Topic 6: Mean-Variance Portfolio Choice and Asset Pricing

Motivation

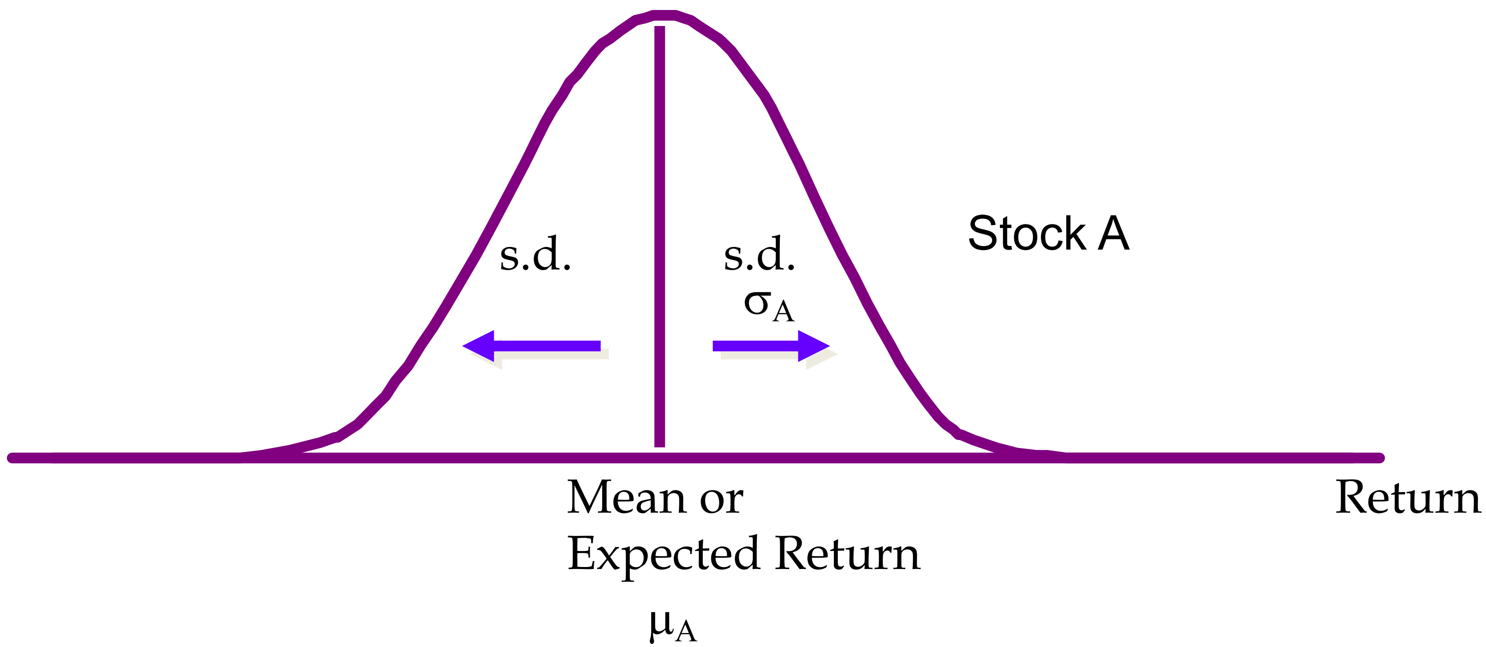
- Mean-variance portfolio analysis
  - Developed by Harry Markowitz in the early 1960's (1990 Nobel Prize in Economics)
  - One of the foundations of modern finance
- Used by all pension plans, endowments, wealthy individuals, banks, insurance companies, ...
- There is an industry of advisors and software makers that implement this approach

Portfolio Theory

- Markowitz showed exactly how an investor can reduce the s.d. of portfolio return by diversification
- If we look a histogram of daily returns of most stocks we observe that they are (close to) normally distributed
- The normal distribution can completely be described by two numbers:
  - Average, mean or “expected return”
  - Standard deviation of return

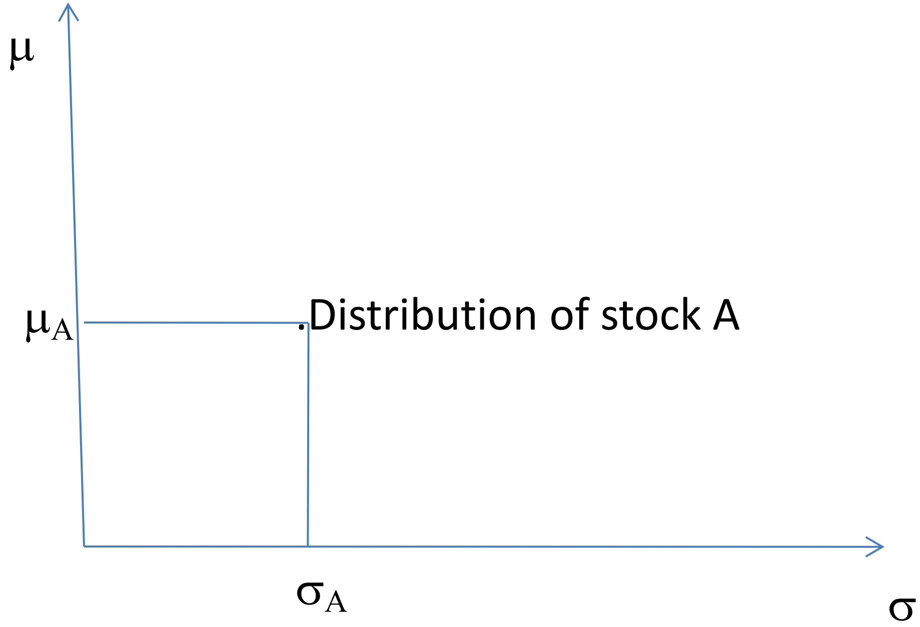
Probability Distribution of Returns

- Normal Distribution is completely described by
  – Mean : Expected return
  – Variance (standard deviation) : dispersion



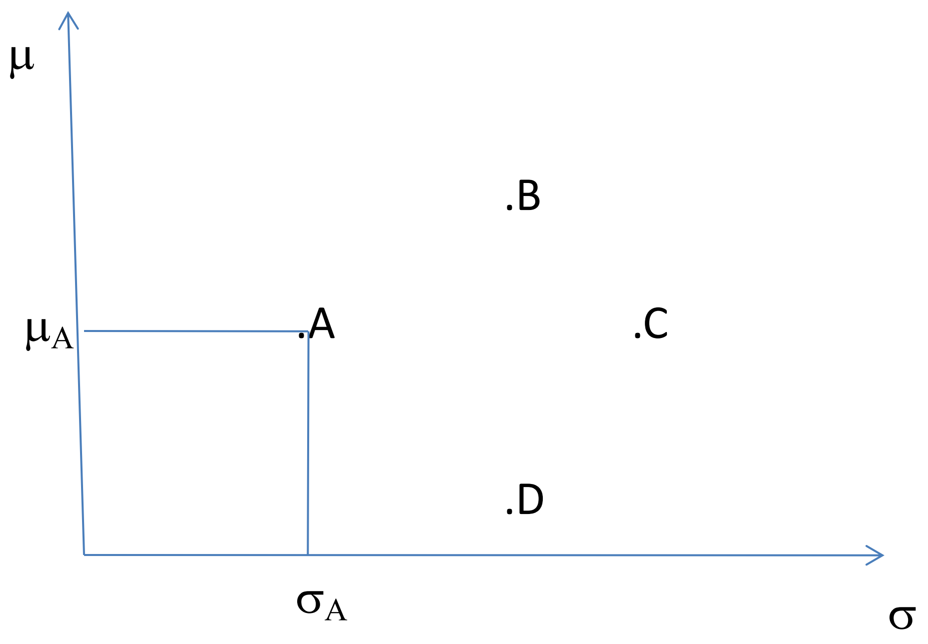
Distribution can be represented by one point in the graph

- Expected return (μ) vs. standard deviation (σ)



If you can only choose one stock?

- Most investors like expected return and dislike uncertainty (risk aversion)

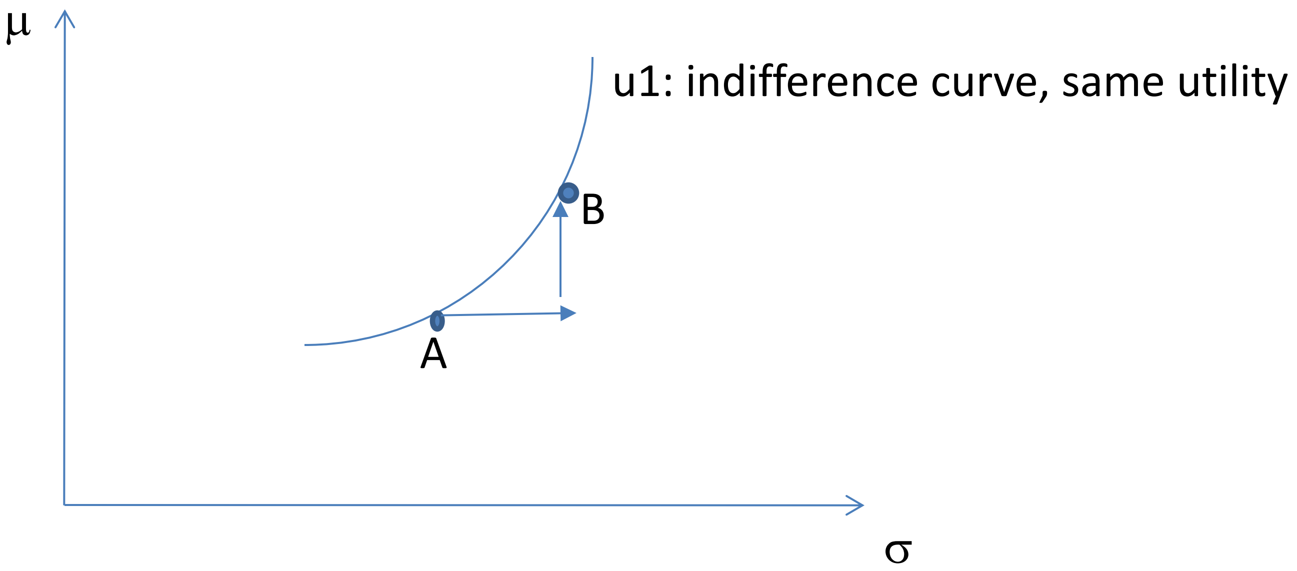


Poll 1: If you can only choose one stock between A and B, which one would you choose?

- A. Stock A
- B. Stock B
- C. None
- D. Depends on your preferences

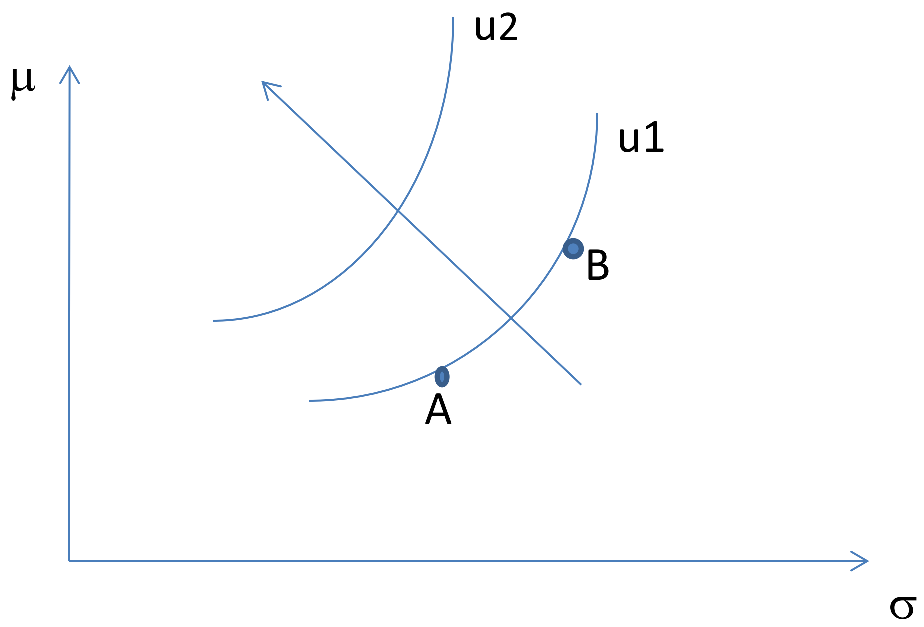
Indifference Curves (for risk averse)

The choice between A and B depends on your preferences
Utility function



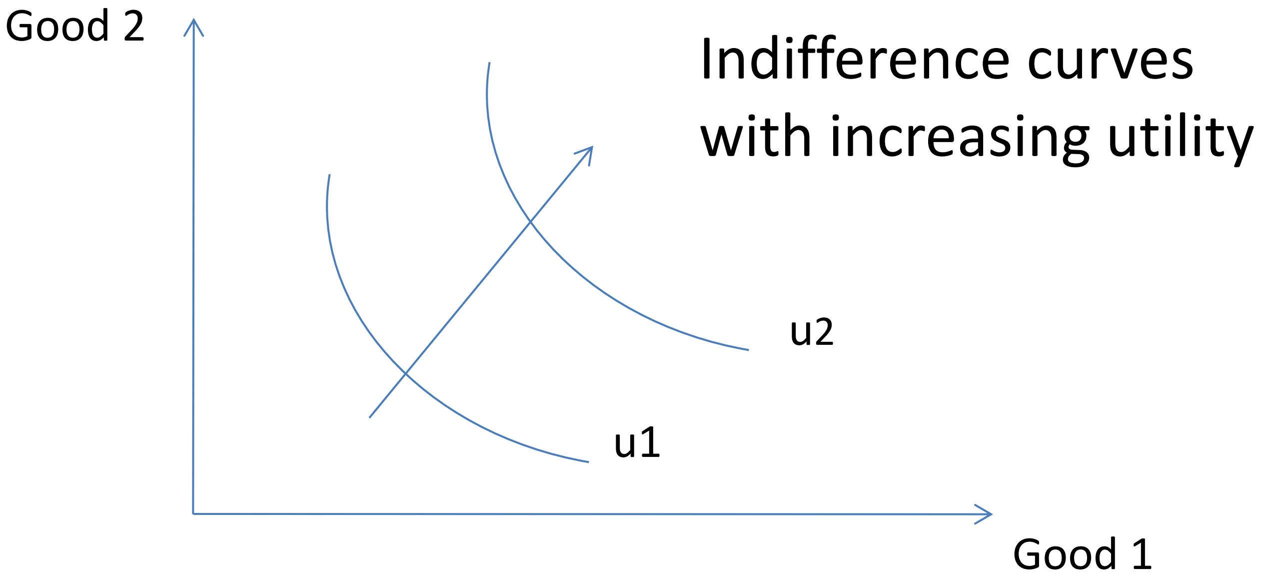
Indifference Curves (for risk averse)

Indifference curves with increasing utility

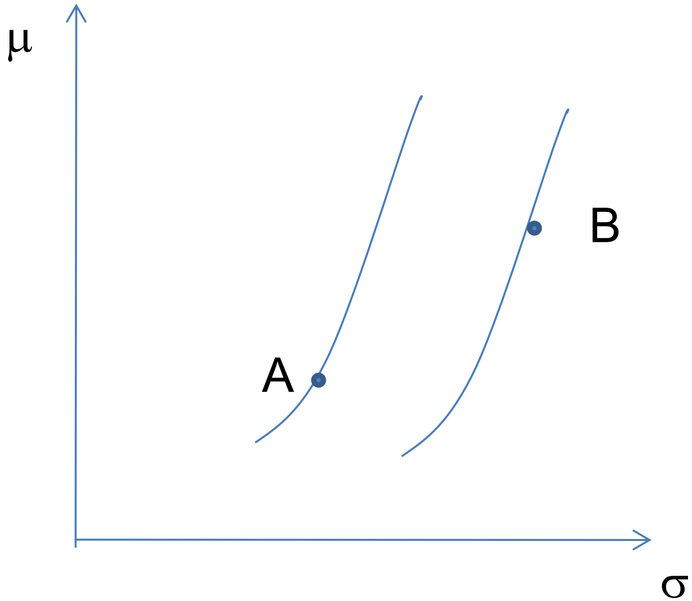


Indifference Curves in Economics

In this case both goods are desirable

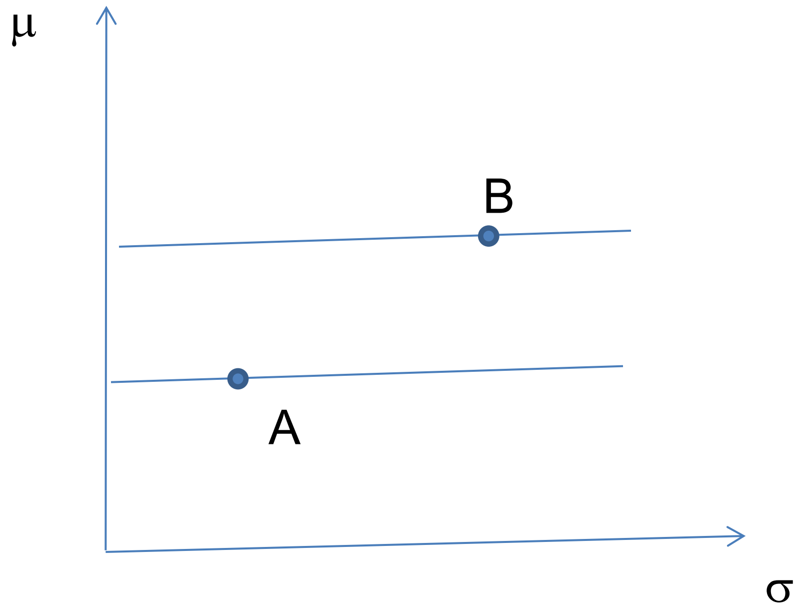


Indifference Curves



Very risk averse investor (steep indifference curves)

Indifference Curves



Risk neutral investor

But, there is no reason to restrict your portfolio to holding one security, you could buy a combination of both (portfolio)

Notation

- Portfolio weights

  w_x and w_y (with $w_x+w_y=1.0$)
- Expected (mean) returns

  $\mu_x=E(r_x)$ and $\mu_y=E(r_y)$
- Variances of returns

  $\sigma^2{}_x=Var(r_x)$ and $\sigma^2{}_y=Var(r_y)$
- Covariance of returns

  $\sigma_{xy} = Cov(r_x, r_y) = \rho_{xy}\,\sigma_x\,\sigma_y$

Portfolio return and expected return

- Portfolio return (random)
  - Dollar value at end of period (plus cash flows) divided by dollar value at beginning of period
  - Can be computed as average of returns on individual securities weighted by their portfolio weights

$$r_p = w_x r_x + w_y r_y$$

- Then the expected return on the portfolio is

$$\mu_p = w_x \mu_x + w_y \mu_y$$

Remember from stats that $E(aX+bY)=aE(X)+bE(Y)$

Portfolio expected return and variance

- The expected return is a weighted average of the expected return on the assets in the portfolio.
- But, the variance is not a weighted average. It depends on how the returns on the assets in the portfolio *covary (correlation, covariance)*.
- Diversification can reduce the variance of a portfolio (extreme example: life insurance company).

Covariance and Correlation

Covariance

$$\sigma_{xy} = \rho_{xy}\,\sigma_x\,\sigma_y$$

Correlation

$$\rho_{xy} \begin{cases} 1 & \text{perfectly correlated} \\ 0 & \text{uncorrelated} \\ -1 & \text{perfectly negatively correlated} \end{cases}$$

Covariance with itself = Variance= $\sigma_{xx} = 1\sigma_x\,\sigma_x = \sigma_x^2$

Determining Covariance and Correlation

- Correlation
  - A measure of the common risk shared by stocks that does not depend on their volatility

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

    - The correlation between two stocks will always be between –1 and +1.

Portfolio variance

- The variance of a portfolio is

$$\sigma^2_p = w_x^2\,\sigma^2_x + w_y^2\,\sigma^2_y + 2w_x\,w_y\,\sigma_{xy}$$

Remember: $Var(aX+bY)=a^2Var(X)+b^2Var(Y)+2abCov(X,Y)$

In Matrix Notation

$$\Omega : 2\times 2\, Variance\ Co\mathrm{var}iance\ matrix\ of\ \ returns = \begin{vmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{vmatrix}$$

$$\mu : 2\ column\ vector\ of\ \ \mathrm{exp}ected\ \ returns = \begin{vmatrix} \mu_x \\ \mu_y \end{vmatrix}$$

$$w : 2\ column\ vector\ of\ \ weights = \begin{vmatrix} w_x \\ w_y \end{vmatrix}$$

$$\mu_P = w_x \mu_x + w_y \mu_y = w^{Tr} \mu \quad scalar\ \ product$$

$$\sigma_P^2 = w_x^2 \sigma_x^2 + w_x w_y \sigma_{xy} + w_y w_x \sigma_{yx} + w_y^2 \sigma_y^2 = w^{Tr} \Omega w$$

$$\sigma_{xy} = \rho_{xy} \sigma_x \sigma_y$$

One risky asset and one risk-free asset

- Individual assets
  - Portfolio weights:

    w in risky asset (stocks)

    (**1-w**) in risk-free asset (T-bill)
  - Expected (mean) returns :

    μ=7.5% and r_f =1.5%
  - Standard deviation of returns :

    σ =20% and 0
  - Covariance of returns : 0

One risky asset and one risk-free asset

$$w_x = w \quad , \quad w_y = 1 - w$$

$$\mu_P = w\mu + (1-w)r_f$$

$$\mu_P = r_f + w(\mu - r_f)$$

$$\mu_P = 0.015 + w(0.075 - 0.015) = 0.015 + 0.06w$$

$$\sigma_P^2 = w^2\sigma^2 + (1-w)^2 0 + 2w(1-w)0 = w^2\sigma^2$$

$$\sigma_P = w\sigma = 0.2w$$

One risky asset and one risk-free asset

- Portfolio expected return

$$\mu_p = w\,\mu + (1\text{-}w)\, r_f = r_f + w(\mu\text{-}\, r_f)$$
$$= 0.015 + 0.06w$$

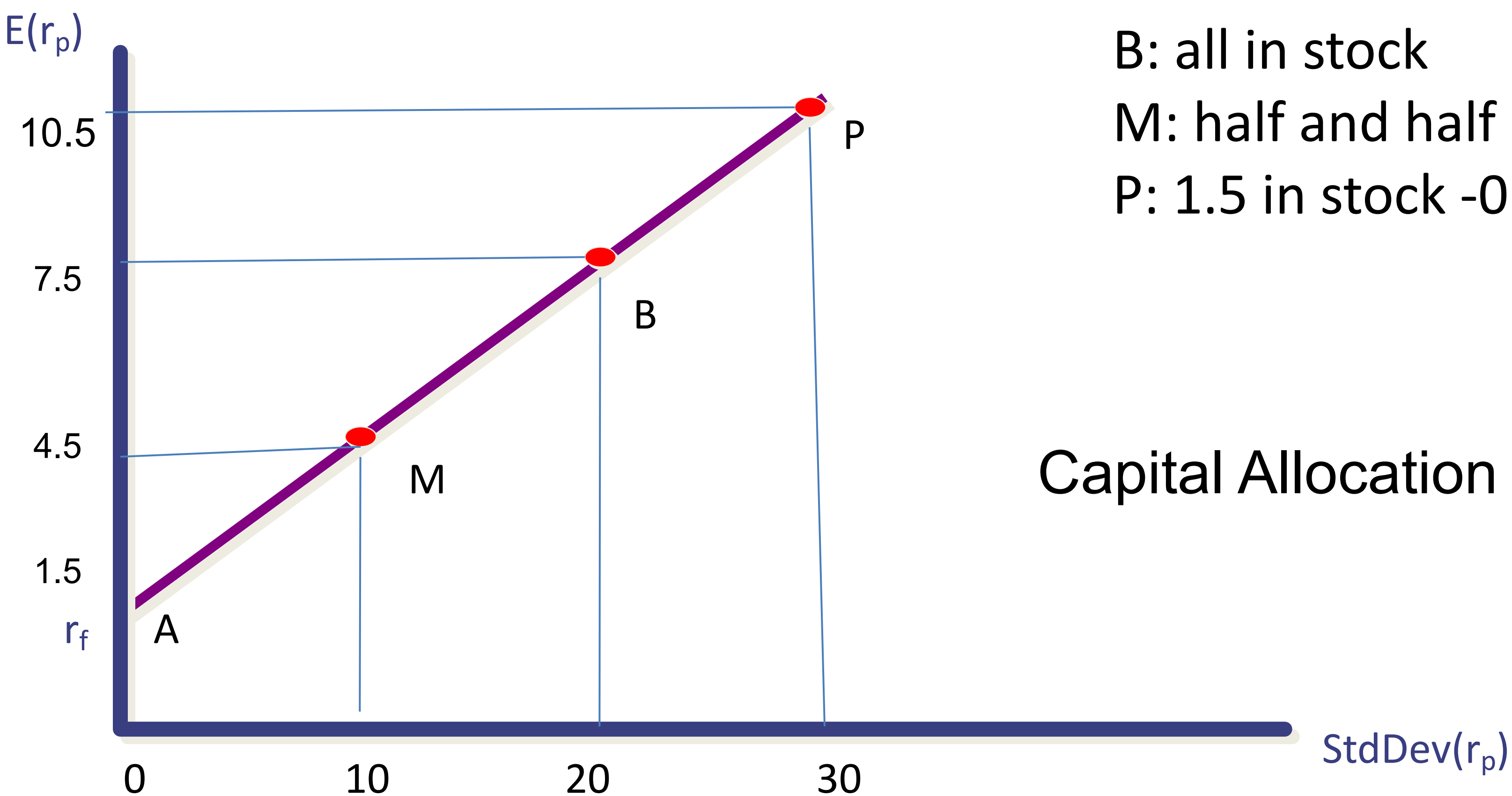
- Portfolio variance

$$\sigma_p{}^2 = w^2\,\sigma^2$$
$$= 0.04w^2$$

Standard deviation

$$\sigma_p = w\sigma = 0.2w$$

One risky asset and one risk-free asset



A: all in risk-free asset

B: all in stock

M: half and half

P: 1.5 in stock -0.5 in r_f

Capital Allocation Line

One risky asset and one risk-free asset

- Lending
  - Invest in the risk free asset
  - Take a long position in the risk free asset
  - Take a positive position in the risk free asset
- Borrowing
  - Take a negative position in the risk free asset
  - Short the risk free asset

Short Selling

- Short a stock
  - Borrow a stock and sell it
  - Take a negative position in the stock
- Short selling involves selling securities you do not own
- Your broker borrows the securities from another client and sells them in the market in the usual way
- Having a negative position in the security (long: positive position)

Short Selling

- At some point in the future you must buy the securities back so they can be replaced in the account of the client
- You must pay dividends and other benefits the owner of the securities receives
- You benefit when the security price goes down

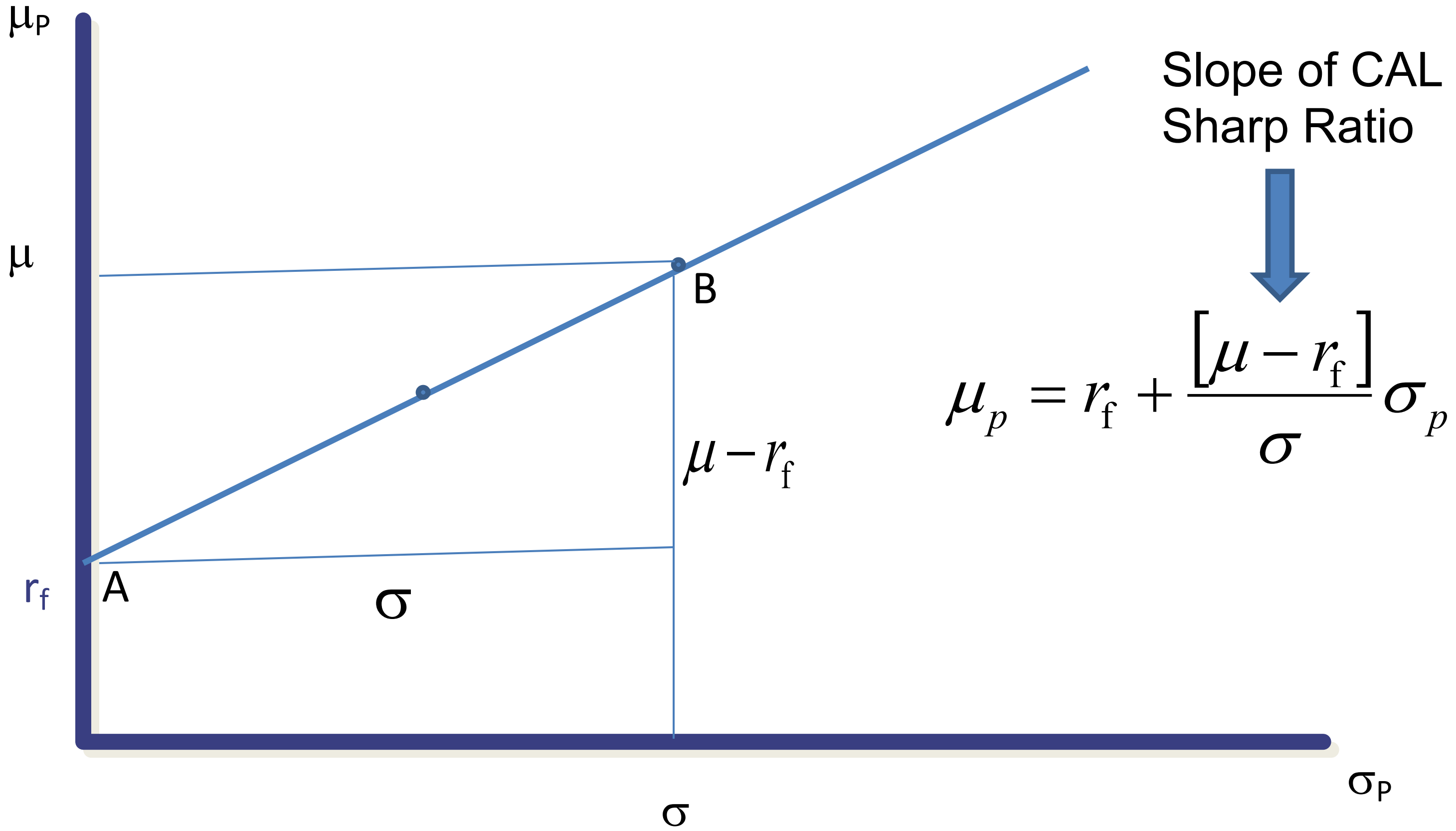
Equation of a Line

$$y = \alpha + \beta x$$

$$\alpha : \text{int}\,ercept \qquad \beta : slope$$

$$\mu_p = r_{\mathrm{f}} + \frac{[\mu - r_{\mathrm{f}}]}{\sigma}\sigma_p$$

Capital Allocation Line



Capital Allocation Line

- Slope of CAL is **Sharpe ratio**: excess return per unit of risk

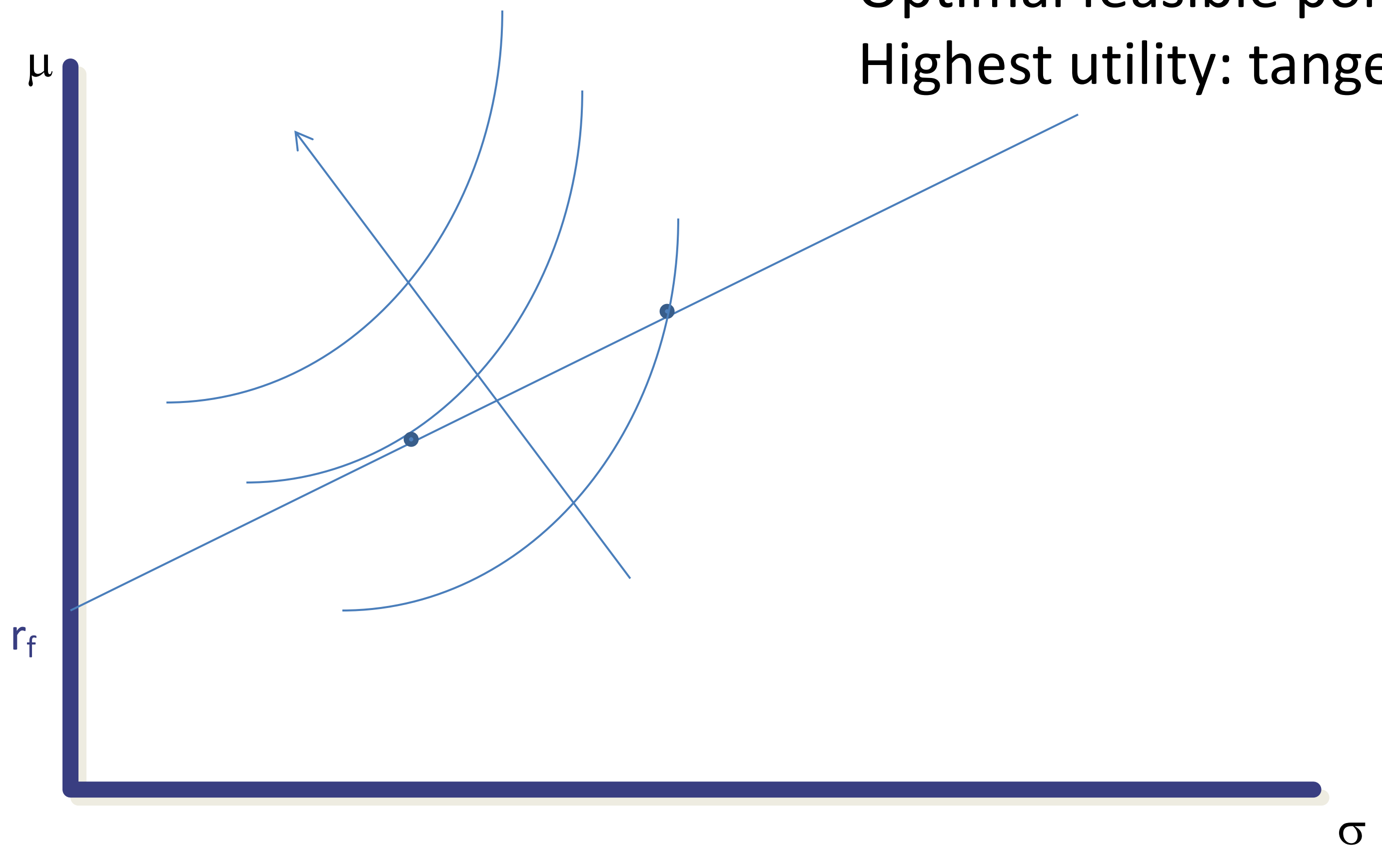
$$S = \frac{\mu - r_{\mathrm{f}}}{\sigma} = \frac{0.06}{0.2} = 0.3$$

- Feasible combinations of mean and standard deviation – Capital Allocation Line

$$\mu_p = r_{\mathrm{f}} + \sigma_p \frac{\left[\mu - r_{\mathrm{f}}\right]}{\sigma} = 0.015 + 0.3\sigma_p$$

Optimal Portfolio on the CAL

Optimal feasible portfolio
Highest utility: tangency point



Mean-variance utility functions

- Utility function (mean-variance)

  $U(r_p) = E(r_p) - \gamma Var(r_p)/2 = \mu_p - \gamma\ \sigma_p^2/2$

  Where γ is coefficient of risk aversion: e.g. $\gamma=4$
- Consistent with quadratic utility function
- Consistent with normal distributions

Quadratic utility function
(unique up to a positive linear transformation)

$$u(W) = W - \frac{b}{2}W^2 = W_0(r_P + 1) - \frac{b}{2}W_0{}^2(r_P + 1)^2$$

$$with \quad W = W_0(r_P + 1)$$

$$E[u(W)] = E[u(r_P)] = U(r_P) = \mu_P - \frac{\gamma}{2}\sigma_P^2$$

We saw that there are other utility functions that are often used in practice

Not as easy to use, but with better properties

Optimal Portfolio

$$U(r_P) = \mu_P - \frac{\gamma}{2}\sigma_P^2$$

$$\mu_P = w\mu + (1-w)r_f, \quad \sigma_P^2 = w^2\sigma^2$$

$$Max_w U(r_P) = w\mu + (1-w)r_f - \frac{\gamma}{2}w^2\sigma^2$$

$$\frac{dU(r_P)}{dw} = \mu - r_f - \gamma w\sigma^2 = 0$$

$$w^* = \frac{\mu - r_f}{\gamma\sigma^2}$$

Portfolio of risky and risk-free assets

- To find optimal portfolio choice

  $\max_w U(r_p) = w\,\mu + (1-w)\,r_f - \gamma\, w^2 \sigma^2/2$

- From first-order conditions

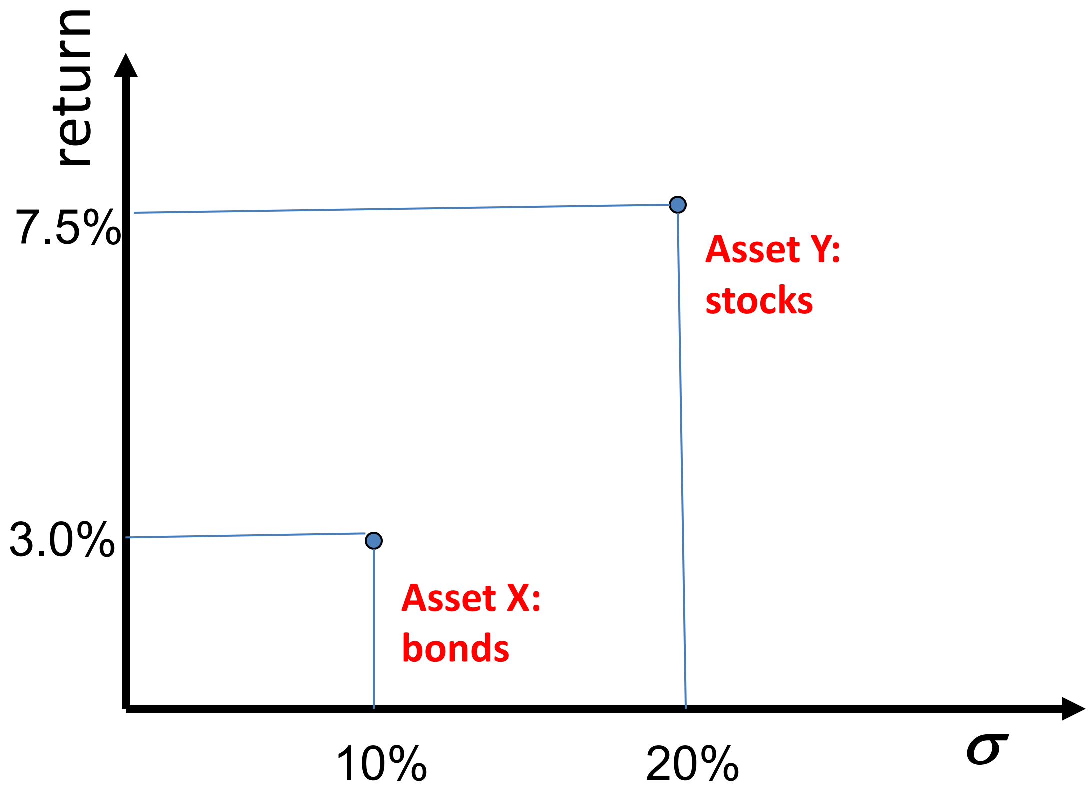
$$w = \frac{\mu - r_\mathrm{f}}{\gamma\sigma^2} = \frac{0.06}{4 \times 0.2^2} = \frac{0.06}{0.16} = 0.375$$

- Invest 37.5% of portfolio in stock and 62.5% in T-bills
- What if the risk aversion coefficient was 2?

Two risky assets

- Two risky assets – bond and stock portfolios (it could also be two stocks)
  - Means of 3.0% and 7.5%
  - Standard deviations of 10% and 20%
  - Correlation of 0.2

Two risky assets



Two risky assets with a correlation of 0.2

Two risky assets

- Consider a portfolio with 1/3 of funds invested in bond and 2/3 of funds invested in stock portfolios
- Portfolio expected return

$$\mu_p = 1/3 \times 0.03 + 2/3 \times 0.075$$
$$= 0.06 = 6.0\%$$

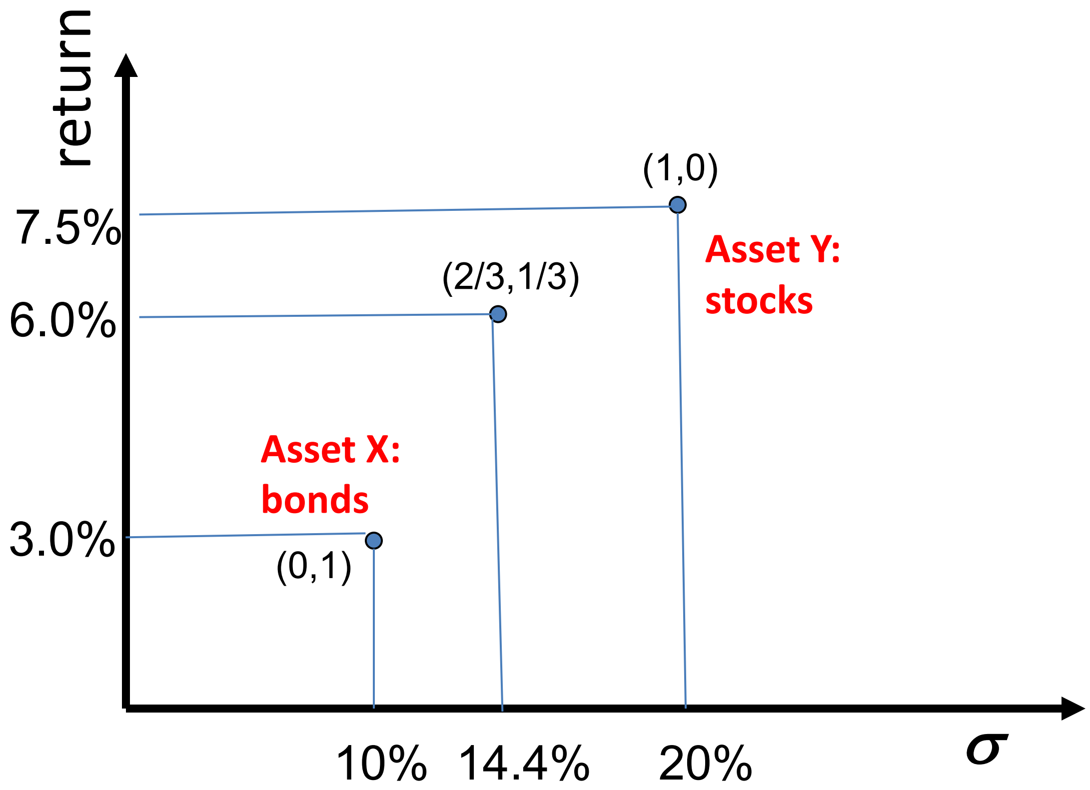
- Portfolio variance= $\boxed{w_x^2 \sigma^2_x + w_y^2 \sigma^2_y + 2 w_x w_y \sigma_{xy}}$ $\boxed{\sigma_{xy} = \rho \sigma_x \sigma_y}$

$$\sigma_p^2 = (1/3)^2 (0.1)^2 + (2/3)^2 (0.2)^2 + 2\,(1/3)(2/3)(0.2 \times 0.1 \times 0.2)$$
$$= 0.0207$$

- Portfolio standard deviation

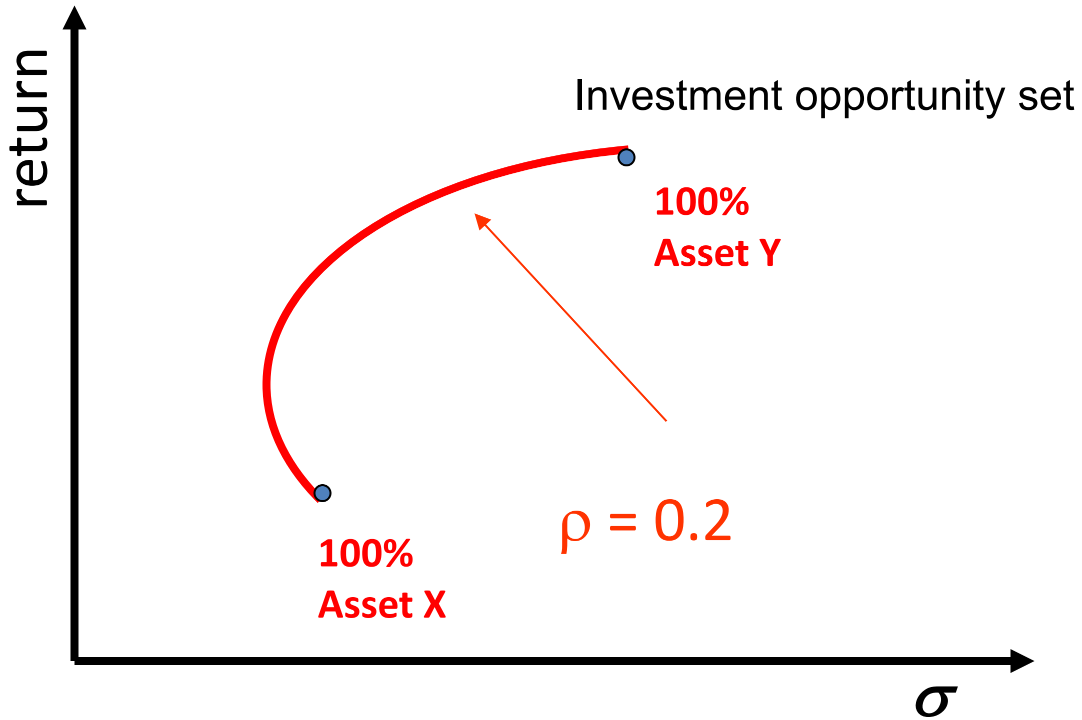
$$\sigma_p = \sqrt{0.0207}$$
$$= 0.1438 = 14.38\%$$

Two risky assets



What happens if the proportions change?

Two risky assets when proportions change



Efficient Portfolios: upper part of the curve. Highest expected return for a given standard deviation.
What happens if the correlation changes?

What happens when the correlation changes? Let's start with Perfect Correlation

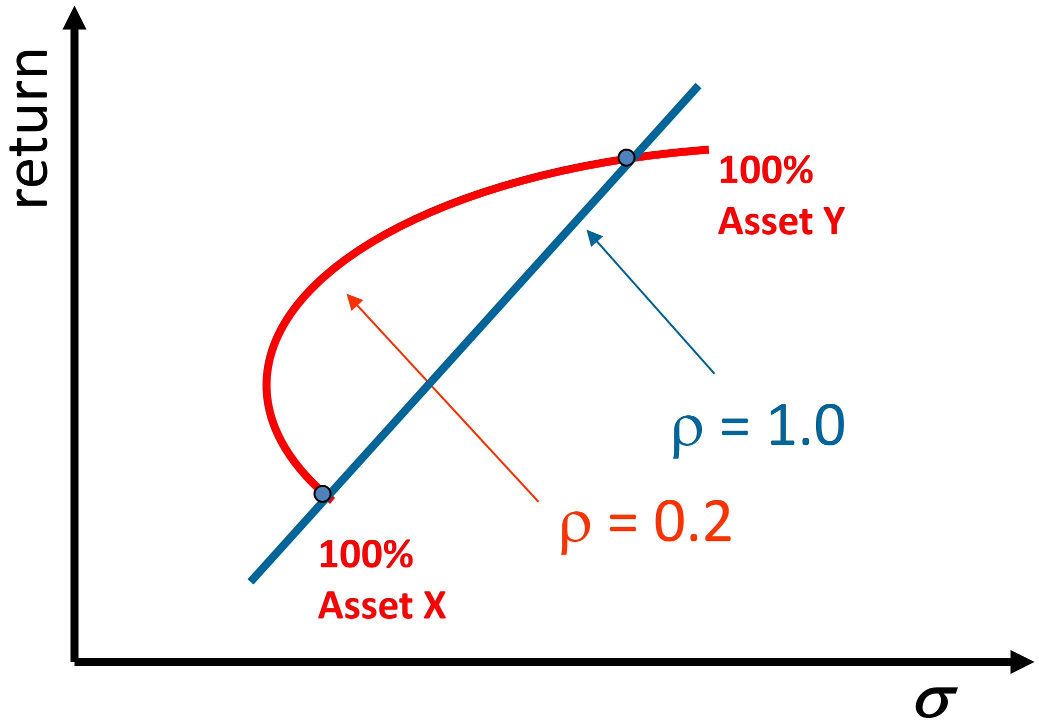
$$\sigma_P^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \rho\sigma_x\sigma_y$$

$$\rho = 1$$

$$\sigma_P^2 = (w_x\sigma_x + w_y\sigma_y)^2$$

$$\sigma_P = w_x\sigma_x + w_y\sigma_y$$

Two-Assets: Different Correlations



Perfectly negative correlation

$$\sigma_P^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \rho\sigma_x\sigma_y$$

$$\rho = -1$$

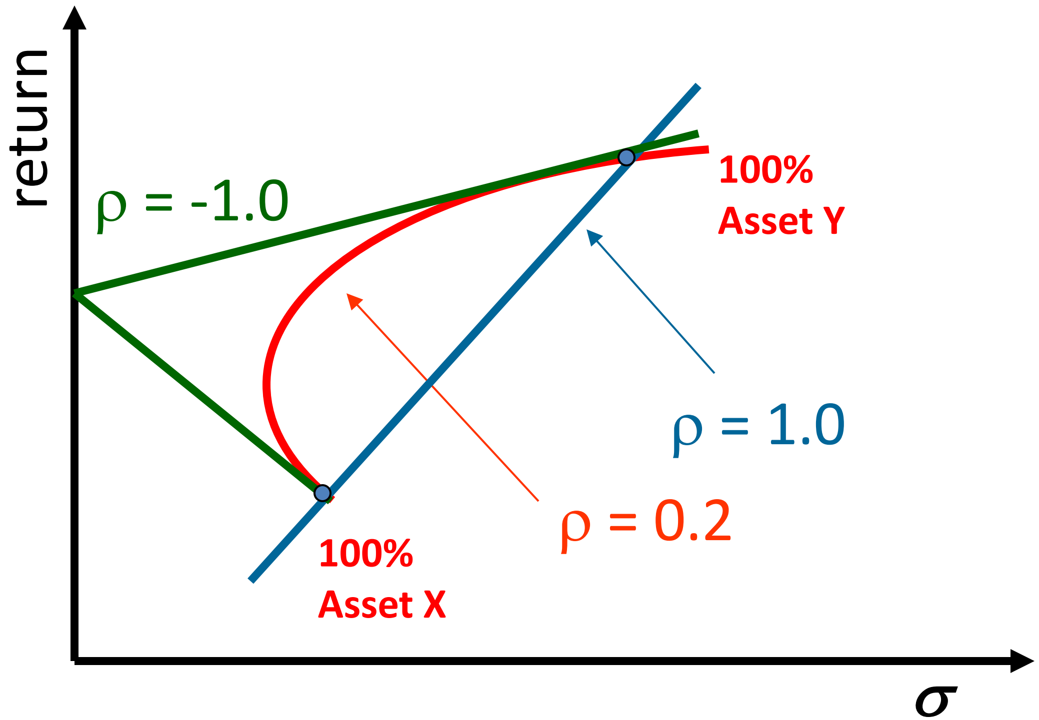
$$\sigma_P^2 = (w_x\sigma_x - w_y\sigma_y)^2$$

$$\sigma_P = w_x\sigma_x - w_y\sigma_y > 0$$

$$\sigma_P = w_y\sigma_y - w_x\sigma_x > 0$$

$$\sigma_P = 0 \ \ when \quad \frac{w_x}{w_y} = \frac{\sigma_y}{\sigma_x}$$

Two-Assets: Different Correlations



Efficient Portfolios: highest expected return for a given σ

Perfect Negative Correlation

$$\sigma_P = 0 \;\; when \quad \frac{w_x}{w_y} = \frac{\sigma_y}{\sigma_x}$$

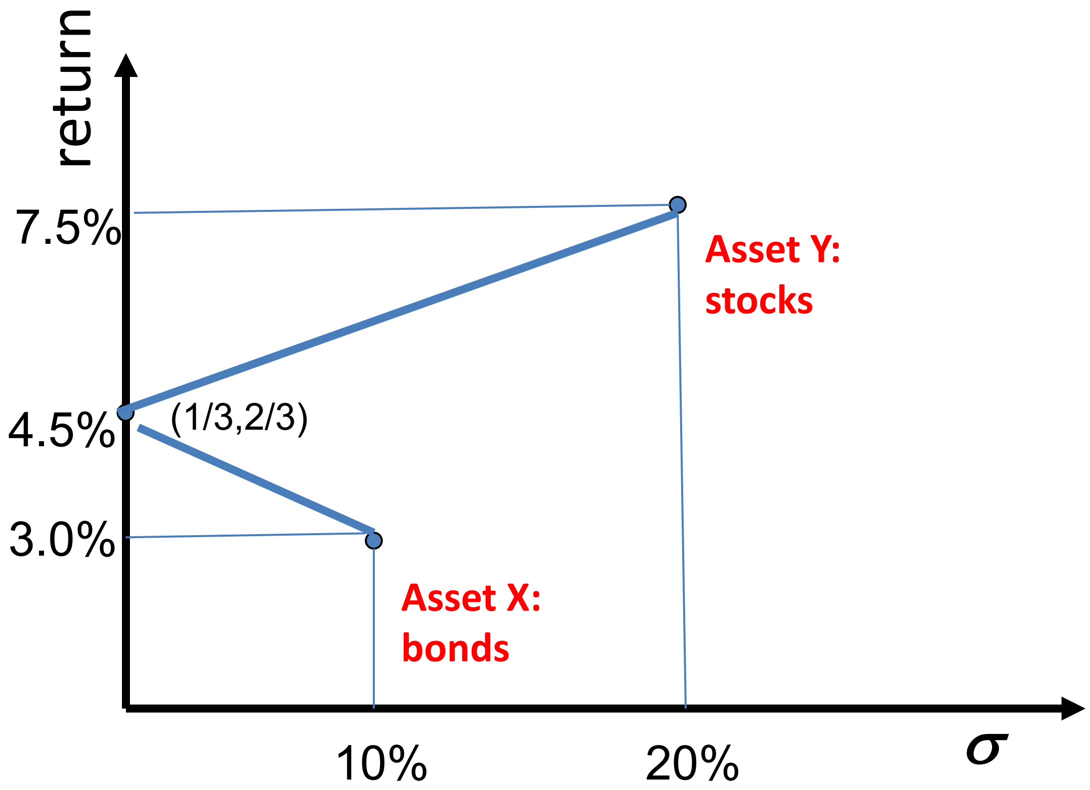
$$\sigma_x = 0.10 \quad \sigma_y = 0.20$$

$$\frac{w_x}{w_y} = \frac{.20}{.10} \quad and \quad w_x + w_y = 1$$

$$w_x = 2/3 \quad w_y = 1/3$$

$$\mu_p = w_x\mu_x + w_y\mu_y = 0.67(0.03) + 0.33(0.075) = 0.045$$

Two risky assets



Two risky assets with a correlation of -1

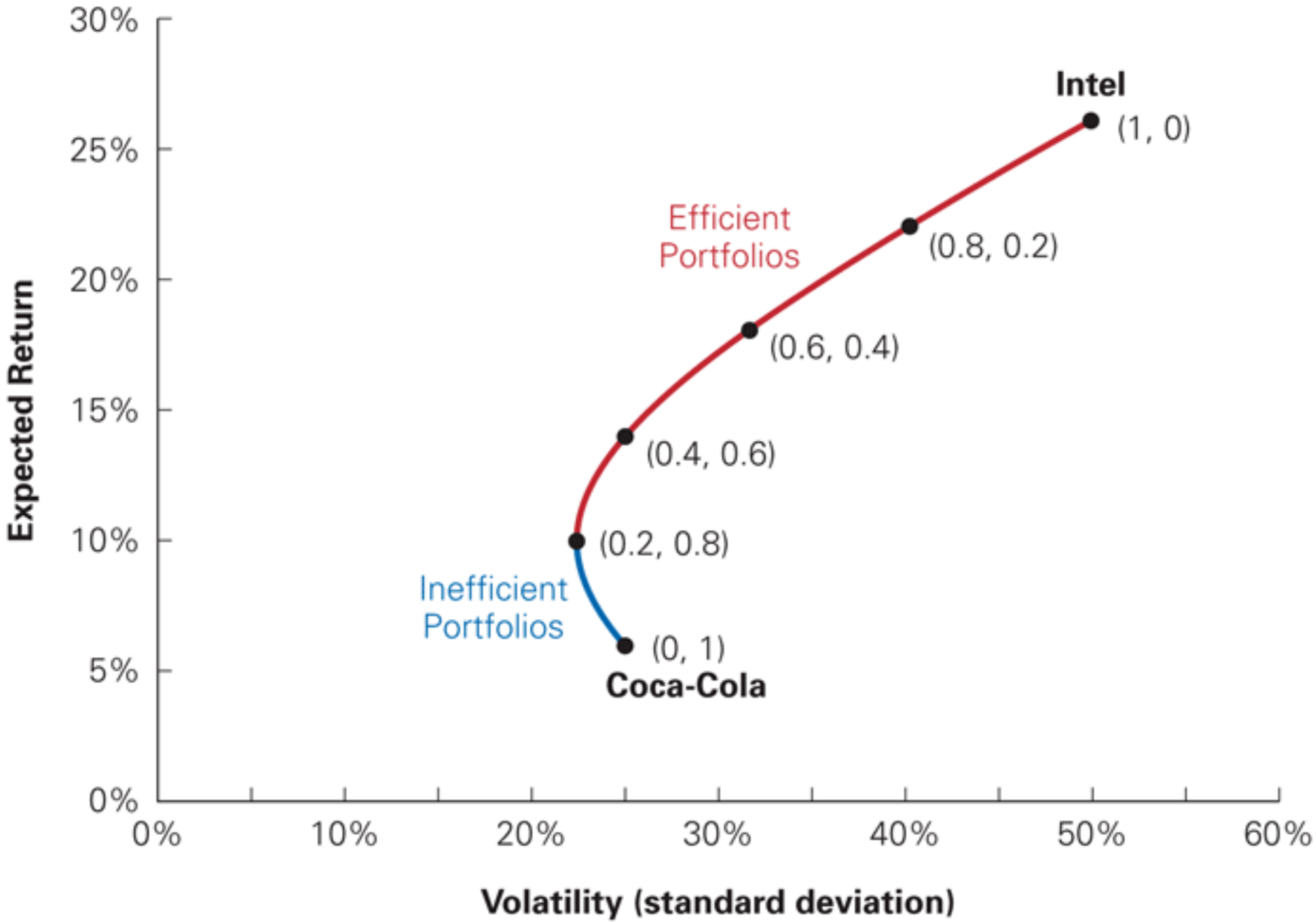
Volatility Versus Expected Return for Portfolios of Intel and Coca-Cola Stock

Expected Returns and Volatility for Different Portfolios of Two Stocks

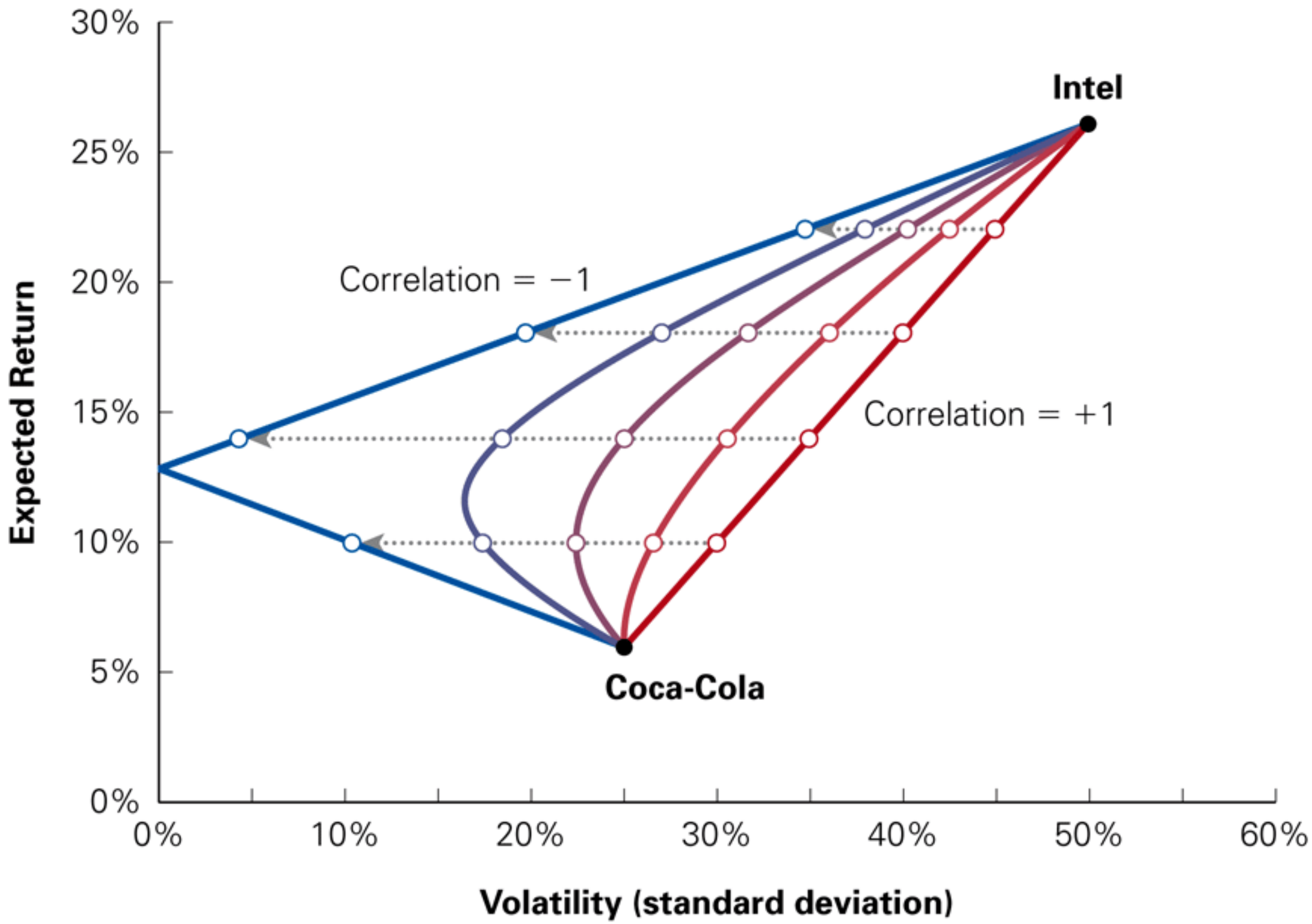
Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.4
0.00	1.00	6.0	25.0

Assume that the returns on these stocks are uncorrelated.

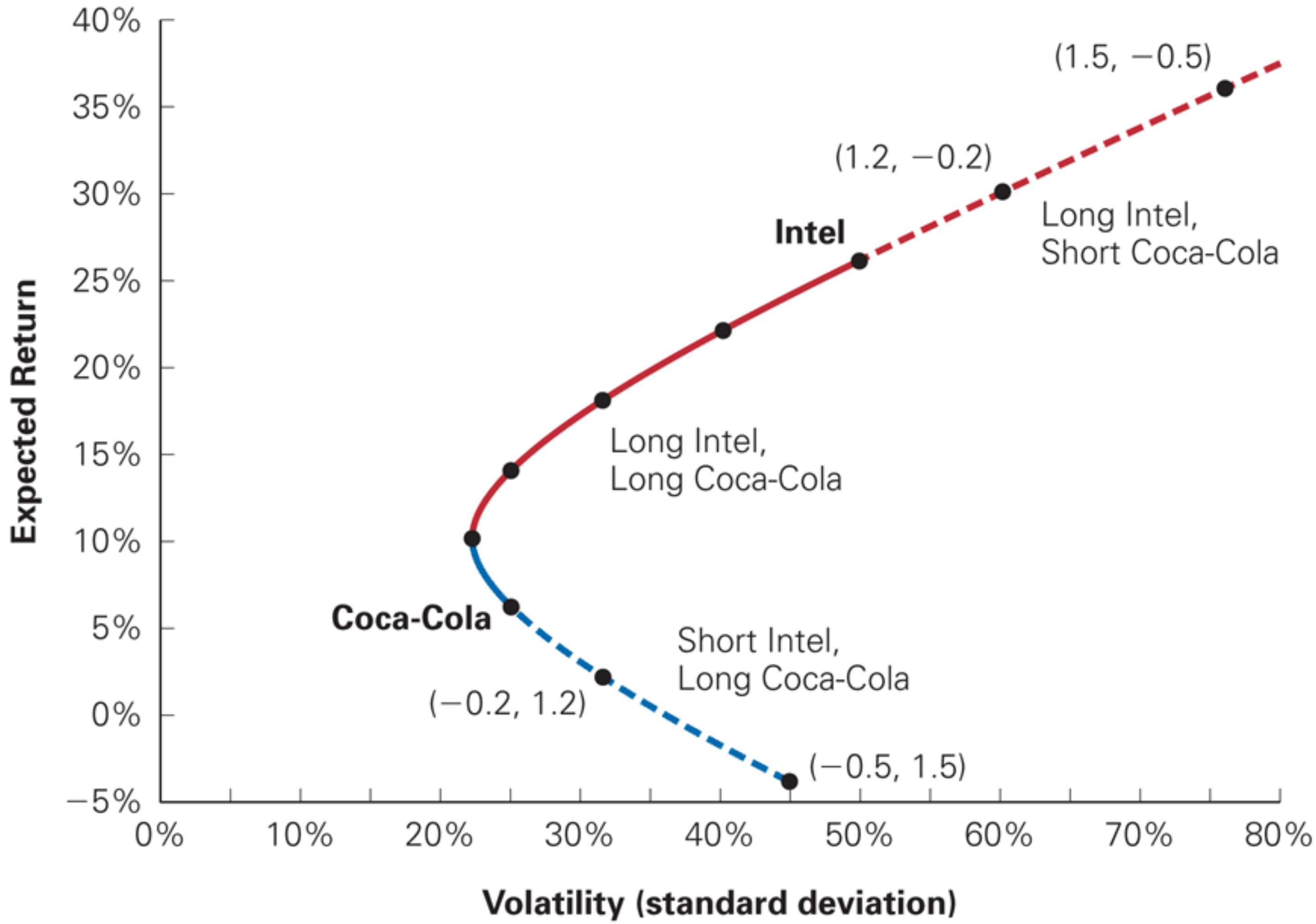
Volatility Versus Expected Return for Portfolios of Intel and Coca-Cola Stock



Effect on Volatility and Expected Return of Changing the Correlation between Intel and Coca-Cola Stock



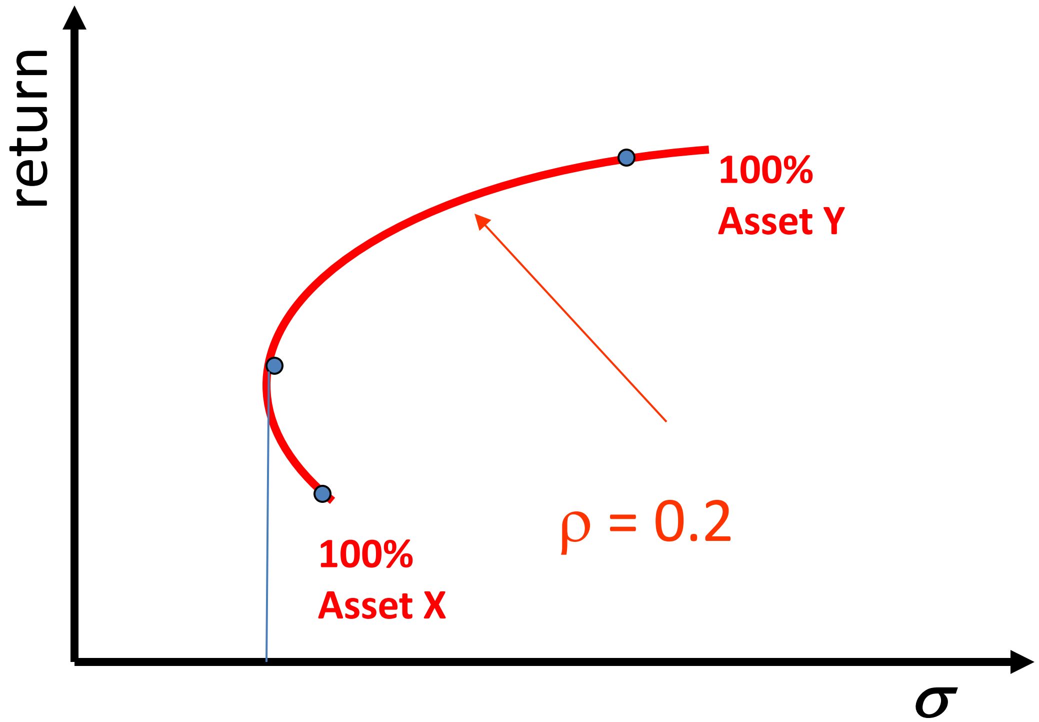
Portfolios of Intel and Coca-Cola Allowing for Short Sales



Correlation and Diversification

- The various combinations of risk and return available all fall on a smooth curve.
- This curve is called an *investment opportunity set* because it shows the possible combinations of risk and return available from portfolios of these two assets.
- A portfolio that offers the highest return for its level of risk is said to be an *efficient portfolio*.
- The undesirable portfolios are said to be *dominated* or *inefficient*.

Minimum Variance Portfolio



Efficient Portfolios: upper part of the curve

Minimum Variance Portfolio

$$\boxed{\sigma_p{}^2 = w_x{}^2 \sigma^2{}_x + w_y{}^2 \sigma^2{}_y + 2 w_x w_y \sigma_{xy}}$$

$$w_x + w_y = 1.0$$

$$\sigma_P^2 = w^2 \sigma_x^2 + (1-w)^2 \sigma_y^2 + 2w(1-w)\sigma_{xy}$$

$$\frac{d\sigma_P^2}{dw} = 2w\sigma_x^2 - 2(1-w)\sigma_y^2 + 2\sigma_{xy} - 4w\sigma_{xy} = 0$$

$$w^* = \frac{\sigma_y^2 - \sigma_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}} = w_x \quad , \quad w_y = 1 - w^*$$

Minimum Variance Portfolio

$$\sigma_x = 0.10 \quad bonds$$

$$\sigma_y = 0.20 \quad stocks$$

$$\rho = 0.2$$

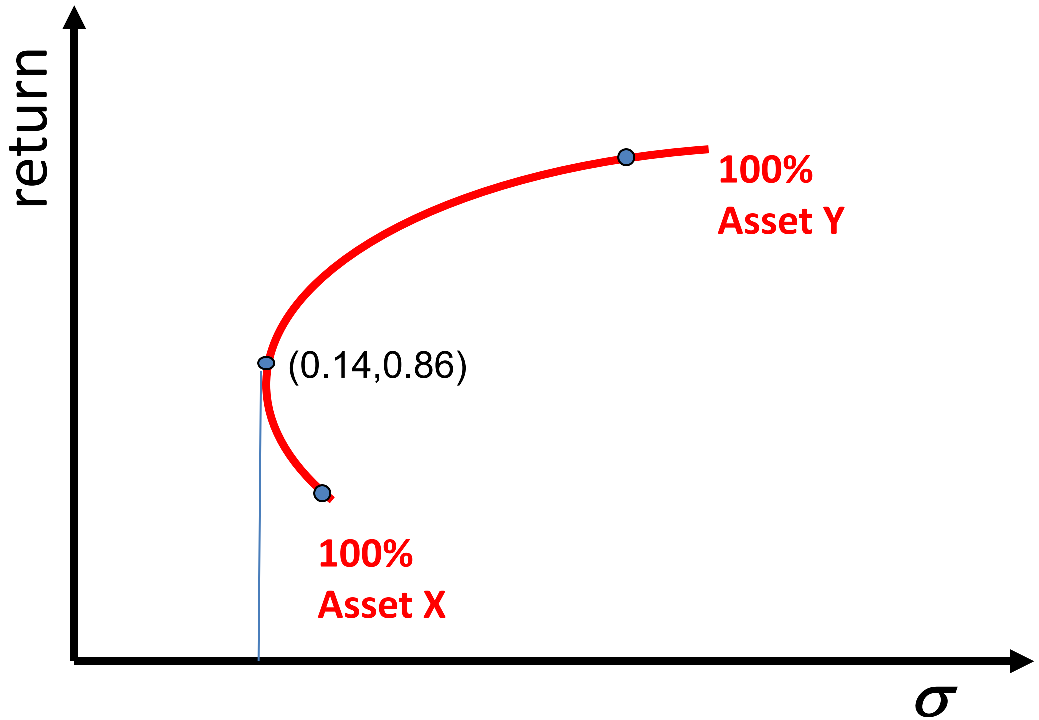
$$\sigma_{xy} = 0.2(0.10)(0.20) = 0.004$$

$$w^* = \frac{\sigma_y^2 - \sigma_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}} = w_x \quad , \quad w_y = 1 - w^*$$

$$w_x = \frac{0.04 - 0.004}{0.01 + 0.04 - 0.008} = \frac{0.036}{0.042} = 0.86$$

$$w_y = 0.14$$

Minimum Variance Portfolio

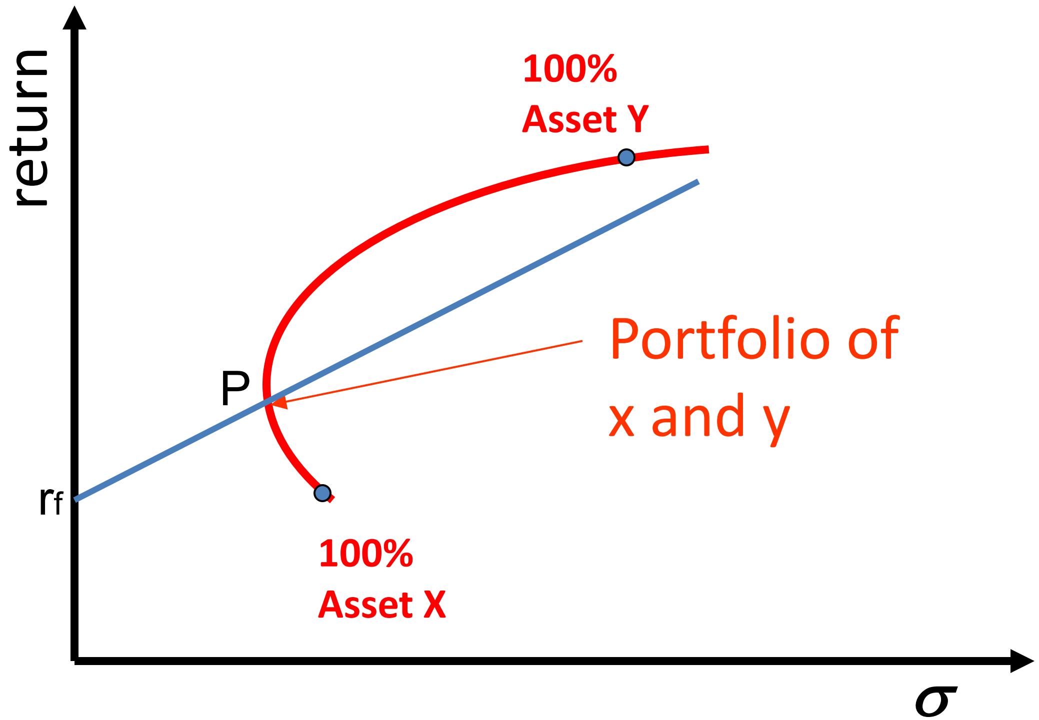


How do we find the expected return and standard deviation of the minimum variance portfolio?

Two risky and one risk-free asset

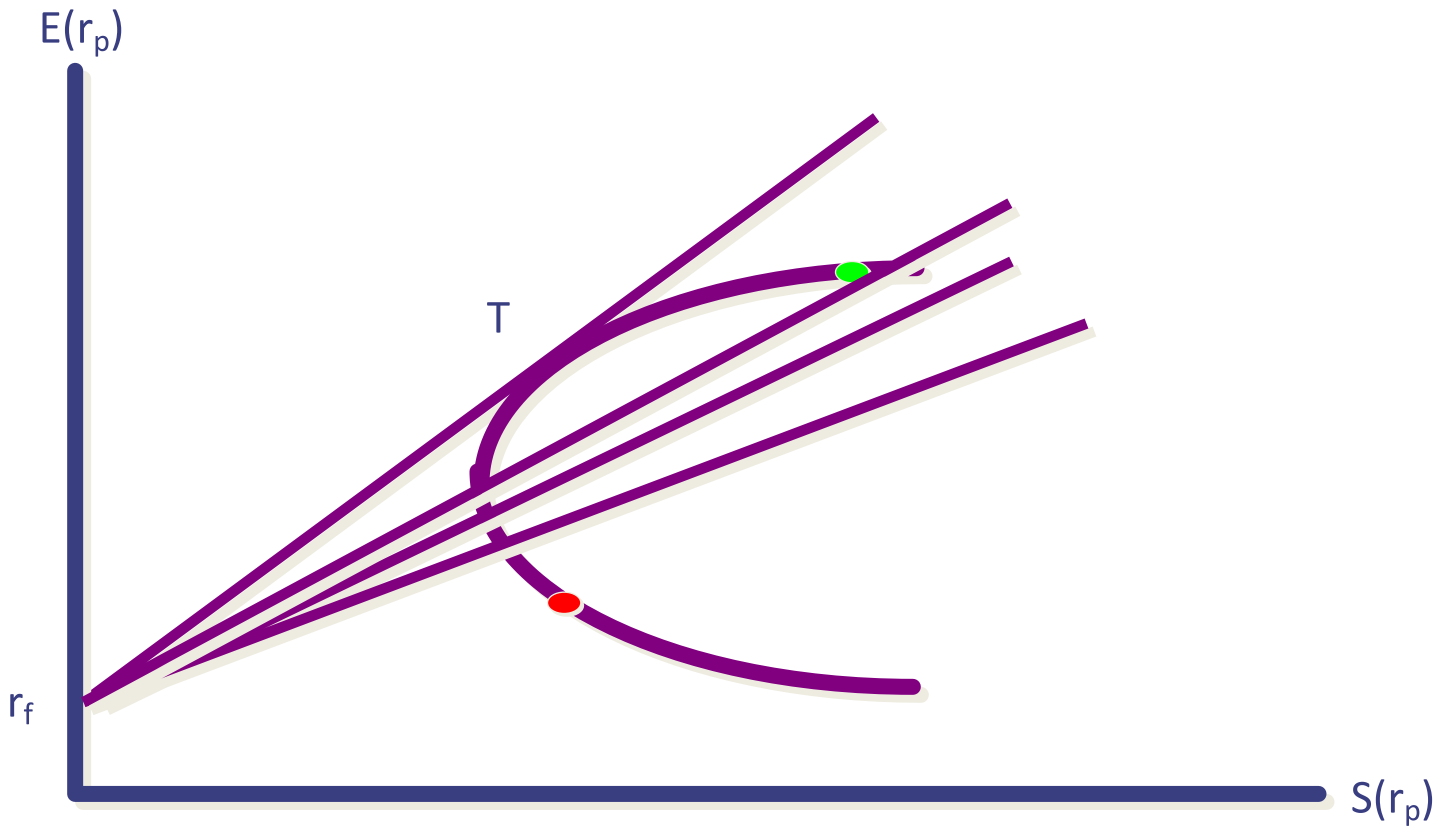
- We now find the portfolio of the two risky assets that can be optimally combined with the risk-free asset
  - This is the *optimal risky portfolio*
- For each risky portfolio, find out the corresponding CAL
- What is the CAL that would give the investor maximum utility?

Two risky assets and one risk-free asset



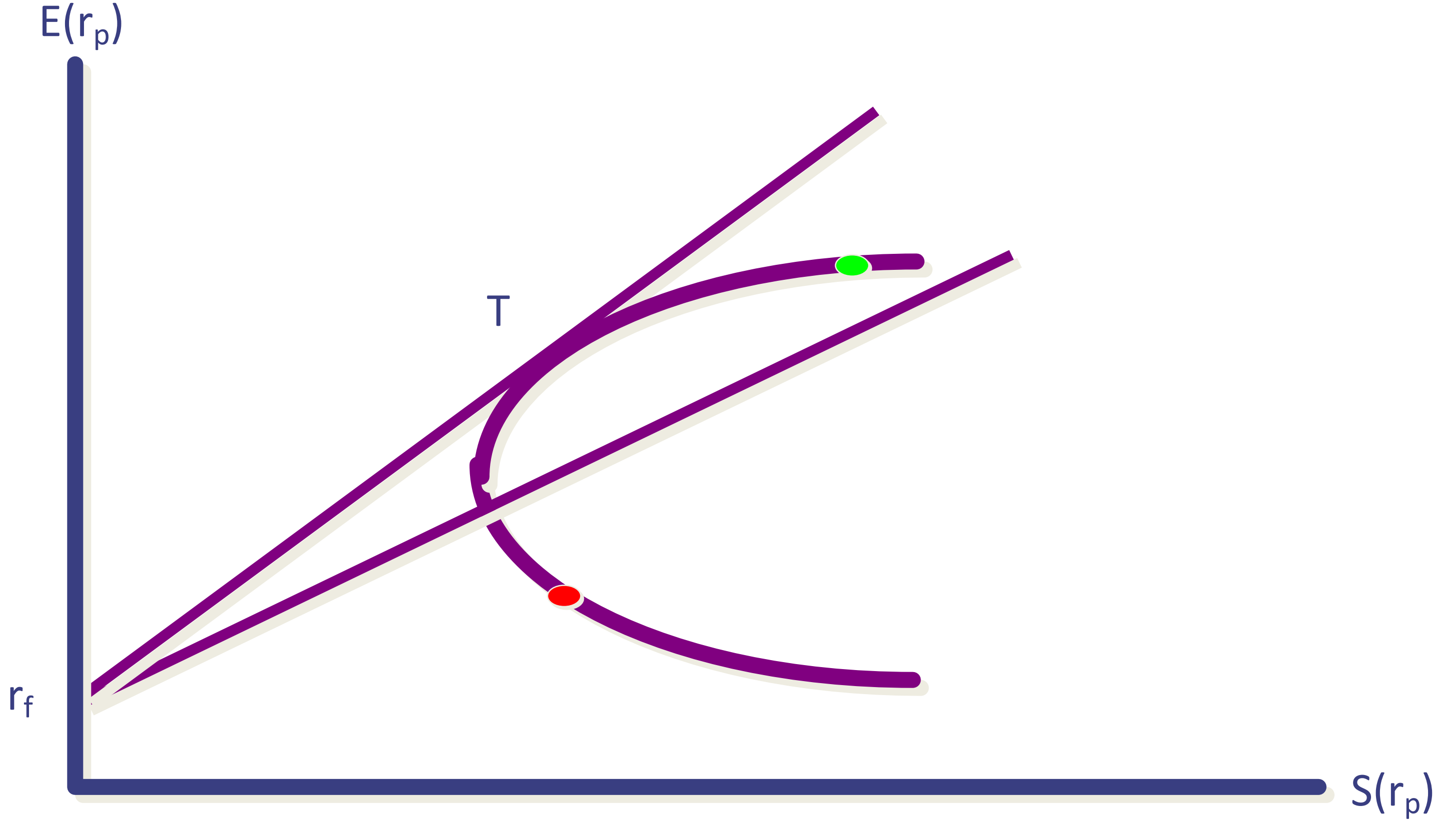
CAL for any portfolio of x and y

Two risky and one risk-free asset



Efficient frontier and tangency portfolio

- Tangency – maximum Sharpe ratio portfolio

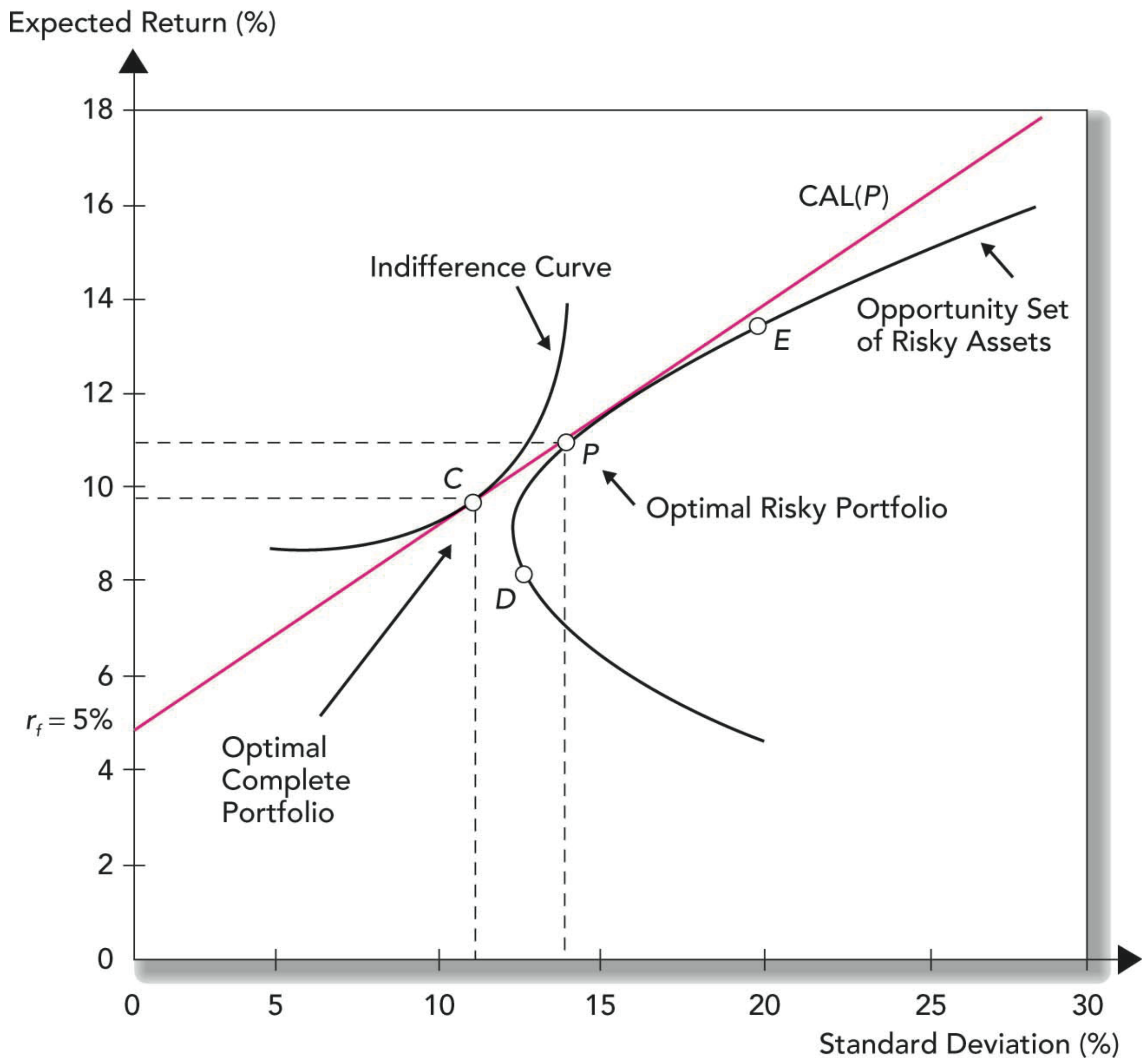


Tangency portfolio: optimal combination of risky securities.
Efficient frontier becomes linear: CAL for T

Optimal Risky Portfolio

- Optimal combination of risky securities
- Tangency Portfolio (the same for all investors!)
- We find the CAL that maximizes the Sharp Ratio
- The efficient frontier becomes linear
- You can separate the determination of the optimal risky portfolio from preferences (Separation Property)

Determination of the Optimal Overall Portfolio



Portfolio weights for tangency portfolio

- We can find the portfolio weights of the tangency portfolio (Max. Sharpe ratio)

$$\max_{w_x, w_y} \frac{\mu_p - r_{\mathrm{f}}}{\sigma_{\mathrm{p}}} = \max_{w_x, w_y} \frac{w_x \mu_x + w_y \mu_y - r_{\mathrm{f}}}{\sqrt{w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2 w_x w_y \sigma_{xy}}}$$

$$\text{s.t.} \quad w_x + w_y = 1$$

$$w_x = \frac{(\mu_x - r_{\mathrm{f}})\sigma_y^2 - (\mu_y - r_{\mathrm{f}})\sigma_{xy}}{(\mu_x - r_{\mathrm{f}})\sigma_y^2 + (\mu_y - r_{\mathrm{f}})\sigma_x^2 - (\mu_x - r_{\mathrm{f}} + \mu_y - r_{\mathrm{f}})\sigma_{xy}}$$

$$w_y = 1 - w_x$$

Max Sharpe Ratio

$$\max_{w_x, w_y} SR = \max_{w_x, w_y} \frac{w_x \mu_x + w_y \mu_y - r_f}{\sqrt{w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2 w_x w_y \sigma_{xy}}}$$

$$\max_{w} SR = \max_{w} \frac{w \ \mu_x + (1 - w \ ) \mu_y - r_f}{\sqrt{w^2 \sigma_x^2 + (1-w)^2 \sigma_y^2 + 2w \ (1 - w \ ) \sigma_{xy}}}$$

$$\frac{\partial \mathrm{SR}}{\partial \mathrm{w}} = 0$$

$$w_x = w^* = \frac{(\mu_x - r_f)\sigma_y^2 - (\mu_y - r_f)\sigma_{xy}}{(\mu_x - r_f)\sigma_y^2 + (\mu_y - r_f)\sigma_x^2 - (\mu_x - r_f + \mu_y - r_f)\sigma_{xy}}$$

$$w_y = 1 - w^*$$

Back to Example

- Two risky assets – bond and stock
  - Means of 3.0% and 7.5%
  - Standard deviations of 10% and 20%
  - Correlation of 0.2
- Assume r_f=1.5%. Find for the Tangency Portfolio:
  - Stock weight
  - Bond weight
  - Expected return
  - Standard deviation
  - Sharp ratio

Example

$$w_x = \frac{\left(\mu_x - r_{\mathrm{f}}\right)\sigma_y^2 - \left(\mu_y - r_{\mathrm{f}}\right)\sigma_{xy}}{\left(\mu_x - r_{\mathrm{f}}\right)\sigma_y^2 + \left(\mu_y - r_{\mathrm{f}}\right)\sigma_x^2 - \left(\mu_x - r_{\mathrm{f}} + \mu_y - r_{\mathrm{f}}\right)\sigma_{xy}}$$

$$w_x = \frac{\left(0.03 - 0.015\right)0.04 - (0.075 - 0.015)0.004}{\left(0.03 - 0.015\right)0.04 + (0.075 - 0.015)0.01 - (0.03 - 0.015 + 0.075 - 0.015)0.004}$$

$$w_x = 0.4$$

$$w_y = 1 - w_x = 0.6$$

Example

- Portfolio expected return

  $\mu_p = 0.4 \times 0.03 + 0.6 \times 0.075$

  $= 0.057 = 5.7\%$

- Portfolio variance= $\boxed{w_x^2 \sigma^2_x + w_y^2 \sigma^2_y + 2 w_x w_y \sigma_{xy}}$ $\boxed{\sigma_{xy} = \rho \sigma_x \sigma_y}$

  $\sigma_p^2 = (0.4)^2 (0.1)^2 + (0.6)^2 (0.2)^2 + 2\,(0.4)(0.6)(0.2 \times 0.1 \times 0.2)$

  $= 0.0180$

- Portfolio standard deviation

  $\sigma_p = \sqrt{0.0180}$

  $= 0.134 = 13.4\%$

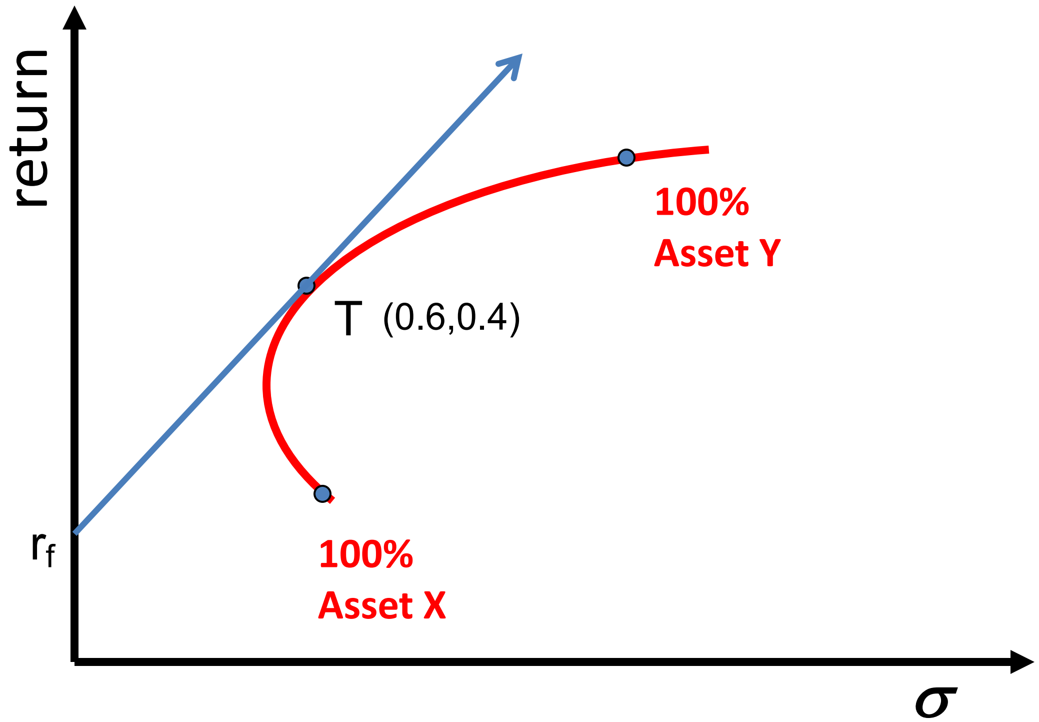
  $S = (5.7 - 1.5)/13.4 = 0.31$

$$S = \frac{\mu_p - r_f}{\sigma_p}$$

The Tangency Portfolio

- In this case (for r_f=1.5%)
  - Stock weight: 0.6
  - Bond weight: 0.4
  - Expected return: 5.7%
  - Std.dev.: 13.4%,
  - Sharp ratio: 0.31

The Tangency Portfolio



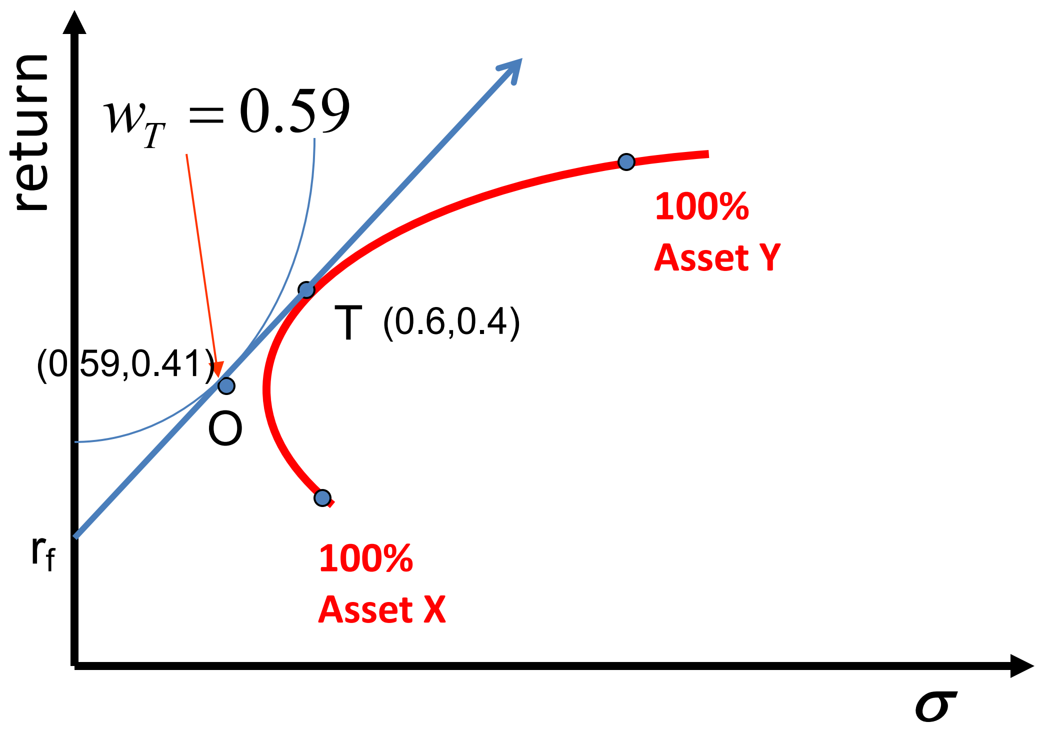
Optimal portfolio of tangency and risk-free asset

- We can figure out the optimal combination of tangency and risk-free asset for an investor with risk aversion $\gamma=4$
- The optimal weight on the tangency portfolio

$$w_T = \frac{\mu_T - r_{\mathrm{f}}}{\gamma\sigma_T^2} = \frac{0.057 - 0.015}{4 \times 0.1339^2} = 0.59$$

- So the weight on the risk-free Tbill is 0.41
- To find the weights on stocks and bonds, multiply the weight on T by the weights that stocks and bonds have in T
  - Weight on stock 0.59 × 0.6 = 0.35 (rounding)
  - Weight on bond 0.59 × 0.4 = 0.24 (rounding)

Optimal portfolio of tangency and risk-free asset

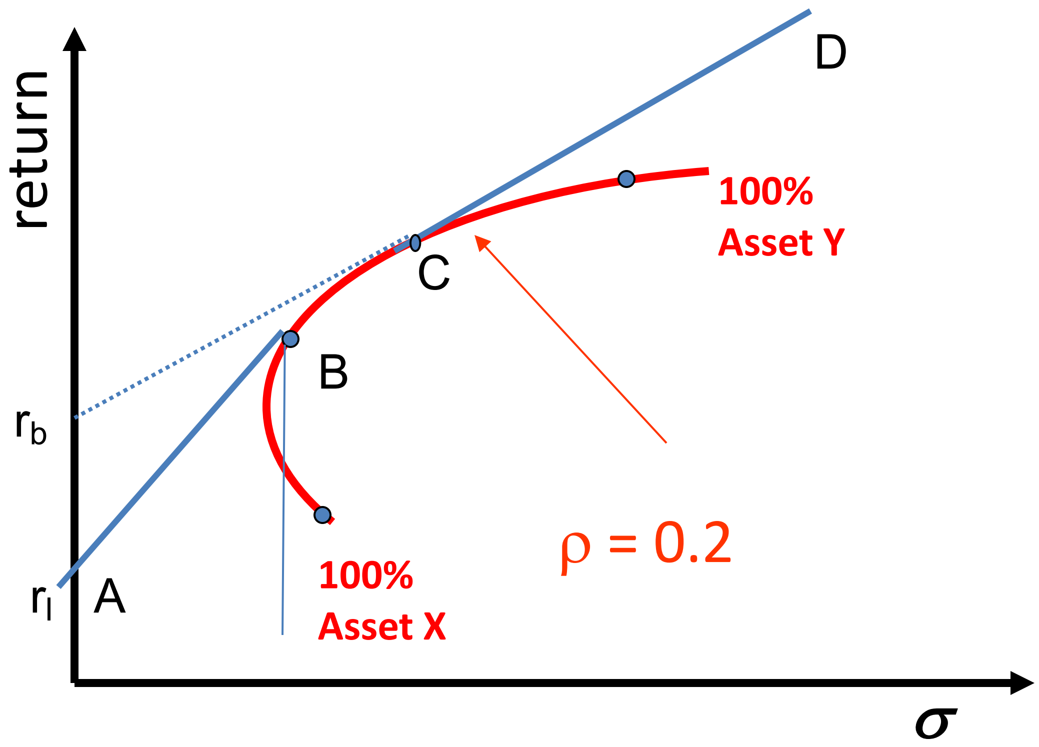


The overall optimal portfolio (O) has 59% in T and 41% in the risk free asset

Is there an unique optimal risky portfolio if the borrowing and the lending rate are different?

- A. Yes
- B. There are two optimal portfolios
- C. There an infinite number of optimal portfolios
- D. Depends on preferences

Opportunity Set when Borrowing and Lending rates are different



Efficient Portfolios: A-B-C-D

Lessons

- Same risky portfolio (tangency portfolio) chosen by all investors regardless of their risk aversion (separation property).
- Depending on risk aversion, investors choose more or less of the tangency portfolio and put the rest in the risk-free asset.
- How do we handle many risky assets?

Annualizing monthly continuously compounded expected returns and variances

If the monthly returns are continuously compounded and independent and identically distributed (iid):

$$r_a = r_1 + r_2 + r_3 + r_2 + \ldots\ldots\ldots\ldots r_{12}$$

$$\mu_a = \mu_1 + \mu_2 + \mu_3 + \mu_4 + \ldots\ldots\ldots\ldots \mu_{12} = 12\mu_m$$

$$\sigma_a^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \ldots\ldots\ldots\ldots \sigma_{12}^2 = 12\sigma_m^2$$

Annualizing monthly continuously compounded data

$$\mu_a = 12\mu_m$$

$$\sigma_a^2 = 12\sigma_m^2$$

$$\sigma_a = \sqrt{12}\sigma_m$$

$$\sigma_{xy,a} = 12\sigma_{xy,m}$$

$$\rho_a = \rho_m$$

Arbitrary Number of Assets

- Consider a portfolio of an arbitrary number *N* of assets with weight w_i in Asset *i.*
- Portfolio mean return and variance of return

$$\mu_P = \sum_{i=1}^{N} w_i \mu_i$$

$$\sum_{i=1}^{N} w_i = 1$$

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Three Assets

$$\mu_P = w_1\mu_1 + w_2\mu_2 + w_3\mu_3$$

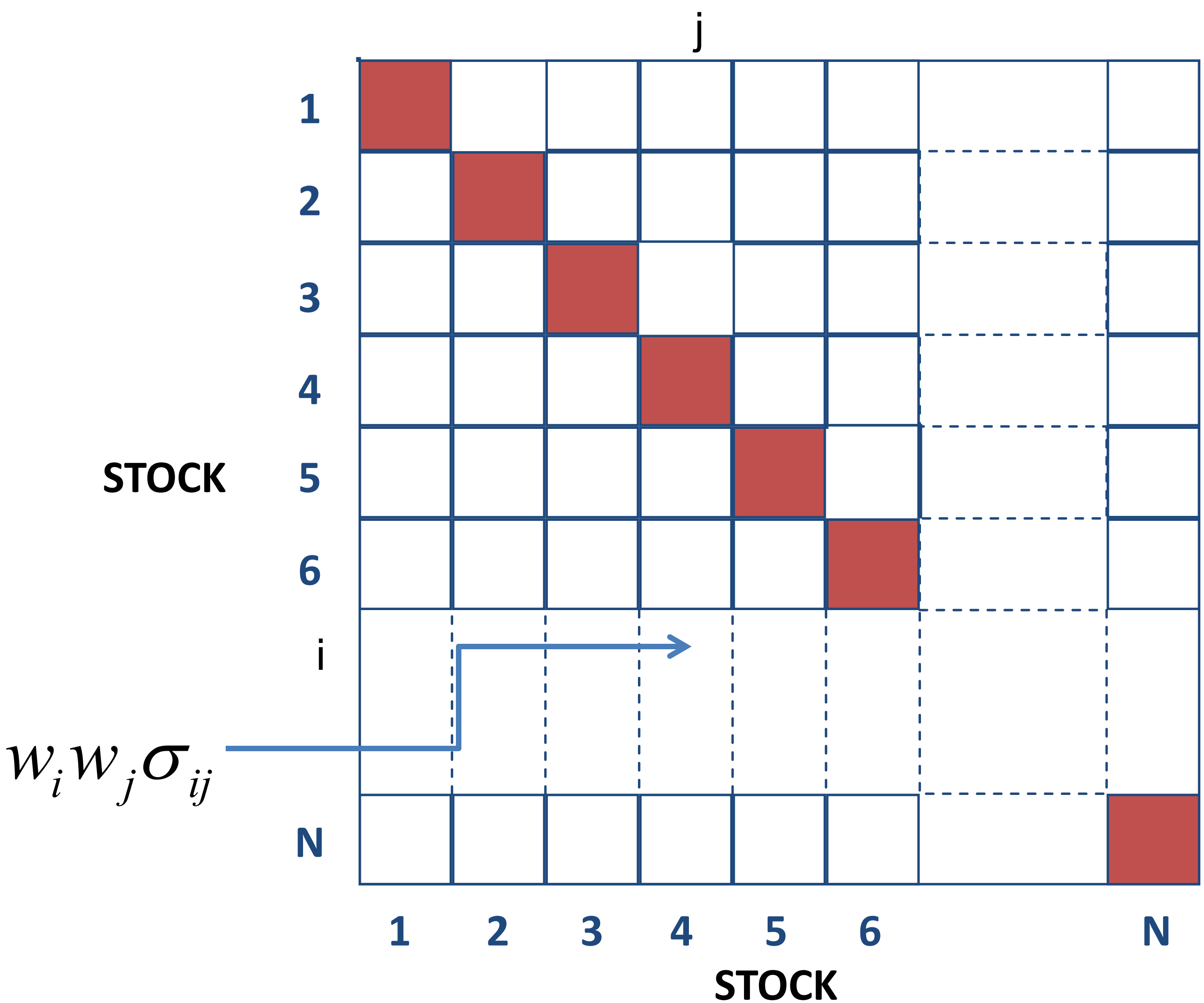
$$w_1 + w_2 + w_3 = 1$$

$$\sigma_P^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j \sigma_{ij}$$

$$\sigma_P^2 = w_1 w_1 \sigma_{11} + w_2 w_2 \sigma_{22} + w_3 w_3 \sigma_{33}$$
$$+ 2w_1 w_2 \sigma_{12} + 2w_1 w_3 \sigma_{13} + 2w_2 w_3 \sigma_{23}$$

$$\sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2$$
$$+ 2w_1 w_2 \sigma_{12} + 2w_1 w_3 \sigma_{13} + 2w_2 w_3 \sigma_{23}$$

Variance and Covariance



The shaded boxes contain variance terms; the remainder contain covariance terms.

Variance of a Portfolio

- If there are N stocks (say 100)
- The number of boxes will be N^2 (10,000)
- The number of variances will be N (100)
- Therefore, the total number of covariances will be: $N^2 – N$. (9,900)
- But, $\sigma_{i,k} = \sigma_{k,i}$ The number of different covariaces is then $(N^2 – N)/2$ (4,950)

In Matrix Notation

$\Omega : N \times N$ *Variance Covariance matrix of returns*

$\vec{\mu} : N$ *column vector of expected returns*

$\vec{w} : N$ *column vector of weights*

$\vec{1} : N$ *column vector of ones*

$$\vec{w}^{Tr}\vec{1} = 1 \quad \textit{scalar product} \quad \Rightarrow \sum w_i = 1$$

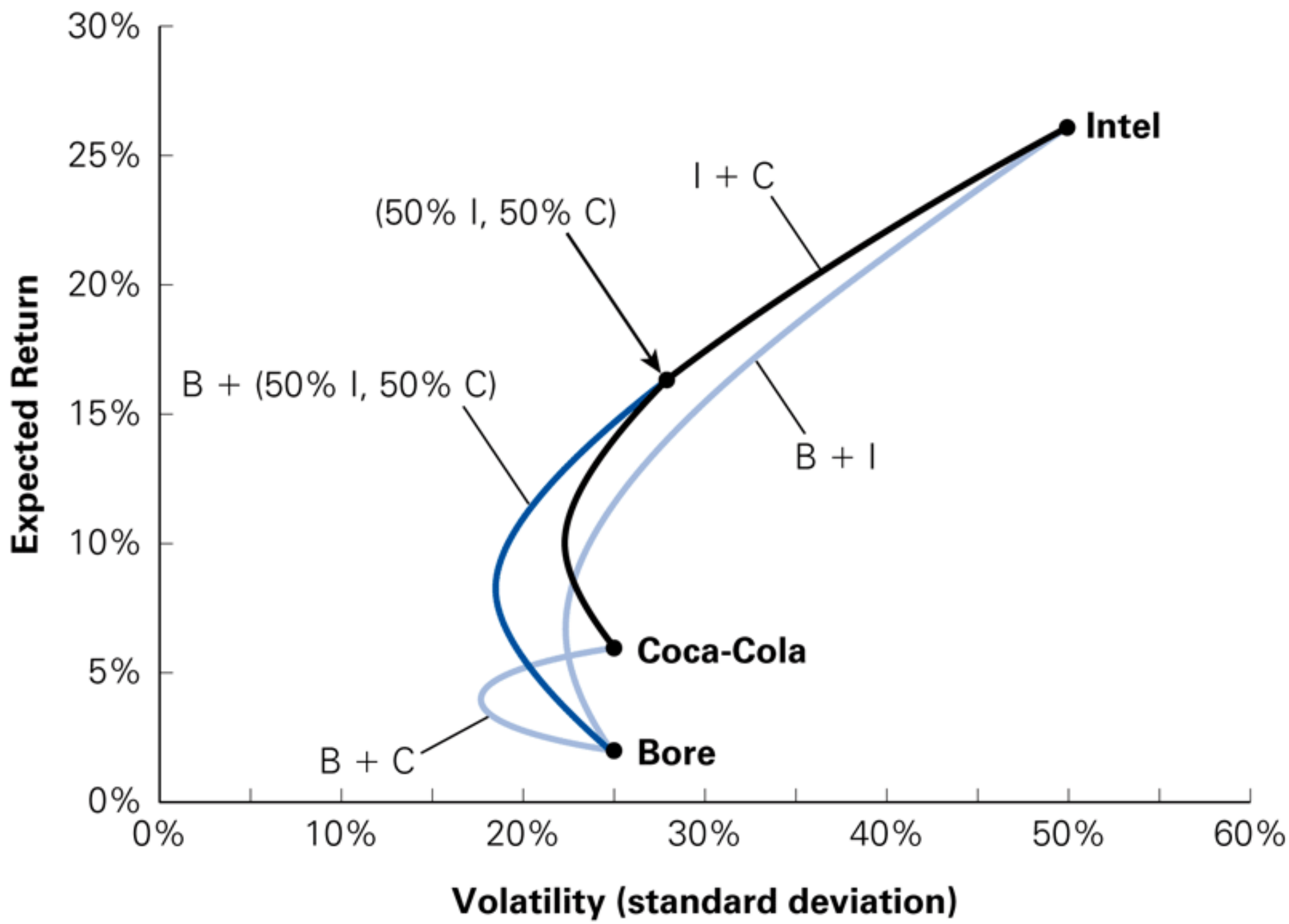
$$\mu_P = \vec{w}^{Tr}\vec{\mu} \quad \textit{scalar product} \quad \Rightarrow \mu_P = \sum w_i \mu_i$$

$$\sigma_P^2 = \vec{w}^{Tr}\Omega\vec{w} \quad \Rightarrow \sigma_P^2 = \sum\sum w_i w_j \sigma_{ij}$$

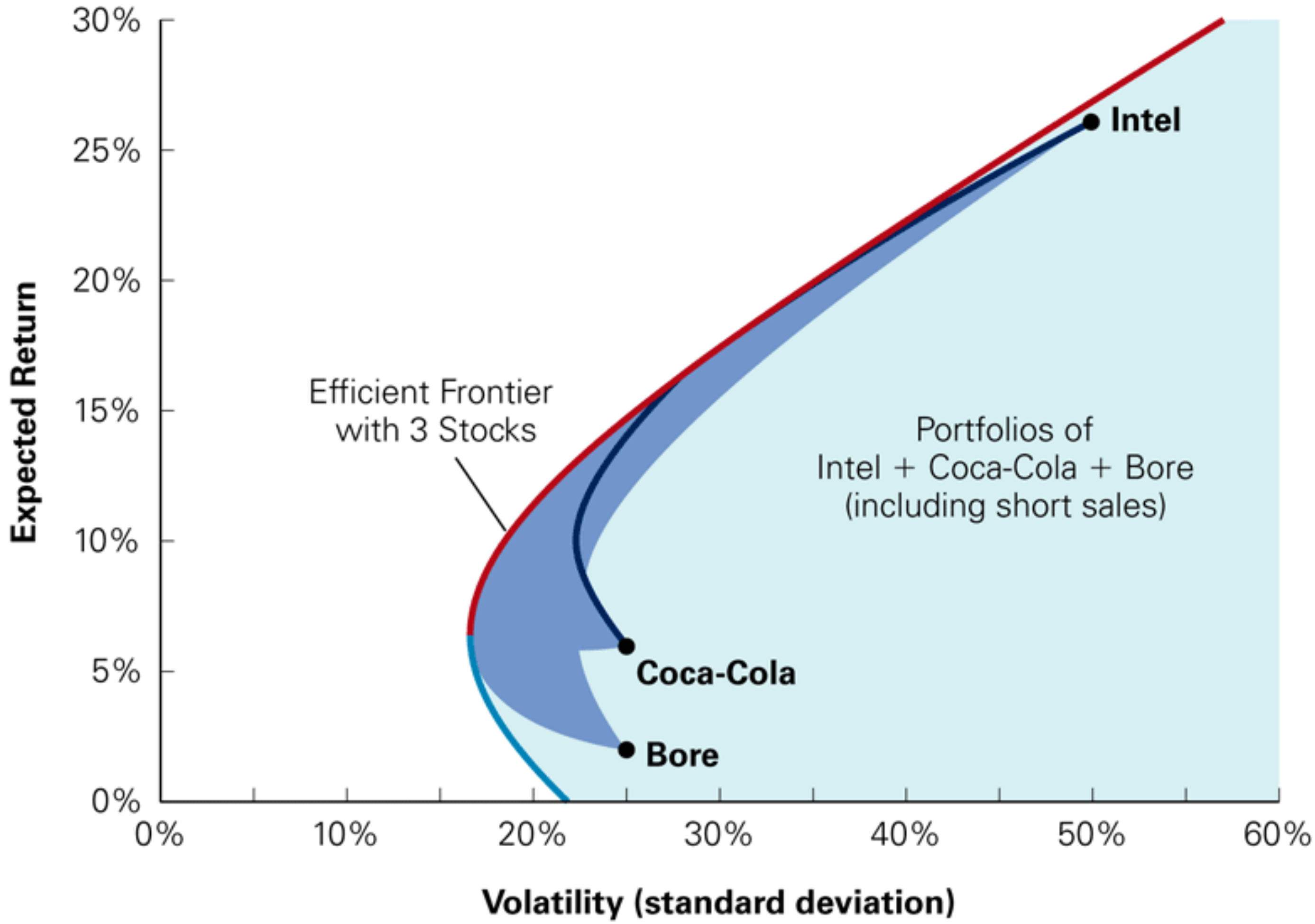
$$\Omega = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

$$\vec{w} = \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} \qquad \vec{\mu} = \begin{vmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{vmatrix} \qquad \vec{1} = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$$

Expected Return and Volatility for Selected Portfolios of Intel, Coca-Cola, and Bore Industries Stocks



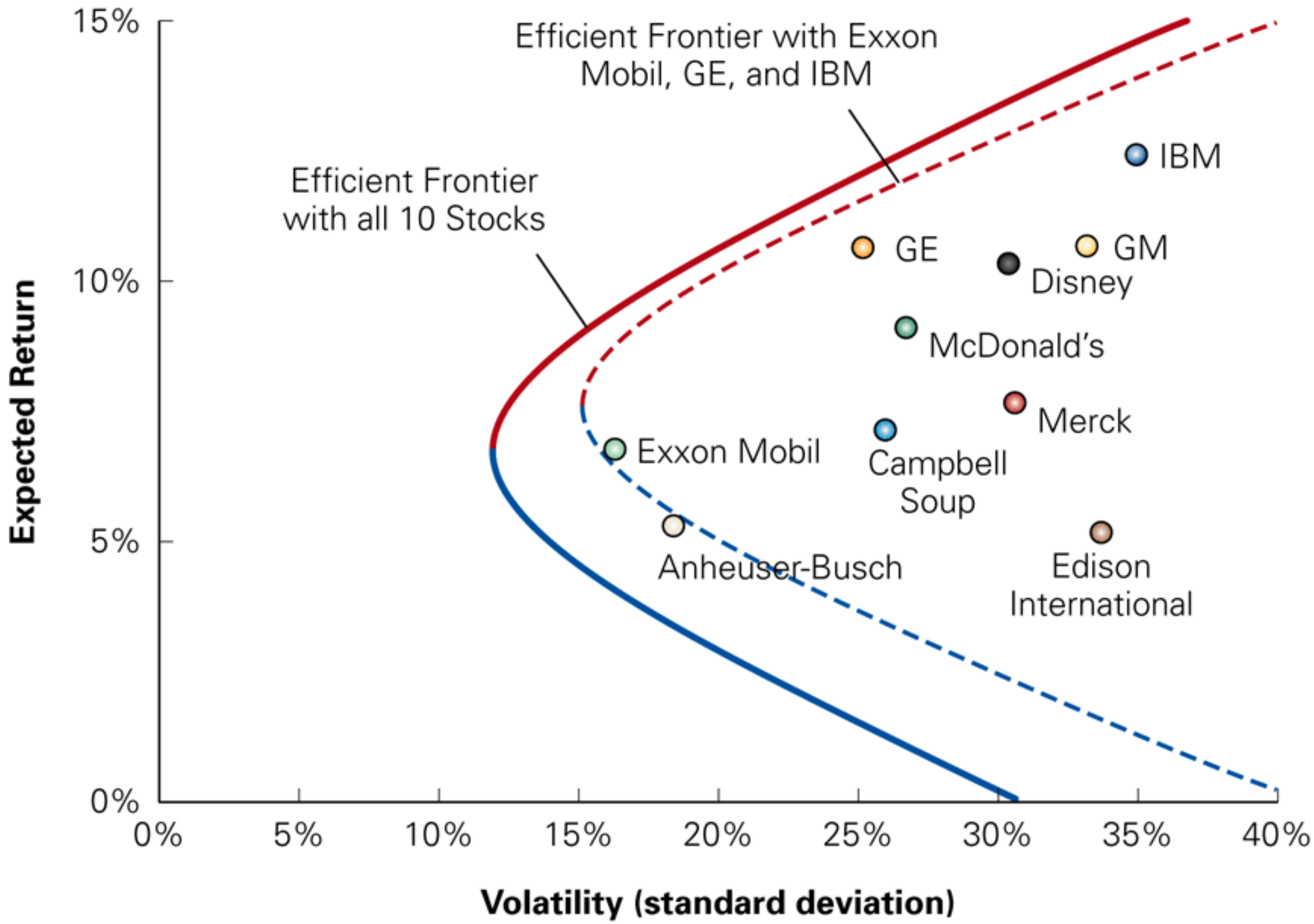
The Volatility and Expected Return for All Portfolios of Intel, Coca-Cola, and Bore Stock



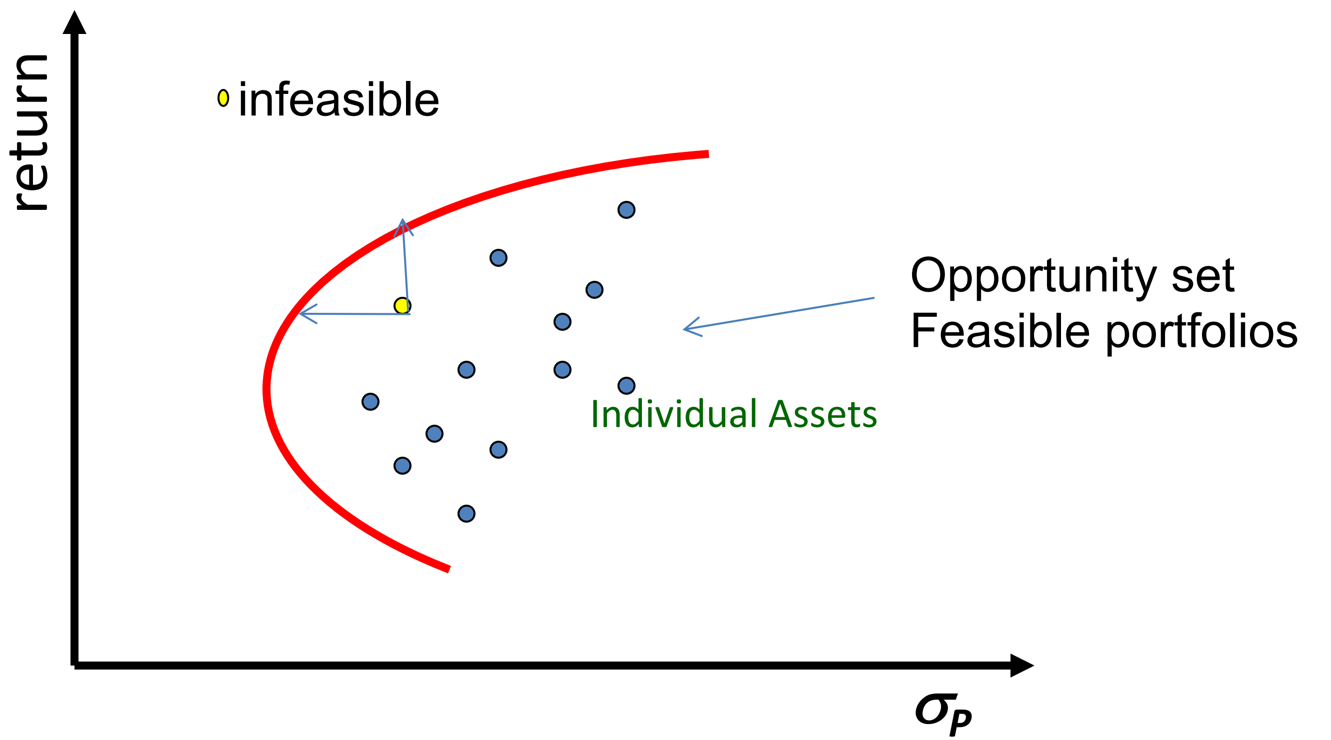
Risk Versus Return: Many Stocks

- The efficient portfolios, those offering the highest possible expected return for a given level of volatility, are those on the northwest edge of the shaded region, which is called the **efficient frontier** for these three stocks.
  - In this case none of the stocks, on its own, is on the efficient frontier, so it would not be efficient to put all our money in a single stock.

Efficient Frontier with Ten Stocks Versus Three Stocks

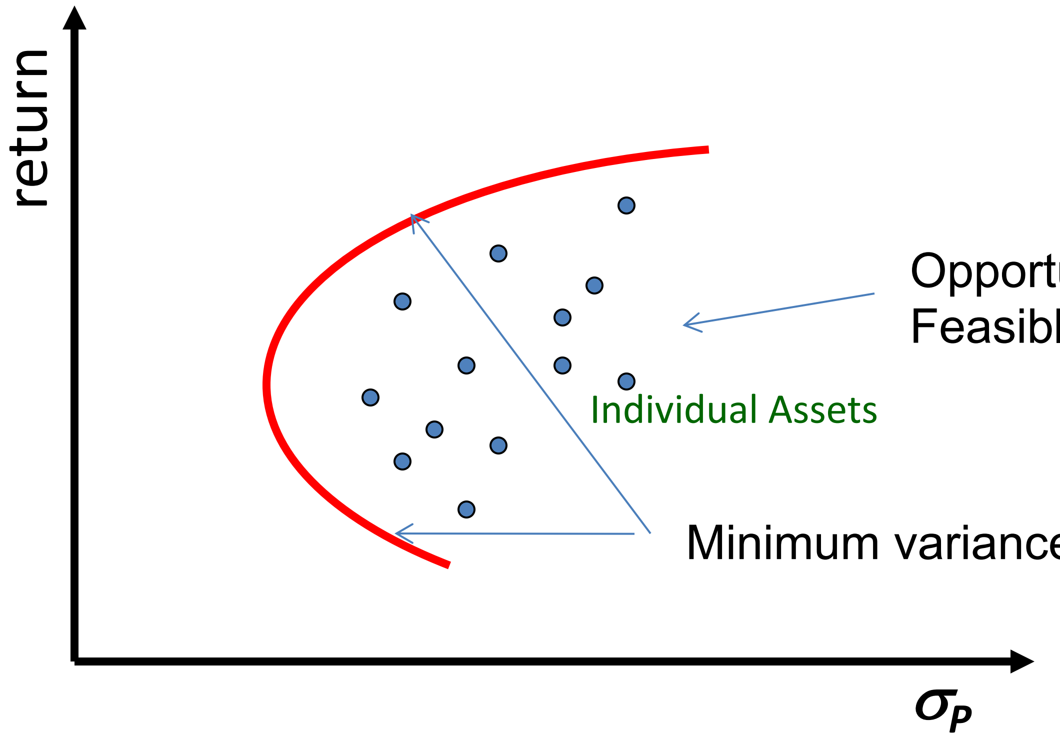


Opportunity Set for Many Securities



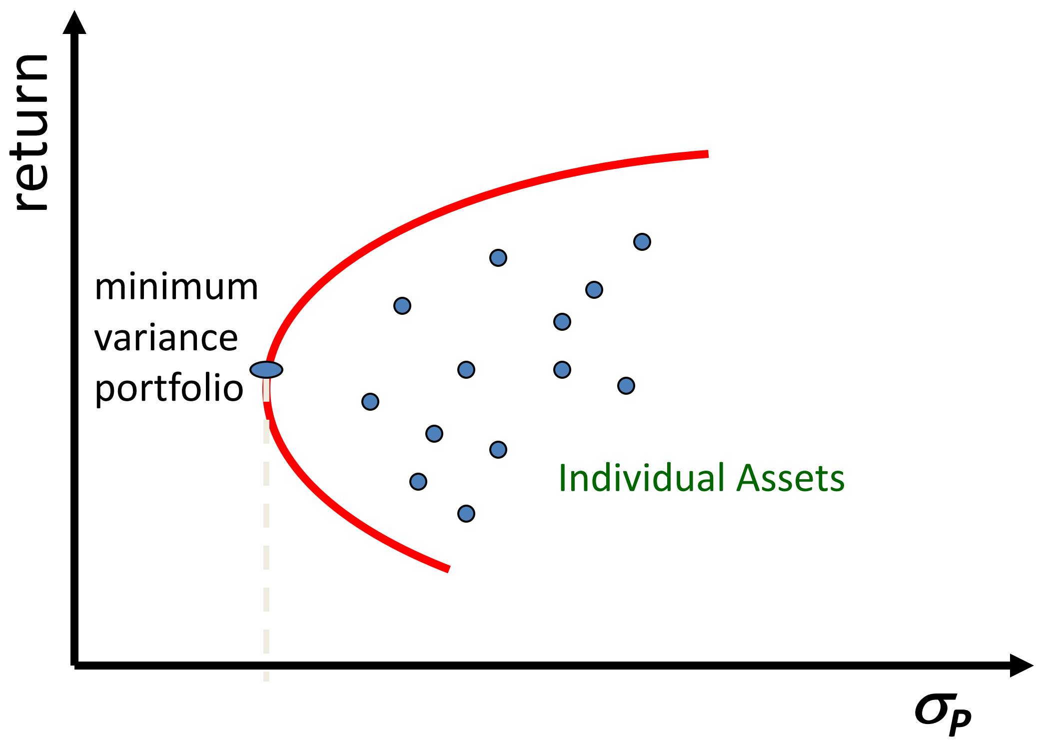
Efficient portfolios are those with the highest expected return for a given standard deviation.

Opportunity Set for Many Securities



We can still identify the *opportunity set* of risk-return combinations of various portfolios, and the envelope of this set has the same shape as before.

Minimum Variance Portfolio

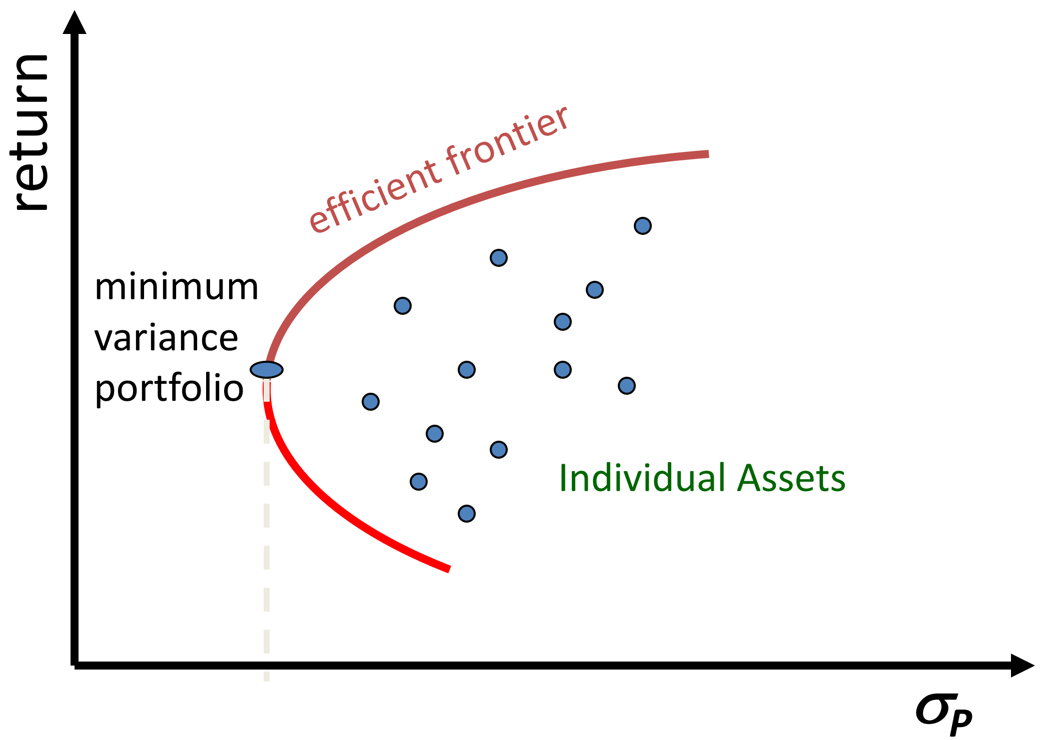


Given the *opportunity set* we can also identify the **minimum variance portfolio**. Now it has many securities.

Efficient Frontier

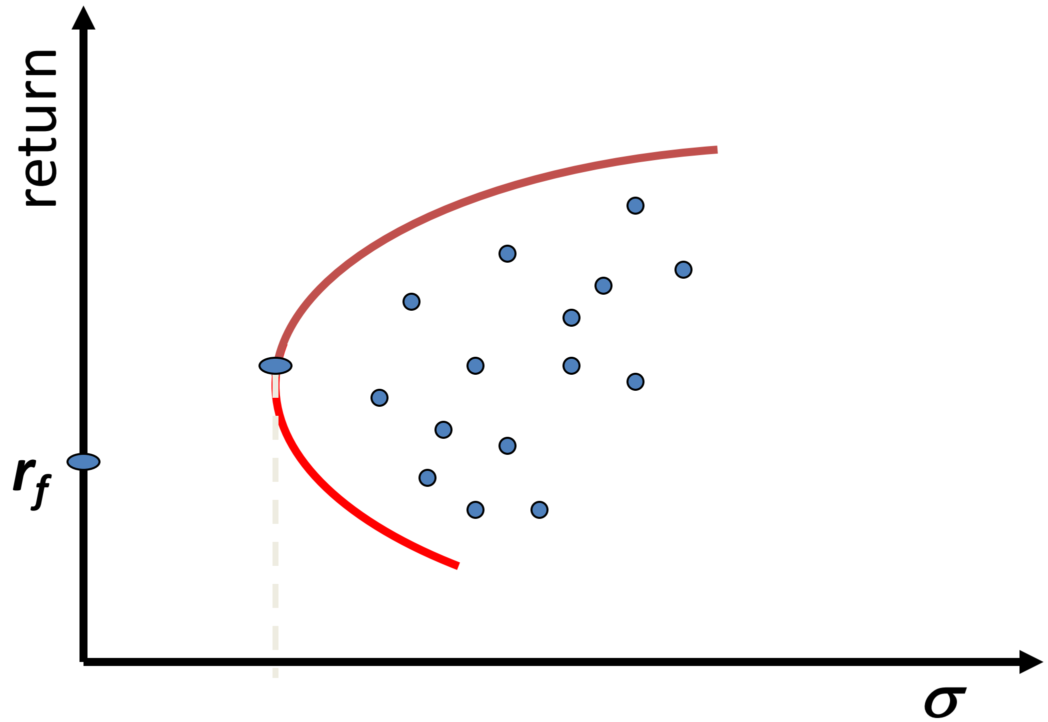
- With more than two risky assets, the investment opportunity set becomes an *area* (rather than a line)
- Efficient Portfolios:
  - Those with the highest expected return for a given variance
- Only the north-west edge of the feasible area will be the *efficient frontier*

Efficient Set for Many Securities



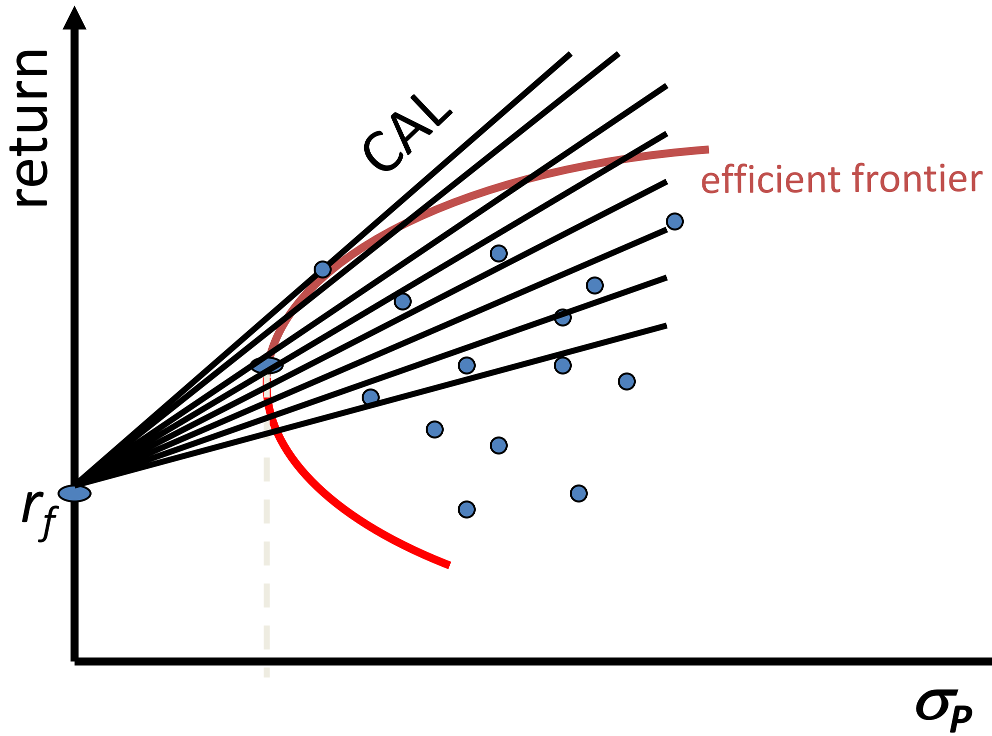
The section of the opportunity set above the minimum variance portfolio is the efficient frontier.

Optimal Risky Portfolio with a Risk-Free Asset



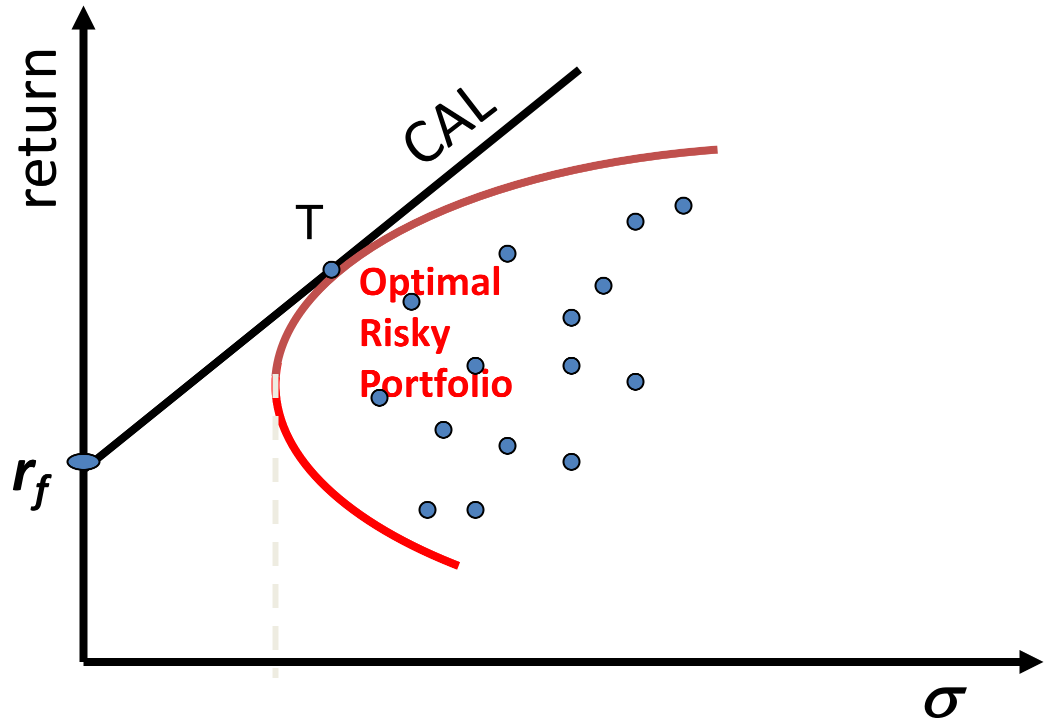
In addition to risky securities, consider a world that also has risk-free securities like T-bills

Riskless Borrowing and Lending



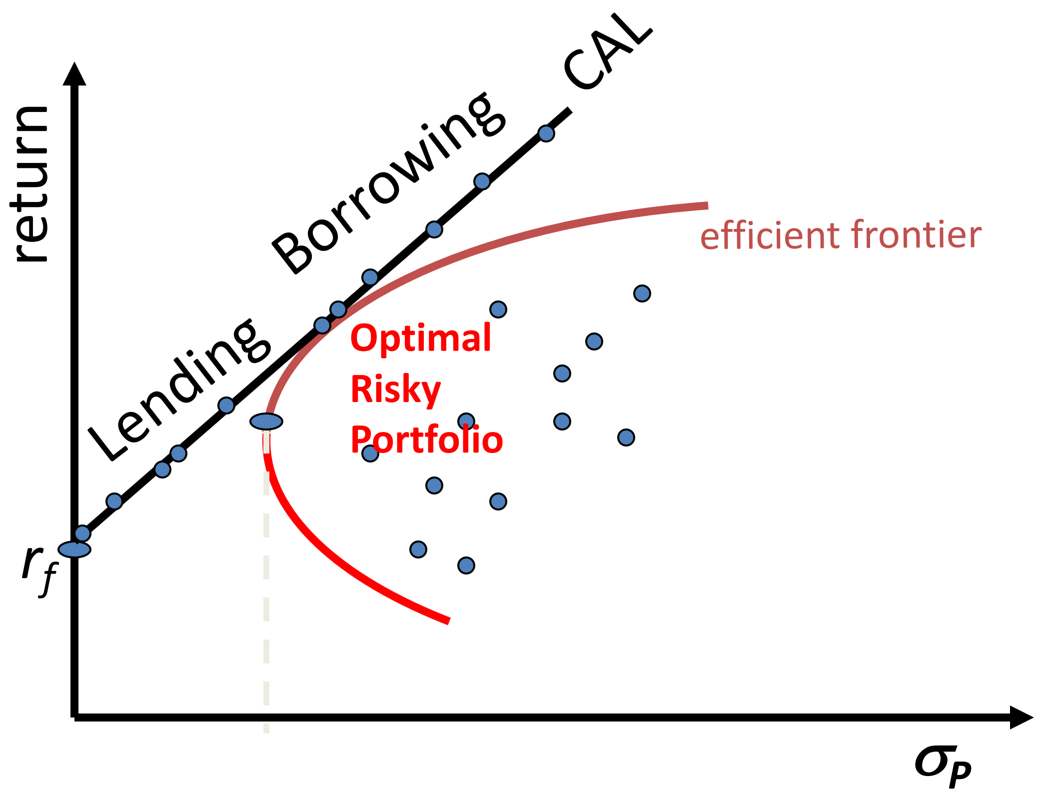
With a risk-free asset available and the efficient frontier identified, we choose the capital allocation line with the steepest slope because that line always has a portfolio with a higher return for the same risk

Riskless Borrowing and Lending



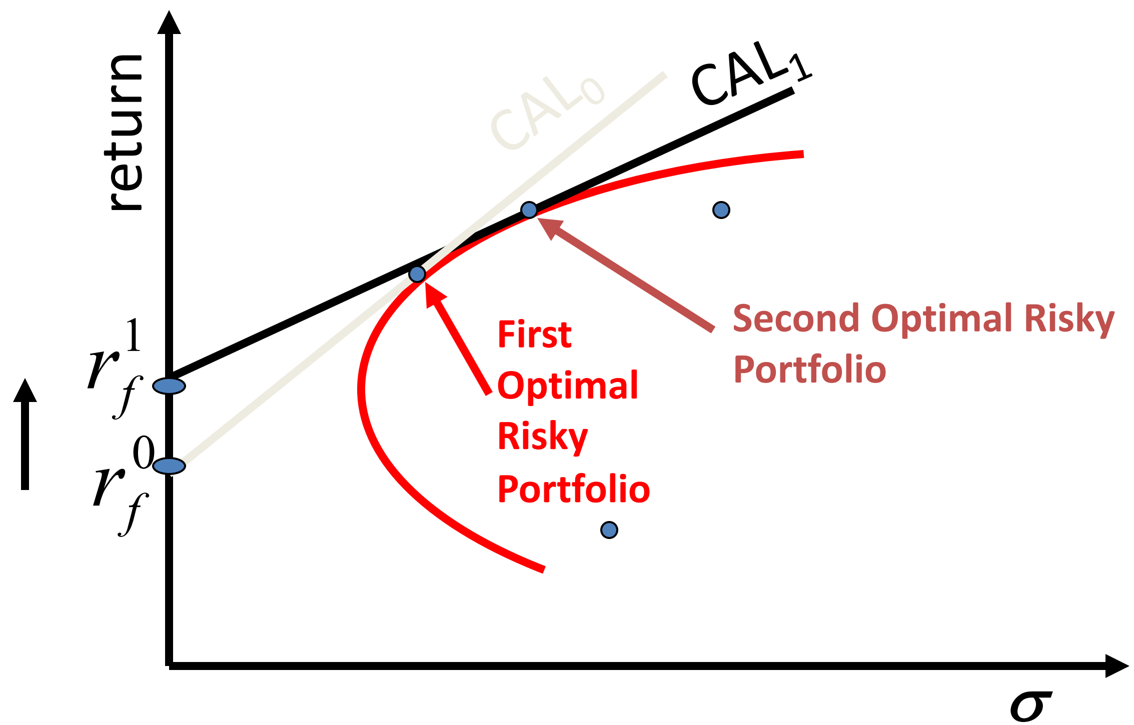
Now investors allocate their money across the T-bills and an optimal portfolio of risky securities: the tangency portfolio for a straight line starting from the risk-free rate point.

The Separation Property



Investor risk aversion is revealed in their choice of where to stay along the capital allocation line: not in their choice of different risky portfolios, or different capital allocation lines

Optimal Risky Portfolio with a Risk-Free Asset



The optimal risky portfolio depends on the risk-free rate as well as the risky assets.

Properties of Covariance: Covariance of a stock with a portfolio

- Return on stock k: r_k
- Return on portfolio P: $r_P = \sum w_i r_i$
- We are interested in the covariance between stock k and portfolio P:

- Cov(r_k, r_P)=σ_{kP}=Cov($r_k, \sum w_i r_i$)= $\sum w_i \sigma_{ki}$

Properties of Covariance:
Covariance of a stock with a portfolio

$$Cov(r_1, w_2 r_2 + w_3 r_3) = w_2 Cov(r_1, r_2) + w_3 Cov(r_1, r_3)$$
$$= w_2 \sigma_{12} + w_3 \sigma_{13}$$

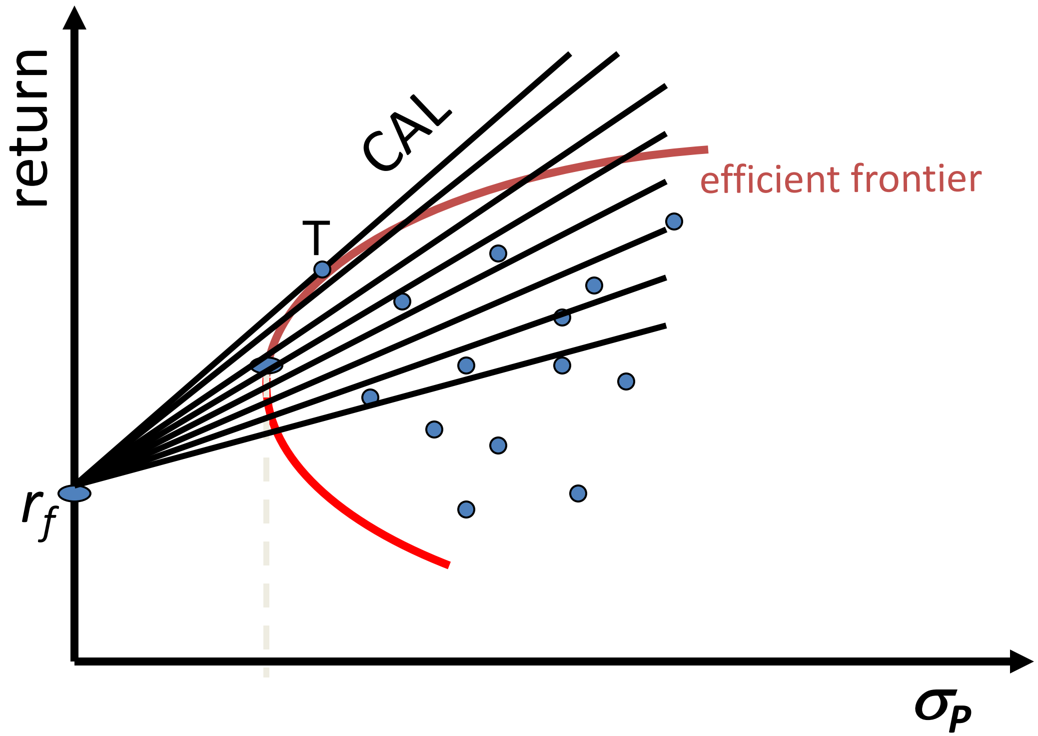
$$Cov(r_1, w_1 r_1 + w_2 r_2) = w_1 \sigma_1^2 + w_2 \sigma_{12}$$

In general :

$$Cov(r_k, r_P) = \sigma_{kP} = Cov(r_k, \sum_{i=1}^{N} w_i r_i) = \sum_{i=1}^{N} w_i \sigma_{ki}$$

The cov of a stock with a portfolio is equal to the weighted average of the cov between the stock and each stock in the portfolio

Determining Tangency Portfolio



With a risk-free asset available and the efficient frontier identified, we choose the capital allocation line with the steepest slope because that line always has a portfolio with a higher return for the same risk. We want to find the feasible w with the highest possible Sharp Ratio.

Determining Tangency Portfolio

- Tangency portfolio will be the one with the highest Sharp Ratio
  - Slope of the investment opportunity set

$$\sum_i w_i r_f = r_f$$

$$\max_{\{w_k\}} SR_P = \max_{\{w_k\}} \frac{\mu_p - r_\mathrm{f}}{\sigma_\mathrm{p}} = \max_{\{w_k\}} \frac{\sum_i w_i \left(\mu_i - r_\mathrm{f}\right)}{\left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{1/2}}$$

Portfolio mean return and variance of return

$$\mu_P = \sum_{i=1}^{N} w_i \mu_i$$

$$\sum_{i=1}^{N} w_i = 1$$

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Tangency portfolio

- First-order conditions for any asset k

$$\frac{\partial SR}{\partial w_k} = 0 \quad k = 1, \cdots N \quad , \quad \sum_{k=1}^{N} w_k = 1$$

Two rules from Calculus

Product rule (derivative of a product):

(f g)'= f' g + f g'

Chain rule:

z=f(y) and y=g(x)

dz/dx =f'(y) g'(x)=dz/dy dy/dx

$$SR_P = \frac{\mu_p - r_\mathrm{f}}{\sigma_\mathrm{p}}$$

$$SR = \sum_i w_i(\mu_i - r_\mathrm{f}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij} \right)^{-1/2}$$

Technical Note

$$\max_{\{w_k\}} SR_P = \max_{\{w_k\}} \frac{\mu_p - r_{\mathrm{f}}}{\sigma_{\mathrm{p}}} = \max_{\{w_k\}} \frac{\sum_i w_i(\mu_i - r_{\mathrm{f}})}{\left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{1/2}}$$

$$SR = \sum_i w_i(\mu_i - r_{\mathrm{f}}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1/2}$$

$$\frac{\partial SR}{\partial w_k} = (\mu_k - r_{\mathrm{f}}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1/2} + \sum_i w_i(\mu_i - r_{\mathrm{f}}) \times (-\frac{1}{2}) \times$$

$$\left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-3/2} \times 2 \times \sum_j w_j \sigma_{kj} = 0$$

$$(\mu_k - r_{\mathrm{f}}) - \sum_i w_i(\mu_i - r_{\mathrm{f}}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1} \times \sum_j w_j \sigma_{kj} = 0$$

First order condition

$$\left(\mu_k - r_{\mathrm{f}}\right) = \sum_i w_i \left(\mu_i - r_{\mathrm{f}}\right) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1} \times \sum_j w_j \sigma_{kj}$$

$$\frac{\mu_k - r_{\mathrm{f}}}{\sum_j w_j^* \sigma_{kj}} = \frac{\sum_i w_i^* \left(\mu_i - r_{\mathrm{f}}\right)}{\sum_i \sum_j w_i^* w_j^* \sigma_{ij}} = \frac{\mu_T - r_{\mathrm{f}}}{\sigma_T^2} \quad , \quad k = 1 \cdots N$$

$$\Leftrightarrow \frac{\mu_k - r_{\mathrm{f}}}{\sigma_{kT}} = \frac{\mu_i - r_{\mathrm{f}}}{\sigma_{iT}} = \frac{\mu_j - r_{\mathrm{f}}}{\sigma_{jT}} = \frac{\mu_T - r_{\mathrm{f}}}{\sigma_T^2} = \frac{1}{K}$$

FOC

$$\frac{\mu_k - r_\mathrm{f}}{\sum_j w_j^* \sigma_{kj}} = \frac{\sum_i w_i^* \left(\mu_i - r_\mathrm{f}\right)}{\sum_i \sum_j w_i^* w_j^* \sigma_{ij}}$$

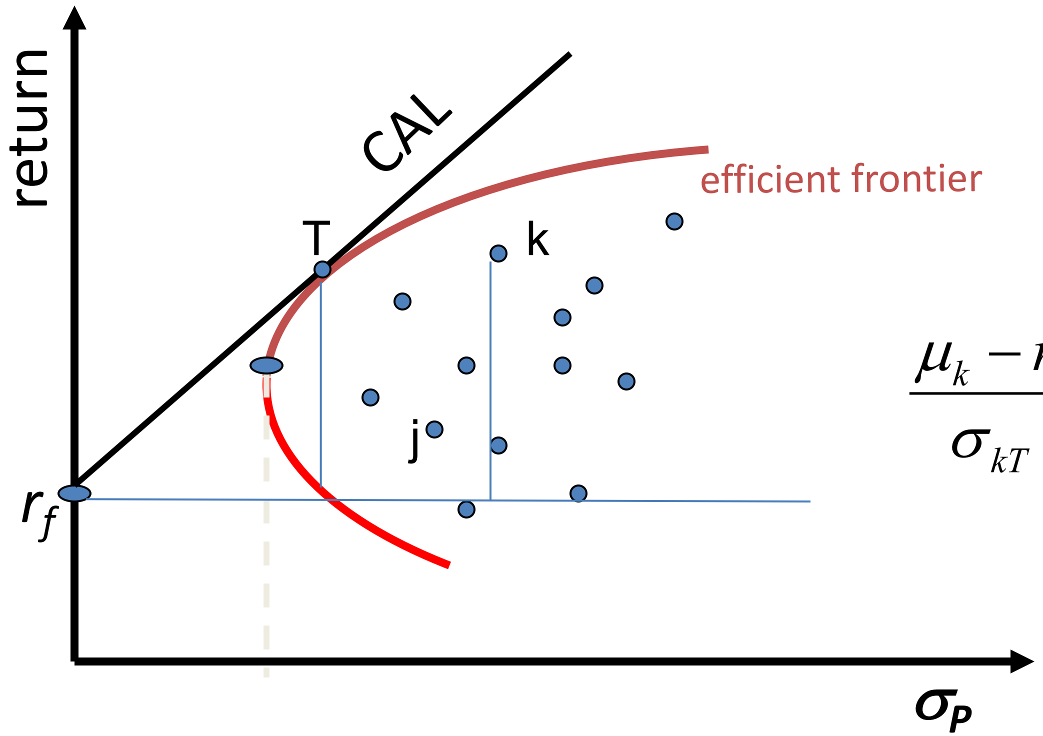
$$\sum_i w_i^* \left(\mu_i - r_\mathrm{f}\right) = \mu_T - r_\mathrm{f}$$

$$\sum_i \sum_j w_i^* w_j^* \sigma_{ij} = \sigma_T^2$$

$$\sum_j w_j^* \sigma_{kj} = \sigma_{kT}$$

$$\Leftrightarrow \frac{\mu_k - r_\mathrm{f}}{\sigma_{kT}} = \frac{\mu_T - r_\mathrm{f}}{\sigma_T^2} \quad , \quad k = 1 \cdots N$$

Property of the Tangency Portfolio



$$\frac{\mu_k - r_f}{\sigma_{kT}} = \frac{\mu_T - r_f}{\sigma_T^2}$$

The excess return on any stock (k) divided by its covariance with the tangency portfolio (T) is constant for all stocks and equal to the excess return on T divided by its variance!

Tangency portfolio weights

- We can rewrite the first-order conditions as

$$\sigma_{kT} = (\mu_k - r_\mathrm{f})\frac{\sigma_T^2}{\mu_T - r_\mathrm{f}}$$

$$\frac{\mu_k - r_\mathrm{f}}{\sigma_{kT}} = \frac{\mu_T - r_\mathrm{f}}{\sigma_T^2}$$

$$\Leftrightarrow \sum_j w_j \sigma_{kj} = (\mu_k - r_\mathrm{f})\frac{\sigma_T^2}{\mu_T - r_\mathrm{f}}$$

$$\Leftrightarrow w_1\sigma_{k1} + w_2\sigma_{k2} + \ldots + w_N\sigma_{kN} = (\mu_k - r_\mathrm{f}) \times \mathrm{K}$$

- Where the constant K is the same for all assets and ensures that weights add up to one

Tangency portfolio weights

- The first order conditions define a system of equations that can be solved for the portfolio weights of the tangency portfolio (set K=1)

$$
\begin{aligned}
&w_1\sigma_{11} + w_2\sigma_{12} + \ldots + w_N\sigma_{1N} = (\mu_1 - r_{\mathrm{f}}) \\
&w_1\sigma_{21} + w_2\sigma_{22} + \ldots + w_N\sigma_{2N} = (\mu_2 - r_{\mathrm{f}}) \\
&\ldots \\
&w_1\sigma_{N1} + w_2\sigma_{N2} + \ldots + w_N\sigma_{NN} = (\mu_N - r_{\mathrm{f}})
\end{aligned}
$$

- Need to rescale the solution so the weights add up to one

In Matrix Notation

$$\vec{\mu}-\vec{r}_f : \quad vector\, of\ excess\, returns$$

$$\Omega\vec{w}_T \propto (\vec{\mu}-\vec{r}_f)$$

$$\vec{w}_T \propto \Omega^{-1}(\vec{\mu}-\vec{r}_f)$$

$$weights\, need\, to\, add\, to\, 1$$

$$\vec{w}_T = \frac{\Omega^{-1}(\vec{\mu}-\vec{r}_f)}{\vec{1}^{tr}\Omega^{-1}(\vec{\mu}-\vec{r}_f)}$$

$$\vec{1}^{tr}\,\vec{w}_T = \frac{\vec{1}^{tr}\Omega^{-1}(\vec{\mu}-\vec{r}_f)}{\vec{1}^{tr}\Omega^{-1}(\vec{\mu}-\vec{r}_f)} = 1$$

Determining Tangency Portfolio

- Consider 3 risky stocks and a risk-free asset.
- Suppose expected returns of Stocks A, B and C are 14%, 8% and 20%, and the associated standard deviations are respectively 6%, 3% and 15%. Let the correlation coefficients be A-B 0.5, A-C 0.2, and B-C 0.4.
- The risk-free rate is 5%
- First, estimate the Variance-Covariance Matrix

Data of Problem

$$\mu_A = 14\% \quad \mu_B = 8\% \quad \mu_C = 20\%$$

$$\sigma_A = 6\% \quad \sigma_B = 3\% \quad \sigma_C = 15\%$$

$$\rho_{AB} = 0.5 \quad \rho_{AC} = 0.2 \quad \rho_{BC} = 0.4$$

$$r_f = 5\%$$

Variance-Covariance Matrix

36	9	18
9	9	18
18	18	225

Determining Tangency Portfolio

- We need to solve the system of 3 equations with three unknowns

$$w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13} = (\mu_1 - r_{\mathrm{f}})$$
$$w_1\sigma_{21} + w_2\sigma_{22} + w_3\sigma_{23} = (\mu_2 - r_{\mathrm{f}})$$
$$w_1\sigma_{31} + w_2\sigma_{32} + w_3\sigma_{33} = (\mu_3 - r_{\mathrm{f}})$$
$$\Omega\vec{w}_T = (\vec{\mu} - \vec{r}_f) \quad \textit{in vector / matrix notation}$$
$$\vec{w}_T = \Omega^{-1}(\vec{\mu} - \vec{r}_f)$$

- Then we need to rescale the solution so the weights add up to one

Determining Tangency Portfolio

- Since the risk free rate is 5%

$$36w_1 + 9w_2 + 18w_3 = 14 - 5 = 9$$

$$9w_1 + 9w_2 + 18w_3 = 8 - 5 = 3$$

$$18w_1 + 18w_2 + 225w_3 = 20 - 5 = 15$$

$$w_1 = \frac{14}{63} \quad , \quad w_2 = \frac{1}{63} \quad , \quad w_3 = \frac{3}{63}$$

$$\textit{finally we normalize the weights to add to } 1 \, (\textit{mult by } \frac{63}{18})$$

$$w_1 = \frac{14}{18} \quad , \quad w_2 = \frac{1}{18} \quad , \quad w_3 = \frac{3}{18}$$

For the Tangency Portfolio

- Recall from the first order conditions, that for every asset k:

$$\frac{\mu_k - r_{\mathrm{f}}}{\sigma_{kT}} = \frac{\mu_T - r_{\mathrm{f}}}{\sigma_T^2} \quad , \quad k = 1 \cdots N$$

$$\mu_k - r_{\mathrm{f}} = \frac{\sigma_{kT}}{\sigma_T^2}\left(\mu_T - r_{\mathrm{f}}\right)$$

$$\mu_k = r_{\mathrm{f}} + \frac{\sigma_{kT}}{\sigma_T^2}\left(\mu_T - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

For the Tangency Portfolio

$$\mu_k = r_{\mathrm{f}} + \frac{\sigma_{kT}}{\sigma_T^2}\left(\mu_T - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

$$let \quad \beta_{kT} = \frac{\sigma_{kT}}{\sigma_T^2}$$

$$\mu_k = r_{\mathrm{f}} + \beta_{kT}\left(\mu_T - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

Risk premium

Relevant measure for risk is the covariance with the tangency portfolio.
Beta is a standardized measure of covariance.
This is a mathematical relation.

For the Tangency Portfolio

- This relation also applies to portfolios

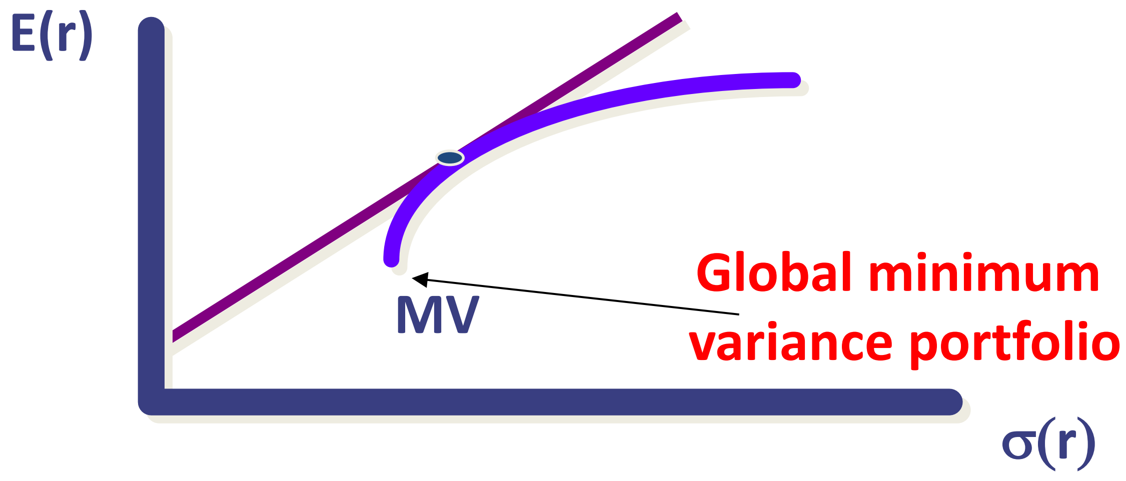
$$\mu_k = r_{\mathrm{f}} + \beta_{kT}\left(\mu_T - r_{\mathrm{f}}\right)$$
$$\sum_k w_k \mu_k = \sum_k w_k r_{\mathrm{f}} + \sum_k w_k \beta_{kT}\left(\mu_T - r_{\mathrm{f}}\right) = r_{\mathrm{f}} \sum_k w_k + \left(\mu_T - r_{\mathrm{f}}\right)\sum_k w_k \beta_{kT}$$
$$\mu_P = r_{\mathrm{f}} + \beta_{PT}\left(\mu_T - r_{\mathrm{f}}\right)$$

- The beta of a portfolio is a weighted average of the betas of the individual assets.

$$\beta_{PT} = \sum_k w_k \beta_{kT}$$

Minimum variance portfolio (MV)

- Has the lowest possible variance
- Weights also solve a system of equations
- Well diversified portfolio



Technical Note

$$\sigma_P^2 = \vec{w}^{Tr}\Omega\vec{w}$$

$$\vec{w}^{Tr}\mathbf{1} = 1$$

$$\min_w L = \vec{w}^{Tr}\Omega\vec{w} - 2\lambda(\vec{w}^{Tr}\mathbf{1} - 1)$$

$$\frac{\partial L}{\partial \vec{w}} = 2\Omega\vec{w} - 2\lambda\mathbf{1} = 0$$

$$\vec{w} = \lambda\Omega^{-1}\mathbf{1}$$

$$\vec{w}_{MVP} \propto \Omega^{-1}\mathbf{1} \quad \Rightarrow \quad \vec{w}_{MVP} = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^{\mathbf{tr}}\Omega^{-1}\mathbf{1}}$$

MVP and Tangency – matrix solution

- Systems of equations for T and MVP better solved with matrix algebra

$$\vec{w}_{MVP} \propto \Omega^{-1}\mathbf{1}$$

$$\vec{w}_T \propto \Omega^{-1}\left(\vec{\mu} - \vec{r}_f\right)$$

Ω : Covariance matrix

μ-r_f : Vector of mean excess returns

1 : Vector of ones

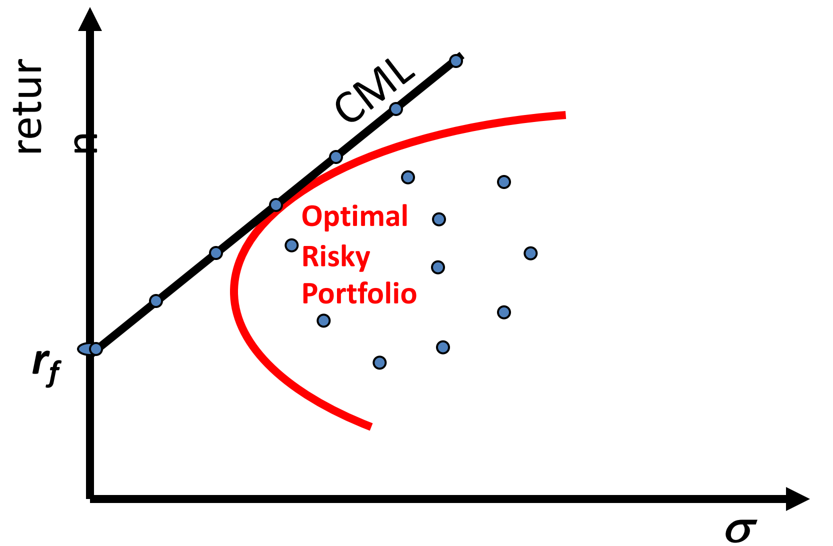
We still need to normalize weights so they add up to one

- Once we have two efficient portfolios, we are back to the two asset case

Capital Market Equilibrium

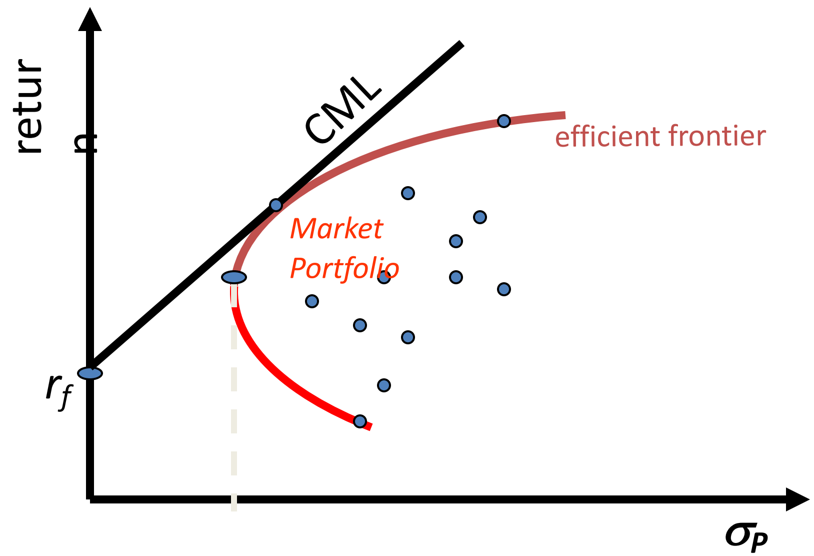
- We learned how to estimate the efficient frontier and the optimal risky portfolio (tangency portfolio) when we have N risky assets.
- Now we want to extend this framework to find out how expected returns are determined in equilibrium.
- For that we will assume that ALL investors have the same expectations about returns and covariances and that to form their portfolio they consider ALL existing securities.

Market Equilibrium



If all investors have the same expectations about risk and return, all investors will have the same CAL because they all hold the same optimal risky portfolio given the risk-free rate. This CAL is the *Capital Market Line (CML).*

Market Equilibrium



With the capital market line identified, all investors choose a point along the line: some combination of the risk-free asset and the common optimal risky portfolio. In equilibrium this portfolio **has to be** the market portfolio *M*.

Intuition

- Every investor solves mean-variance problem and holds a combination of risk-free asset and portfolio of risky assets (tangency portfolio)
- With homogeneous expectations, tangency portfolio is the same for all investors
- In equilibrium the sum of all investors’ desired portfolios must equal the supply of assets
- Aggregate supply of asset is the market portfolio
- Market portfolio has to be the tangency portfolio

Intuition

- If everyone in the economy holds an efficient portfolio, how should securities be priced so that demand equals supply?
- If, for given expected returns, variances, and covariances, no investor wants to hold Xerox, what will happen?
  - Price (and expected return) of Xerox needs to adjust
- Equilibrium
  - Every investor is happy with her portfolio
  - Supply of assets is equal to demand for assets

Market Equilibrium

- The market portfolio is the same for all investors. *The Separation Property* means that investors can *separate* their risk aversion from their choice of the market portfolio.
- If all investors have the same set of expectations, they will choose the same optimal risky portfolio. All investors will always hold this portfolio. In market equilibrium, demand will equal supply. This optimal risky portfolio will then be the market portfolio, the value-weighted portfolio of all risky assets.
- Then with borrowing or lending, the investor selects a point along the CML, the Capital Market Line.

The Market Portfolio

- Everybody will want to invest in Portfolio M and borrow or lend to be somewhere on the CML
  - Therefore this portfolio must include all risky assets (or else some assets would have no demand)

- Because the market is in equilibrium, all assets are included in this portfolio in proportion to their market value

The Market Portfolio

- Firm i number of shares: Ni
- Price per share: Pi
- Market capitalization of firm i: Ni Pi

- Market capitalization of the whole market:

$$\sum Ni\ Pi$$

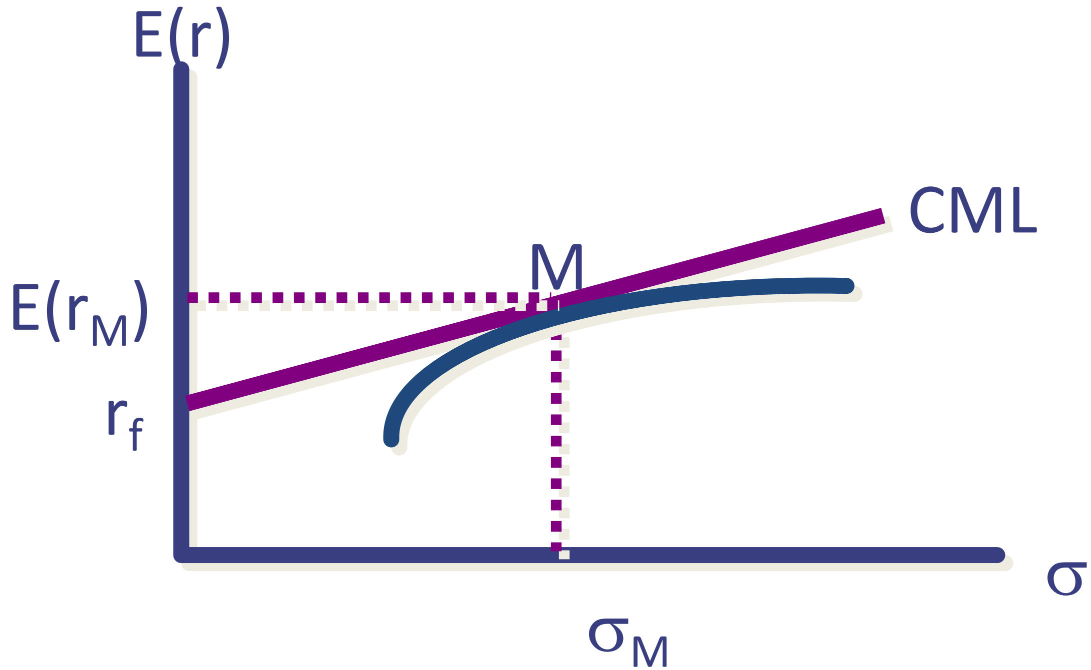
The Market Portfolio

- Portfolio of all risky assets that exist in the economy
- Portfolio weights

$$w_{Mi} = \frac{N_i \times P_i}{\sum_j N_j \times P_j}$$

  where N_i is the number of shares and P_i the market price of company i
- In theory it should include all assets
  - Real estate
  - Human capital

Capital Market Line



Capital Market Line

- The CML is a straight line with intercept r_f and slope equal to the Sharpe Ratio of the market portfolio

$$Slope = \frac{E[r_M] - r_f}{\sigma_M}$$

- The equation for the CML

$$E[r_P] = r_f + \frac{E[r_M] - r_f}{\sigma_M}\sigma_P$$

- All efficient portfolios lie on the CML

For Market (Tangency) Portfolio

- For the true market portfolio M, it must be true that for each individual asset or portfolio

$$\mu_k = r_{\mathrm{f}} + \frac{\sigma_{kM}}{\sigma_M^2}\left(\mu_M - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

$$let \quad \beta_k = \frac{\sigma_{kM}}{\sigma_M^2}$$

$$\mu_k = r_{\mathrm{f}} + \beta_k\left(\mu_M - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

- The relevant measure of risk is the covariance of the asset or portfolio with the market (beta)

CAPM and SML

- This is the capital asset pricing model (CAPM)

$$\mu_k = r_\mathrm{f} + \beta_k\left(\mu_M - r_\mathrm{f}\right)$$
$$E[r_k] = r_\mathrm{f} + \beta_k\left(E[r_M] - r_\mathrm{f}\right)$$

- A plot of this relation is called the Security Market Line (SML)
- Gives the price of risk for individual assets and portfolios
- Just as the CML gives the price of risk for efficient portfolios

Security Market Line