Preconditioned iterative solvers for discretisations with space-time adaptivity

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Physical problem

- hyperbolic PDE in conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

$$(x, y, t) \in \Omega \times [0, t_{\mathrm{end}}]$$

flux functions $f = f(u)$, $g = g(u)$

- Backward Differentiation Formula BDF-2 in time

- finite volume method in space

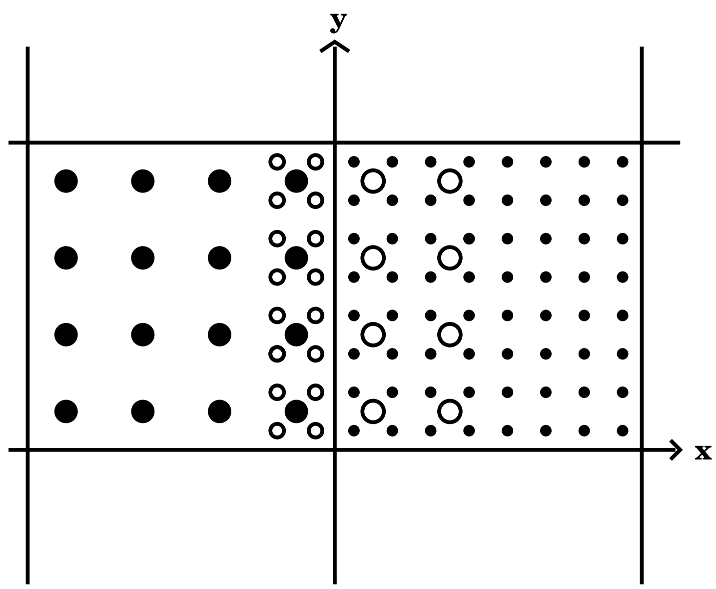
  – spatial grid partitioned into N_b predetermined blocks

  – each block (uniformly) partitioned into $n_b^x \times n_b^y$ cells

  – grid size at block boundaries may jump by at most a factor of two

Discretisation at block boundaries

- two layers of ghost cells around each block
- no jump in grid size: ghost cell values given by direct transfer
- jump in grid size: interpolation needed
  - coarse grid ghost cell: area-weighted averaging
  - fine grid ghost cell: more sophisticated interpolation scheme
- interpolation used ensures stability, second-order accuracy



Adaptivity

- computational grid and timesteps adjusted to meet criteria on local discretisation errors

- discretisation error $\tau = \tau_t + \tau_s$

- adaptivity in time:

  choice of Δt^n based on estimate of τ_t at t^n, computed via comparison of explicit predictor and implicit BDF-2

- adaptivity in space:

  compare error in approximation on a fine grid and next coarser grid

  $|\tau_s| > \epsilon$ in at least one cell $\Rightarrow$ refinement in a block

  $|\tau_s| < 0.1\epsilon$ in all cells $\Rightarrow$ coarsening in a block

Linear system

- non-linear PDE: solve by Newton iteration

- linear PDE (or at each Newton step): large sparse linear system

$$\begin{bmatrix} A & B \\ C & I \end{bmatrix} \begin{bmatrix} \underline{u}^I \\ \underline{u}^G \end{bmatrix} = \begin{bmatrix} \underline{c}^I \\ \underline{c}^G \end{bmatrix}$$

$\underline{u}^I$: internal unknowns, $\underline{u}^G$: ghost cell unknowns

- block Gaussian elimination: solve

$$(A - BC)\underline{u}^I = \underline{c}^I - B\underline{c}^G$$

- preconditioned GMRES convergence based on spectrum of

$$M^{-1}(A - BC)$$

and condition number of eigenvector matrix

Preconditioning

- block preconditioner with semi-Toeplitz blocks

- constant coefficients: preconditioner

$$M = A \equiv \texttt{bl\_diag}(A_1, \ldots, A_{N_b})$$

- each A_b has decomposition

$$A_b = (I_{n_b^y} \otimes S_{n_b^x} \otimes I_{n_c})\, T_b\, (I_{n_b^y} \otimes S_{n_b^x} \otimes I_{n_c})^H$$

T_b is block tridiagonal with block-diagonal blocks
$S_{n_b^x}$ is a modified sine matrix

- solve a preconditioner system using banded solves and FFTs

- variable coefficients: average entries over diagonals of A_b

Spectral analysis

- unit square, periodic boundary conditions, $P = Q = I$, p strips, same n^x, n^y in each strip

- aim: examine eigenvalues of

$$M^{-1}(A - BC) = I - A^{-1}BC$$

- observation

$$\begin{array}{ccc} & & 2pn^y \text{ eigenvalues} \\ & & \text{of } I - CA^{-1}B \\ pn^x n^y \text{ eigenvalues of} & & \\ I - A^{-1}BC & \equiv & + \\ & & \\ & & pn^y(n^x - 2) \text{ ones} \end{array}$$

- theorem: eigenvalues $\lambda^{\pm}_{\ell j}$ of $I - CA^{-1}B$ are given by

$$\lambda^{\pm}_{\ell,j} = 1 + a_j \cos\left(\frac{2\pi(\ell-1)}{p}\right) \pm i\sqrt{b_j^2 + a_j^2 \sin^2\left(\frac{2\pi(\ell-1)}{p}\right)}$$

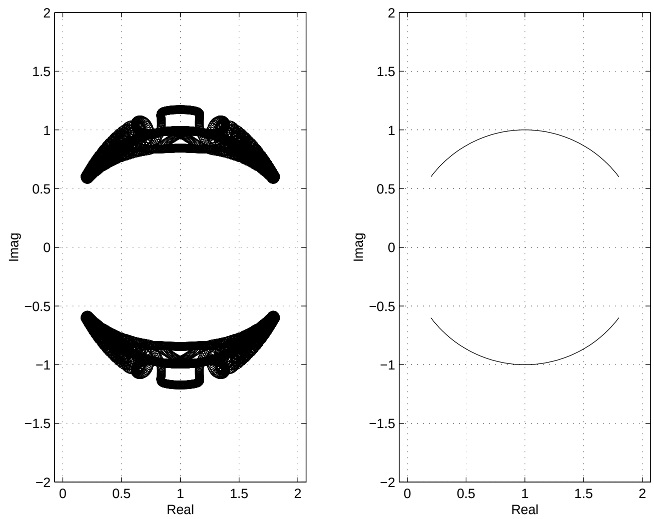
$$\ell = 1, \ldots, p, \qquad j = 1, \ldots, n^y_k$$

Asymptotic spectrum

- grid quotient $\phi = \dfrac{n^y}{n^x}$

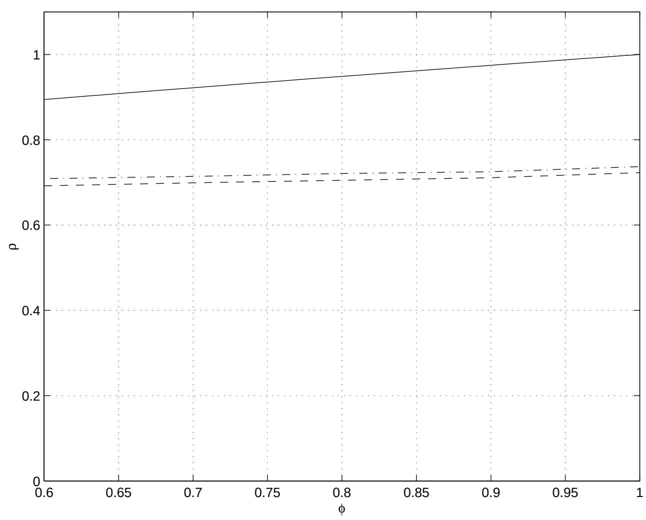
- as $n^x \to \infty$, eigenvalues lie on curve segment

$$\tilde{\lambda}(\zeta) = 1 - \zeta \pm i\sqrt{1-\zeta^2}\ ,\ -\phi \le \zeta \le \phi$$



- asymptotic convergence factor ρ satisfies

$$\rho \le \frac{\sqrt{2+2\phi}}{2}$$



Numerical results

- FORTRAN 90 parallel code

- SunFire 15k parallel cluster, 12 parallel processors

- GMRES(100)

- GMRES iteration terminated when
$$\frac{\|r^{(k)}\|}{\|r^{(0)}\|} < 5e-5$$

- no time adaptivity:
$$\Delta t = 0.005$$

- results averaged over 150 timesteps

Linearised Euler equations

- test problem:

$$u_t + \begin{bmatrix} 1 & c & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_x + \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{bmatrix} u_y = 0$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \Omega = [0, 12] \times [0, 4]$$

- boundary conditions:

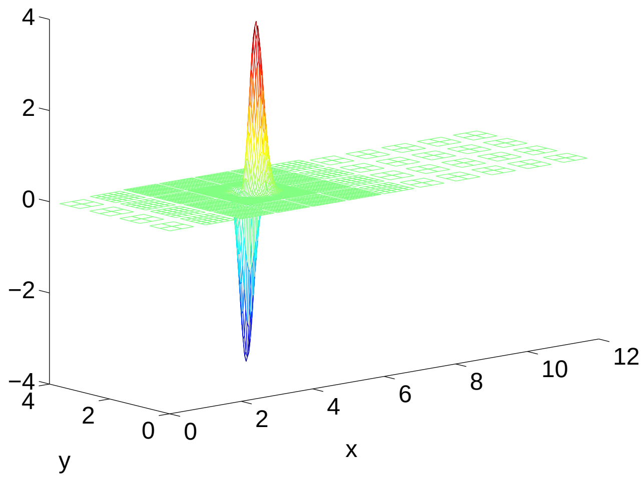
  – periodic at inflow and outflow boundaries
  – u_3 is zero at upper and lower boundaries

- initial distribution:

$$u(x, y, 0) = \begin{bmatrix} 0 \\ -2\frac{y-y_c}{\sigma} \quad e^{-\frac{1}{\sigma}\left((x-x_c)^2+(y-y_c)^2\right)} \\ 2\frac{x-x_c}{\sigma} \quad e^{-\frac{1}{\sigma}\left((x-x_c)^2+(y-y_c)^2\right)} \end{bmatrix}$$

modified Gaussian pulse, centre (x_c, y_c), width σ

- smooth solution with two different timescales



u_2 after one period with $c = 100$

Effect of increasing c

	$M = I$				$M = A$		
c	k	CPU	cells		k	CPU	cells
2	2.00	0.30	8.54e4		5.14	6.32	8.54e4
4	3.88	0.41	8.81e4		6.50	7.11	8.81e4
8	7.88	0.84	8.95e4		7.60	8.89	8.95e4
16	15.55	2.04	9.07e4		8.66	9.16	9.07e4
32	30.20	7.74	9.17e4		10.61	12.75	9.17e4
64	61.81	36.80	9.92e4		15.76	15.90	9.28e4
128	434.60	594.32	4.37e5		26.40	48.62	1.40e5