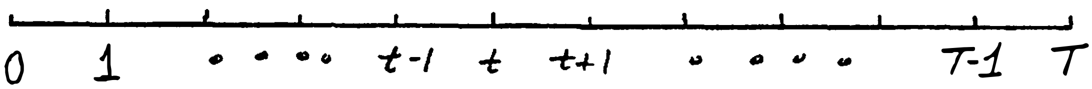
Discrete Time Intertemporal Portfolio Selection

The Problem:

- We have an investor who lives T periods (time of death is known):



- Investor's Portfolio Problem is:

$$\max E[U(C_0, C_1, ..., C_{T-1}, W_T)]$$

- Assumptions:
  - T (time of death) is known.
  - Utility Function is Additively Separable:

$$U(C_0, C_1, ..., C_{T-1}, W_T) = \sum_{t=0}^{T-1} U(C_t, t) + B(W_T, T)$$

- At time t the investor:
  - Receives wealth W_t from from previous period's investments.
  - Receives incomes Y_t. (We assume this is zero.)
  - Consumes C_t.
  - Invests remainder ($I_t = W_t + Y_t - C_t$) in risky and riskless assets.

- At time $t+1$ the investor:
  - Receives $W_{t+1} = I_t \cdot \tilde{Z}_t^*$ from previous period's investments. ...

$$\tilde{Z}_t^* = \sum_{i=0}^{n} w_i^* \tilde{Z}_{i,t}$$

where

$$w_i(t) \equiv \frac{N_i(t)P_i(t)}{I(t)}$$

$$\tilde{Z}_{i,t} = \frac{\tilde{P}_i(t+1)}{P_i(t)}.$$

Investments:

- Investor has the opportunity to invest in risky assets ($i = 1, \ldots, n$) and the risk free asset ($i = 0$).

$$\sum_{i=0}^{n} w_i \tilde{Z}_i = w_0 R + \sum_{i=1}^{n} w_i \tilde{Z}_i$$

$$= \left(1 - \sum_{i=1}^{n} w_i\right) R + \sum_{i=1}^{n} w_i \tilde{Z}_i$$

$$= \sum_{i=1}^{n} (\tilde{Z}_i - R) + R$$

- Alternately, we can think that the investor puts everything into the risk-free asset and then borrows (at rate R) to make all risky investments. Therefore, we have:

$$\tilde{W}_{t+1} = (W_t + Y_t - C_t) \left[\sum_{i=1}^{n} w_i(\tilde{Z}_i - R) + R\right]$$

Question:

- In order to maximize his/her lifetime utility,
  - How much should the investor consume at time t?
  - What is the investor's optimal asset portfolio at time t?

Difficulty:

- We can't treat this as a one-period problem because consumption and investments at time t will (in general) depend on (and affect) future choices, investment opportunities, future optimal portfolios, *etc.*.

Solution

- We chain backwards, using dynamic programming techniques, defining the **derived utility of wealth**.

$$J(W_t, t) = \max E_t \left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T) \right]$$

  which is the investor's expected utility from current and all future consumption (and bequest), given optimizing behavior, as a function of his/her current wealth W_t.

- To derive J, start at final period:

$$J(W_T, T) = B(W_T, T)$$

- Now move back one period:

$$J(W_{T-1}, T-1) = \max_{C_{T-1}, w_i} \left[U(C_{T-1}, T-1) + E_{T-1}B(\tilde{W}_T, T)\right]$$

where

$$\tilde{W}_T = (W_{T-1} - C_{T-1})[\sum_{i=1}^{n} w_i(\tilde{Z}_i - R) + R]$$

  - For any W_{T-1}, the investor chooses C_{T-1} and w_i so as to maximize J.
  - This yields $J(W_{T-1}, T-1)$ as a function of W_{T-1} alone.

- Having thus defined $J(W_{T-1}, T-1)$, we can chain back to $T-2$:

$$J(W_{T-2}, T-2) = \max_{C_{T-2}, w_i} \left[U(C_{T-2}, T-2) + E_{T-2}J(\tilde{W}_{T-1}, T-1)\right]$$

where

$$\tilde{W}_{T-1} = (W_{T-2} - C_{T-2})[\sum_{i=1}^{n} w_i(\tilde{Z}_i - R) + R]$$

which similarly defines $J(W_{T-2}, T-2)$

- We therefore have for arbitrary time t:

$$J(W_t, t) = \max_{C_t, w_{i,t}} [U(C_t, t) + E_t[J(W_{t+1}, t+1)]]$$

  - the *Bellman Equation* for $J(W_t, t)$.

Solving for J:

At time t the investor solves the following problem (in terms of J):

$$
\begin{aligned}
& J(W_t, t) \\
&= \max_{C_t, \underline{w}} [U(C_t, t) + E_t J(\tilde{W}_{t+1}, t+1)] \\
&\equiv \max_{C_t, \underline{w}} G(C, \underline{w}; W_t) \qquad (1)
\end{aligned}
$$

The first order conditions for a maximum are that:

$$
\frac{\partial G}{\partial C} = 0 \qquad \frac{\partial G}{\partial w_i} = 0 \quad \forall i
$$

This leads to:

$$
\frac{\partial G}{\partial C} = U_{C_t}(C_t, t) - E_t[J_{W_{t+1}}(\cdot)\overbrace{(\sum_{i=1}^{n} w_i^*[\tilde{Z}_i - R] + R)}^{\tilde{Z}^*}] = 0
$$

$$
\frac{\partial G}{\partial w_i} = E_t[J_{W_{t+1}}(\cdot)(\tilde{Z}_i - R)] = 0 \quad \forall i
$$

These two equations become:

$$
U_C = E_t[J_{W_{t+1}} \cdot \tilde{Z}^*]
$$

$$
E_t[J_{W_{t+1}} \cdot \tilde{Z}_i] = RE_t[J_{W_{t+1}}]
$$

Multiply the second by w_i^* and sum over i to get:

$$
E_t[J_{W_{t+1}} \cdot \tilde{Z}^*] = RE_t[J_{W_{t+1}}] = U_{C_t}
$$

Notes: PhD4: Discrete Time Portfolio Solution

$$\tilde{W}_{t+1} = (W_t - C_t)\underbrace{\left[\sum_{i=1}^{n} w_i (Z_i - R) + R\right]}_{\tilde{Z}}$$

$$\frac{dW_{t+1}}{dC} = -\tilde{Z}^*$$

$$\frac{dW_{t+1}}{dw_i} = (W_t - C_t)(\hat{Z}_i - R)$$

$$\frac{dW_{t+1}}{dW_t} = \hat{Z}^*$$

Log utility

$$J(W,T) = \delta^{-T} \ln W = B(W,T)$$

$$J_{W_T} = \delta^{-T}\frac{1}{W_T} = \frac{\delta^{-T}}{(W_{T-1} - C^*)\cdot Z^*}$$

$$R\, E_{T-1}\left[\frac{1}{Z^*}\right] = E_{T-1}\left[\frac{Z_i^*}{Z^*}\right] = E_{T-1}\left[\frac{\sum w_i^* Z_i}{Z^*}\right] = 1$$

To formally develop the envelope condition, use equation (1) evaluated at $G(C^*, \underline{w}^*; W)$. We can write:

$$J_{W_t} = \frac{\partial G}{\partial C^*}\frac{\partial C^*}{\partial W_t} + \sum \frac{\partial G}{\partial w_i^*}\frac{\partial w_i^*}{\partial W_t} + \frac{\partial G}{\partial W_t}$$

Then for $C = C^*$ and $w_i = w_i^*$:

$$J_{W_t} = E_t[J_{W_{t+1}} \cdot \tilde{Z}^*] = E_t[J_{W_{t+1}} \cdot \tilde{Z}_i] = RE_t[J_{W_{t+1}}] = U_{C_t}$$

This is the complete **envelope condition.** It says that all of the following are equal:

1. The Marginal Utility of wealth
2. The MU of Consumption
3. The MU of Investment in Risky Assets
4. The MU of Investment in the Risk Free Asset

Second Order Conditions:

We have assumed second order conditions would hold; to show this differentiate envelope condition w.r.t W_t

$$\frac{\partial^2 U(C_t, t)}{\partial C_t \partial C_t} \cdot \frac{\partial C_t^*}{\partial W_t} = \frac{\partial^2 J(W_t, t)}{\partial W_t \partial W_t}$$

or

$$U_{CC} C_W^* = J_{WW}$$

- Since U is concave in C, and since, for time additive utility consumption is a normal good, the second order condition for J is satisfied.

Example 1: Log Utility Function, $Y \equiv 0$

$$U(C,t) = \rho^{-t} \ln C$$

$$B(W,T) = \rho^{-T} \ln W$$

At time $T-1$ we have:

$$U_C = \frac{\rho^{-(T-1)}}{C^*} = E_{T-1}\left[\frac{\rho^{-T}}{(W_{T-1} - C^*)\tilde{Z}^*}\tilde{Z}^*\right] = \frac{\rho^{-T}}{W_{T-1} - C^*}$$

therefore:

$$C^*_{T-1} = \frac{\rho}{1+\rho} W_{T-1}$$

C^* is independent of the returns available (*i.e.*, the income and substitution effects just cancel). This characteristic is peculiar to the log utility function and is termed *myopia*. Also:

- For zero rate of time preference ($\rho = 1$), $C = \frac{W}{2}$.
- For regular time preference ($\rho > 1$), $C > \frac{W}{2}$.

From the return condition:

$$E_{T-1}\left[\frac{\rho^{-T}}{(W_{T-1} - C)\tilde{Z}^*}\tilde{Z}_i\right] = RE_{T-1}\left[\frac{\rho^{-T}}{(W_{T-1} - C)\tilde{Z}^*}\right]$$

$$E_{T-1}\left[\frac{\tilde{Z}_i}{\tilde{Z}^*}\right] = RE_{T-1}\left[\frac{1}{\tilde{Z}^*}\right] = 1$$

Portfolio is independent of W and C, and is the same as for a log utility investor in a single period world (recall that log-utility exhibits one-fund separation). Consumption and portfolio decisions are thus *separable.*

For time $T-1$ we can write the derived utility of wealth function as:

$$J(W, T-1) = \rho^{1-T} \ln C^*_{T-1} + E_{T-1}\rho^{-T} \ln[\tilde{Z}^*(W_{T-1} - c^*_{t-1})]$$

Substituting in for C^*_{T-1}:

$$= \rho^{1-T} \ln\left(\frac{\rho}{1+\rho}W\right) + E_{T-1}(\rho^{-T} \ln \tilde{Z}^*) + \rho^{-T} \ln[W \overbrace{(1 - \frac{\rho}{1+\rho})}^{\frac{1}{1+\rho}}]$$

or:

$$J(W, T-1) = \rho^{-T}(1+\rho) \ln W + \phi(T-1)$$

where ϕ is independent of W.

We can see that the envelope condition is satisfied:

$$J_W = \frac{\rho^{-T}(1+\rho)}{W}$$

and, recalling that $C^* = \frac{\rho}{1+\rho}W$:

$$U_C = \frac{\rho^{1-T}}{C^*} = \frac{\rho^{1-T}(1+\rho)}{\rho W} = \frac{\rho^{-T}(1+\rho)}{W} = J_W$$

When the problem is extended to time $T - s$, we can show:

$$C^*_{T-s} = \frac{\rho}{\sum_{i=0}^{s} \rho^i} W_{T-s}$$

$$E_t \left[\frac{\tilde{Z}_i}{\tilde{Z}^*} \right] = RE_t \left[\frac{1}{\tilde{Z}^*} \right] = 1$$

$$J(W, T - s) = \rho^{-T} \left(\sum_{i=0}^{s} \rho^i \right) \ln W_{T-s} + \phi(T - s)$$

where $\phi(T - s)$, again, does not affect the maximization process.

So, for log utility, at any time period, we can conclude:

1. The consumption decision is independent of future investment opportunities.
2. Investment decisions are independent of future investment opportunities, and of current wealth and consumption.

Example 2: Power Utility Function, $Y \equiv 0$

$$U(C,t) = \rho^{-t}\frac{C^\gamma}{\gamma}$$

$$B(W,T) = \rho^{-T}\frac{W^\gamma}{\gamma}$$

For $T-1$ we can write (from the first order conditions):

$$U_C = \rho^{1-T}C^{*\gamma-1} = E_{T-1}[\rho^{-T}[(W-C^*)\tilde{Z}^*]^{\gamma-1}\tilde{Z}^*]$$

$$\rho^{1-T}C^{*\gamma-1} = \rho^{-T}(W-C^*)^{\gamma-1}E_{T-1}[\tilde{Z}^{*\gamma}]$$

therefore:

$$C^*_{T-1} = W_{T-1}\cdot\underbrace{\phi(\tilde{Z}^*,\rho,\gamma)}_{\phi(T-1)}$$

So consumption is a constant fraction of wealth (as for log utility), but now depends on future investment opportunities.

The FOC's for the portfolio weights are:

$$E_{T-1}[\rho^{-T}[(W-C_*)\tilde{Z}^*]^{\gamma-1}\tilde{Z}_i] = RE_{T-1}[\rho^{-T}[(W-C_*)\tilde{Z}^*]^{\gamma-1}]$$

$$E_{T-1}[\tilde{Z}^{*\gamma-1}\tilde{Z}_i] = RE_{T-1}[\tilde{Z}^{*\gamma-1}]$$

From equation (1) we can write:

$$J(W,T-1) = \rho^{1-T}\frac{C^{*\gamma}}{\gamma} + \rho^{-T}E_{T-1}\frac{[(W-C^*)\tilde{Z}^*]^\gamma}{\gamma}$$

$$= \rho^{1-T}\frac{C^{*\gamma}}{\gamma} + \rho^{-T}(W - C^*)^\gamma E_{T-1}\tilde{Z}^{*\gamma}$$

$$J(W,T-1) = \rho^{1-T}\frac{W^\gamma}{\gamma}\varphi(T-1)$$

So J is still a power function, but now is state dependent. An easier way to see this is from the envelope condition:

$$J_W = U_C = \rho^{1-T}C_*^{\gamma-1} = \rho^{1-T}\tilde{\phi}^{\gamma-1}W^{\gamma-1}$$
$$J_W = \rho^{1-T}W^{\gamma-1}\tilde{\varphi}(T-1)$$

Integrating gives $J(W,T-1)$.

Similar results are obtained for all preceeding periods:

$$C_t^* = W_t \cdot \tilde{\phi}(t)$$
$$J(W,t) = \rho^{-t}\frac{W^\gamma}{\gamma}\tilde{\varphi}(t)$$

where $\tilde{\phi}(t)$ and $\tilde{\varphi}(t)$ depend on current and future investment opportunities but not on wealth.

At time t we have:

$$E_t[\underbrace{\rho^{-(t+1)}\tilde{\varphi}(t+1)(W_t - C_t)^{\gamma-1}\tilde{Z}_t^{*\gamma-1}}_{J_W(W_{t+1},t+1)}(\tilde{Z}_i - R)] = 0$$

$$E_t[\tilde{\varphi}(t+1)\tilde{Z}_t^{*\gamma-1}(\tilde{Z}_i - R)] = 0$$

The optimal portfolio will depend on current and future investment opportunities through (respectively) $\tilde{Z}^*$ and $\tilde{\varphi}(t+1)$.

1. if $\tilde{\varphi}$ is constant or independent of $\tilde{Z}_i$ the portfolio decision is as in the one period model.
   (a) So for power utility (unless $\gamma = 0$), future investment opportunities affect both consumption and portfolio choice.
2. For HARA Utility
   (a) Consumption and portfolio choices are stochastic because they depend on wealth.
   (b) With constant opportunity set (or statistically independent) we get "partial myopia," where the risky assets are held in constant proportions.
3. ICAPM (Merton).

Notes: PhD4: Discrete Time Portfolio Solution

$$\tilde{W}_{t+1} = (W_t - C_t)\underbrace{\left[\sum_{i=1}^{n} \omega_i (Z_i - R) + R\right]}_{\tilde{Z}}$$

$$\frac{dW_{t+1}}{dC} = -\tilde{Z}^*$$

$$\frac{dW_{t+1}}{d\omega_i} = (W_t - C_t)(\hat{Z}_i - R)$$

$$\frac{dW_{t+1}}{dW_t} = \hat{Z}^*$$

Log utility

$$J(W,T) = \rho^{-T} \ln W = B(W,T)$$

$$J_{W_T} = \rho^{-T}\frac{1}{W_T} = \frac{\rho^{-T}}{(W_{T-1} - C^*)\cdot Z^*}$$

$$R\, E_{T-1}\left[\frac{1}{Z^*}\right] = E_{T-1}\left[\frac{Z_i}{Z^*}\right] = E_{T-1}\left[\frac{\sum \omega_i^* Z_i}{Z^*}\right] = 1$$