|_{L^2}^p] + \mathbb{E}[\sup_{t\in[0,T]} \|X_t^h - X_t^{h,N}\|_{L^2}^p] + \mathbb{E}[\sup_{t\in[0,T]} \|X_t^{h,N} - X_t^{M,N}\|_{L^2}^p].$$

Starting with the first term in the expression

Lemma 4.0.1.2. *Let (1.0.2) hold. Then for any $p \ge 1$, $\epsilon \in (0,1/2)$, there exists a constant $C =$*

[1]One can choose $\beta > 2\tilde{m}+1$, then the $\sup \omega(x)|g^{(i)}(x)|$ is bounded using (4.0.1.1)

$C(T,\epsilon,p,m,K,\mathscr{M})$ *such that*

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^h - X_t^{h,N}\|_{L^2}^p \le CN^{-p(1/2-\epsilon)}.$$

Proof. First, similar to (4.0.1.2), one splits the difference by introducing $R^h := X^h - O$ and $R^{h,N} := X^{h,N} - O^N$, then trivially $\|X_t^h - X_t^{h,N}\|_{L^2}^p \lesssim \|R_t^h - R_t^{h,N}\|_{L^2}^p + \|O_t^h - O_t^{h,N}\|_{L^2}^p$. The OU process bound is proved in (2.0.0.5). For the R bound, one can apply definition (4.0.1), by taking $R^h - R^{h,N}$ as F, $H = u_0 - \Pi_N u_0$, and $G = g_h(X^h) - \Pi_N(g_h(X^{h,N}))$. Then the bound satisfies the following:

$$\|R_t^h - R_t^{h,N}\|_{L^2}^p \le \|u_0 - \Pi_N u_0\|_{L^2}^p + p\int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2}\langle g_h(X_s^h) - \Pi_N(g_h(X_s^{h,N})), R_s^h - R_s^{h,N}\rangle ds.$$

Now, one can add and substract the term $g_h(X_s^{h,N})$, so

$$G_t = g_h(X_t^h) - g_h(X_t^{h,N}) + g_h(X_t^{h,N}) - \Pi_N g_h(X_t^{h,N})$$

Splitting the inner product, the first part of the inner product taken with $R^h - R^{h,N}$, and using its definition, becomes $\langle g_h(X_t^h) - g_h(X_t^{h,N}), R_s^h - R_s^{h,N}\rangle = \langle g_h(X_t^h) - g_h(X_t^{h,N}),\, X_t^h - X_t^{h,N}\rangle - \langle g_h(X_t^h) - g_h(X_t^{h,N}),\, O_t - O_t^N\rangle$. So the whole thing becomes

$$\begin{aligned}\|R_t^h - R_t^{h,N}\|_{L^2}^p \le{}& \|u_0 - \Pi_N u_0\|_{L^2}^p \\ &+ p\int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2}\langle g_h(X_s^h) - g_h(X_s^{h,N}),\, X_s^h - X_s^{h,N}\rangle ds \\ &- p\int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2}\langle g_h(X_s^h) - g_h(X_s^{h,N}),\, O_s^N - O_s\rangle ds \\ &+ p\int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2}\langle g_h(X_s^{h,N}) - \Pi_N(g_h(X_s^{h,N})),\, R_s^h - R_s^{h,N}\rangle ds\end{aligned}$$

In this expression, for the second inner product, using the condition from (4.0.1.1), one has

$$\|g_h(X_t^h) - g_h(X_t^{h,N})\| \le C\,(1 + \|X_t^h\|_{L^\infty}^m + \|X_t^{h,N}\|_{L^\infty}^m)\,\|X_t^h - X_t^{h,N}\|$$

Then, using Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\langle g_h(X_t^h) - g_h(X_t^{h,N}),\, O_t - O_t^N\rangle\right| &\le \|g_h(X_t^h) - g_h(X_t^{h,N})\|\,\|O_t - O_t^N\| \\ &\lesssim (1 + \|X_t^h\|_{L^\infty}^m + \|X_t^{h,N}\|_{L^\infty}^m)\|X_t^h - X_t^{h,N}\|\,\|O_t - O_t^N\|\end{aligned} \tag{4.11}$$

Similarly, by Cauchy -Schwartz and Youngs inequality for the first and last parts of the expression, with

$p_1 = p/(p-1)$ and $p_2 = p$, results in

$$
\begin{aligned}
\|R_t^h - R_t^{h,N}\|_{L^2}^p \lesssim \|u_0 - \Pi_N u_0\|_{L^2}^p \\
&+ \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2} \|X_s^h - X_s^{h,N}\|_{L^2}^2 \, ds \\
&+ (1 + \|X^h\|_{C_T L^\infty}^{2\tilde{m}} + \|X^h\|_{C_T L^\infty}^{2\tilde{m}}) \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2} \|X_s^h - X_s^{h,N}\|_{L^2} \|O_t - O_t^N\|_{L^2} \, ds \\
&+ \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^p \, ds + \int_0^t \|g_h(X_s^{h,N}) - \Pi_N(g_h(X_s^{h,N}))\|_{L^2}^p \, ds.
\end{aligned}
$$

Now, using the fact that $\|\Pi_N u - u\|_{L^2} \lesssim N^{-\alpha}\|u\|_{H^\alpha}$[2], $\alpha \geq 0$,(recall that $H^\alpha = B_{2,2}^\alpha$) and that $\|X_t^h - X_t^{h,N}\|_{L^2}^p \lesssim \|R_t^h - R_t^{h,N}\|_{L^2}^p + \|O_t^h - O_t^{h,N}\|_{L^2}^p$, one further obtains

$$
\begin{aligned}
\|R_t^h - R_t^{h,N}\|_{L^2}^p \lesssim N^{-\alpha p}\|u_0\|_{H^\alpha} + \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^p ds \\
&+ (1 + \|X^h\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}}) \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-2} \|O_t - O_t^N\|_{L^2}^2 ds \\
&+ (1 + \|X^h\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}}) \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^{p-1} \|O_t - O_t^N\|_{L^2} ds \\
&+ N^{-p\alpha}\|g_h(\tilde{X}^{h,N})\|_{C_T H^\alpha}^p.
\end{aligned}
$$

Finally, one can apply the Young's inequality twice with $p_1 = p/(p-1)$ and $p_2 = p$ and for $p_1 = p/(p-2)$ and $p_2 = p/2$ to achieve

$$
\begin{aligned}
\|R_t^h - R_t^{h,N}\|_{L^2}^p \lesssim N^{-\alpha p}\|u_0\|_{H^\alpha} + \int_0^t \|R_s^h - R_s^{h,N}\|_{L^2}^p ds \\
&+ (1 + \|X^h\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}})^p \|O - O^N\|_{C_T L^\infty}^p \\
&+ N^{-p\alpha}\|g_h(\tilde{X}^{h,N})\|_{C_T H^\alpha}^p.
\end{aligned}
$$

Using the embedding of Besov spaces $\mathcal{C}^\beta \hookrightarrow H^{\beta'}$ for $\beta > \beta'$,for any $\alpha \in (0, 1/2)$ and $\epsilon' \in (0, 1/2 - \alpha)$ one has

$$
\|g_h(X^{h,N})\|_{C_T H^\alpha} \lesssim \|g_h(X^{h,N})\|_{C_T \mathscr{C}^{\alpha+\epsilon'}} \lesssim (1 + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}})\|X^{h,N}\|_{C_T \mathscr{C}^{\alpha+\epsilon'}},
$$

where again, in the last inequality one uses the bound for g_h' via (4.0.1.1). Following this bound and applying the Grönwall inequality, as well as (2.7) the expression becomes

$$
\begin{aligned}
\|R_t^h - R_t^{h,N}\|_{L^2}^p \lesssim N^{-p\alpha}\|u_0\|_{\mathscr{C}^{1/2}} + N^{-p\alpha}\|O\|_{C_T \mathscr{C}^{\alpha+\epsilon'}}^p (1 + \|X^h\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}})^p \\
&+ N^{-p\alpha}(1 + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}})^p \|X^{h,N}\|_{C_T \mathscr{C}^{\alpha+\epsilon'}}^p.
\end{aligned}
$$

Due to Corollary (4.0.1.1), claim follows directly. □

[2] $\|u - \Pi_N u\|_{L^2}^2 = \sum_{|k|>N} |\hat{u}_k|^2 \leq N^{-2\alpha}\|u\|_{H^\alpha}{}^2$

Lemma 4.0.1.3. *Let (1.0.2) hold and let $p \geq 1$. Then there exists a constant $C = C(T, K, m, p, \mathscr{M})$ such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|u_t - X_t^h\|_{L^2}^p \leq CM^{-p}.$$

Proof. Again, one can use (4.0.1) with $F = u - X^h$, $G = f(u) - g_h(X^h)$, and $H = 0$. Applying the variational estimate, then using the fact that $g_0(u) = f(u)$, one can use the bounds on g_h and $g_h - g_0$ from (4.0.1.1), to obtain for $p \geq 2$, $t \in [0, T]$,

$$\begin{aligned}\|u_t - X_t^h\|_{L^2}^p &\leq p \int_0^t \|u_s - X_s^h\|_{L^2}^{p-2} \langle u_s - X_s^h, g_h(u_s) - g_h(X_s^h)\rangle ds \\ &\quad + p \int_0^t \|u_s - X_s^h\|_{L^2}^{p-2} \langle u_s - X_s^h, g_0(u_s) - g_h(u_s)\rangle ds \\ &\lesssim \int_0^t \|u_s - X_s^h\|_{L^2}^p ds + h(1 + \|u\|_{C_T L^\infty}^{4m+2}) \int_0^t \|u_s - X_s^h\|_{L^2}^{p-1} ds.\end{aligned}$$

Then, via Young's inequality with $p_1 = p$ and $p_2 = p/(p-1)$ to the second term, applying Grönwall's inequality, taking expectations, and using the a priori bounds from (2.0.0.6), the claim follows. □

It now remains to prove the last bound, which is substantially more involved that it was in the bounded case. In particular, since g_h has polynomial growth, the bound on the inner product is not trivial, so the Grönwall inequality can't be applied at the level of mild formulation of the solution. So, one has to switch to the variational formulation, and estimate the remainder term via stochastic sewing lemma. Consider the following result

Lemma 4.0.1.4. *Let (1.0.2) hold. Then for any $p \geq 1$, $\epsilon \in (0, 1/4)$, there exists a constant $C = C(T, \epsilon, p, K, \mathscr{M})$ such that the following bound holds*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t^{h,N} - X_t^{M,N}\|_{L^2}^p \leq C\big(M^{p(-1+\epsilon)} + N^{p(-1/2+\epsilon)}\big).$$

To prove the result, completely analogous to the bounded case, one introduces a supporting result, similar to how (3.0.1.2) was used.

Proposition 4.0.1.2. *Let (1.0.2) hold. Then for any $p \geq 1$, $\epsilon \in (0, 1/4)$, there exists a constant $C = C(T, \epsilon, p, K, \mathscr{M})$ such that the following bound holds*

$$\mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t P_{t-s}^N [g_h(X_s^{M,N}) - g_h(X_{k_M(s)}^{M,N})] ds \right\|_{L^\infty}^p \leq CM^{p(-1+\epsilon)}.$$

Proof. Proposition (4.0.1.2) To prove the proposition, one once again uses the Stochastic Sewing Lemma (2.0.0.1). First, note that like in the Proposition (3.0.1.2), to prove the statement it is sufficient to show that for any $j \geq -1$, $x \in \mathbb{T}$, $0 \leq s \leq t \leq R \leq T$, one has

$$\mathbb{E}\Big| \int_s^t \Delta_j P^N_{R-r}[g_h(R^{M,N}_r + O^N_r) - g_h(R^{M,N}_{k_M(r)} + O^N_{k_M(r)})]dr\Big|^p \leq C2^{-jp\epsilon} M^{p(-1+2\epsilon)} |t-s|^{1/4+\epsilon/2}. \tag{4.12}$$

To apply the sewing lemma, set

$$A_{st} = \int_s^t \mathbb{E}_s[\Delta_j P^N_{R-r}(g_h(\mathbb{E}_s R^{M,N}_r + O^N_r) - g_h(\mathbb{E}_s R^{M,N}_{k_M(r)} + O^N_{k_M(r)}))](x)dr,$$

for $j \geq -1$, $x \in \mathbb{T}$, $R \leq T$. So by the lemma, it is sufficient to show that for all $0 \leq s < u < t < R$ such that $|t-s| \leq |R-t|$, one has

$$\|A_{ut}\|_{L^p(\Omega)} \lesssim 2^{-j\epsilon} M^{-1+2\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)^{1/2+\epsilon} \tag{4.13}$$

$$\|\mathbb{E}_s \delta A_{s,u,t}\|_{L^p(\Omega)} \lesssim 2^{-j\epsilon} M^{-1+2\epsilon}(R-t)^{-1/4-\epsilon/2}(t-s)^{1+\epsilon}. \tag{4.14}$$

Let the shorthand notation be $\mathscr{R}_{u,r} = \mathbb{E}_u R^{M,N}_r$. As with the bounded case, to prove (4.0.2), consider two cases: $|t-u| < 3M^{-1}$ and $|t-u| > 3M^{-1}$. In the first case, the result is completely analogous to (3.0.1.2), where $\tilde{O}$ is replaced $\mathscr{R}_{u,\cdot} + O^N$, as long as one has the analogue for the bound (2.8) for $\mathscr{R}_{u,\cdot} + O^N$. To that end,first notice that using the Jensens inequality and for any $q \geq 1$

$$\big(\mathbb{E}\|\mathscr{R}_{s,r} - \mathscr{R}_{s,k_M(r)}\|^q_{L^\infty}\big)^{1/q} \leq \big(\mathbb{E}\|R^{M,N}_r - R^{M,N}_{k_M(r)}\|^q_{L^\infty}\big)^{1/q} \lesssim_q (r - k_M(r))^{1-\epsilon}. \tag{4.15}$$

so naturally, the time difference bound in Hölder space gives:

$$\big(\mathbb{E}\|\mathscr{R}_{u,r} + O^N_r - (\mathscr{R}_{u,k_M(r)} + O^N_{k_M(r)})\|^q_{\mathscr{C}^{-1/2+\epsilon}}\big)^{1/q} \lesssim_q (r - k_M(r))^{1/2-\epsilon}, \tag{4.16}$$

where triangle inequality and the a priori bound on O^N are also used. Applying the same bound on the smoothed version, in combination with (2.9) where $\theta = 1 - 4\epsilon$, $r - k_M(r)$ is in place of s, and $k_M(r) - u$ in place of t, one gets

$$\big(\mathbb{E}\|\mathscr{R}_{u,r} + P_{r-u}O^N_u - (\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O^N_u)\|^q_{\mathscr{C}^{-1/2+\epsilon}}\big)^{1/q} \lesssim_q (r - k_M(r))^{1-2\epsilon}(k_M(r) - u)^{-1/2+\epsilon}, \tag{4.17}$$

By applying the same methods as in (3.0.1.2), i.e. the mean value theorem, (3.16) is replaced by

$$\mathbb{E} \sup_{r \in [0,R]} \|g'_h(\lambda(\mathscr{R}_{u,r} + O^N_r) + (1-\lambda)(\mathscr{R}_{u,k_M(r)} + O^N_{k_M(r)}))\|^q_{\mathscr{C}^\theta} \lesssim \mathbb{E}\big((1 + \|X^{M,N}\|^{2\tilde{m}-1}_{C_T L^\infty})\|X^{M,N}\|_{C_T\mathscr{C}^\theta}\big)^q \lesssim_{q,\theta} 1,$$

for $\theta \in (0, 1/2)$,which follows from the bound for g''_h from (4.0.1.1). Thus, the rest of the proof following from (3.0.1.2), the bound is proved in the case with $|t-u| < 3M^{-1}$.

For the second case, one, again, further splits the integral domain into $\int_u^{t'}$ and $\int_{t'}^t$, such that $|t'-u| < 3M^{-1}$. Then it remains only to prove the bound for the second integral. As for the bounded case, recall that for $r \geq t'$

we have $k_M(r) - u \geq (r-u)/2$ and that $\mathscr{R}_{u,r}$ and $\mathscr{R}_{u,k_M(r)}$ are $\mathcal{F}_u$-measurable. Thus, similarly to (3.18) that the time integral from t' to t equals

$$\begin{aligned}
&\int_{t'}^{t} \Delta_j P_{R-u}\mathbb{E}_u[(g_h(\mathscr{R}_{u,r} + O_r^N) - g_h(\mathscr{R}_{u,k_M(r)} + O_{k_M(r)}^N))]dr \\
&\quad = \int_{t'}^{t} \Delta_j P_{R-u}(P^{\mathbb{R}}_{Q^N(r-u)} g_h)(\mathscr{R}_{u,r} + P_{r-u}O_u^N) \\
&\qquad\qquad\qquad\qquad - (P^{\mathbb{R}}_{Q^N(k_M(r)-u)} g_h)(\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)dr \\
&\quad = \int_{t'}^{t} \Delta_j P^N_{R-r}[(P^{\mathbb{R}}_{Q^N(r-u)} g_h)(\mathscr{R}_{u,r} + P_{r-u}O_u^N) \\
&\qquad\qquad\qquad\qquad - (P^{\mathbb{R}}_{Q^N(r-u)} g_h)(\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)](x)dr \\
&\qquad + \int_{t'}^{t} \Delta_j P^N_{R-r}[[(P^{\mathbb{R}}_{Q^N(r-u)} g_h) - (P^{\mathbb{R}}_{Q^N(k_M(r)-u)} g_h)](\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)](x)dr, \\
&\quad =: I_1 + I_2,
\end{aligned}$$

where the variance is again $Q^N(t-s) = \mathbb{E}\Big(\int_s^t \int_{\mathbb{T}} p^N_{t-r}(x-y)\xi(dr,dy)\Big)^2$. Each integral is to be proved separately. Starting with I_1, one uses identical logic as in (3.21), however, a modification of (2.9) is used again, given by (4.17), as well as the bound

$$\mathbb{E}\|(P^{\mathbb{R}}_{Q^N(r-u)} g_h)'(\lambda(P_{r-u}O_u^N + \mathscr{R}_{u,r}) + (1-\lambda)(P_{k_M(r)-u}O_u^N + \mathscr{R}_{u,k_M(r)}))\|^q_{\mathscr{C}^\theta} \lesssim_{\theta,q} 1$$

for $\theta \in (0,1/2)$, $q \geq 1$, in place of (3.20), the proof of which is also the same. This gives

$$(\mathbb{E}|I_1|^p)^{1/p} \lesssim 2^{-j\epsilon} M^{-1+2\epsilon}(R-t)^{-1/4-\epsilon/2}(t-u)^{1/2+\epsilon}, \tag{4.18}$$

hence, proving the first bound. For the I_2,once again, modify the expression (3.23), via the following bound

$$\begin{aligned}
&\mathbb{E}\bigg|\int_{t'}^{t} \Delta_j P^N_{R-r}[[(P^{\mathbb{R}}_{Q^N(r-u)} g_h) - (P^{\mathbb{R}}_{Q^N(k_M(r)-u)} g_h)](\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)](x)dr\bigg|^p \\
&\lesssim 2^{-j\epsilon p}\mathbb{E}\bigg(\int_{t'}^{t} (R-r)^{-\epsilon/2}\|[(P^{\mathbb{R}}_{Q^N(r-u)} g_h) - (P^{\mathbb{R}}_{Q^N(k_M(r)-u)} g_h)](\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)\|_{L^\infty}dr\bigg)^p \\
&\lesssim 2^{-j\epsilon p}\mathbb{E}\bigg(\int_{t'}^{t} (R-r)^{-\epsilon/2}\|(P^{\mathbb{R}}_{Q^N(r-u)} g_h) - (P^{\mathbb{R}}_{Q^N(k_M(r)-u)} g_h)\|_{C^0_\omega}\|\omega^{-1}(\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)\|_{L^\infty}dr\bigg)^p,
\end{aligned}$$

where in the last step one uses the modification of canonical norm to weighted spaces[3]. One can apply the kernel bounds (4.10) and as in bounded case use (3.22), to get

$$\begin{aligned}
\|(P^{\mathbb{R}}_{Q^N(r-u)} g_h) - (P^{\mathbb{R}}_{Q^N(k_M(r)-u)} g_h)\|_{C^0_\omega} &\lesssim (Q^N(r-u) - Q^N(k_M(r)-u))\|g_h\|_{C^2_\omega} \\
&\lesssim (k_M(r)-u)^{-1/2+\epsilon}(r-k_M(r))^{1-\epsilon}\|g_h\|_{C^2_\omega}
\end{aligned}$$

and

$$\|\omega^{-1}(\mathscr{R}_{u,k_M(r)} + P_{k_M(r)-u}O_u^N)\|_{L^\infty} \lesssim (1 + \|O^N\|^2_{C_T L^\infty} + \|R^{M,N}\|^2_{C_T L^\infty})^{\beta/2}.$$

[3] $\|f\|_{C^k_\omega} := \|\omega f\|_{C^k_b} = \sum_{|\alpha|\leq k}\sup_{x\in\mathbb{R}^d}|D^\alpha(\omega(x)f(x))|$, so in particular $\|f\|_{C^0_\omega} = \sup_x \omega(x)|f(x)|$, which implies $|f(y)| \leq \omega(y)^{-1}\|f\|_{C^0_\omega}$.

Applying the bounds together with a priori bound on X the bound on I_2 becomes

$$(\mathbb{E}|I_2|^p)^{1/p} \lesssim 2^{-j\epsilon} M^{-1+\epsilon}(R-t)^{-\epsilon/2}(t-u)^{1/2+\epsilon}. \tag{4.19}$$

and completes the proof of (4.0.2).

It remains to prove (4.14). To prove the claim one needs to consider bounds constructed in (A.0.1). Recall that

$$\mathbb{E}_s\delta A_{sut} = \Delta_j \int_u^t \mathbb{E}_s\mathbb{E}_u P^N_{R-r}\big(g_h(\mathscr{R}_{s,r}+O^N_r) - g_h(\mathscr{R}_{s,k_M(r)}+O^N_{k_M(r)}) \\ - (g_h(\mathscr{R}_{u,r}+O^N_r) - g_h(\mathscr{R}_{u,k_M(r)}+O^N_{k_M(r)}))\big)(x)dr,$$

is to be bounded. To use established bounds, start by decomposing the integrand as

$$\begin{aligned}
g_h(\mathscr{R}_{s,r}+O^N_r) &- g_h(\mathscr{R}_{s,k_M(r)}+O^N_{k_M(r)}) - (g_h(\mathscr{R}_{u,r}+O^N_r) - g_h(\mathscr{R}_{u,k_M(r)}+O^N_{k_M(r)}))\\
&= (\mathscr{R}_{s,r}-\mathscr{R}_{u,r})\int_0^1 g_h'\Big(\lambda(\mathscr{R}_{s,r}+O^N_r)+(1-\lambda)(\mathscr{R}_{u,r}+O^N_r)\Big)\,d\lambda\\
&\quad + (\mathscr{R}_{s,k_M(r)}-\mathscr{R}_{u,k_M(r)})\int_0^1 g_h'\Big(\lambda(\mathscr{R}_{s,k_M(r)}+O^N_{k_M(r)})+(1-\lambda)(\mathscr{R}_{u,k_M(r)}+O^N_{k_M(r)})\Big)\,d\lambda\\
&= (\mathscr{R}_{s,r}-\mathscr{R}_{u,r})\int_0^1 \Big[g_h'\Big(\lambda(\mathscr{R}_{s,r}+O^N_r)+(1-\lambda)(\mathscr{R}_{u,r}+O^N_r)\Big) - g_h'\Big(\lambda(\mathscr{R}_{s,k_M(r)}+O^N_{k_M(r)})\\
&\quad + (1-\lambda)(\mathscr{R}_{u,k_M(r)}+O^N_{k_M(r)})\Big)\Big]\,d\lambda\\
&\quad + (\mathscr{R}_{s,k_M(r)}-\mathscr{R}_{s,r}+\mathscr{R}_{u,r}-\mathscr{R}_{u,k_M(r)})\\
&\qquad \times \int_0^1 g_h'\Big(\lambda(\mathscr{R}_{s,k_M(r)}+O^N_{k_M(r)})+(1-\lambda)(\mathscr{R}_{u,k_M(r)}+O^N_{k_M(r)})\Big)\,d\lambda\\
&=: J_1+J_2.
\end{aligned}$$

where, one uses the integral mean value theorem and adds and subtracts relevant terms. Estimating each integral separately, starting with J_2, recall that $\|\Delta_j P^N_{R-r}\varphi\|_{L^\infty} \lesssim 2^{-j\epsilon}(R-r)^{-\epsilon/2}\|\varphi\|_{L^\infty}$,, then using the fact that $r\in[u,t] \implies (R-r)^{-\epsilon/2}\le (R-t)^{-\epsilon/2}$ one has

$$\left(\mathbb{E}\left|\Delta_j\int_u^t \mathbb{E}_s[P^N_{R-r}\mathbb{E}_u[J_2]](x)dr\right|^p\right)^{1/p} \lesssim 2^{-j\epsilon}(R-t)^{-\epsilon/2}\Big(\mathbb{E}\Big|\int_u^t \|\,\mathbb{E}_s\mathbb{E}_u J_2(r)\|_{L^\infty}\,dr\Big|^p\Big)^{1/p}.$$

Since one also has that $\|J_2(r)\|_{L^\infty} \le \int_0^1 \left\|g_h'(\cdot)\right\|_{L^\infty} d\lambda \left\|\mathscr{R}_{s,k_M(r)}-\mathscr{R}_{s,r}+\mathscr{R}_{u,r}-\mathscr{R}_{u,k_M(r)}\right\|_{L^\infty}$, via applying Cauchy-Schwartz and Hölder, as well as (4.0.1.1) achieves

$$\left(\mathbb{E}\left|\Delta_j\int_u^t \mathbb{E}_s[P^N_{R-r}\mathbb{E}_u[J_2]](x)dr\right|^p\right)^{1/p} \lesssim 2^{-j\epsilon}(R-t)^{-\epsilon/2}(1+\mathbb{E}\|R^{M,N}\|^{2\tilde{m}p}_{C_TL^\infty}+\mathbb{E}\|O^N\|^{2\tilde{m}p}_{C_TL^\infty})^{1/2p}\\ \times\left(\mathbb{E}\left(\int_u^t \|\mathscr{R}_{s,k_M(r)}-\mathscr{R}_{s,r}+\mathscr{R}_{u,r}-\mathscr{R}_{u,k_M(r)}\|_{L^\infty}dr\right)^{2p}\right)^{1/2p}$$

Denoting $B(r) := \mathscr{R}_{s,k_M(r)}-\mathscr{R}_{s,r}+\mathscr{R}_{u,r}-\mathscr{R}_{u,k_M(r)}$, and using the bounds in (A.0.1), one has $\|B(r)\|_{L^q} \lesssim (r-k_M(r))^{1-\epsilon}$ and $\|B(r)\|_{L^q} \lesssim |r-s|^{1-\epsilon}$, so $\|B(r)\|_{L^q} \lesssim \min\big((r-k_M(r))^{1-\epsilon},\,(r-s)^{1-\epsilon}\big)$ or

$$\|B(r)\|_{L^{2p}} \lesssim \min\big((r-k_M(r))^{1-\epsilon/2},\,(r-s)^{1-\epsilon/2}\big).$$

So, using this bound, and interpolating similar to proposition (3.0.1.2)[4], one has

$$\left(\mathbb{E}\left|\Delta_j \int_u^t \mathbb{E}_s[P^N_{R-r}\mathbb{E}_u[J_2]](x)dr\right|^p\right)^{1/p} \lesssim 2^{-j\epsilon}(R-t)^{-\epsilon/2}\int_u^t \min((r-k_M(r))^{1-\epsilon/2},(r-s)^{1-\epsilon/2})dr \tag{4.20}$$

$$\lesssim 2^{-j\epsilon}(R-t)^{-\epsilon/2}(t-s)^{1+\epsilon/2}M^{-1+\epsilon}, \tag{4.21}$$

which finishes the bound for J_2. To estiamte J_1, however, once again consider cases where $|t-u| \leq 3M^{-1}$ and $|t-u| > 3M^{-1}$, like for A_{ut}. Following once again the proof of (3.0.1.2), via (3.21) using (4.16) instead of (2.8), but in this case (A.8) contributes additional factor from the bound $(r-s)^{3(1-\epsilon)/4}$, producing

$$\left(\mathbb{E}\left|\Delta_j \int_u^t \mathbb{E}_s[P^N_{R-r}\mathbb{E}_u[J_1]](x)dr\right|^p\right)^{1/p} \lesssim 2^{-j\epsilon}M^{-1+\epsilon}(R-t)^{-\epsilon/2}(t-u)^{5/4+\epsilon/4}. \tag{4.22}$$

For $|t-u| > 3M^{-1}$ again splttin into u to t' to t. It suffices to show the second bound. Like in the case of J_2, using the definition of dyadic partition, Cauchy-Schwartz and Hölder, one achieves

$$\Big(\mathbb{E}\big|\Delta_j\int_{t'}^t \mathbb{E}_s\big[P^N_{R-r}\mathbb{E}_u J_1(r)\big](x)\,dr\big|^p\Big)^{1/p} \lesssim 2^{-j\epsilon}(R-t)^{-1/4-\epsilon/2}\Big(\mathbb{E}\Big(\int_{t'}^t \|\mathscr{R}_{s,r}-\mathscr{R}_{u,r}\|^2_{C^{1/2-\epsilon/2}}\,dr\Big)^p\Big)^{1/2p}$$
$$\times\Big(\mathbb{E}\Big(\int_{t'}^t\Big(\int_0^1 \big\|g'_h(\cdot_r)-g'_h(\cdot_{k_M(r)})\big\|_{C^{-1/2+\epsilon}}\,d\lambda.\Big)^2\,dr\Big)^p\Big)^{1/2p}.$$

By (A.8) one has that $\Big(\mathbb{E}\big(\int_{t'}^t\|\mathscr{R}_{s,r}-\mathscr{R}_{u,r}\|^2_{C^{1/2-\epsilon/2}}\,dr\big)^p\Big)^{1/2p} \lesssim (t-s)^{\frac{3}{4}(1-\epsilon)}$, then, finally, exactly as in the estimate of the time–integral on $[t',t]$ done earlier for g_h when bounding I_2, but now applied to $g'_h \in C^2_\omega$, one obtains

$$\Big(\mathbb{E}\int_{t'}^t\|(P^{\mathbb{R}}_{Q^N(r-u)}g'_h)(\cdot_r)-(P^{\mathbb{R}}_{Q^N(k_M(r)-u)}g'_h)(\cdot_{k_M(r)})\|^{2p}_{C^{-1/2+\epsilon}}\,dr\Big)^{1/2p} \lesssim \int_{t'}^t (r-k_M(r))^{1-2\epsilon}\,(k_M(r)-u)^{-1/2+\epsilon}\,dr,$$

So for any fixed $\lambda \in [0,1]$, together all of the bounds as well as (A.8) and (4.17), yield

$$\left(\mathbb{E}\left|\Delta_j \int_{t'}^t \mathbb{E}_s[P^N_{R-r}\mathbb{E}_u[J_1]](x)dr\right|^p\right)^{1/p} \lesssim 2^{-j\epsilon}(t-s)^{3(1-\epsilon)/4}(R-t)^{-1/4-\epsilon/2}\int_{t'}^t (r-k_M(r))^{1-2\epsilon}(k_M(r)-u)^{-1/2+\epsilon}dr$$
$$\lesssim 2^{-j\epsilon}(t-s)^{5/4+\epsilon/4}(R-t)^{-1/4-\epsilon/2}M^{-1+2\epsilon}.$$

The bounds for J_1 and J_2 make condition (4.14) hold, Lemma (2.0.0.1) applies and yields (4.12), as long as one shows

$$\mathcal{A}_t = \int_0^t \Delta_j P^N_{R-r}[g_h(R^{M,N}_r + O^N_r) - g_h(R^{M,N}_{k_M(r)} + O^N_{k_M(r)})](x)dr, \tag{4.23}$$

i.e. the conditions (2.22) and (2.23) must hold. To prove the conditions, split $\mathcal{A}_{st} - A_{st}$ into $\mathcal{A}_{st} - \tilde{A}_{st}$ and $\tilde{A}_{st} - A_{st}$,where $\tilde{A}_{st}$ is gievn by

$$\tilde{A}_{s,t} = \int_s^t \mathbb{E}_s[\Delta_j P^N_{R-r}(g_h(R^{M,N}_r + O^N_r) - g_h(R^{M,N}_{k_M(r)} + O^N_{k_M(r)}))](x)dr.$$

[4] $\min(A,B) \leq A^{1-\theta}B^\theta, \qquad A,B \geq 0,\ \theta \in (0,1)$ for $\theta = \epsilon/(1-\epsilon) \in (0,1)$

Consider first the bound on $A_{s,t} - \widetilde{A}_{s,t}$. In particular by writing the difference using the previous mean-value estimates, one has that

$$g_h(\mathscr{R}_r + O_r^N) - g_h(R_r + O_r^N) = (\mathscr{R}_r - R_r) \int_0^1 g_h'\big(\lambda(\mathscr{R}_r + O_r^N) + (1-\lambda)(R_r + O_r^N)\big)\, d\lambda,$$

with the same bound in $k_M(r)$. Using the Lemma(4.0.1.1) ($|g_h'(z)| \lesssim 1 + |z|^{2\tilde{m}}$), and using the apriori bounds on $X^{M,N}$,

$$\Big\| \int_0^1 g_h'\big(\lambda(\mathscr{R}_r + O_r^N) + (1-\lambda)(R_r + O_r^N)\big)\, d\lambda \Big\|_{L^{2p}(\Omega)} \lesssim \Big(1 + \mathbb{E}\|X^{M,N}\|_{C_T L^\infty}^{2p(2\tilde{m}+1)}\Big)^{1/(2p)},$$

again with the analogous claim for $k_M(r)$. Then,

$$\begin{aligned}
(\mathbb{E}|A_{st} - \tilde{A}_{st}|^p)^{1/p} &\lesssim (t-s)(1 + \mathbb{E}\|X^{M,N}\|_{C_T L^\infty}^{2p(2\tilde{m}+1)})^{1/2p} \\
&\quad \times \sup_{r \in [s,t]} \Big((\mathbb{E}\|R_r^{M,N} - \mathscr{R}_{s,r}\|_{L^\infty}^{2p})^{1/2p} + (\mathbb{E}\|R_{k_M(r)}^{M,N} - \mathscr{R}_{s,k_M(r)}\|_{L^\infty}^{2p})^{1/2p} \Big) \\
&\lesssim (t-s)^{2-\epsilon},
\end{aligned}$$

where in the last line used (A.6) and (A.9) were used. Which proves (2.23). To prove (2.22), exactly the same bound is applied, but without (A.6) and (A.9), instead using a priori bounds to get $\|\mathscr{R}_r^{M,N} - R_r^{M,N}\|_{L^{2p}(\Omega)} \le 2$, giving that

$$\big(\mathbb{E}|\mathcal{A}_{st} - \tilde{A}_{st}|^p\big)^{1/p} \le 4(t-s)(1 + \mathbb{E}\|X^{M,N}\|_{C_T L^\infty}^{p(2\tilde{m}+1)})^{1/p},$$

The same argument proves (2.22), (2.23) for $\mathcal{A}_{st} - \tilde{A}_{st}$ with R in place of $\mathscr{R}$. □

Using the proposition, one can prove (4.0.1.4).

Proof. Lemma (4.0.1.4) One begins by writing the difference in terms in the lemma as

$$\begin{aligned}
X_t^{h,N} - X_t^{M,N} = R_t^{h,N} - R_t^{M,N} &= \int_0^t P_{t-s}^N g_h(X_s^{h,N}) ds - \int_0^t P_{t-k_M(s)}^N g_h(X_{k_M(s)}^{M,N}) ds && (4.24) \\
&= \int_0^t P_{t-s}^N [g_h(X_s^{h,N}) - g_h(X_s^{M,N})] ds + \int_0^t P_{t-s}^N [g_h(X_s^{M,N}) - g_h(X_{k_M(s)}^{M,N})] ds && (4.25) \\
&\quad - \int_0^t [P_{t-k_M(s)}^N - P_{t-s}^N] g_h(X_{k_M(s)}^{M,N}) ds, && (4.26)
\end{aligned}$$

where one simply adds and subtracts $P_{t-s}^N[g_h(X_s^{M,N})]$ and $P_{t-s}^N[g_h(X_{k_M(s)}^{M,N})]$. Now bound each term separately. The middle term is immediately given by the proposition just proved ((4.0.1.2)). The bound of the last term is direct consequence of the semigroup bound and (4.0.1.1). In particular,

$$\begin{aligned}
\Big\| \int_0^t [P_{t-k_M(s)}^N - P_{t-s}^N] g_h(X_{k_M(s)}^{M,N}) ds \Big\|_{L^\infty} &= \Big\| \int_0^t P_{t-s}^N (P_{s-k_M(s)}^N - \mathrm{Id}) g_h(X_{k_M(s)}^{M,N}) ds \Big\|_{L^\infty} \\
&\lesssim \int_0^t (s - k_M(s))^{1-\epsilon} (t-s)^{-1+\epsilon} \|g_h(X_{k_M(s)}^{M,N})\|_{L^\infty} ds \\
&\lesssim M^{-1+\epsilon}(1 + \|X^{M,N}\|_{C_T L^\infty}^{2\tilde{m}+1}). && (4.27)
\end{aligned}$$

The first term in the expression, which would usually be bounded via Grönwall inequality, does not satisfy the required properties to apply the result, due to the fact the g_h has non-constant growing Lipschitz constant. As a result, one again considers the variational formulation, and uses the energy inequality to bound the term. Define $\bar{R}_t^{M,N} := \int_0^t P_{t-s}^N g_h(X_s^{M,N})ds$, and apply the (4.0.1) with $F = R^{h,N} - \bar{R}^{M,N}$, $G = g_h(X^{h,N}) - g_h(X^{M,N})$, $H = 0$. The energy inequality gives immedeatly that $\|R_t^{h,N} - \bar{R}_t^{M,N}\|_{L^2}^p \le p\int_0^t \|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-2}\langle g_h(X_s^{h,N}) - g_h(X_s^{M,N}), R_s^{h,N} - \bar{R}_s^{M,N}\rangle ds$, so by adding and substracting terms, gives

$$\begin{aligned}\|R_t^{h,N} - \bar{R}_t^{M,N}\|_{L^2}^p \le p\int_0^t &\|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-2}\langle g_h(X_s^{h,N}) - g_h(X_s^{M,N}), R_s^{h,N} - R_s^{M,N}\rangle ds\\ &+ p\int_0^t \|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-2}\langle g_h(X_s^{h,N}) - g_h(X_s^{M,N}), R_s^{M,N} - \bar{R}_s^{M,N}\rangle ds\\ &+ p\int_0^t \|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-2}\langle (\Pi_N - \mathrm{Id})(g_h(X_s^{h,N}) - g_h(X_s^{M,N})), R_s^{h,N} - \bar{R}_s^{M,N}\rangle ds.\end{aligned} \tag{4.28}$$

Using the bound (4.0.1.1), where we replace $X^{h,N} - X^{M,N} = R^{h,N} - R^{M,N}$ in the first two terms, analogous to method used in proof (4.0.1.2) the result becomes

$$\begin{aligned}\|R_t^{h,N} - \bar{R}_t^{M,N}\|_{L^2}^p \lesssim \int_0^t &\|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-2}\|R_s^{h,N} - R_s^{M,N}\|_{L^2}^2 ds\\ &+ (1 + \|X^{h,N}\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{M,N}\|_{C_T L^\infty}^{2\tilde{m}})\\ &\qquad \times \int_0^t \|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-2}\|R_s^{h,N} - R_s^{M,N}\|_{L^2}\|R_s^{M,N} - \bar{R}_s^{M,N}\|_{L^2} ds\\ &+ \int_0^t \|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^{p-1}\|(\Pi_N - \mathrm{Id})(g_h(X^{h,N}) - g_h(X_s^{M,N}))\|_{L^2} ds.\end{aligned}$$

Again, as in (4.0.1.2), one uses Young's inequality several times to separate the integrands

$$\begin{aligned}\|R_t^{h,N} - \bar{R}_t^{M,N}\|_{L^2}^p \lesssim &\int_0^t \|R_s^{h,N} - \bar{R}_s^{M,N}\|_{L^2}^p ds + \int_0^t \|R_s^{h,N} - R_s^{M,N}\|_{L^2}^p ds\\ &+ [1 + \|X^{M,N}\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{h,M}\|_{C_T L^\infty}^{2\tilde{m}}]^p \int_0^t \|R_s^{M,N} - \bar{R}_s^{M,N}\|_{L^\infty}^p ds\\ &+ \int_0^t \|(\Pi_N - \mathrm{Id})(g_h(X^{h,N}) - g_h(X_s^{M,N}))\|_{L^2}^p ds.\end{aligned}$$

Here, recognize the last term from (4.0.1.2), so following the same procedure as before, for $\alpha \in (0, 1/2)$ and $\epsilon' \in (0, 1/2 - \alpha)$ apply Grönwall's inequality to get

$$\|R_t^{h,N} - \bar{R}_t^{M,N}\|_{L^2}^p \lesssim \int_0^t \|R_s^{h,N} - R_s^{M,N}\|_{L^2}^p ds + K_* \int_0^t \|R_s^{M,N} - \bar{R}_s^{M,N}\|_{L^\infty}^p\, ds + K_* N^{-\alpha}, \tag{4.29}$$

where, the notation K_* is introduced, denoting the random variables that have their moments bounded by (4.0.1.1) and (4.0.1.2). Note, that the RV K_* may change based on the context[5]. Since $\bar{R}_t^{M,N} - R_t^{M,N} = (R_t^{M,N} - \bar{R}_t^{M,N}) - (R_t^{h,N} - \bar{R}_t^{M,N})$, which are both bounded, in (4.0.1.2) and in (4.27). Thus, one has that

$$\mathbb{E} \sup_{t\in[0,T]} \|R_t^{M,N} - \bar{R}_t^{M,N}\|_{L^\infty}^p \lesssim M^{p(-1+\epsilon/2)} \tag{4.30}$$

[5]In the case of this particular inequality above one has $K_* = [1 + \|X^{M,N}\|_{C_T L^\infty}^{2\tilde{m}} + \|X^{h,M}\|_{C_T L^\infty}^{2\tilde{m}}]^p$

for any $p \geq 1$. Then, combining everything (4.27) and (4.29) in (4.24) and using (4.30) one has

$$\|R_t^{h,N} - R_t^{M,N}\|_{L^2}^p \lesssim K_* \big(M^{p(-1+\epsilon)} + N^{-\alpha p} + \sup_{t\in[0,T]} \|R_t^{M,N} - \bar{R}_t^{M,N}\|_{L^\infty}^p \big) + \int_0^t \|R_s^{h,N} - R_s^{M,N}\|_{L^2}^p ds.$$

Applying Grönwall inequality

$$\sup_{t\in[0,T]} \|R_t^{h,N} - R_t^{M,N}\|_{L^2}^p \lesssim K_* \big(M^{p(-1+\epsilon)} + N^{-\alpha p} + \sup_{t\in[0,T]} \|R_t^{M,N} - \bar{R}_t^{M,N}\|_{L^\infty}^p \big),$$

and taking expectations and applying Hölder's inequality for $\mathbb{E}(K_* \sup_{t\in[0,T]} \|R_t^{M,N} - \bar{R}_t^{M,N}\|_{L^\infty}^p)$[6], the claimed result is achieved

$$\mathbb{E} \sup_{t\in[0,T]} \|R_t^{h,N} - R_t^{M,N}\|_{L^2}^p \lesssim M^{p(-1+\epsilon)} + N^{-\alpha p}.$$

□

[6] $\mathbb{E}[K_* E_*^p] \leq \left(\mathbb{E}[K_*^{1+\delta}] \right)^{\frac{1}{1+\delta}} \left(\mathbb{E}[E_*^{p(1+\delta)/\delta}] \right)^{\frac{\delta}{1+\delta}} \lesssim M^{p(-1+\epsilon/2)}$

Chapter 5

Conclusion

In this work, improved convergence rates of the numerical approximation of the stochastic reaction diffusion equation (1.1) were derived. The results are shown in two scenarios: globally Lipschitz f with bounded derivatives up to order 2 and maximally polynomially growing non-linearity with one-sided Lipschitz assumption, a driving example for which was the Allen-Cahn non-linearity $f(u) = u - u^3$. In both cases, functional bounds were obtained and shown to be of order $M^{-1+\epsilon}$ in time and $N^{-1/2+\epsilon}$ in space.

In the first case, a spectral Galerkin approximation was used in space and an exponential explicit Euler scheme was used for time discretisation. The L^∞ error bounds for the scheme were proved, in particular, for any $\epsilon > 0$ and $p \in [1, \infty)$

$$\big(\mathbb{E} \sup_{t\in[0,T]} \|u_t - V_t^{M,N}\|^p_{L^\infty(\mathbb{T})}\big)^{1/p} \leq C\big(N^{-1/2+\epsilon} + M^{-1+\epsilon}\big). \tag{5.1}$$

In the second case, as mentioned before, since explicit schemes blow up under superlinear drift, the splitting numerical scheme is used for temporal discretisations, and the following variational bound in space $(L^2)/$ uniform in time was achieved

$$\Big(\mathbb{E} \sup_{t\in[0,T]} \|u_t - X_t^{M,N}\|^p_{L^2(\mathbb{T})}\Big)^{1/p} \lesssim N^{-1/2+\epsilon} + M^{-1+\epsilon},$$

for any $\epsilon > 0$ and $p \in [1, \infty)$.

As mentioned in the introduction, the improved rates of convergence are based on two main structural ideas - construction of bounds in weaker topologies and approximation of the error via the Stochastic Sewing Lemma. While in the first case the construction of bounds is relatively straightforward, for the growing non-linearity, in addition to the introduction of auxiliary g_h process, one also switches to the variational formulation to implement the Grönwall inequality. Other crucial ingredients were used, like weighted C^k_ω spaces with polynomially decaying weights $\omega(x) = (1 + |x|^2)^{-\beta/2}$, which allowed to tame polynomial growth of g_h and its derivatives, resulting in Lipschitz properties in the weighted norm, so one could implement the semigroup estimates of the form $\|(P^{\mathbb{R}}_t - P^{\mathbb{R}}_s)u\|_{C^0_\omega} \lesssim |t-s|\, \|u\|_{C^2_\omega}$,

In both cases, the second main idea relied on the Stochastic Sewing Lemma to approximate the time-discretisation remainder

$$\int_0^t P^N_{t-s}\big(g_h(X^{M,N}_s) - g_h(X^{M,N}_{k_M(s)})\big)\, ds,^{1}$$

which controls the temporal rate of approximation of the numerical solution. In summary, to position these results in the existing literature, while for globally bounded f, the error estimates of $N^{-1/2}$ in space and M^{-1} in time have been already known, for truly non-linear drifts the derivation was unknown, since the direct application of Grönwall in the mild formulation is not possible, making the approach reviewed in this work[2] novel, as it recovers the same convergence rate by combining weighted composition estimates, switching to variational formulation, as well as using SSL to effectively use cancellations, produced by the oscillatory nature of OU process.

There are, however, both restrictions and limitations of the results reviewed in this analysis. To begin with, in the superlinear regime, the error is measured in $L^2(\mathbb{T})$ rather than $L^\infty(\mathbb{T})$, which, unlike for the case with bounded f does not directly yield pointwise convergence. Additionally, the results were derived under the assumption that the initial condition satisfies $u_0 \in \mathscr{C}^{1/2}$, which might not necessarily be the case in many applications, and certain estimates, like in the construction of a priori bounds, break down if the initial condition has lower regularity (like L^2). Lastly, some identities in the proofs rely on using a uniformly spaced grid. One would require a more general framework to deal with general boundaries/domains, where techniques like adaptive grids are required.

Naturally, however, there are a variety of extensions to the presented work. As mentioned in [1], one can consider constructing analogous bounds for stochastic equations with multiplicative space-time white noise. Additionally, one can consider other classes of equations, for instance, those for which nonlinearities depend on the gradient of the solution, like the stochastic Burgers' equation. Yet another type of equation to consider, or rather a generalisation to Allen-Cahn, could be Cahn–Hilliard type equations[3]. Finally, one can attempt to move to more general types of non-linearities (not necessarily polynomially growing), using, for instance, a different type of weighted function spaces to tame the non-linear term. In terms of the numerics, as mentioned in [1], one can extend the current work by considering different discretisation schemes, like tamed schemes and implicit Euler approximation.

[1] $\int_0^t P^N_{t-s}\big(f(O_s) - f(O_{k_M(s)})\big)\, ds$ in the first scenario

[2] Following [1]

[3] A recent paper has proved the exact same convergence bounds in L^2 for Cahn–Hilliard SPDE on a 2-d Torus in [12] . However, the results were shown for the bounded non linearity G, so one could consider proving the result for the polynomially growing non-linearity reviewed in this work

Appendix A

Missaleneous results

Theorem A.0.1 (Kolmogorov Continuity Theorem). *Let $(X_t)_{t\geq 0}$ be a d-dimensional stochasitc process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in separable Banack space with $s, t \in [0,1], s < t$. If for any constants$C >$ $0, \epsilon > 0, p > 1$*

$$\mathbb{E}\left[\|X_t - X_s\|^p\right] < C|t-s|^{1+\epsilon}$$

then there exists a modification[a] of the process, denoted $(\hat{X})_{t\geq 0}$ such that $(\hat{X})_{t\geq 0}$ is $\gamma-$ Hölder in time for every $\gamma \in [0, \epsilon/p]$.

[a] A process $(\hat{X}_t)_{t\geq 0}$ is a modification of the process $(X_t)_{t\geq 0}$ if for every $t \geq 0, \mathbb{P}\left(X_t = \hat{X}_t\right) = 1$. Importantly, modfication has the same distribution as the original process

Proof. Consider first that the process X is stochastically uniformly continuous since, for any $\gamma > 0$, one has (using Markov inequality)

$$\mathbb{P}(\|X(t) - X(s)\| \geq \gamma) \leq \gamma^{-p}\mathbb{E}\|X(t) - X(s)\|^p \leq C\gamma^{-p}|t-s|^{1+\epsilon}.$$

Since for a continuous process with values in a separable Banach space,there exists a mea- surable modification(See Proposition 3.2 [6] for details), one has

$$(t, s, \omega) \mapsto \frac{\|X(t,\omega) - X(s,\omega)\|^p}{|t-s|^{1+\epsilon}}$$

is $\mathscr{B}([0,T] \times [0,T]) \times \mathcal{F}$-measurable, via Fubini theorem and prior inequality:

$$\begin{aligned}\int_\Omega \int_0^T \int_0^T \frac{\|X(t,\omega) - X(s,\omega)\|^p}{|t-s|^{1+\epsilon}}\, dt\, ds\, \mathbb{P}(d\omega) &= \int_0^T \int_0^T \frac{\mathbb{E}\|X(t) - X(s)\|^p}{|t-s|^{1+\epsilon}}\, dt\, ds \\ &\leq C\int_0^T \int_0^T |t-s|^{-\epsilon}\, dt\, ds < \infty \quad \text{since } \frac{1+\epsilon}{p} > 1.\end{aligned}$$

This implies that $X(\omega) \in W^{\alpha,p}(0,T)$, $\mathbb{P}$-a.s. Choose any $\alpha \in \left(\frac{1}{p}, \frac{1+\epsilon}{p}\right) \subset (0,1)$. Then a.s. $X(\cdot) \in W^{\alpha,p}(0,T;E) =$ $B^\alpha_{p,p}(0,T;E)$. By (2.0.0.1) with $d = 1$, $(p_1, r_1) = (p,p)$ and $(p_2, r_2) = (\infty, \infty)$, we have the embedding

$B^{\alpha}_{p,p} \hookrightarrow B^{\alpha-1/p}_{\infty,\infty} = C^{\alpha-1/p}$ (for non-integer $\alpha - 1/p$); hence X admits a C^{γ}-modification for every $\gamma < \alpha - \frac{1}{p}$, and letting $\alpha \uparrow \frac{1+\epsilon}{p}$ yields any $\gamma < \epsilon/p$.

Let

$$X_h(t,\omega) := \frac{1}{h}\int_{(t-h)\vee 0}^{t} X(s,\omega)\,ds, \quad (t,\omega) \in [0,T]\times\Omega,$$

which is measurable with respect to $\omega \in \Omega$, and by the Fubini theorem for Bochner integrable functions, and for almost all $\omega \in \Omega$, $X_h(\cdot,\omega)$ is a continuous function on $[0,T]$. Therefore, the process X_h is measurable.

Let

$$\tilde{X}(t,\omega) := \begin{cases} \lim_{h\to 0} X_h(t,\omega), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{X}$ is also a measurable process and $\tilde{X}(t,\omega)$ is equal to $X(t,\omega)$ for almost all $\omega \in \Omega$ and equal to $X(t,\cdot)$ almost surely with respect to the first variable. So $X(\cdot,\omega) = \tilde{X}(\cdot,\omega)$ for almost all t and ω. Thus

$$\mathbb{P}(X(t) = \tilde{X}(t)) = 1 \quad \forall\, t \in [0,T].$$

Since both X and $\tilde{X}$ are stochastically continuous, we have:

$$\mathbb{P}(X(t) = \tilde{X}(t)) = 1 \quad \forall\, t \in [0,T].$$

$\square$

Theorem A.0.2 (Gaussian Hypercontractivity)**.** *The result due to [13, Corollary 1.63]. Let $p \geq 2$. Then the operator*[a] $I_n : f \to I_n(f)$ *from* $L^2(\xi)$ *to* $L^p(\xi)$ *is continuous and*

$$\|I_n(f)\|_p \leq (p-1)^{n/2}\|f\|_2$$

[a] The integral operator I_n is defined rigorously in [13, Remark 1.3.5] via Hermite polynomials

Proposition A.0.2.1. *For $\theta \in (0,1/2)$, $q \geq 1$ and $\lambda \in (0,1)$*

$$\mathbb{E}\sup_{r\in[0,R]} \|f'(\lambda\tilde{O}_r + (1-\lambda)\tilde{O}_{k_M(r)})\|^q_{\mathscr{C}^\theta} \lesssim \mathbb{E}\|f'\|^q_{C^1_b}\|\tilde{O}\|^q_{C_T\mathscr{C}^\theta} \lesssim_{q,\theta} 1.$$

Proof. Recall first that the mean value theorem guarantees that

$$\sup_{c\in\mathbb{T}}|f'(c)| = \frac{|f(x)-f(y)|}{|x-y|} \implies |f(x)-f(y)| \leq \|f'\|_\infty|x-y| \tag{A.1}$$

and that the Hölder seminorm for composition of functions is given by

$$\|g\circ h\|_{\mathscr{C}^\theta} := \|g\circ h\|_{L^\infty} + \sup_{x\neq y}\frac{|g(h(x)) - g(h(y))|}{|x-y|^\theta}.$$

Hence, since by (A.1) $\sup_{x\neq y}\frac{|g(h(x))-g(h(y))|}{|x-y|^\theta}\leq \|g'\|_\infty \sup_{x\neq y}\frac{|h(x)-h(y)|}{|x-y|^\theta}$, one has that

$$\begin{aligned}\|f'(\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)})\|_{\mathscr{C}^\theta} &\leq \|f'\|_\infty+\|f''\|_\infty\|\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)}\|_{\mathscr{C}^\theta}\\ &\leq \|f'\|_{C^1_b}(1+\|\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)}\|_{\mathscr{C}^\theta})\end{aligned}$$

Finally, as $\|\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)}\|_{\mathscr{C}^\theta}\leq \lambda\|\tilde{O}_r\|_{C^\theta}+(1-\lambda)\|\tilde{O}_{k_M(r)}\|_{\mathscr{C}^\theta}\lesssim \|\tilde{O}\|_{C_t\mathscr{C}^\theta}$, one has that

$$\|f'(\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)})\|_{\mathscr{C}^\theta}\lesssim \|f'\|_{C^1_b}(1+\|\tilde{O}\|_{C_t\mathscr{C}^\theta})$$

Taking supremum in $r\in[0,R]$, raising to the q-th power, applying expectation, and using that $\|f'\|_{C^1_b}<\infty$

$$\mathbb{E}\sup_{r\in[0,R]}\|f'(\lambda\tilde{O}_r+(1-\lambda)\tilde{O}_{k_M(r)})\|^q_{\mathscr{C}^\theta}\lesssim \|f'\|^q_{C^1_b}\left(\mathbb{E}\|\tilde{O}\|^q_{C_T\mathscr{C}^\theta}\right),$$

It remains to show the second bound. The bound is satisfied, since $\tilde{O}$ satisfies uniform moment bounds in $C_T\mathscr{C}^\theta$ same as O by Proposition 2.0.0.5), i.e. $\mathbb{E}\|\tilde{O}\|^q_{C_T\mathscr{C}^\theta}<\infty$ and as$\|f'\|_{C^1_b}<\infty$ the bound follows. □

Proof of (4.0.1.1). Following the proof of [1, Lemma 5.1], by definition one has that

$$\Phi_h(z)=z+\int_0^h f(\Phi_s(z))ds,\quad h\geq 0\qquad g_h(z)=\frac{1}{h}\int_0^h f(\Phi_s(z))ds,\quad h>0.$$

Immediately one has, using one sided Lipshitz condition

$$\begin{aligned}(\Phi_h(x)-\Phi_h(y))^2&=(x-y)^2+2\int_0^h(f(\Phi_s(x))-f(\Phi_s(y)))(\Phi_s(x)-\Phi_s(y))ds\\ &\leq (x-y)^2+2K\int_0^h(\Phi_s(x)-\Phi_s(y))^2ds\end{aligned}$$

and hence by Grönwall's inequality

$$(\Phi_h(x)-\Phi_h(y))^{\leq}e^{Kh}(x-y)^2\implies|\Phi_h(x)-\Phi_h(y)|\leq e^{Kh/2}|x-y|\implies|\partial_z\Phi_s(z)|\leq e^{Ks/2}\leq e^{Kh/2},\quad s\in[0,h],$$

proving the first identity. Additionally, one has that, again, using one sided Lipschitz condition that

$$\begin{aligned}\Phi_h(z)^2=z^2+2\int_0^h f(\Phi_s(z))\Phi_s(z)ds&=z^2+2\int_0^h(f(\Phi_s(z))-f(0))\Phi_s(z)ds+2\int_0^h f(0)\Phi_s(z)ds\\ &\leq z^2+2K\int_0^h\Phi_s(z)^2ds+hf(0)^2+\int_0^h\Phi_s(z)^2ds,\end{aligned}$$

which again, using Grönwall'sl, implies that

$$\sup_{s\in[0,h]}|\Phi_s(z)|^2\leq e^{(2K+1)h}(h|f(0)|^2+|z|^2)\implies\sup_{s\in[0,h]}|\Phi_s(z)|\leq e^{(2K+1)h/2}(h^{1/2}|f(0)|+|z|)\leq Ke^{(2K+1)h/2}(1+|z|),\tag{A.2}$$

where one uses that $|f(0)|\leq K$ for $h\leq 1$. Furthermore, we have that

$$(\partial_z^2\Phi_h(z))^2=1+2\int_0^h(\partial f)(\Phi_s(z))(\partial_z^2\Phi_s(z))^2ds+2\int_0^h(\partial^2 f)(\Phi_s(z))(\partial_z\Phi_s(z))^2\partial_z^2\Phi_s(z)ds$$

applying Young's inequality for $p=2$, the bounds on f,(A.2), and again Lipschitz condition on Φ_s, there exist constants $C'(K,m), C(K,m) > 1$, with

$$
\begin{aligned}
(\partial_z^2\Phi_h(z))^2 &\le 1 + 2K\int_0^h (\partial_z^2\Phi_s(z))^2 ds + \int_0^h ((\partial^2 f)(\Phi_s(z))(\partial_z\Phi_s(z))^2)^2 ds + \int_0^h (\partial_z^2\Phi_s(z))^2 ds \\
&\le 1 + (2K+1)\int_0^h (\partial_z^2\Phi_s(z))^2 ds + hC'(K,m)e^{hC(K,m)}(1+|z|^{2(2m-1)}).
\end{aligned}
$$

So, again, Grönwall's inequality implies for $h \le 1$[1],

$$
\sup_{s\in[0,h]} |\partial_z^2\Phi_s(z)| \le C'(K,m)e^{C(K,m)h}(1+|z|^{2m-1}). \tag{A.3}
$$

Using a similar strategy for $\partial_z^3\Phi_h(z)$

$$
\begin{aligned}
(\partial_z^3\Phi_h(z))^2 = 1 &+ 2\int_0^h (\partial f)(\Phi_s(z))(\partial_z^3\Phi_s(z))^2 ds + 6\int_0^h (\partial^2 f)(\Phi_s(z))\partial_z\Phi_s(z)\partial_z^2\Phi_s(z)\partial_z^3\Phi_s(z) ds \\
&+ 2\int_0^h (\partial^3 f)(\Phi_s(z))(\partial_z\Phi_s(z))^3\partial_z^3\Phi_s(z) ds
\end{aligned}
$$

one has taht there are $C'(K,m), C(K,m)$ and $\tilde{m} \ge m$, $h \le 1$, such that

$$
\sup_{s\in[0,h]} |\partial_z^3\Phi_s(z)| \le C'(K,m)e^{C(K,m)h}(1+|z|^{2\tilde{m}-3}). \tag{A.4}
$$

As a result, since one clearly has

$$
\begin{aligned}
\partial_z g_h(z) &= \frac{1}{h}\int_0^h (\partial f)(\Phi_s(z))\partial_z\Phi_s(z) ds, \\
\partial_z^2 g_h(z) &= \frac{1}{h}\int_0^h (\partial^2 f)(\Phi_s(z))(\partial_z\Phi_s(z))^2 ds + \frac{1}{h}\int_0^h (\partial f)(\Phi_s(z))\partial_z^2\Phi_s(z)) ds, \\
\partial_z^3 g_h(z) &= \frac{1}{h}\int_0^h (\partial^3 f)(\Phi_s(z))(\partial_z\Phi_s(z))^3 ds + 3\frac{1}{h}\int_0^h (\partial^2 f)(\Phi_s(z))(\partial_z^2\Phi_s(z))\partial_z\Phi_s(z) ds \\
&\quad + \frac{1}{h}\int_0^h (\partial f)(\psi_s(z))\partial_z^3\Phi_s(z)) ds.
\end{aligned}
$$

In combination with (A.2), (A.3) and (A.4) one achieves that

$$
|g_h(z)| \le K(1 + \sup_{s\in[0,1]} |\Phi_s(z)|^{2m+1}) \le K^2 e^{(2K+1)h/2}(1+|z|^{2m+1})
$$

and

$$
|\partial_z g_h(z)| \le K e^{Kh/2}(1 + \sup_{s\in[0,1]} |\Phi_s(z)|^{2m}) \le C'(K,m)e^{hC(K,m)}(1+|z|^{2m}).
$$

with $K^2 e^{(2K+1)h/2} \le K^2 e^{(2K+1)/2}$ for $h \in [0,1]$. Similarly, but using (A.3), (A.4), for derrivatives of order $i = 2,3$, there are constants $\tilde{m} \ge m$, $K(h,m) > 1$, such that

$$
|\partial_z^i g_h(z)| \le K(h,m)(1+|z|^{2\tilde{m}-i}),
$$

[1]constants need not be the same

where $K(h,m) \leq \tilde{K}$ for $h \in [0,1]$ for a constant $\tilde{K} > 1$. Also, since $\partial_h \partial_z \Phi_h(z) = (\partial f)(\Phi_h(z))\partial_z \Phi_h(z)$ by Grönwall's inequality and using again that $\partial f \leq K$, one obtains

$$\partial_z \Phi_h(z) \leq e^{\int_0^h (\partial f)(\Phi_s(z))ds} \leq e^{hK} \implies \partial_z g_h(z) = \frac{1}{h}(\partial_z \Phi_h(z) - 1) \leq \frac{1}{h}(e^{hK} - 1) \leq K.$$

Lastly, sing the bounds on f and (A.2)

$$\begin{aligned} |g_h(z) - g_0(z)| = |g_h(z) - f(z)| &= \frac{1}{h}\left| \int_0^h f(\Phi_s(z)) - f(z)ds \right| \\ &\leq K^2 e^{Kh/2}(1 + |z|^{2m+1}) \sup_{s \in [0,h]} |\Phi_s(z) - z| \end{aligned}$$

and

$$|\Phi_s(z) - z| \leq sK^2 e^{Kh/2}(1 + |z|^{2m+1}).$$

Finally giving

$$|g_h(z) - g_0(z)| \leq hK^4 e^{Kh}(1 + |z|^{2(2m+1)}), \quad h \in [0,1].$$

and concluding the proof of Lemma (4.0.1.1) □

A.0.1 Proposition (4.0.1.2) supplementary results

Notice that for any $|\cdot|$, $q \geq 1$, and any random variables X, Y, such that Y is $\mathcal{F}_s$-measurable, one has

$$(\mathbb{E}|X - \mathbb{E}_s X|^q)^{1/q} \leq 2(\mathbb{E}|X - Y|^q)^{1/q} \tag{A.5}$$

This comes from the fact that $X - \mathbb{E}_s(X) = (X - Y) - \mathbb{E}_s(X_Y)$, so denoting $Z := X - Y$, via Minkowski inequality one has

$$(\mathbb{E}|X - \mathbb{E}_s X|^q)^{1/q} \leq (\mathbb{E}|Z|^q)^{1/q} + (\mathbb{E}(\mathbb{E}_s|Z|)^q)^{1/q} \leq 2(\mathbb{E}|Z|^q)^{1/q}$$

Using this estimate in combination with (4.0.1.2), one directly obtains

$$\left(\mathbb{E}\|R_r^{M,N} - \mathscr{R}_{s,r}\|_{L^\infty}^q\right)^{1/q} \leq 2\left(\mathbb{E}\|R_r^{M,N} - R_s^{M,N}\|_{L^\infty}^q\right)^{1/q} \lesssim_q |r - s|^{1-\epsilon}. \tag{A.6}$$

By using triangle inequality, for any $s \leq u$ this inequality leads to

$$(\mathbb{E}\|\mathscr{R}_{s,r} - \mathscr{R}_{u,r}\|_{L^\infty}^q)^{1/q} \lesssim_q |r - s|^{1-\epsilon}. \tag{A.7}$$

Using a similar strategy, one can construct an analogue of this bound in $\mathscr{C}^{1/2-\epsilon/2}$, since (A.5) holds for any norm, by applying the fact that $\|\mathscr{R}_{s,r} - \mathscr{R}_{u,r}\|_{\mathscr{C}^\alpha} \leq |r - s|^{1-\alpha/2-\epsilon}\|R^{M,N}\|^p_{C_T^{1-\alpha/2-\epsilon}\mathscr{C}^\alpha}$ in combination with the apriori bound, one has

$$(\mathbb{E}\|\mathscr{R}_{s,r} - \mathscr{R}_{u,r}\|_{\mathscr{C}^{1/2-\epsilon/2}}^q)^{1/q} \lesssim_q |r - s|^{3(1-\epsilon)/4}, \tag{A.8}$$

When constructing analogous bounds for $k_M(r)$ instead of r, it is not trivial. In particular, substituting $k_M(r)$ in place of r in (A.7) does not give the required order, since one may have $|k_M(r)-s| \gg |r-s|$.Thus, one has to consider two cases, $s \le k_M(r)-M^{-1}$ and $s \ge k_M(r)-M^{-1}$.In the first case, $|k_M(r)-s| \le |r-s|$ and the bound follows from (A.6) plugging $k_M(r)$ in place of r. In the second case, like in the proof of [14, Lemma 4.7], within the last mesh interval before $k_M(r)$, the conditional expectation essentially becomes $\mathscr{R}_{s,k} = \mathbb{E}_s R^{M,N}_k = R^{M,N}_k$, thus $E\big(\|R^{M,N}_{k_M(r)} - \mathscr{R}_{s,k_M(r)}\|^q_{L^\infty}\big)^{1/q} = 0$, and so overall one has

$$\mathbb{E}\big(\|R^{M,N}_{k_M(r)} - \mathscr{R}_{s,k_M(r)}\|^q_{L^\infty}\big)^{1/q} \lesssim_q |r-s|^{1-\epsilon}. \tag{A.9}$$

Again, like in the previous statement, via triangle inequality one obtains additional bound for $s \le u$,

$$(\mathbb{E}\|\mathscr{R}_{s,k_M(r)} - \mathscr{R}_{u,k_M(r)}\|^q_{L^\infty})^{1/q} \lesssim_q |r-s|^{1-\epsilon}. \tag{A.10}$$

Bibliography

[1] A. Djurdjevac, M. Gerencsér, and H. Kremp. Higher order approximation of nonlinear spdes with additive space-time white noise, 2024.

[2] I. Gyöngy. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise ii. *Potential Analysis*, 11(1):1–37, 1999.

[3] A. Jentzen and P. E. Kloeden. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space–time noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 465(2102):649–667, 2009.

[4] K. Lê. A stochastic sewing lemma and applications. *Electronic Journal of Probability*, 25(none), Jan. 2020.

[5] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2nd edition, 2010.

[6] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2nd edition, 2014.

[7] M. Hairer. An introduction to stochastic pdes, 2009. Lecture notes, available at http://www.hairer.org/notes/SPDEs.pdf.

[8] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, volume 343 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, Heidelberg, 2011.

[9] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum of Mathematics, Pi*, 3:e6, 2015.

[10] S. Cerrai. *Second Order PDE's in Finite and Infinite Dimension: A Probabilistic Approach*, volume 1762 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, 2001.

[11] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Optimal rate of convergence for approximations of spdes with nonregular drift. *SIAM Journal on Numerical Analysis*, 61(2):1103–1137, 2023.

[12] D. Blömker, C. Ling, and J. Rimmele. Numerical approximation of cahn-hilliard type nonlinear spdes with additive space-time white noise, 2025.

[13] V. I. Bogachev. *Gaussian Measures*, volume 62 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, Rhode Island, 1998.

[14] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Strong rate of convergence of the euler scheme for sdes with irregular drift driven by levy noise, 2024.