On Preconditioning for Finite Element Equations on Irregular Grids

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Abstract

Preconditioning methods are widely used in conjunction with the conjugate gradient method for solving large sparse symmetric linear systems arising from the discretisation of self-adjoint linear elliptic partial differential equations.

1 Introduction

When faced with choosing a method for solving a partial differential equation on an irregular domain, one possibility is certainly the finite element method which has many attractive approximation properties (see e.g. [1]). Other choices would be

- use finite differences
- give up and go home.

As with other numerical methods for solving partial differential equations, using the finite element method involves some form of **discretisation** of the problem domain. In the light of such observations,
we consider the general topic of solving finite element equations with particular reference to producing pretty papers.

2 Eigenvalue bounds

2.1 Analysis

In order to analyse irregular grids we introduce lots of equations so we can number them. Suppose that the finite element grid contains numbered elements $e = 1, \ldots, E$ with V_e local unknowns on each element, giving a total of N global unknowns. The coefficient matrix A can be written as

$$A = L^T [A_e] L \qquad \text{for any matrix } A \tag{2.1}$$

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where $[A_e]$ represents an $[E.V_e \times E.V_e]$ block diagonal matrix whose $[V_e \times V_e]$ diagonal blocks are the element-calculated coefficient matrices. We can now use (2.1) later.

2.2 Result

A useful result not contained in Wathen [2] is that if $\{B_e\}$ is any set of $[V_e \times V_e]$ symmetric positive definite matrices and a preconditioner B is formed from

$$B = \left(\frac{a^2 + b^2 - c}{\theta\alpha} \right) \frac{\partial x}{\partial t} \hat{\alpha} \underline{v} \qquad\qquad \forall \hat{\alpha} \in [0, 1]$$

then the eigenvalues of the global matrix $B^{-1}A$ must satisfy

$$\min_e \lambda_{\min}(B_e^{-1}A_e) \le \lambda(B^{-1}A) \le \max_e \lambda_{\max}(B_e^{-1}A_e). \tag{2.2}$$

3 Conclusion

This is a load of rubbish, and the phrase

"Why bother?"

comes to mind. We list the reasons why here:

1. To enhance the education of the human race.
2. To avoid getting the sack.

A Appendix

In this case the stiffness matrix is

$$K_e = \frac{1}{4S_e} \begin{bmatrix} a & -b & b-a \\ -b & c & b-c \\ b-a & b-c & a-2b+c \end{bmatrix} \tag{A.1}$$

where S_e is the element area, $a = y_2^2$, $b = y_1 y_2$ and $c = x_1^2 + y_1^2$.

References

[1] G. Strang and G.J. Fix, *An Analysis of the Finite Element Method,* Prentice-Hall, 1973.

[2] A.J. Wathen, *Spectral Bounds and Preconditioning Methods,* MAFELAP VI (1987), J.R. Whiteman, ed., Academic Press, 1988, pp. 157-168.