Multigrid Solution of Discrete Convection-Diffusion Equations

Alison Ramage
Dept of Mathematics
University of Strathclyde
Glasgow, Scotland

Howard Elman
Dept of Computer Science
University of Maryland
College Park, MD, USA

Convection-Diffusion in 2D

$$\begin{aligned} -\epsilon\nabla^2 u(x,y) + \mathbf{w}.\nabla u(x,y) &= f(x,y) \quad \text{in} \quad \Omega \in \mathbb{R}^2 \\ u(x,y) &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

divergence-free convective velocity ('wind') $\mathbf{w}$

diffusion parameter $\epsilon << 1$

discretisation parameter h

mesh Péclet number $P_h = \dfrac{\|\mathbf{w}\| h}{2\epsilon}$

Boundary Layers and Oscillations

- Galerkin finite element method

$$\epsilon(\nabla u_h, \nabla v_h) + (\mathbf{w} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

- oscillations observed in discrete solutions for $P_h > 1$
- Streamline Diffusion method

$$\begin{aligned} \epsilon(\nabla u_h, \nabla v_h) \; &+ \; (\mathbf{w} \cdot \nabla u_h, v_h) + \frac{\delta h}{||\mathbf{w}||}(\mathbf{w} \cdot \nabla u_h, \mathbf{w} \cdot \nabla v_h) \\ &= \; (f, v_h) + \frac{\delta h}{||\mathbf{w}||}(f, \mathbf{w} \cdot \nabla v_h) \quad \forall v_h \in V_h \end{aligned}$$

$$P_h \leq 1: \qquad \delta = 0 \qquad \text{Galerkin FEM}$$

$$P_h > 1: \quad \delta = \frac{1}{2} - \frac{\epsilon}{h} \quad \text{Streamline Diffusion}$$

Multigrid and Convection-Diffusion

- MG: decompose grid functions into two subspaces

  approximate inverse operator
  for components in subspace 1

  smoothing iteration
  rapidly reduces error components in subspace 2

- approximation: choice of discretisation
- smoothing: choice of relaxation method
- multigrid can be implemented effectively for convection-diffusion problems

Multigrid Method

- two-grid method: N_f (fine grid), N_c (coarse grid)
- coefficient matrices: A_f (fine grid), A_c (coarse grid)
- prolongation: bilinear interpolation P
- restriction: transpose of prolongation P^T
- smoothing: line Gauss-Seidel S_A
- ν steps of pre-smoothing, no post-smoothing
- two-grid iteration matrix $M = (I - PA_c^{-1}P^T A_f)S_A^\nu$
- convergence: $\|\mathbf{e}_k\| \leq \|M\|^k \|\mathbf{e}_0\|$

Convergence Analysis

- various successful approaches
  - perturbation arguments
    Bank (1981), Mandel (1986), Bramble, Pasciak and Xu (1988), Wang (1993)
  - matrix-based methods
    Reusken (2002), Olishanskii and Reusken (2002)

- AIM: bound $\|M\|_2 = \|(I - PA_c^{-1}P^TA_f)S_A^\nu\|_2$
  - write $M = (A_f^{-1} - PA_c^{-1}P^T)(A_fS_A^\nu) = M_AM_S$ and bound $\|M_A\|_2$, $\|M_S\|_2$ separately
  - bound $\|M\|_2$ directly

Coefficient Matrix

$$A = \begin{bmatrix} M_1 & M_2 & & & 0 \\ M_3 & M_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & M_3 & M_1 & M_2 \\ 0 & & & M_3 & M_1 \end{bmatrix}$$

eigenvectors and eigenvalues:

$$\begin{aligned} M_1 \mathbf{v}_j &= \lambda_j \mathbf{v}_j, \quad \lambda_j = m_{1c} + 2m_{1r} \cos \tfrac{j\pi}{N_f} \\ M_2 \mathbf{v}_j &= \sigma_j \mathbf{v}_j, \quad \sigma_j = m_{2c} + 2m_{2r} \cos \tfrac{j\pi}{N_f} \\ M_3 \mathbf{v}_j &= \gamma_j \mathbf{v}_j, \quad \gamma_j = m_{3c} + 2m_{3r} \cos \tfrac{j\pi}{N_f} \end{aligned}$$

$$\mathbf{v}_j = \sqrt{\frac{2}{N_f}} \left[\sin \frac{j\pi}{N_f}, \quad \sin \frac{2j\pi}{N_f}, \quad \ldots, \sin \frac{(N_f - 1)j\pi}{N_f} \right]^T$$

Transformation: Coefficient Matrix

$$\hat{V}_f = \left[\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_{n_f}\right] \qquad M_1 \hat{V}_f = \hat{V}_f \Lambda, \;\; M_2 \hat{V}_f = \hat{V}_f \Sigma, \;\; M_3 \hat{V}_f = \hat{V}_f \Gamma$$

$$V_f = \mathtt{diag}(\hat{V}_f, \ldots, \hat{V}_f), \quad \text{permutation } \Pi_f$$

$$\Pi_f^T \left[V_f^T A_f V_f\right] \Pi_f = T_f = \begin{bmatrix} T_1 & & & & 0 \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_{n_f-1} & \\ 0 & & & & T_{n_f} \end{bmatrix}$$

$$T_j = \mathtt{tridiag}(\gamma_j, \lambda_j, \sigma_j)$$

fine grid: $A_f = Q_f T_f Q_f^T \qquad Q_f = V_f \Pi_f$

coarse grid: $A_c = Q_c T_c Q_c^T \qquad Q_c = V_c \Pi_c$

Transformation: Smoothing Matrix

block matrix splitting: $A_f = D_A - L_A - U_A$

Gauss-Seidel smoothing matrix:

$$S_A = (D_A - L_A)^{-1} U_A = I - (D_A - L_A)^{-1} A_f$$

transformation:

$$S_A = Q_f S_T Q_f^T$$

where $S_T = I - (D_T - L_T)^{-1} T_f$ is block-diagonal

Transformation: Prolongation Matrix

2D prolongation matrix: $P = L \otimes L$

$$L^T = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & & & & \ddots & & \\ & & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

transformation: $Q_f = (I_f \otimes \hat{V}_f)\Pi_f, \quad Q_c = (I_c \otimes \hat{V}_c)\Pi_c$

$$\bar{P} = Q_f^T P Q_c = \mathcal{A}^T \otimes L$$

$$\mathcal{A} = \begin{bmatrix} \alpha_1 & & & & & & & \alpha_{N_f-1} \\ & \alpha_2 & & & & & \alpha_{N_f-2} & \\ & & \ddots & & & \iddots & & \\ & & & \alpha_{n_c} & 0 & \alpha_{N_f-n_c} & & \end{bmatrix}$$

Transformation: Iteration Matrix (1)

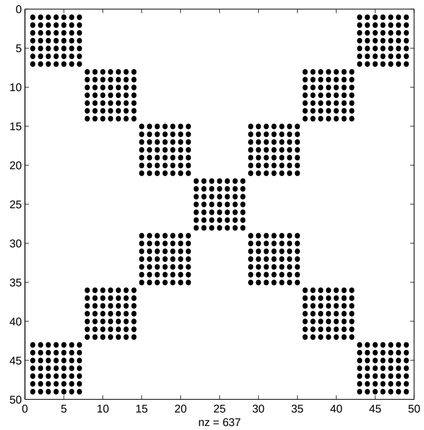
$$
\begin{aligned}
M &= (I - PA_c^{-1}P^TA_f)S_A^\nu \\
&= (I - PQ_cT_c^{-1}Q_c^TP^TQ_fT_fQ_f^T)S_A^\nu \\
&= Q_f(I - \bar{P}T_c^{-1}\bar{P}^TT_f)Q_f^T(Q_fS_TQ_f^T)^\nu \\
&= Q_f\left(I - \bar{P}T_c^{-1}\bar{P}^TT_f\right)S_T^\nu Q_f^T \\
\Rightarrow M &= Q_f\bar{M}Q_f^T
\end{aligned}
$$

where $\bar{M} = \left(I - \bar{P}T_c^{-1}\bar{P}^TT_f\right)S_T^\nu$

Q_f is orthogonal:

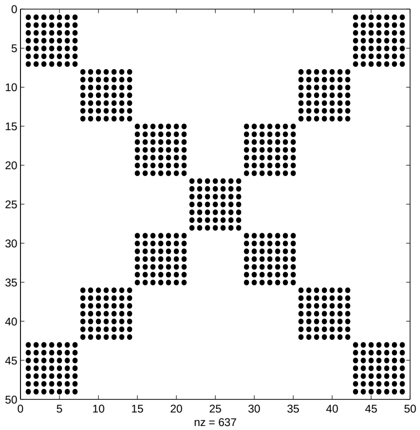
$$\|M\|_2 = \|\bar{M}\|_2$$

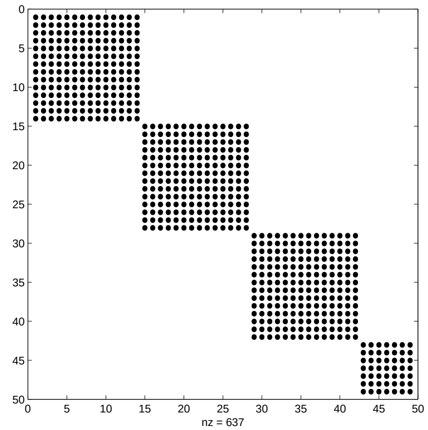
Transformed Iteration Matrix (2)



$$\begin{bmatrix} \mathbf{B_1} & & & & & & \mathbf{C_1} \\ & \mathbf{B_2} & & & & \mathbf{C_2} & \\ & & \mathbf{B_3} & & \mathbf{C_3} & & \\ & & & \mathbf{B_4} & & & \\ & & \mathbf{C_5} & & \mathbf{B_5} & & \\ & \mathbf{C_6} & & & & \mathbf{B_6} & \\ \mathbf{C_7} & & & & & & \mathbf{B_7} \end{bmatrix}$$

Transformed Iteration Matrix (2)





$$\begin{bmatrix} \mathbf{B_1} & & & & & & \mathbf{C_1} \\ & \mathbf{B_2} & & & & \mathbf{C_2} & \\ & & \mathbf{B_3} & & \mathbf{C_3} & & \\ & & & \mathbf{B_4} & & & \\ & & \mathbf{C_5} & & \mathbf{B_5} & & \\ & \mathbf{C_6} & & & & \mathbf{B_6} & \\ \mathbf{C_7} & & & & & & \mathbf{B_7} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{B_1} & \mathbf{C_1} & & & & & \\ \mathbf{C_7} & \mathbf{B_7} & & & & & \\ & & \mathbf{B_2} & \mathbf{C_2} & & & \\ & & \mathbf{C_6} & \mathbf{B_6} & & & \\ & & & & \mathbf{B_3} & \mathbf{C_3} & \\ & & & & \mathbf{C_5} & \mathbf{B_5} & \\ & & & & & & \mathbf{B_4} \end{bmatrix}$$

$$\|\bar{M}\|_2 = \max\left\{ \max_{j=1,\dots,n_c} \left\| \begin{bmatrix} B_j & C_j \\ C_k & B_k \end{bmatrix} \right\|_2, \|B_{N_c}\|_2 \right\}, \quad k = N_f - j$$

The Story So Far...

- $n_f^2 \times n_f^2$ two-grid iteration matrix M
- Fourier transformation converts 2D problem to a set of n_f problems with 1D structure
- $\|M\|_2$ can be found from norms of N_c smaller problems

n_c of size $2n_f \times 2n_f$, 1 of size $n_f \times n_f$

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- IDEA: analyse semiperiodic version of the problem

  n_c of size $2N_f \times 2N_f$, 1 of size $N_f \times N_f$

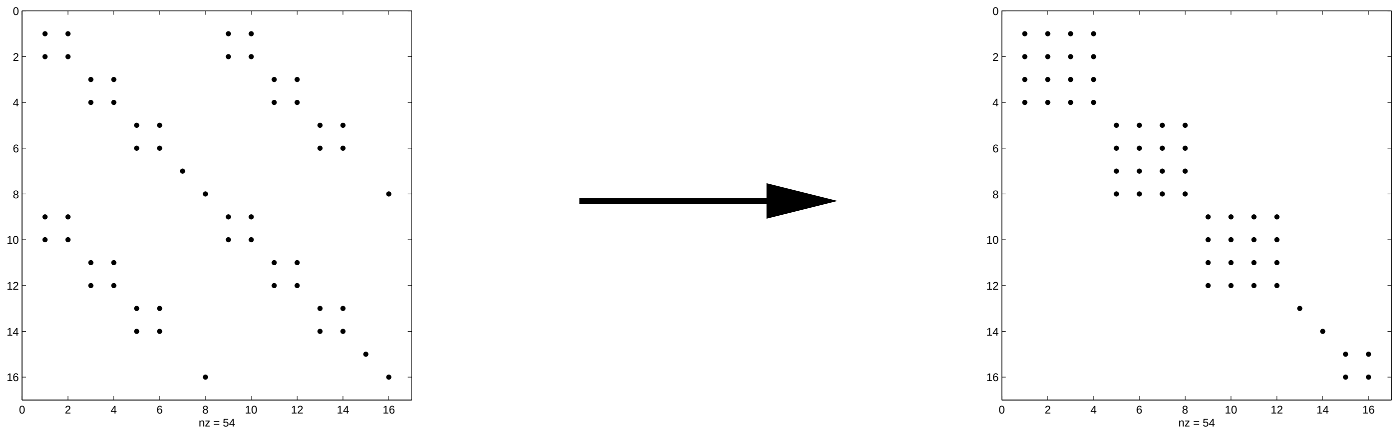
- gain insight into Dirichlet problem behaviour?

Semiperiodic problem

- B_j, C_j are replaced by periodic versions, e.g.

$$B_j^{per} = [I - \bar{P}_j^{per} (T_c^{per})_j^{-1} (\bar{P}_j^{per})^T (T_f^{per})_j] \, S_j^{per}$$

- transform using coarse grid periodic eigenvectors
- B_j^{per}, C_j^{per} become block diagonal with 2×2 blocks
- permute into block diagonal form



- 2-norm of M^{per} given by max 2-norm of 4×4 blocks

Analytic result

- with semiperiodic approximation, when $P_h > 1$

$$\|M^{per}\|_2 = \frac{\sqrt{3 + \cos(2\pi h)}}{\sqrt{2}(5^\nu)}$$

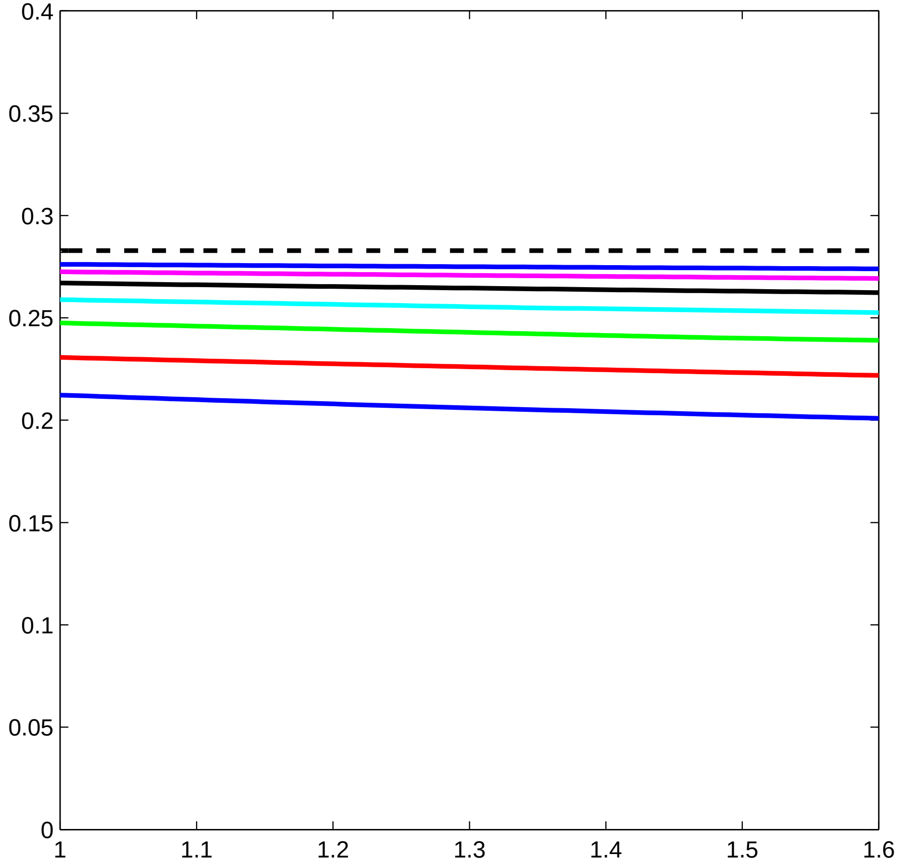
- as h is small in practice,

$$\|M^{per}\|_2 \simeq \frac{\sqrt{2}}{5^\nu}$$

- when $P_h < 1$, analysis is more detailed but good approximations to $\|M^{per}\|_2$ can be derived

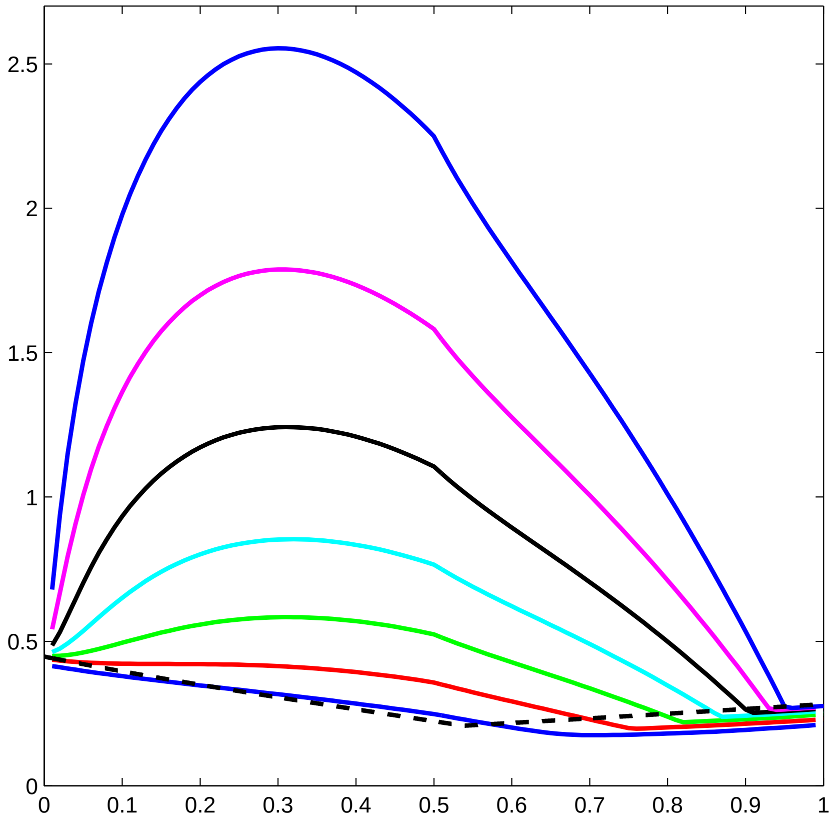
Question: Does this semiperiodic analysis correctly predict Dirichlet problem behaviour?

Model Problem Results (1)



- $\|M\|_2$ vs P_h
- $P_h \geq 1$ only
- semiperiodic: dashed line
- Dirichlet: solid lines
- h fixed for each line
- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- semiperiodic: $\frac{\sqrt{2}}{5} \simeq 0.28$
- Dirichlet $\rightarrow \frac{\sqrt{2}}{5}$

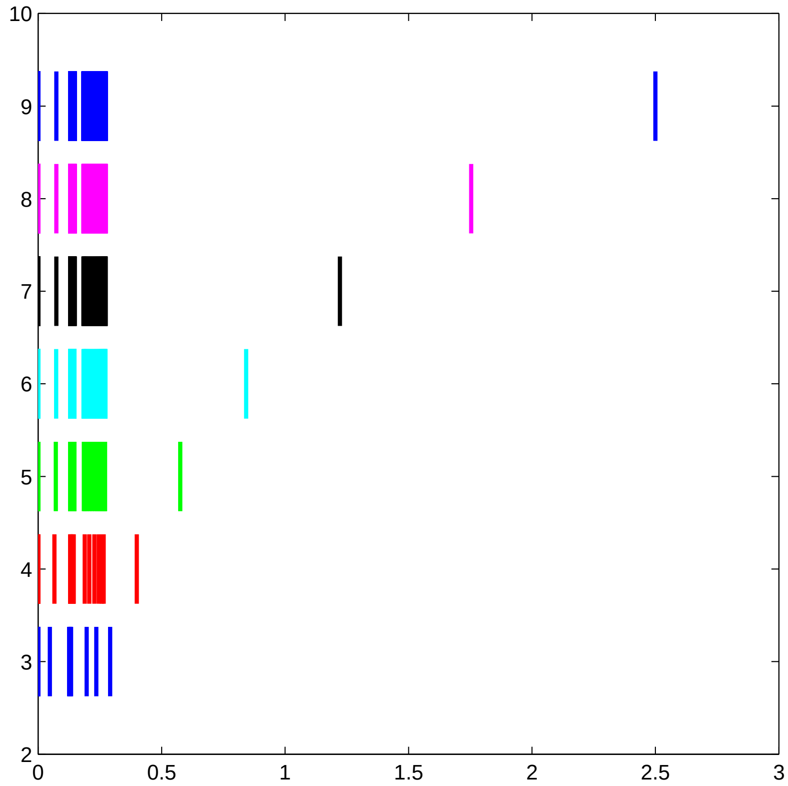
Model Problem Results (2)



- $\|M\|_2$ vs P_h
- $P_h < 1$ only
- semiperiodic: dashed line
- Dirichlet: solid lines
- h fixed for each line
- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- not a good match
- MG may diverge!

Observations

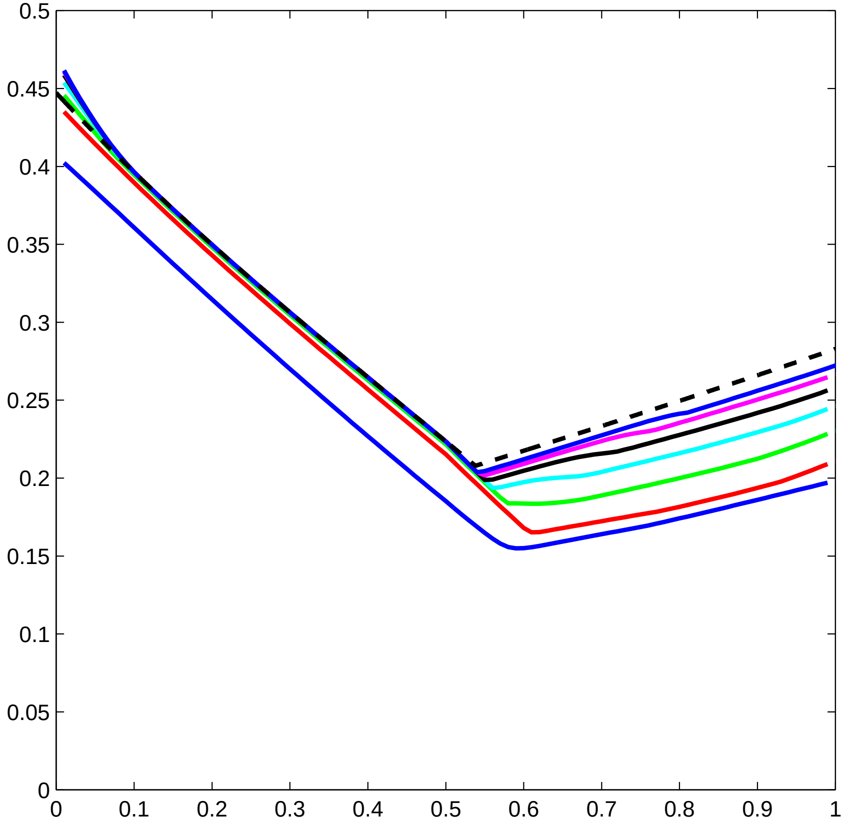
- $\|M\|_2 = \sqrt{|\lambda_1(M^*M)|}$
- for $P_h < 1$, matrix blocks have one 'bad' eigenvalue



$\sqrt{|\lambda(\mathcal{M}_1^*\mathcal{M}_1)|}$ for fixed $P_h = 0.38$

Alternative Bound?

- artificially ‘remove’ this eigenvalue: use $\sqrt{|\lambda_2(\mathcal{M}_i^*\mathcal{M}_i)|}$



- $P_h < 1$ only
- semiperiodic: $\|M^{per}\|_2$
- Dirichlet: $\sqrt{|\lambda_2(\mathcal{M}_1^*\mathcal{M}_1)|}$

Outlying eigenvalue

- in practice, the effect of this outlying eigenvalue is transient

- the eigenvector corresponding to the outlying eigenvalue is large only on grid lines very close to the inflow boundary

- after a few MG iterations, it is smooth and so is easily eliminated by coarse grid correction

- these effects do not have an impact on practical MG performance

MG Iteration Counts

- MG-like convergence for any value of P_h

	ϵ										
h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$
$\frac{1}{4}$	5	5	5	5	5	4	4	3	2	2	2
$\frac{1}{8}$	7	7	6	6	5	5	4	4	3	2	2
$\frac{1}{16}$	7	7	7	6	5	5	5	4	4	3	2
$\frac{1}{32}$	7	7	7	7	6	5	5	4	4	3	3
$\frac{1}{64}$	7	7	7	7	6	5	5	4	4	4	3
$\frac{1}{128}$	7	6	6	6	6	6	5	4	4	4	3

$P_h < 1$ $\quad\quad$ $P_h \geq 1$

Further Remarks

- Separate approximation and smoothing matrices:
  - semiperiodic analysis for smoothing matrix norm is representative of Dirichlet problem behaviour for all values of P_h,
  - semiperiodic analysis for approximation matrix norm is representative of Dirichlet problem behaviour for $P_h \geq 1$: for $P_h < 1$, one 'bad' eigenvalue again causes trouble.

- Replacing the Dirichlet condition by a Neumann condition on the outflow boundary leads to similar computational results.

Conclusions

- Linear algebra can be used to give a useful insight into convergence of two-grid iteration.
- We have obtained bounds on the multigrid convergence factor for a problem with semiperiodic boundary conditions.
- Boundary effects associated with a Dirichlet condition on the inflow boundary appear to be transient.
- Semiperiodic analysis gives an accurate description of MG behaviour for the full Dirichlet problem.