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Using Logic to Formalize Reasoning

def. A proposition is a statement about the world that is definitely true or definitely false.

ex. Parents tell you "rocks are soft or fire is hot."
You accept as true.
You later discover rocks are not soft.
Conclude fire is hot.
Let p = "rocks are soft"
q = "fire is hot".
Reasoning:

$$\begin{array}{l} p \vee q \\ \underline{\neg p \qquad} \\ \quad q \end{array}$$

Conditional Statement

def. The conditional $p \Rightarrow q$.
"If p then q"
"p implies q"
"p is sufficient for q"
"q only if p"

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Proposition.

$$\begin{aligned} p \Rightarrow q &\equiv \neg p \vee q \\ &\equiv \neg q \Rightarrow \neg p \end{aligned}$$

proof of first equivalence.

p	q	$p \Rightarrow q$	$\neg p \vee q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

proof of second equivalence.

$$\begin{aligned} \neg q \Rightarrow \neg p &\equiv \neg(\neg q) \vee \neg p \\ &\equiv q \vee \neg p \\ &\equiv \neg p \vee q \end{aligned}$$

def. The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. These are logically equivalent.

def. The converse of $p \Rightarrow q$ is $q \Rightarrow p$. These are not the same.

def. The biconditional $p \Leftrightarrow q$ ("p if and only if q")

p	q	$p \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Proposition.

$$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$$
$$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$$
$$\equiv (\neg p \wedge \neg q) \vee (p \wedge q)$$
$$\equiv \neg p \Leftrightarrow \neg q$$

ex. Let A be an $n \times n$ matrix. A is invertible $\Leftrightarrow \det A \neq 0$.

Problem.
Prove that $f = (p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$ is a tautology.
proof.

$$f \equiv (\neg p \vee q) \Rightarrow ((\neg r \vee p) \Rightarrow (\neg r \vee q))$$
$$\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q))$$
$$\equiv (\neg\neg p \vee \neg q) \vee ((\neg\neg r \vee \neg p) \vee (\neg r \vee q))$$
$$\equiv (p \wedge \neg q) \vee ((r \wedge \neg p) \vee (\neg r \vee q))$$
$$\equiv ((p \wedge \neg q) \vee q) \vee ((r \wedge \neg p) \vee \neg r)$$
$$\equiv ((p \vee q) \wedge (\neg q \vee q)) \vee ((r \vee \neg r) \wedge (\neg p \vee \neg r))$$
$$\equiv ((p \vee q) \wedge 1) \vee (1 \wedge (\neg p \vee \neg r))$$
$$\equiv (p \vee q) \vee (\neg p \vee \neg r)$$
$$\equiv (p \vee \neg p) \vee q \vee \neg r$$
$$\equiv 1 \vee q \vee \neg r$$
$$\equiv 1$$

Quantifiers and Predicates

ex. Let $P(n)$ mean "n is prime" where $n \in \mathbb{N}$.
$P(3)$ is a true proposition.
$P(4)$ is a false proposition.

def. a predicate is a statement involving variables that becomes true/false when you substitute values for all variables.

ex. $P(n)$ is a predicate.

ex. $E(x,y)$: "x equals y" is a predicate.

def. Universal quantifier, "$\forall$" means "for all".
Existential quantifier, "$\exists$" means "there exists".

ex. $\exists n \in \mathbb{N}\ P(n)$
"There exists a natural number n such that n is prime."
This is a true proposition. $n=3$ is prime.

ex. $\forall n \in \mathbb{N}\ P(n)$
"all natural numbers are prime."
This is a false proposition. $n=4$ is not prime.

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ex. $\forall n \in \mathbb{N} \; P(n) \Rightarrow \neg P(n+1)$.

This is false.

Rather: $\forall n \in \mathbb{N} \; n \geqslant 3 \quad P(n) \Rightarrow P(n+1)$.

If n is prime and $n \geqslant 3$, then n must be odd. Hence, $n+1$ is even and therefore not prime.