Exact inversion with diagonal error covariance matrix

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Given the following identity that can be derived from the Woodbury matrix lemma,

$$(\mathbf{I} + \mathbf{S}^{\mathrm{T}}\mathbf{C}_{dd}^{-1}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{C}_{dd}^{-1} = \mathbf{S}^{\mathrm{T}}(\mathbf{S}\mathbf{S}^{\mathrm{T}} + \mathbf{C}_{dd})^{-1}, \tag{1}$$

or if $\mathbf{C}_{dd} \equiv \mathbf{I}$

$$(\mathbf{I} + \mathbf{S}^{\mathrm{T}}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}(\mathbf{S}\mathbf{S}^{\mathrm{T}} + \mathbf{I})^{-1}. \tag{2}$$

We start by defining the prior ensemble of N model realizations

$$\mathbf{X} = \Big(\mathbf{x}_1^{\mathrm{f}},\ \mathbf{x}_2^{\mathrm{f}},\ \dots\ ,\ \mathbf{x}_m^{\mathrm{f}}\Big), \tag{3}$$

and we define the zero-mean (i.e., centered) anomaly matrix as

$$\mathbf{A} = \mathbf{X}\Big(\mathbf{I}_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\Big)/\sqrt{N-1}, \tag{4}$$

where $\mathbf{1} \in \Re^N$ is defined as a vector with all elements equal to 1, $\mathbf{I}_N$ is the N-dimensional identity matrix, and the projection $\mathbf{I}_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ subtracts the mean from the ensemble.

Write the EnKF analysis equation as

$$\mathbf{X}^{\mathrm{a}} = \mathbf{X}^{\mathrm{f}} + \mathbf{A}^{\mathrm{f}}\mathbf{S}^{\mathrm{T}}\big(\mathbf{S}\mathbf{S}^{\mathrm{T}} + \mathbf{C}_{dd}\big)^{-1}\mathbf{D} \tag{5}$$

where we define the measurements of the predicted model with the mean subtracted

$$\mathbf{S} = \mathbf{h}\big(\mathbf{X}^{\mathrm{f}}\big)\Big(\mathbf{I}_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\Big)/\sqrt{N-1} \tag{6}$$

$$\mathbf{D} = \mathbf{d}\mathbf{1}^{\mathrm{T}} + \mathbf{E} - \mathbf{h}\big(\mathbf{X}^{\mathrm{f}}\big) \tag{7}$$

With a diagonal $\mathbf{C}_{dd} = \mathbf{I}$ we can write the analysis equation as

$$\begin{aligned}
\mathbf{X}^{\mathrm{a}} &= \mathbf{X}^{\mathrm{f}} + \mathbf{A}^{\mathrm{f}}\mathbf{S}^{\mathrm{T}}\big(\mathbf{S}\mathbf{S}^{\mathrm{T}} + \mathbf{I}\big)^{-1}\mathbf{D} && (8)\\
&= \mathbf{X}^{\mathrm{f}} + \mathbf{X}^{\mathrm{f}}\Big(\mathbf{I}_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\Big)\mathbf{S}^{\mathrm{T}}\big(\mathbf{S}\mathbf{S}^{\mathrm{T}} + \mathbf{I}\big)^{-1}\mathbf{D}/\sqrt{N-1} && (9)\\
&= \mathbf{X}^{\mathrm{f}}\Big(\mathbf{I} + \mathbf{S}^{\mathrm{T}}\big(\mathbf{S}\mathbf{S}^{\mathrm{T}} + \mathbf{I}\big)^{-1}\widetilde{\mathbf{D}}\Big) && (10)\\
&= \mathbf{X}^{\mathrm{f}}\Big(\mathbf{I} + \big(\mathbf{I} + \mathbf{S}^{\mathrm{T}}\mathbf{S}\big)^{-1}\mathbf{S}^{\mathrm{T}}\widetilde{\mathbf{D}}\Big) && (11)\\
&= \mathbf{X}^{\mathrm{f}}\Big(\mathbf{I} + \big(\mathbf{Z}\boldsymbol{\Lambda}\mathbf{Z}^{\mathrm{T}}\big)^{-1}\mathbf{S}^{\mathrm{T}}\widetilde{\mathbf{D}}\Big) && (12)\\
&= \mathbf{X}^{\mathrm{f}}\Big(\mathbf{I} + \mathbf{Z}\boldsymbol{\Lambda}^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\widetilde{\mathbf{D}}\Big). && (13)
\end{aligned}$$

In Eq. (9) we inserted Eq. (4) for $\mathbf{A}^{\mathrm{f}}$, then in Eq. (10) we have used that $\mathbf{1}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} = 0$ and we have defined $\widetilde{\mathbf{D}} = \mathbf{D}/\sqrt{N-1}$. In Eq. (11) we have used the Woodbury lemma in Eq. (2) which reduces the dimension of the inverse matrix from $m \times m$ to $N \times N$. By forming the matrix $\mathbf{I} + \mathbf{S}^{\mathrm{T}}\mathbf{S}$ and computing its eigenvalue decomposition we obtain the final result for the analysis in Eq. (13), which can be computed to a cost $\mathcal{O}(mN^2)$.

This scheme is implemented as option $mode = 10$.