Oscillations in Discrete Solutions of Convection-Diffusion Equations

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Incompressible Navier-Stokes Equations

- steady-state problem

$$\begin{aligned} -\epsilon\nabla^2\underline{u} + (\underline{u}.\nabla)\underline{u} + \nabla p &= \underline{f} \\ -\nabla.\underline{u} &= 0 \end{aligned}$$

on $\Omega \in \mathbb{R}^2$ with suitable boundary conditions on $\partial\Omega$

velocity vector $\underline{u}$, pressure p, viscosity ϵ

- linearisation, discretisation (e.g. finite element method): coefficient matrix

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$

(1,1) block represents a convection diffusion problem

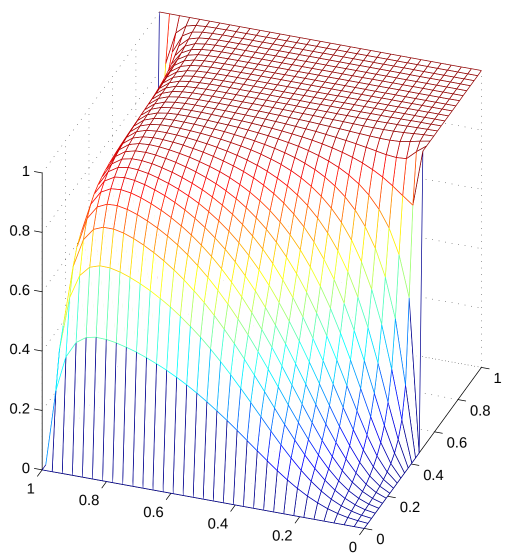
- understanding convection diffusion problems should aid development of fast Navier-Stokes solvers

Convection-Diffusion in 2D

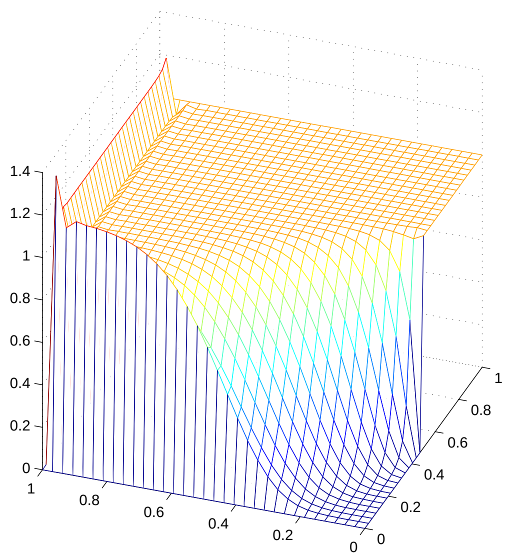
$$-\epsilon\nabla^2 u(x,y) + \underline{w}.\nabla u(x,y) = f(x,y) \quad \text{in} \quad \Omega \in \mathbb{R}^2$$
$$u(x,y) = g \quad \text{on} \quad \partial\Omega$$

- exponential and characteristic boundary layers
- discretise with Galerkin FEM, square bilinear elements
- oscillations observed in discrete solutions for large P_e

$$P_e = \frac{h\|\underline{w}\|}{2\epsilon}$$



$P_e = 0.5$



$P_e = 2$

1D: oscillations if and only if $P_e > 1$

2D: when and why do oscillations occur?

2D Model Problem

- grid-aligned flow with $f = 0$

$$A\underline{u} = \underline{f}$$

$$A = \begin{bmatrix} M_1 & M_2 & & & 0 \\ M_3 & M_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & M_3 & M_1 & M_2 \\ 0 & & & M_3 & M_1 \end{bmatrix}$$

- eigenvectors and eigenvalues

$$M_1\underline{v}_j = \lambda_j\underline{v}_j, \qquad M_2\underline{v}_j = \sigma_j\underline{v}_j, \qquad M_3\underline{v}_j = \gamma_j\underline{v}_j,$$

$$\underline{v}_j = \sqrt{\frac{2}{N}}\left[\sin\frac{j\pi}{N}, \quad \sin\frac{2j\pi}{N}, \quad \ldots, \sin\frac{(N-1)j\pi}{N}\right]^T$$

- matrix blocks simultaneously diagonalisable via sine transforms

$$V = [\underline{v}_1\underline{v}_2\ldots\underline{v}_{N-1}], \qquad \mathcal{V} = \mathtt{diag}(V, \ldots, V)$$

$$M_1V = V\Lambda, \qquad M_2V = V\Sigma, \qquad M_3V = V\Gamma$$

Transformation

$$\mathcal{V}^T A \mathcal{V} = \mathcal{T} = \begin{bmatrix} \Lambda & \Sigma & & & 0 \\ \Gamma & \Lambda & \Sigma & & \\ & \ddots & \ddots & \ddots & \\ & & \Gamma & \Lambda & \Sigma \\ 0 & & & \Gamma & \Lambda \end{bmatrix}$$

- permute into tridiagonal form

$$P^T \mathcal{T} P = T = \begin{bmatrix} T_1 & & & & 0 \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_{N-2} & \\ 0 & & & & T_{N-1} \end{bmatrix}$$

$$T_i = \mathtt{tridiag}(\gamma_i, \lambda_i, \sigma_i)$$

- solution $\quad \underline{u} = \mathcal{V} P \underline{y} \quad$ where $\quad T \underline{y} = P^T \mathcal{V}^T \underline{f} \equiv \underline{\hat{f}}$

$$N - 1 \text{ block systems } T_i \underline{y}_i = \underline{\hat{f}}_i$$

- discrete solution value at point (jh, kh)

$$u_{jk} = \sqrt{\frac{2}{N}} \sum_{i=1}^{N-1} \sin \frac{ij\pi}{N} y_{ik}$$

where y_{ik} is the kth entry of vector $\underline{y}_i$

Solving for $\underline{y}_i$

- three-term recurrence

$$\gamma_i y_{i(k-1)} + \lambda_i y_{ik} + \sigma_i y_{i(k+1)} = \hat{f}_{ik}$$

- auxiliary equation roots

$$\mu_1(i) = \frac{-\lambda_i + \sqrt{\lambda_i^2 - 4\sigma_i\gamma_i}}{2\sigma_i}, \quad \mu_2(i) = \frac{-\lambda_i - \sqrt{\lambda_i^2 - 4\sigma_i\gamma_i}}{2\sigma_i}$$

- recurrence relation solution

$$y_{ik} = c_1\mu_1^k + c_2\mu_2^k \quad + \quad \frac{\hat{f}_{ik}}{\sigma_i + \lambda_i + \gamma_i}$$

or

$$y_{ik} = F_b(i) + [F_t(i) - F_b(i)]\, G_1(i,k) + [F_s(i) - F_b(i)]\, G_2(i,k)$$

where

$F_b(i)$ depends on bottom boundary conditions
$F_t(i)$ depends on top boundary conditions
$F_s(i)$ depends on side boundary conditions

and

$$G_1(i,k) = \frac{\mu_1^k - \mu_2^k}{\mu_1^N - \mu_2^N}$$

$$G_2(i,k) = (1 - \mu_1^k) - (1 - \mu_1^N)\left[\frac{\mu_1^k - \mu_2^k}{\mu_1^N - \mu_2^N}\right]$$

Galerkin Bilinear Finite Elements

- computational molecule

$$
\begin{array}{lccccc}
M_2: & -\left(\frac{\epsilon}{3}-\frac{h}{12}\right) & & -\left(\frac{\epsilon}{3}-\frac{h}{3}\right) & & -\left(\frac{\epsilon}{3}-\frac{h}{12}\right) \\
 & & \nwarrow & \uparrow & \nearrow & \\
M_1: & -\frac{\epsilon}{3} & \longleftarrow & \frac{8\epsilon}{3} & \longrightarrow & -\frac{\epsilon}{3} \\
 & & \swarrow & \downarrow & \searrow & \\
M_3: & -\left(\frac{\epsilon}{3}+\frac{h}{12}\right) & & -\left(\frac{\epsilon}{3}+\frac{h}{3}\right) & & -\left(\frac{\epsilon}{3}+\frac{h}{12}\right)
\end{array}
$$

- auxiliary equation roots

$$
\mu_{1,2} = \frac{-\left[\dfrac{4-C_i}{2+C_i}\right]\dfrac{1}{P_e} \pm \sqrt{1+\dfrac{3(5+C_i)(1-C_i)}{(2+C_i)^2}\dfrac{1}{P_e^2}}}{1-\left[\dfrac{1+2C_i}{2+C_i}\right]\dfrac{1}{P_e}}
$$

where

$$
P_e = \frac{h}{2\epsilon}, \qquad C_i = \cos\frac{i\pi}{N}
$$

Theorem. For any P_e, there exists i such that $G_1(i,k)$ and $G_2(i,k)$ are oscillatory functions of k.

G_1, G_2 oscillate when $\mu_2 < 0 \Leftrightarrow i > 2N/3$

Full Discrete Solution $\underline{u}$

$$\underline{u}_{j,:} = \sqrt{\frac{2}{N}} \sum_{i=1}^{N-1} \sin \frac{ij\pi}{N} \underline{y}_{i,:}$$

- **Test Problem I** $f_t = 1$, $f_b = f_l = f_r = 0$
  solution is nearly zero everywhere: exponential layer along the top boundary

$$\underline{u}_{j,:} = \sqrt{\frac{2}{N}} \sum_{i=1}^{N-1} \sin \frac{ij\pi}{N} \left[-F_t(i) G_1(i,:)\right] = \frac{2}{N} \sum_{i=1}^{N-1} d_{ij} G_1(i,:)$$

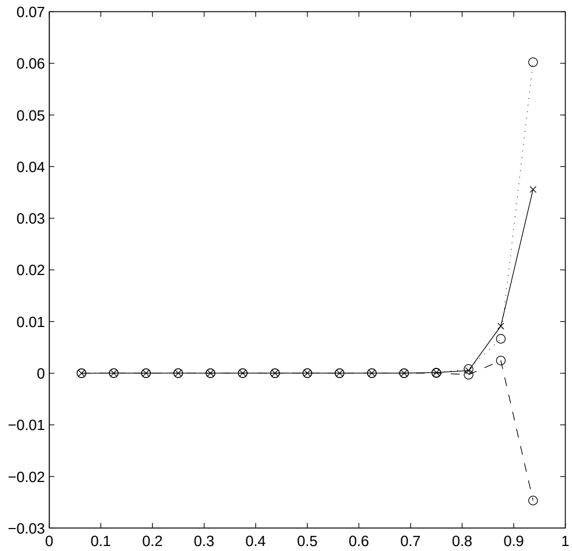
where $|d_{ij}|$ decreases as i increases

- split into smooth and oscillatory parts

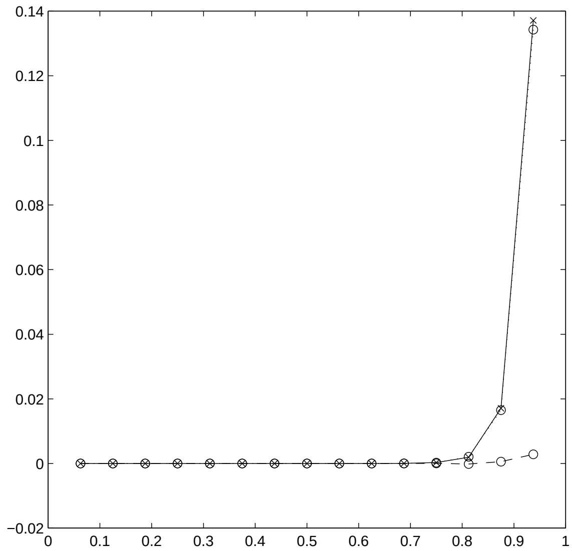
$$\underline{u}_{j,:} = \frac{2}{N} \sum_{i=1}^{i^*-1} d_{ij} G_1(i,:) + \frac{2}{N} \sum_{i=i^*}^{N-1} d_{ij} G_1(i,:) = S_{\text{smooth}} + S_{\text{osc}}$$

- for $P_e > 1$, $S_{\text{smooth}} = 0$ so $\underline{u}_{j,:}$ is oscillatory **but . . .**
  $\underline{u}_{j,:}$ may be oscillatory even for $P_e < 1$

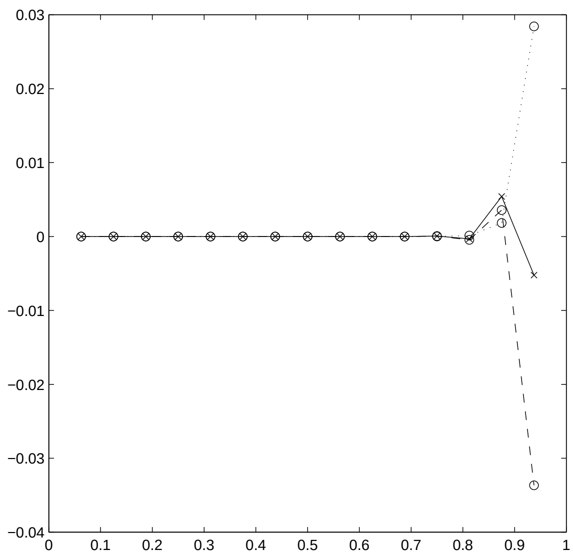
S_{smooth} : dotted line, o
S_{osc} : dashed line, o
$\underline{u}_{j,:}$: solid line, x



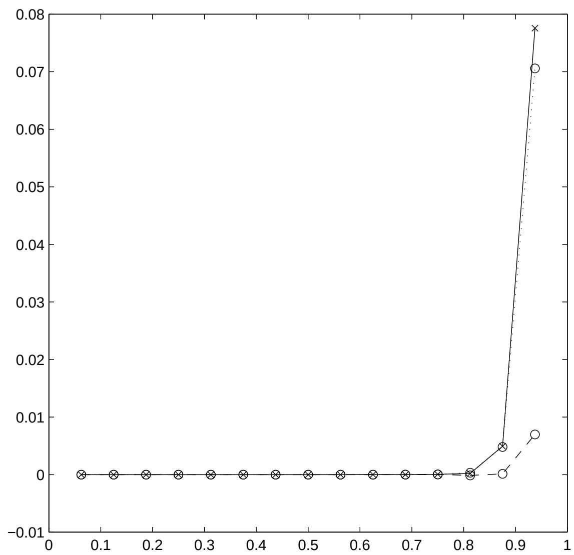
(a) $j = 1$: $P_e = 0.75$.



(b) $j = 3$: $P_e = 0.75$.



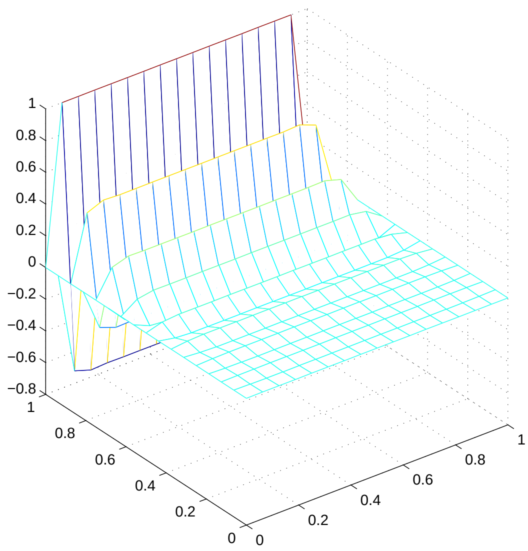
(c) $j = 1$: $P_e = 0.85$.



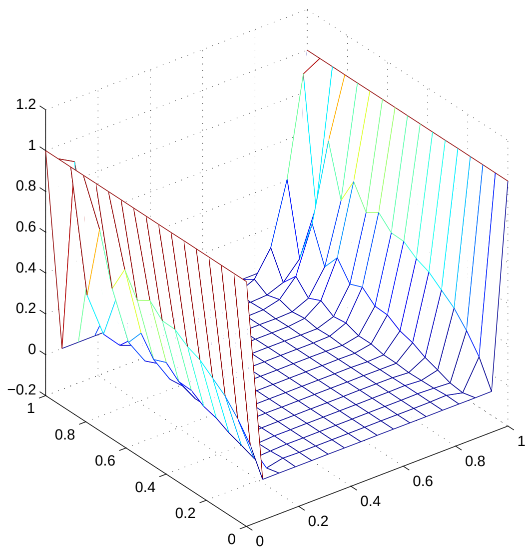
(d) $j = 3$: $P_e = 0.85$.

Test Problems: $P_e = 5$

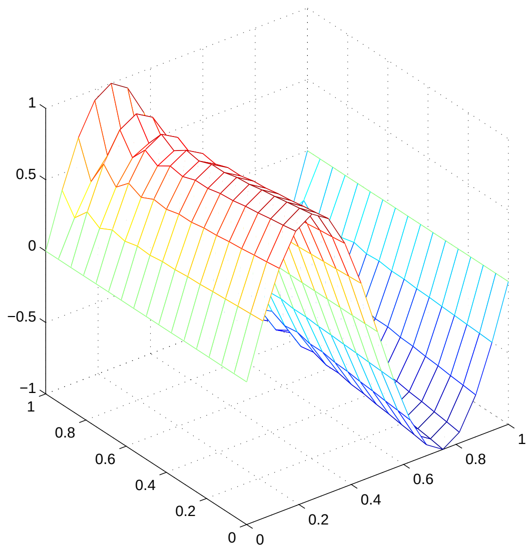
$f_t(x) = 1$
$f_b(x) = f_l(y) = f_r(y) = 0$
oscillations caused by G_1
no G_2 present



$f_r(y) = f_l(y) = 1$
$f_b(x) = f_t(x) = 0$
oscillations caused by G_2
no G_1 present



$f_b(x) = f_t(x) = \sin(2\pi x)$
$f_r(y) = f_l(y) = 0$
oscillations caused by G_2
no G_1 present



Streamline Diffusion

- computational molecule

$$
\begin{array}{llcccc}
M_2: & -\frac{1}{12}\left[(2\delta-1)\,h+4\epsilon\right] & & -\frac{1}{3}\left[(2\delta-1)\,h+\epsilon\right] & & -\frac{1}{12}\left[(2\delta-1)\,h+4\epsilon\right] \\
 & & \nwarrow & \uparrow & \nearrow & \\
M_1: & \frac{1}{3}\left(\delta h-\epsilon\right) & \leftarrow & \frac{4}{3}\left(\delta h+2\epsilon\right) & \rightarrow & \frac{1}{3}\left(\delta h-\epsilon\right) \\
 & & \swarrow & \downarrow & \searrow & \\
M_3: & -\frac{1}{12}\left[(2\delta+1)\,h+4\epsilon\right] & & -\frac{1}{3}\left[(2\delta+1)\,h+\epsilon\right] & & -\frac{1}{12}\left[(2\delta+1)\,h+4\epsilon\right]
\end{array}
$$

- auxiliary equation roots

$$
\mu_{1,2} = \frac{-2\delta - \left[\dfrac{4-C_i}{2+C_i}\right]\dfrac{1}{P_e} \pm \sqrt{1 + \dfrac{12\delta(1-C_i)}{(2+C_i)}\dfrac{1}{P_e} + \dfrac{3(5+C_i)(1-C_i)}{(2+C_i)^2}\dfrac{1}{P_e^2}}}{-2\delta + 1 - \left[\dfrac{1+2C_i}{2+C_i}\right]\dfrac{1}{P_e}}
$$

where

$$
P_e = \frac{\|w\|h}{2\epsilon}, \qquad C_i = \cos\frac{i\pi}{N}
$$

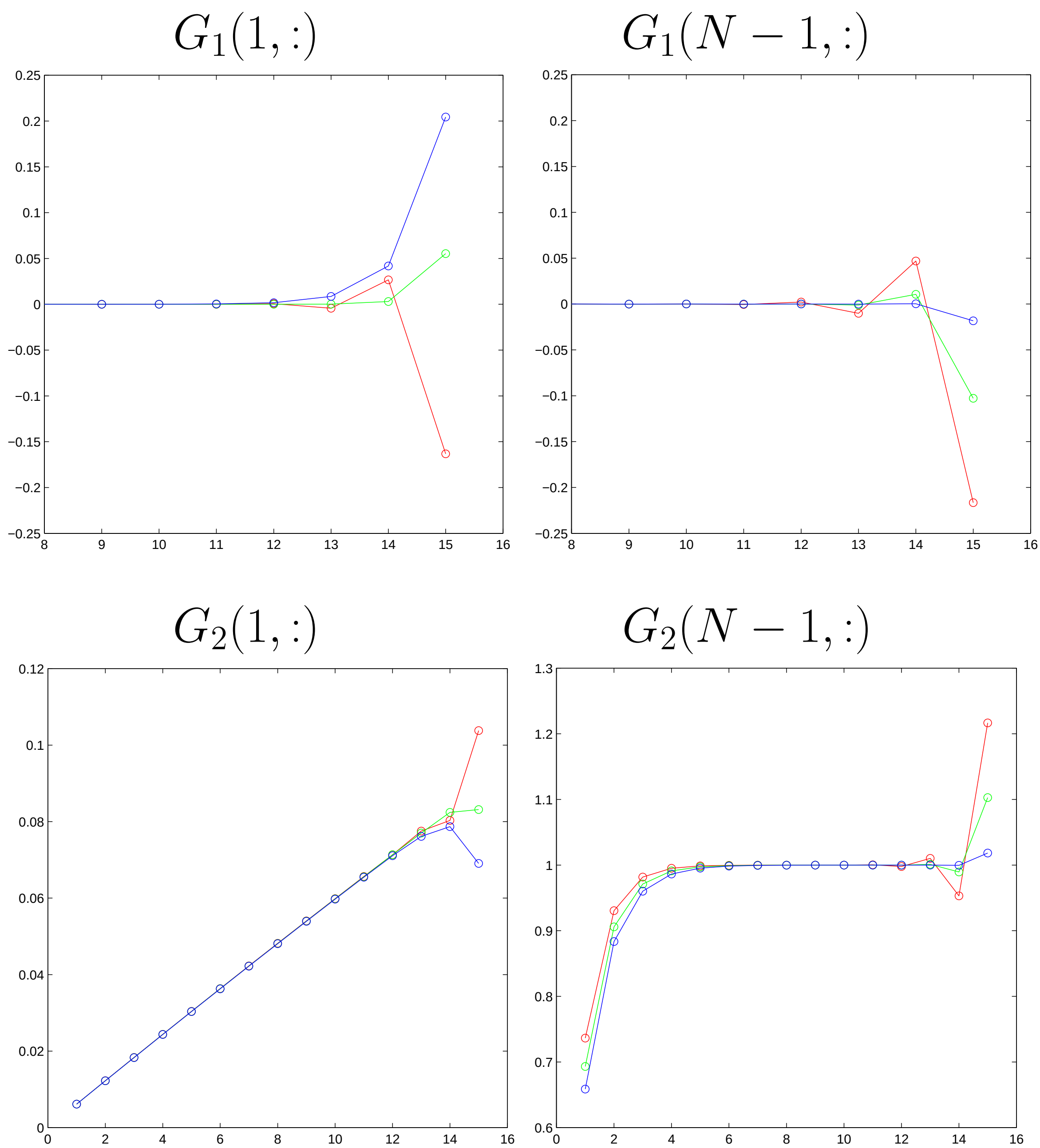
- If $P_e > 1$, then for each $i \in \{1, \ldots, N-1\}$ there exists a parameter

$$
\delta_i^c = \frac{1}{2}\left(1 - \left[\frac{1+2C_i}{2+C_i}\right]\frac{1}{P_e}\right)
$$

such that $\delta > \delta_i^c$ implies that $G_1(i,k)$ and $G_2(i,k)$ are non-oscillatory functions of k.

Effects of changing δ on G_1 and G_2

$\delta = 0.2$: red, $\delta = 0.4$: green, $\delta = 0.6$: blue



Bounds on δ_i^c

$$\delta_i^c = \frac{1}{2}\left(1 - \left[\frac{1+2C_i}{2+C_i}\right]\frac{1}{P_e}\right)$$

- extremal values: put $C_i = 1$, $C_i = -1$

$$\delta_* = \frac{1}{2}\left(1 - \frac{1}{P_e}\right), \quad \delta^* = \frac{1}{2}\left(1 + \frac{1}{P_e}\right)$$

$$\delta_* < \delta_i^c < \delta^*$$

- split $\underline{u}_{j,:}$ into smooth and oscillatory parts

$$\delta < \delta_* \Rightarrow S_{\text{smooth}} = 0, \qquad \delta > \delta^* \Rightarrow S_{\text{osc}} = 0$$

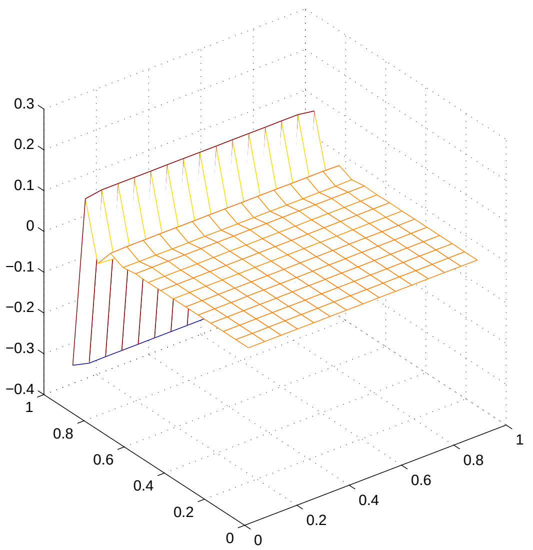
- **Result:** if $\delta > \delta^*$, then $\underline{y}_{i,:}$ is a non-oscillatory function of k for all i and $\underline{u}_{j,:}$ is a non-oscillatory function of k for all j.

- define $\delta_s \in (\delta_*, \delta^*)$:

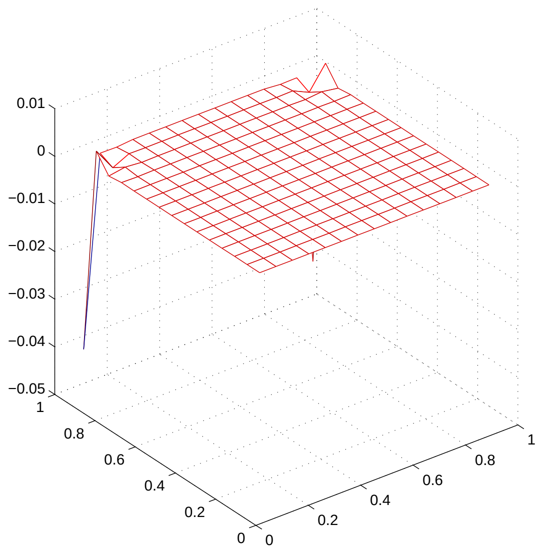
$$\delta_s = \text{smallest } \delta \text{ such that } \underline{u}_{j,:} \text{ is non-oscillatory}$$

Test Problem I: $f_t = 1,\ f_b = f_l = f_r = 0$

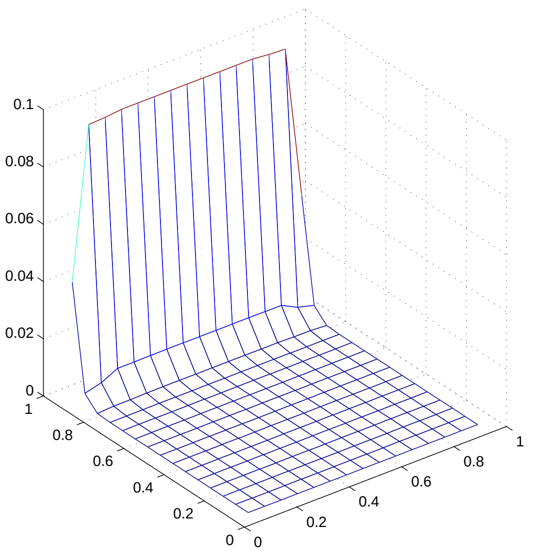
$N = 16, \quad P_e = 2$



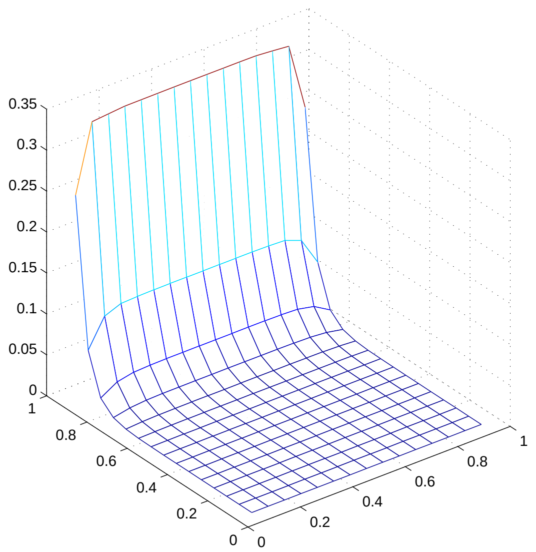
(a) $\delta = 0$.



(b) $\delta = \delta_* = 0.25$.



(c) $\delta = \delta_{\mathrm{s}} = 0.354$.



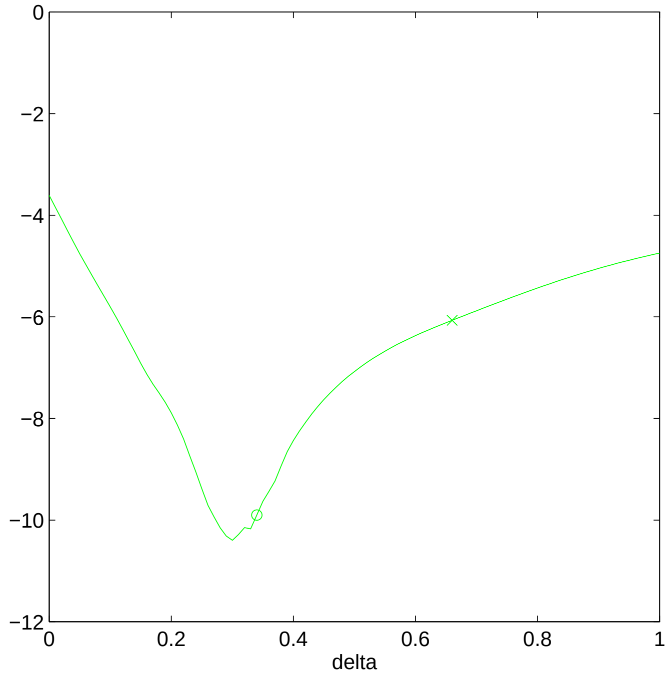
(d) $\delta = \delta^* = 0.75$.

Observations

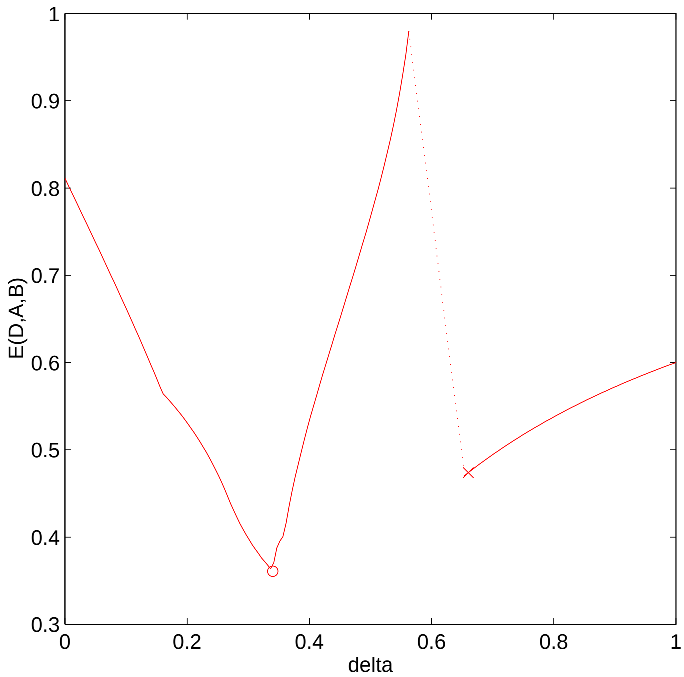
- no single optimal choice of δ: a different value is best for different choices of error norm
- guaranteeing a smooth solution with $\delta = \delta^*$ leads to over-stabilisation
- 'ideal' choice $\delta = \delta_s$ for removing oscillations is difficult to identify in practice
- recommended value:

$$\delta = \delta_* = \frac{1}{2}\left(1 - \frac{1}{P_e}\right)$$

  - easy to compute
  - leads to small errors (although there will be some oscillations)
  - leads to fast iterative solution
    Fischer, Ramage, Silvester and Wathen, CMAME 1999



GMRES residual reduction
$h = 1/16$, $\epsilon = 1/100$



asymptotic convergence estimate
$h = 1/16$, $\epsilon = 1/100$

Summary

- Transforms based on Fourier analysis can be used to construct closed-form solutions for 2D convection-diffusion model problems.

- These solutions provide insight into why the Galerkin method produces oscillations.

- We can examine multi-dimensional issues, e.g. the effects of different types of boundary layers.

- The observations help to explain the success of the streamline upwinding method.

- The analysis can be applied to other discretisations, e.g. finite difference approximations, the artificial diffusion method.

- The model problem analysis gives guidelines for problems with varying wind direction.

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