IFISS : A Matlab Toolbox for Modelling Incompressible Flow

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IFISS

Incompressible **F**low & **I**terative **S**olver **S**oftware

`www.manchester.ac.uk/ifiss`
`www.cs.umd.edu/~elman/ifiss.html`

Google search for
IFISS

H.C. Elman, D.J. Silvester and A.J. Wathen

Finite Elements and Fast Iterative Solvers: with applications in incompressible fluid dynamics

Oxford University Press, Oxford, 2005.

Overview

Four underlying problems on two-dimensional domains:

- Diffusion equation

$$\nabla^2 u = f$$

- Convection-Diffusion equation

$$-\epsilon \nabla^2 u + w \cdot \nabla u = f$$

- Stokes equations

$$\begin{aligned} -\nabla^2 u + \text{grad}\, p &= f \\ -\text{div}\, u &= 0 \end{aligned}$$

- Navier-Stokes equations

$$\begin{aligned} -\nu \nabla^2 u + (u \cdot \text{grad})\, u + \text{grad}\, p &= f \\ -\text{div}\, u &= 0 \end{aligned}$$

Two Main Components

Finite element discretisations

- Bilinear/biquadratic elements on rectangles
- Streamline upwinding for convection-diffusion equation
- Mixed finite elements for Stokes/Navier-Stokes equations with stable and stabilised elements
- A posteriori error estimation

Iterative solution of discrete (linearised) systems

- Preconditioned Krylov subspace methods

CG	MINRES
GMRES	BiCGStab(2)

- Problem-appropriate preconditioners

Preconditioned Solvers

- Diffusion equation

CG	+	IC
MINRES		Multigrid

- Convection-Diffusion equation

GMRES	+	ILU
BiCGStab(2)		Multigrid

- Stokes equations

MINRES	+	Block preconditioning with inner multigrid

- Navier-Stokes equations

GMRES	+	Pressure convection-diffusion
BiCGStab(2)		Least squares commutator with inner multigrid

Other Key Features

- several problem domains — square, L-shaped, step
- graphical displays — grid, solution, error estimate
- user access to data and problem structure
- full user access to code
- user can change problem features:
  - domain
  - boundary conditions
  - solvers

Selected Features: Convection-Diffusion

$$-\epsilon\nabla^2 u + w\cdot\nabla u = f$$

- Galerkin FEM

$$\epsilon(\nabla u_h, \nabla v_h) + (w\cdot\nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

- Petrov-Galerkin FEM (streamline diffusion)

$$\begin{aligned}\epsilon(\nabla u_h, \nabla v_h) \;&+\; (w\cdot\nabla u_h, v_h) + \frac{\delta h}{||w||}(w\cdot\nabla u_h, w\cdot\nabla v_h)\\ &=\; (f, v_h) + \frac{\delta h}{||w||}(f, w\cdot\nabla v_h) \quad \forall v_h \in V_h\end{aligned}$$

- parameter δ generated automatically

Elman and Ramage SINUM 40 (2002), Math. Comp. 72 (2003)

Fischer, Ramage, Silvester and Wathen BIT 38 (1998)

Selected Features: Stokes

$$\begin{aligned} -\nabla^2 u + \mathsf{grad}\, p &= f \\ -\mathsf{div}\, u &= 0 \end{aligned}$$

- Stable elements

$Q_2 - Q_1$	biquadratic velocities bilinear pressures
$Q_2 - P_{-1}$	biquadratic velocities discontinuous linear pressures

- Stablilised elements

$Q_1 - P_0$	bilinear velocities constant pressures
$Q_1 - Q_1$	bilinear velocities bilinear pressures

Selected Features: Stokes

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- Ideal block preconditioner:
  Poisson solve
  Mass matrix solve
  $$\begin{bmatrix} A & 0 \\ 0 & M_S \end{bmatrix}$$

- Iterated preconditioner:
  One GMG V-cycle for Poisson solve
  Mass matrix solve
  $$\begin{bmatrix} Q_A & 0 \\ 0 & M_S \end{bmatrix}$$

- convergence rates independent of h

  Silvester and Wathen SINUM 31 (1994)

- natural norm for problem $\sqrt{|u_h - u_h^k|_1^2 + \|p_h - p_h^k\|_0^2}$

  Elman, Silvester and Wathen Numer. Math. 90 (2002)

Selected Features: Navier-Stokes

$$\begin{aligned} -\nu\nabla^2 u + (u\cdot \text{grad})\, u + \text{grad}\, p &= f \\ -\text{div}\, u &= 0 \end{aligned}$$

$A \equiv$ Laplacian

$B \equiv$ divergence

$N \equiv$ convection

$W \equiv$ velocity derivatives

- Picard

$$\begin{bmatrix} \nu A + N & 0 & B_x^T \\ 0 & \nu A + N & B_y^T \\ B_x & B_y & 0 \end{bmatrix} \begin{bmatrix} \delta u_x \\ \delta u_y \\ p \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{g} \end{bmatrix}$$

- Newton

$$\begin{bmatrix} \nu A + N + W_{xx} & W_{xy} & B_x^T \\ W_{yx} & \nu A + N + W_{yy} & B_y^T \\ B_x & B_y & 0 \end{bmatrix} \begin{bmatrix} \delta u_x \\ \delta u_y \\ p \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{g} \end{bmatrix}$$

Selected Features: Navier-Stokes

preconditioner $\quad P = \begin{bmatrix} M_F & B^T \\ 0 & -M_S \end{bmatrix}$

$M_F \sim$ convection-diffusion operator $\quad F$

$M_S \sim$ Schur complement $\quad S = BF^{-1}B^T + \frac{1}{\nu}C$

- Pressure convection-diffusion preconditioning

  Silvester et al. JCAM 128 (2001), Kay et al. SISC 24 (2002)

- Least squares commutator preconditioning

  Elman SISC 20 (1999), Elman et al. SISC 27 (2006)

Sample Problem: Diffusion

- temperature distribution in a plate
- L-shaped domain
- uniform heating

  constant source function $f(x, y) = 1$

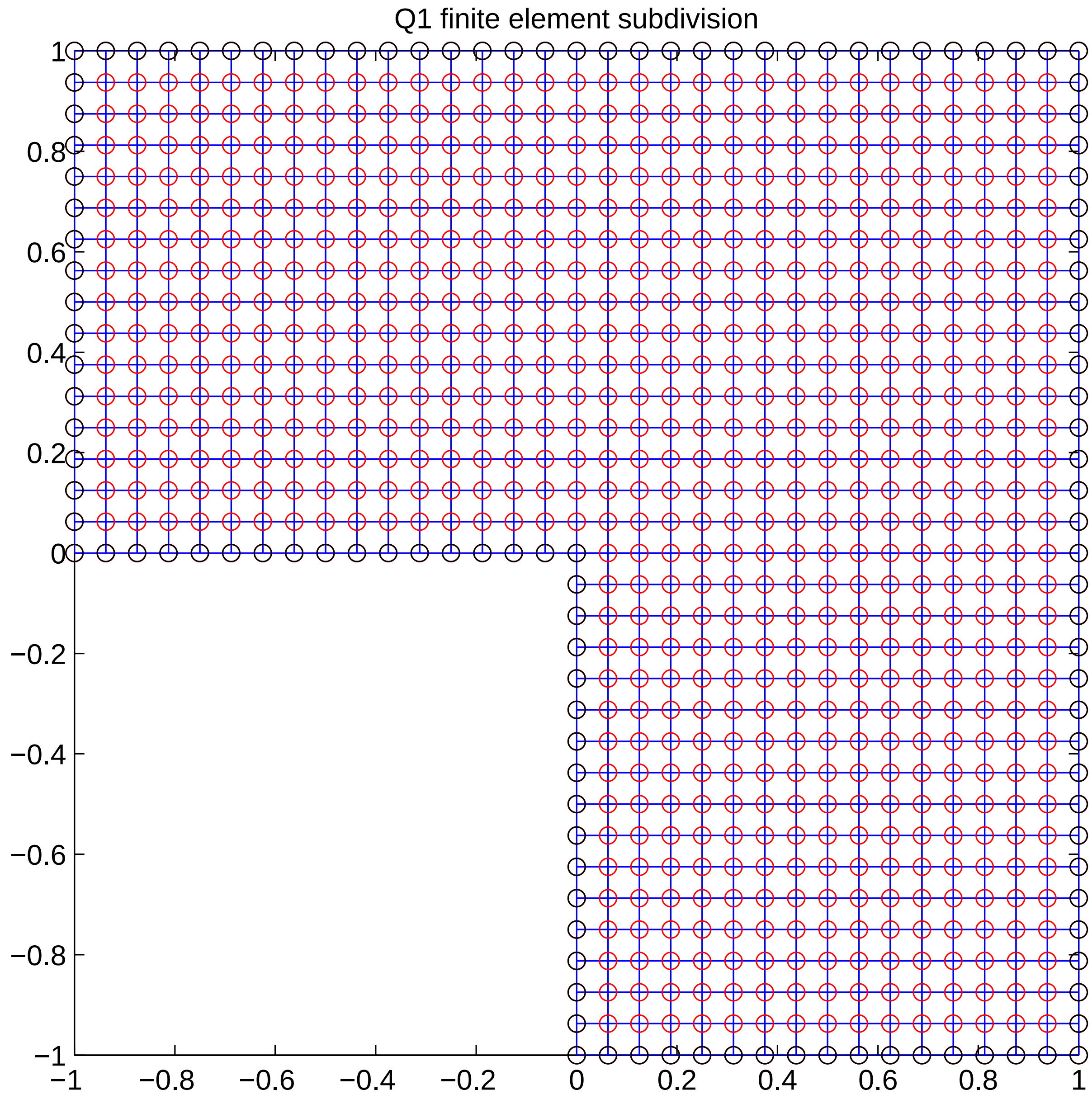
- edges kept ice-cold

  zero Dirichlet boundary conditions everywhere

- underlying solution has a singularity at the origin

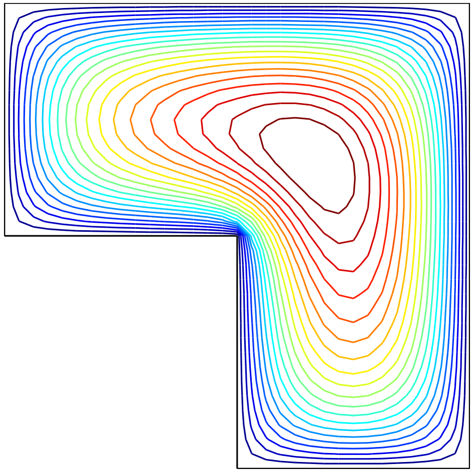
```
specification of reference Poisson problem.
choose specific example
     1 Square domain, constant source
     2 L-shaped domain, constant source
     3 Square domain, analytic solution
     4 L-shaped domain, analytic solution
    2
Grid generation for a simple L-shaped domain.
grid parameter:  3 for underlying 8x8 grid (default is 4)
    5
grid stretch factor (default is 1)
    1
Q1/Q2 approximation 1/2?  (default Q1)
    1
setting up Q1 diffusion matrices...  done
system saved in ell_diff.mat ...
solving linear system ...  done
Galerkin system solved in 5.693e-03 seconds
computing Q1 element flux jumps...  done
computing Q1 interior residuals...  done
computing local error estimator...  done
estimated global error (in energy):  3.466590e-02
plotting solution and estimated errors...  done
```

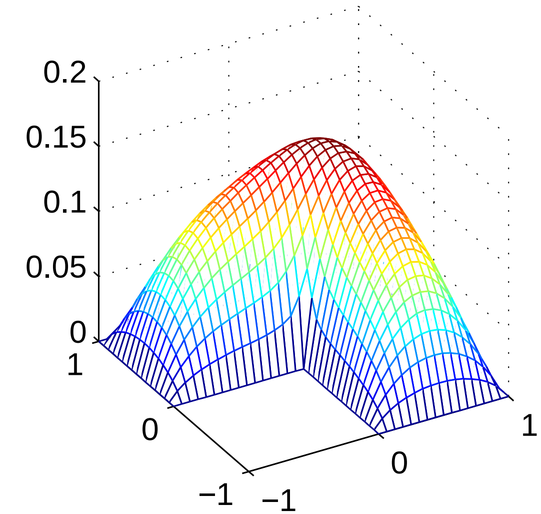
Sample Results: Diffusion



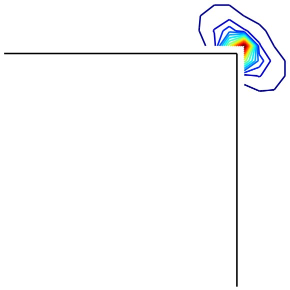
Sample Results: Diffusion

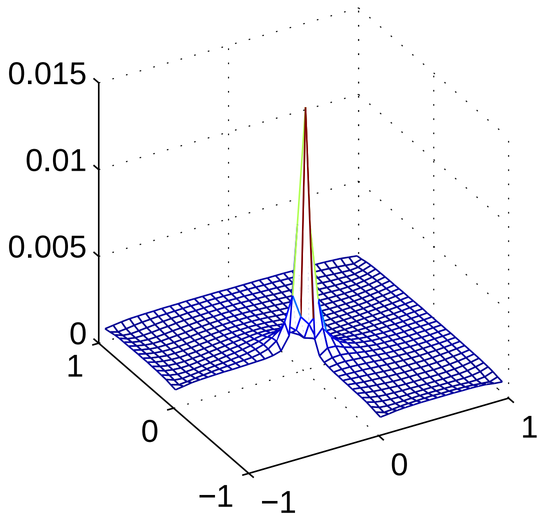
Finite Element Solution





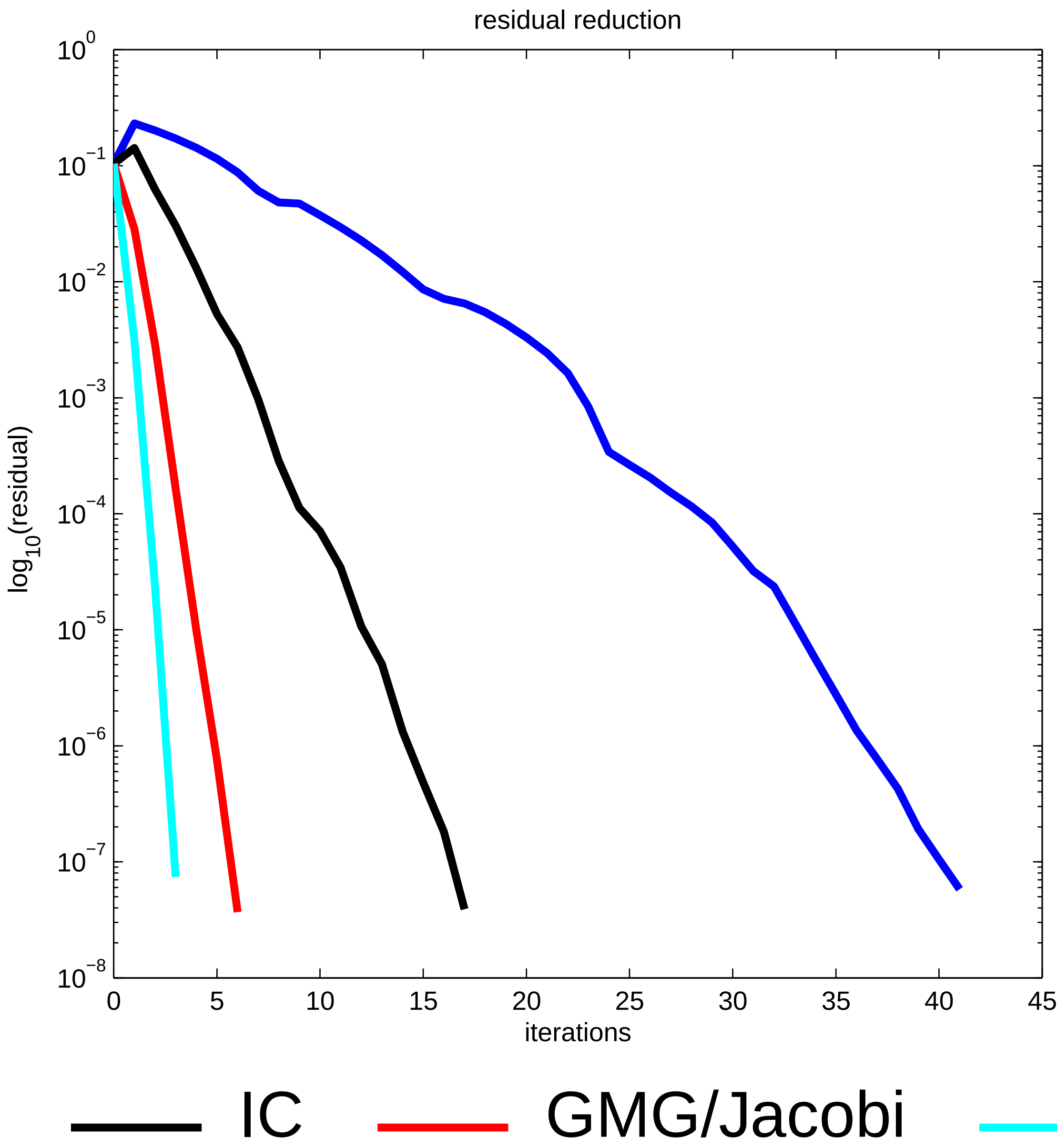
Estimated Error





```
discrete diffusion system ...
PCG/MINRES? 1/2 (default PCG)
    1
tolerance?  (default 1e-6)
    1.0000e-06
maximum number of iterations?  (default 100)
    100
preconditioner:
 0 none
 1 diagonal
 2 incomplete cholesky
 3 geometric multigrid
default is geometric multigrid
    0
PCG iteration ...
convergence in 41 iterations
 k log10(||r_k||/||r_0||)
 0 0.0000
...
 41 -6.2529
Bingo!
 3.6958e-01 seconds
use new (enter figno) or existing (0) figure, default is 0    1
colour (b,g,r,c,m,y,k):  enter 1-7 (default 1)    1
```

Sample Results: Diffusion



Sample Problem: Convection-Diffusion

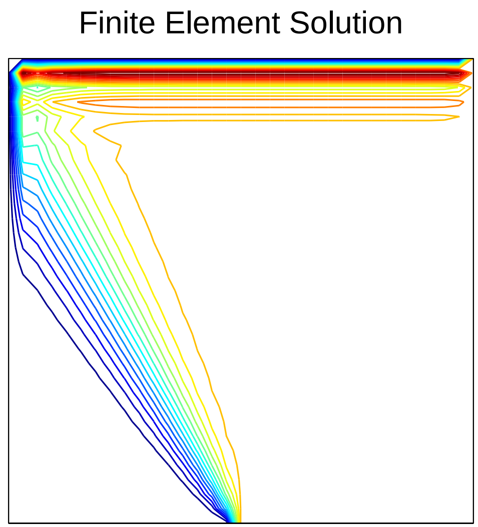
- square domain
- constant wind at angle of 30° to the left of vertical

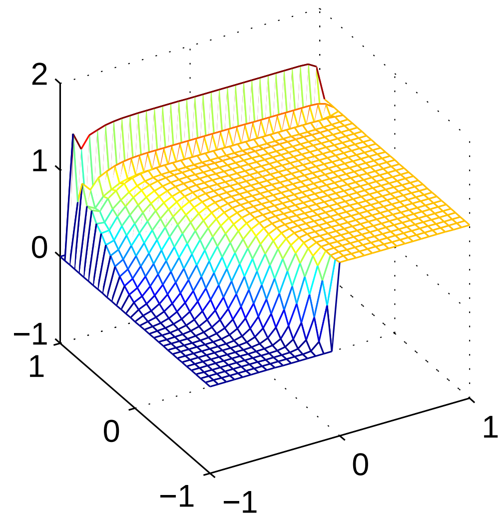
$$\mathbf{w} = \left(-\sin\frac{\pi}{6},\ \cos\frac{\pi}{6}\right)$$

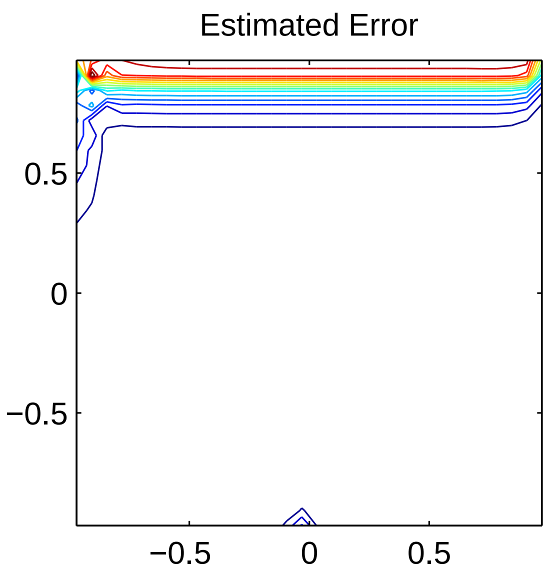
- Dirichlet boundary conditions:
  - 0 on left and top boundaries
  - 1 on the right boundary
  - jump discontinuity (from 0 to 1) on the bottom boundary at $(0, -1)$
- solution features:
  - exponential boundary layer near the top boundary
  - internal layer as discontinuity smeared by the presence of diffusion

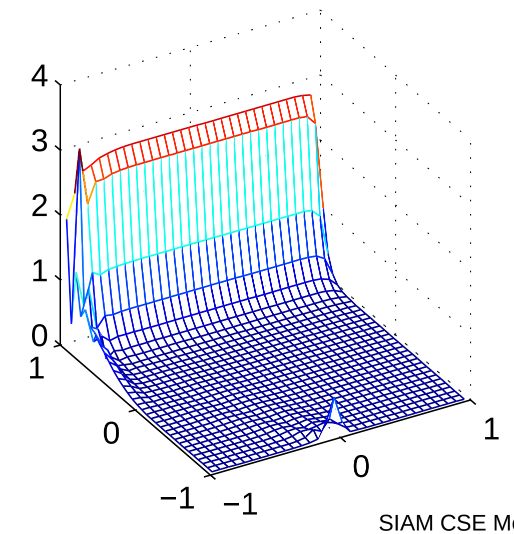
Sample Results: Convection-Diffusion

Galerkin



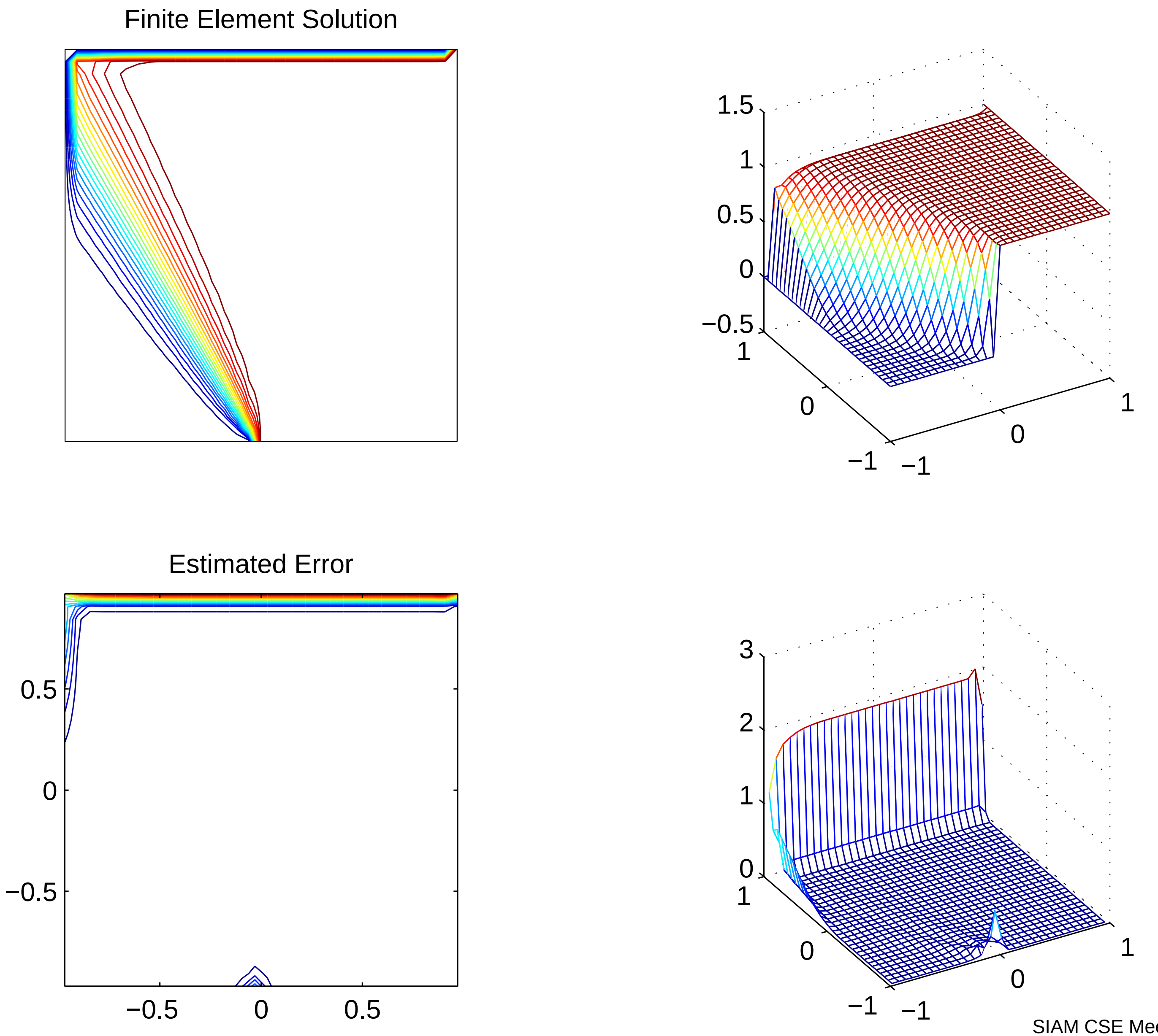




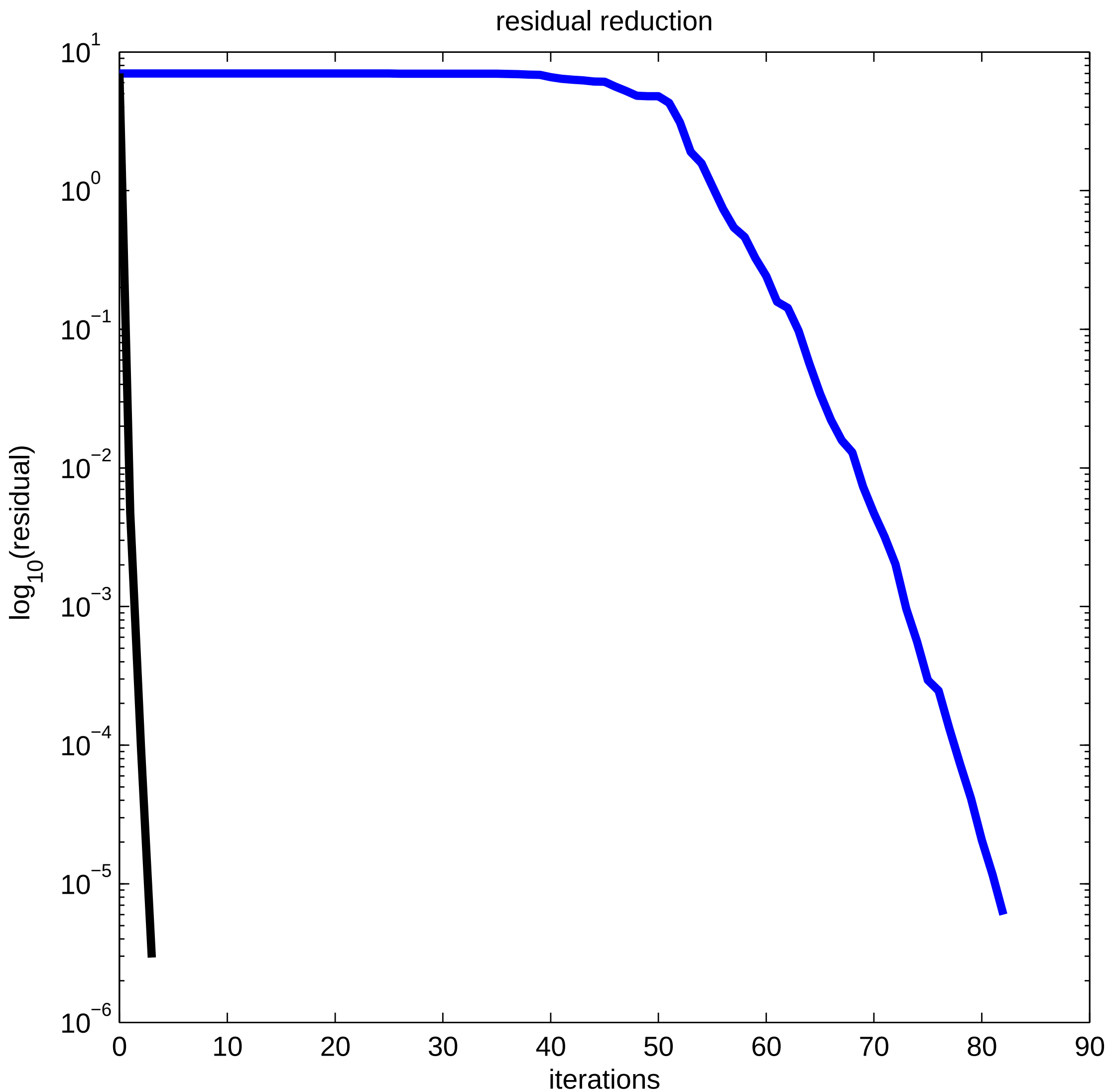


Sample Results: Convection-Diffusion

Petrov-Galerkin



Sample Results: Convection-Diffusion



GMG/GS — Galerkin — Petrov-Galerkin

Sample Problem: Stokes

- classical test problem used in fluid dynamics

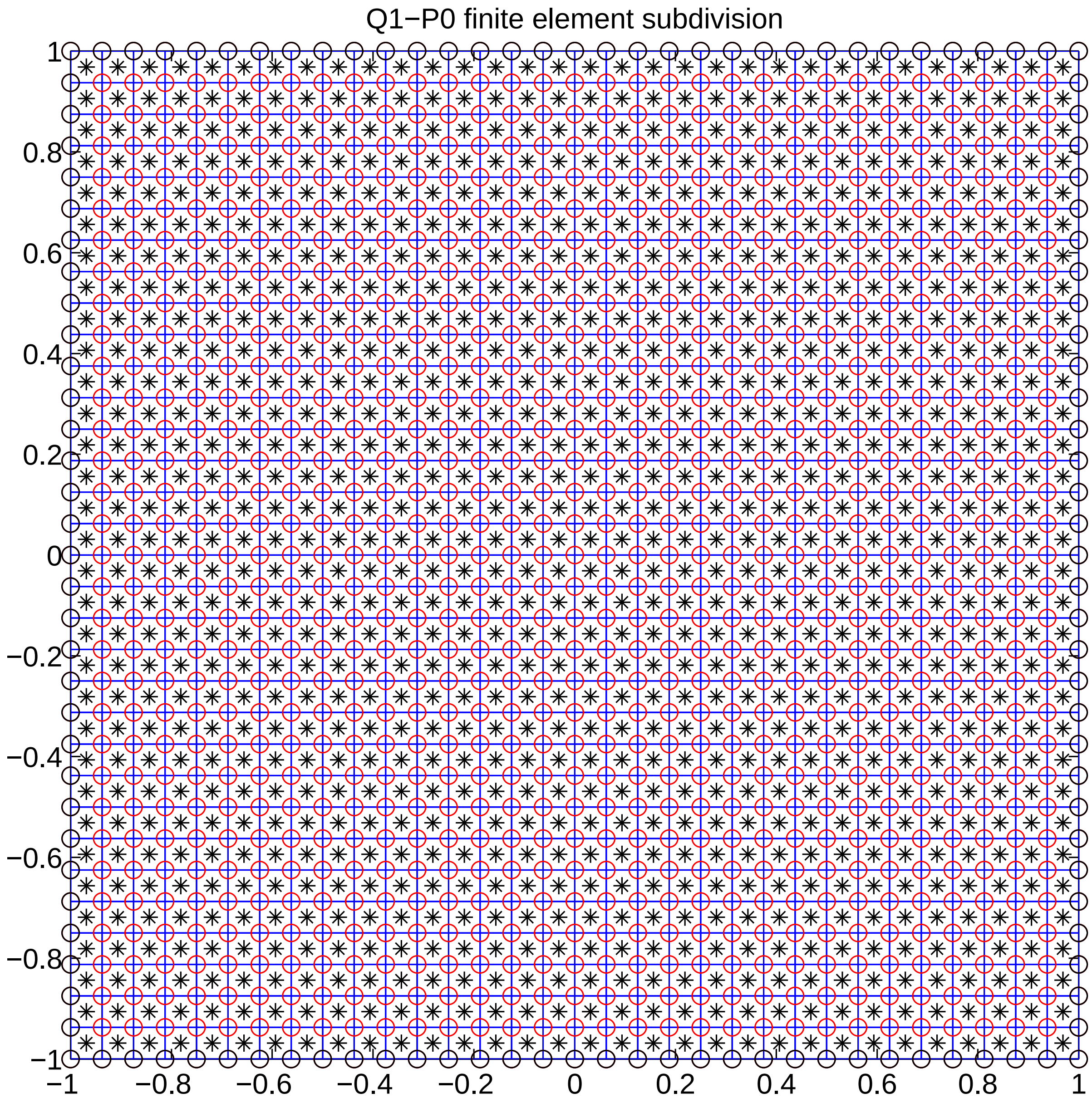
  lid-driven cavity

- square cavity $[-1, 1] \times [-1, 1]$
- flow induced by lid moving from left to right
- Dirichlet no-flow boundary condition on side and bottom boundaries
- different choices of nonzero horizontal velocity on the lid give rise to different computational models

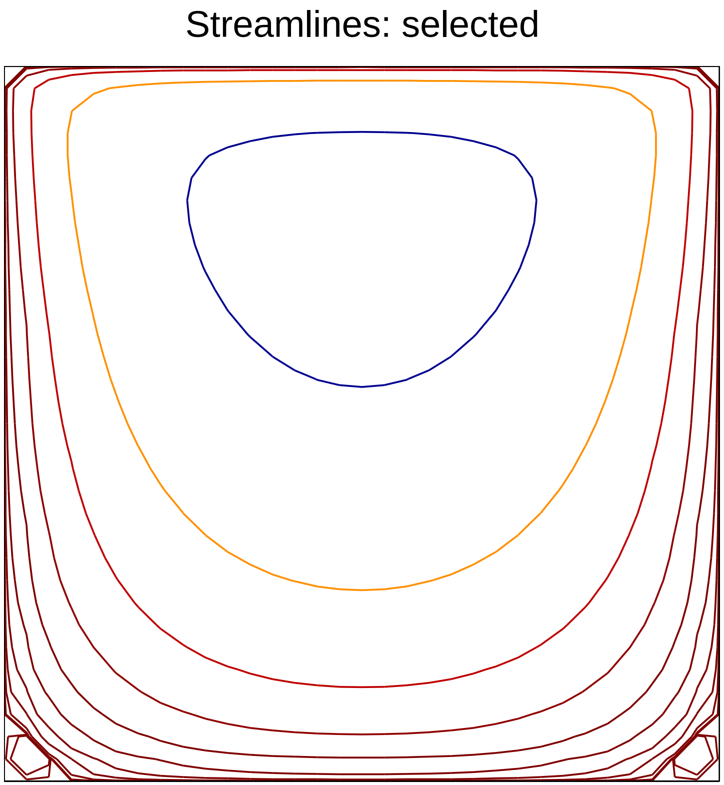
$$\{y = 1; -1 \leq x \leq 1| \quad u_x = 1 - x^4\}$$

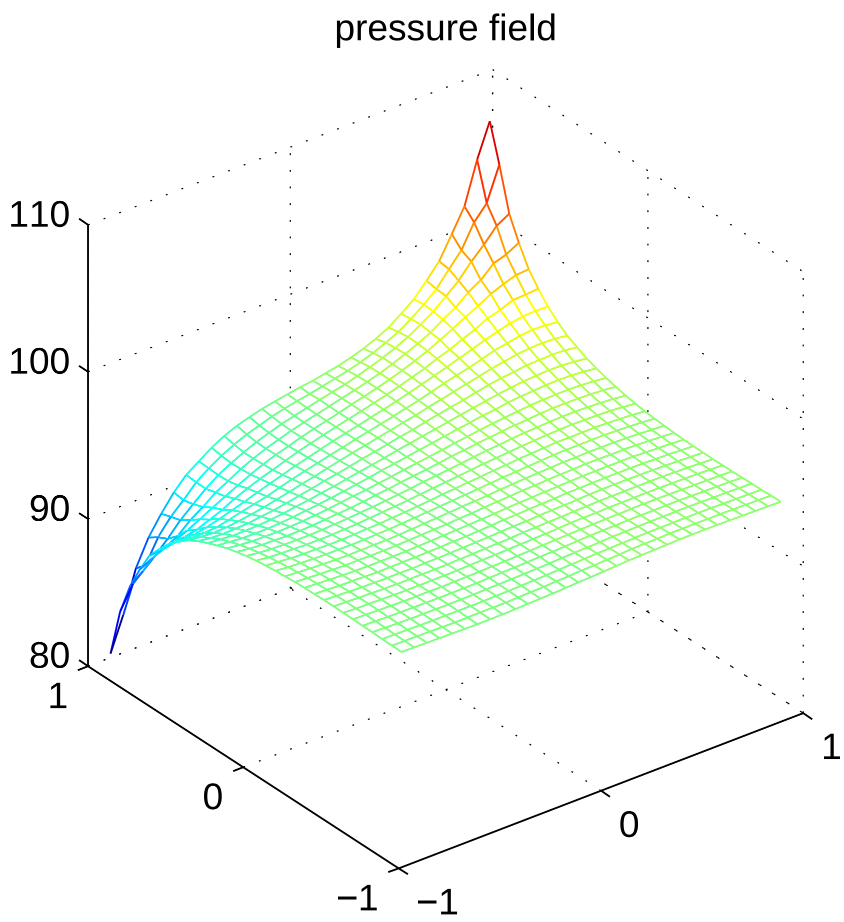
regularised cavity

Sample Results: Stokes

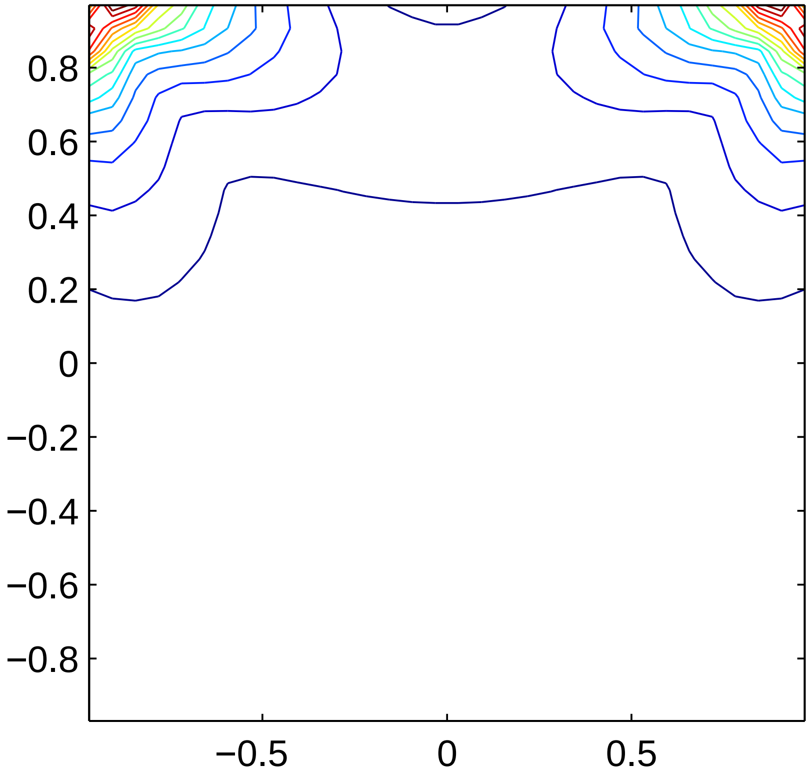


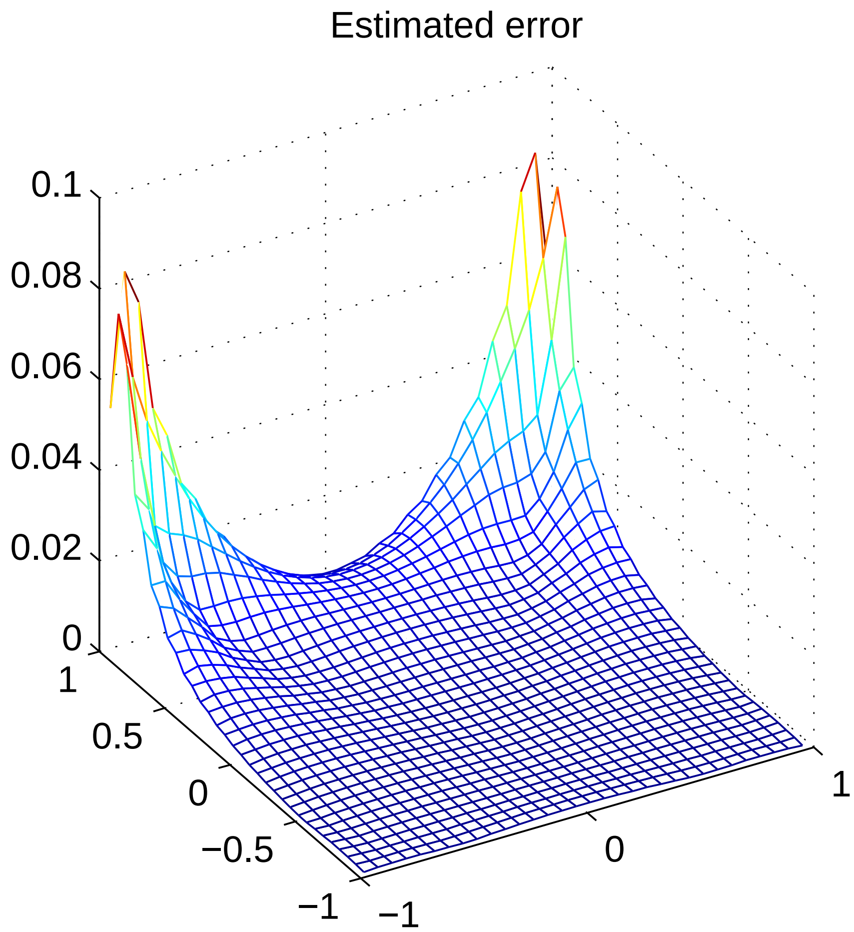
Sample Results: Stokes



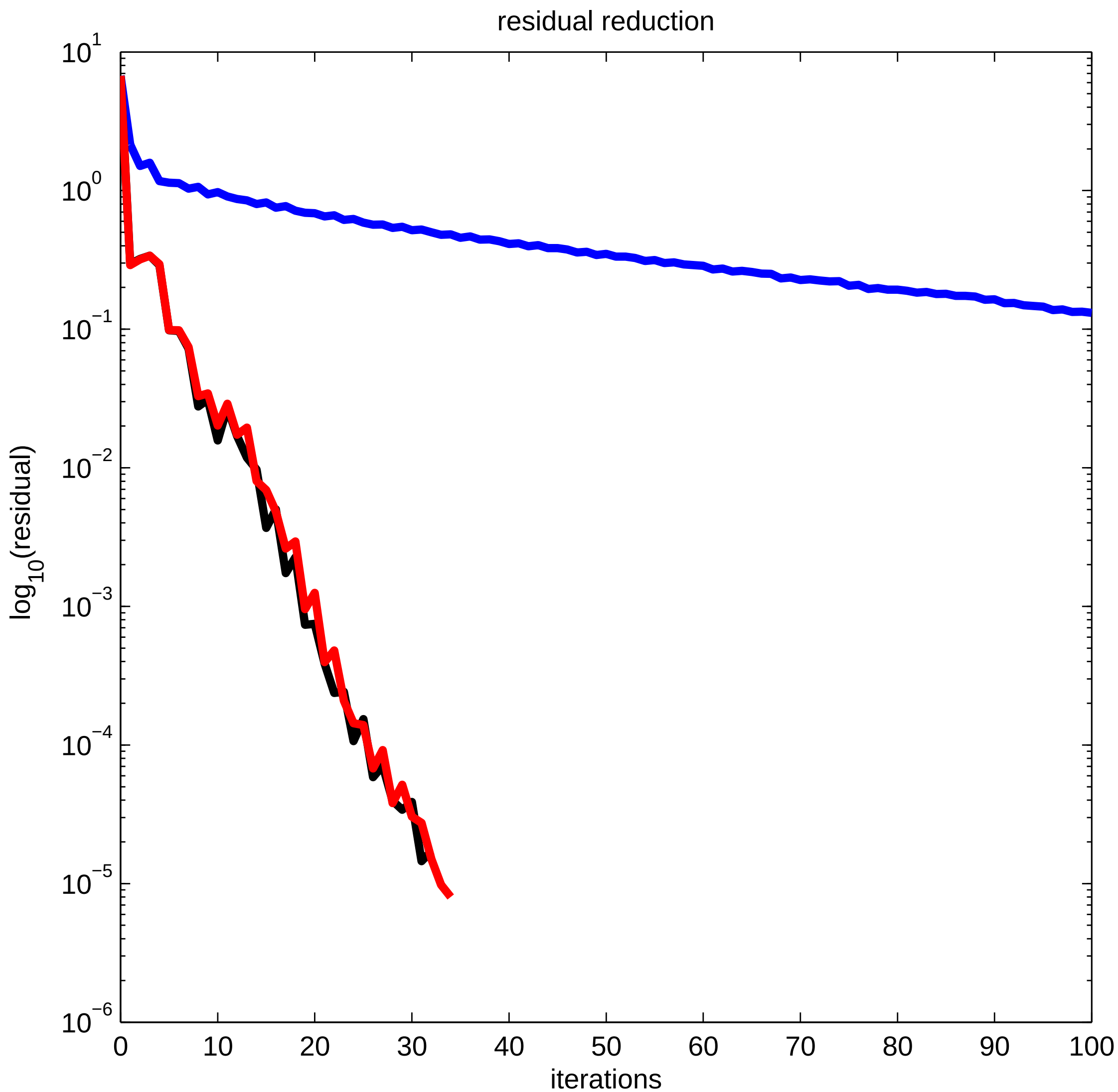


Sample Results: Stokes





Sample Results: Stokes



diag — ideal block — GMG/Jacobi block

Sample Problem: Navier-Stokes

- flow over a step
- step of user-specified length (for high Reynolds number flow, longer steps required to allow flow to fully develop)
- boundary conditions:
  - Poiseuille flow profile on inflow boundary
  - no-flow condition on top and bottom walls
  - Neumann condition at outflow boundary (zero mean outflow pressure)
- singularity at the origin

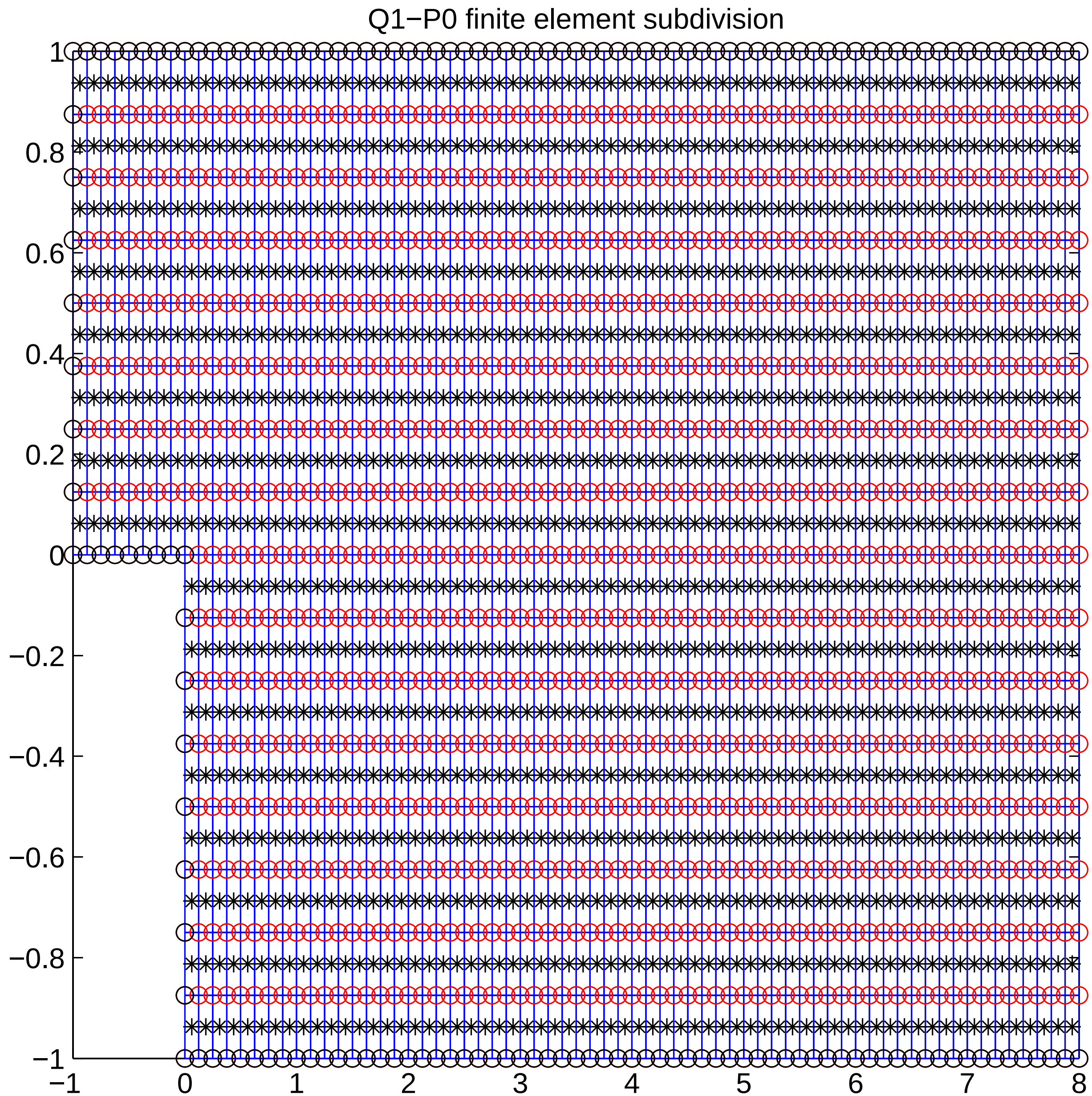
Navier-Stokes Batch Mode File

NSQ1P0_batch.m

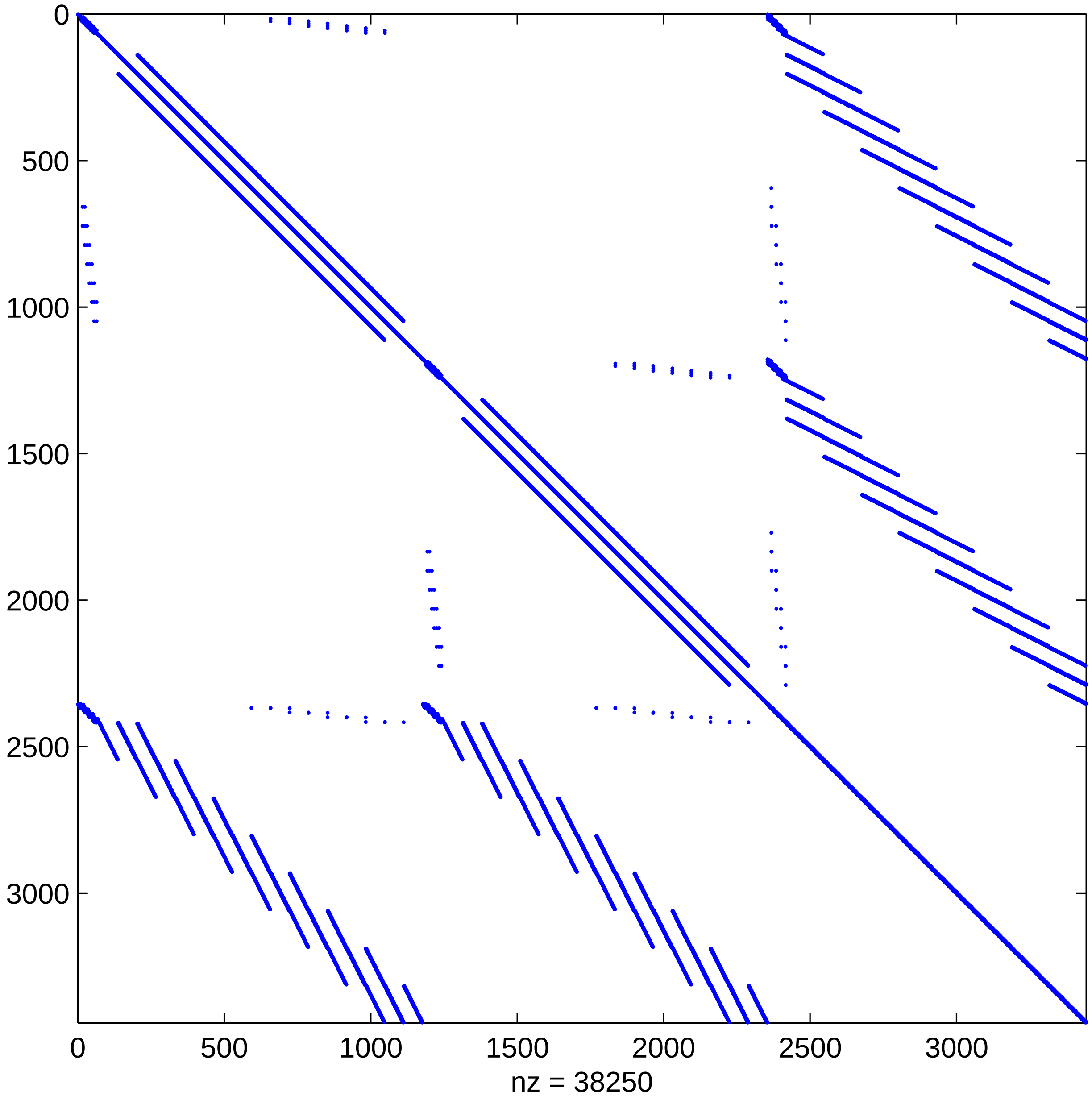
```
2       % Step problem
8       % Location of outflow boundary
4       % Grid parameter
1       % Uniform outflow
2       % Q1-P0
.01     % Viscosity
3       % Hybrid nonlinear iteration
2       % Number of Picard steps
4       % Number of Newton steps
1.d-5   % Nonlinear tolerance
0.25    % Stokes stabilisation parameter
```

```
batchmode('NSQ1P0');
```

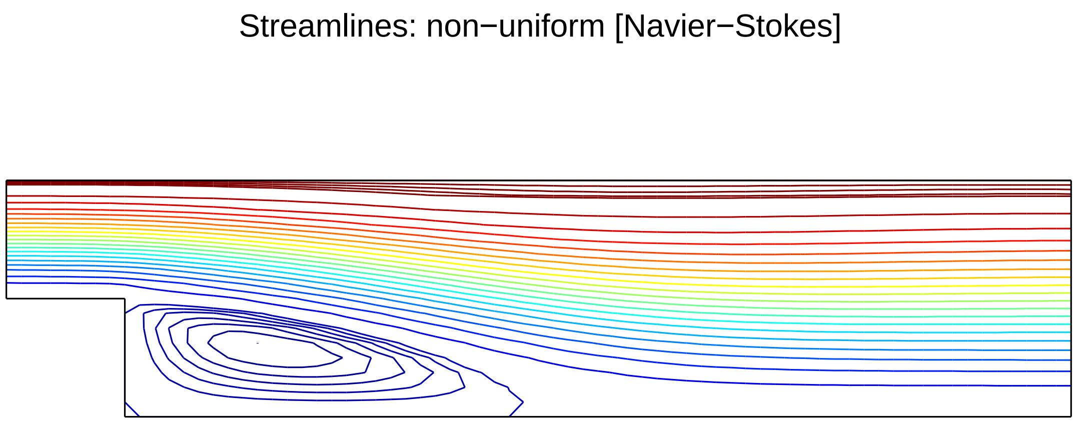
Sample Results: Navier-Stokes

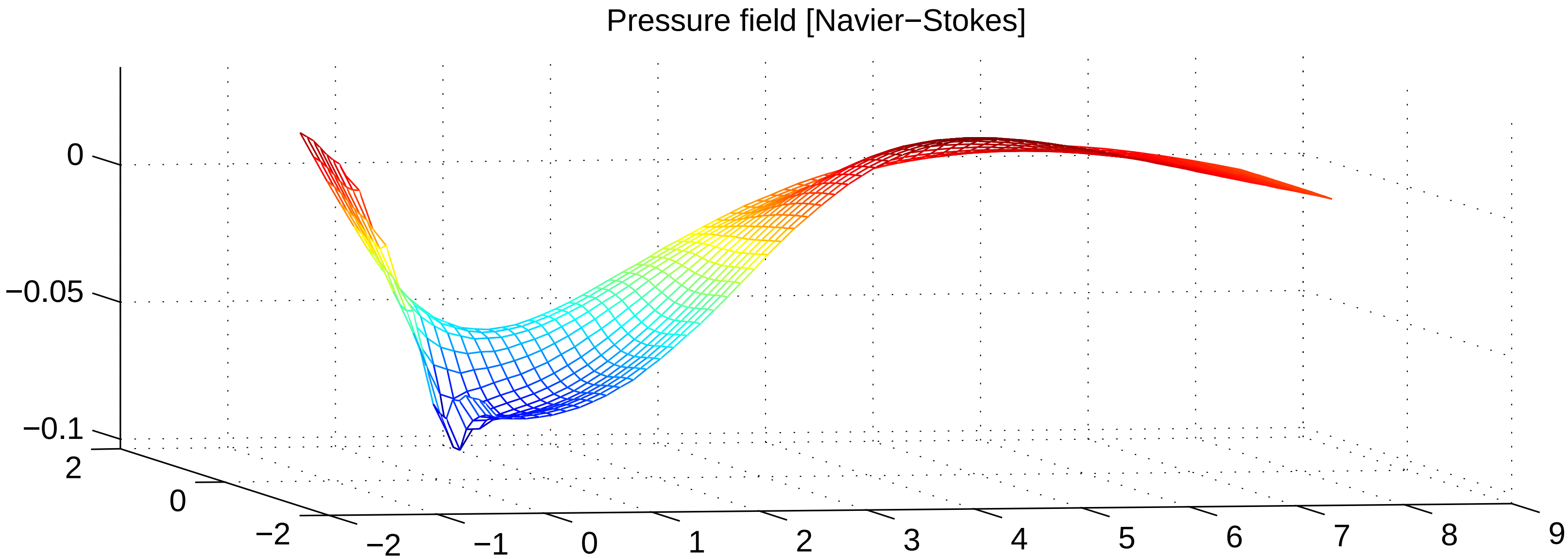


Sample Results: Navier-Stokes

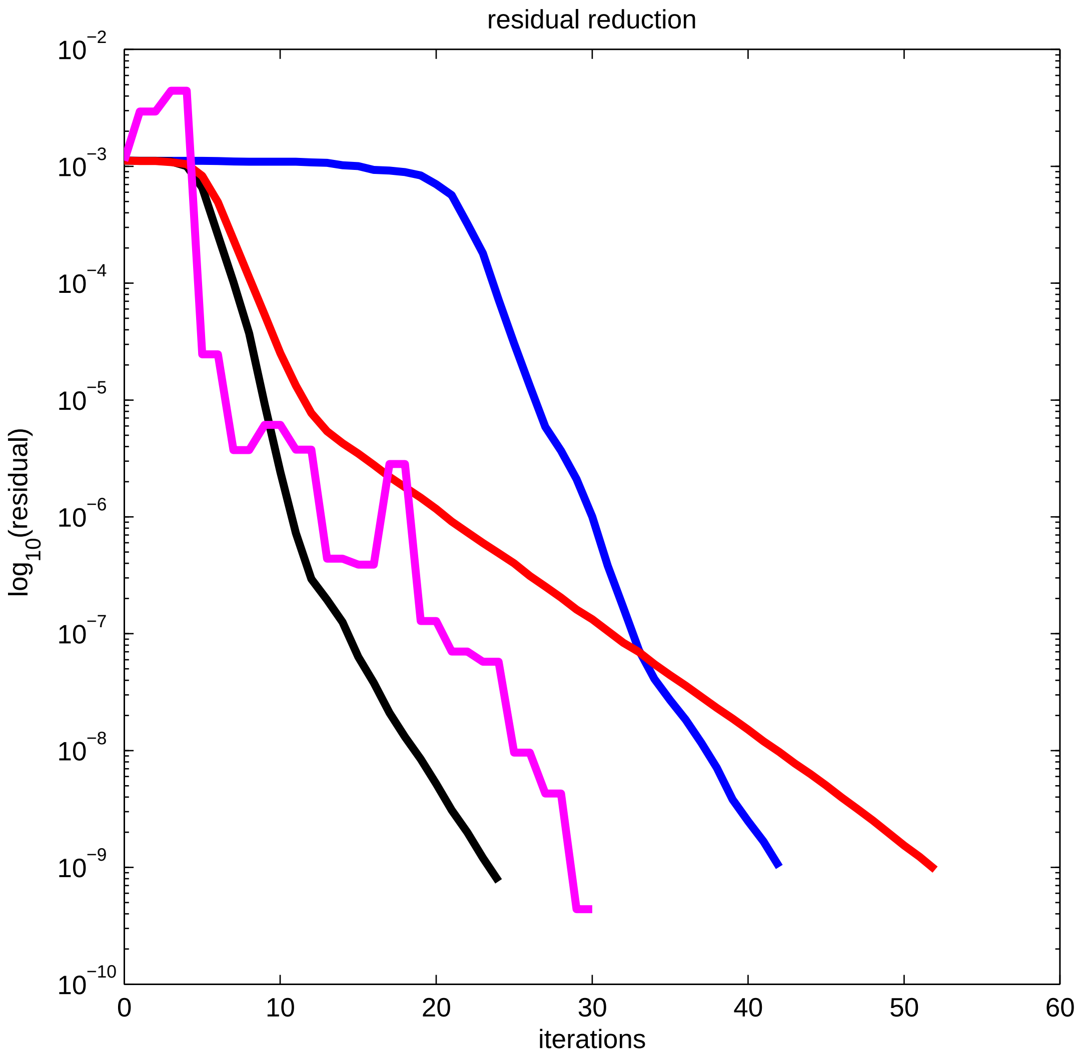


Sample Results: Navier-Stokes





Sample Results: Navier-Stokes



GMRES/ideal PCD GMRES/ideal LSC

GMRES/LSC+GMG BiCGStab(2)/LSC+GMG

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Stay tuned for applications/examples/extensions!