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Propositional Logic

def. A boolean variable is a variable which only takes a value of 0 ("false") or 1 ("true").

Let p, q, r be boolean variables.
Define the following:

(1) Negation, $\neg p$

p	$\neg p$
0	1
1	0

(2) Conjunction "and", $p \wedge q$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

(3) Disjunction "or", $p \vee q$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

ex. Evaluate $\neg(p \wedge (q \vee \neg r))$ if $p = 1$, $q = 0$, $r = 0$.

$$
\begin{aligned}
\neg(1 \wedge (0 \vee \neg 0) &= \neg(1 \wedge (0 \vee 1)) \\
&= \neg(1 \wedge 1) \\
&= \neg 1 \\
&= 0
\end{aligned}
$$

Note: Precedence of operators; use brackets to clarify.
'$\neg$' has highest precedence.

def. A formula is an expression involving boolean variables and operators that is syntactically correct.

def. A truth table for a formula f gives the value of f for all possible variable truth value assignments.

ex. Write a truth table for $\neg((\neg(p \vee q)) \vee q)$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg((\neg(p \vee q)) \vee q)$
0	0	0	1	1
0	1	1	0	1
1	0	1	0	0
1	1	1	0	1

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def. Two formulas f_1, f_2 are logically equivalent if they have the same truth table.
Denote as $f_1 \equiv f_2$.

def. If $f \equiv 1$ (f evaluates to 1 for any values assigned to its variables), f is called a tautology.

ex. $(\neg p) \vee p$ is a tautology.

def. If $f \equiv 0$, f is called a contradiction.

ex. $(\neg p) \wedge p$ is a contradiction.

def. If f is neither, f is called a contingency.

Simplifying Boolean Formulas

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

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ex. Simplify $(\neg(p \vee q)) \vee q$.

$(\neg(p \vee q)) \vee q$	$\equiv (\neg p \wedge \neg q) \vee q$	De morgan
	$\equiv q \vee (\neg p \wedge \neg q)$	Commutative
	$\equiv (q \vee \neg p) \wedge (q \vee \neg q)$	Distributive
	$\equiv (q \vee \neg p) \wedge 1$	Negation
	$\equiv q \vee \neg p$	Identity

Boolean Satisfiability

Given a formula f, can you assign truth values to make f evaluate to 1? If so, f is called satisfiable.

ex. Show that $f = (p \vee q) \wedge (\neg q \vee \neg r) \wedge (p \vee r)$ is satisfiable.

	p	q	r	$p \vee q$	$\neg q \vee \neg r$	$p \vee r$	f
	0	0	0	0	1	0	0
	0	0	1	0	1	1	0
	0	1	0	1	1	0	0
	0	1	1	1	0	1	0
✓	1	0	0	1	1	1	1
✓	1	0	1	1	1	1	1
✓	1	1	0	1	1	1	1
	1	1	1	1	0	1	0

If f has n variables, there are 2^n rows in the truth table.