Topic 13: Introduction to Options

A Brief Review of Options: Terminology

- **Call** gives its holder the right to purchase an asset for a specified price (**exercise or strike price**) on or before some specified date (**expiration or maturity date**)
- **Put** gives its holder the right to sell an asset for a specified price on or before some specified date
- Options are contracts between two counterparties (zero sum game, zero net supply)
  - The buyer – takes a long position
  - The seller, or writer – takes a short position

Terminology

- **American** option can be exercised at any time before expiration or maturity
- **European** option can only be exercised on the expiration or maturity date

- American options are more valuable
- Most exchange-traded options are American

Types of options

- Stock Options
- Index Options
- Futures Options
- Foreign Currency Options
- Interest Rate Options
- Commodity Options

- Options can be customized (over-the-counter) or exchange traded
  - CBOE, CME are big exchanges

Notation

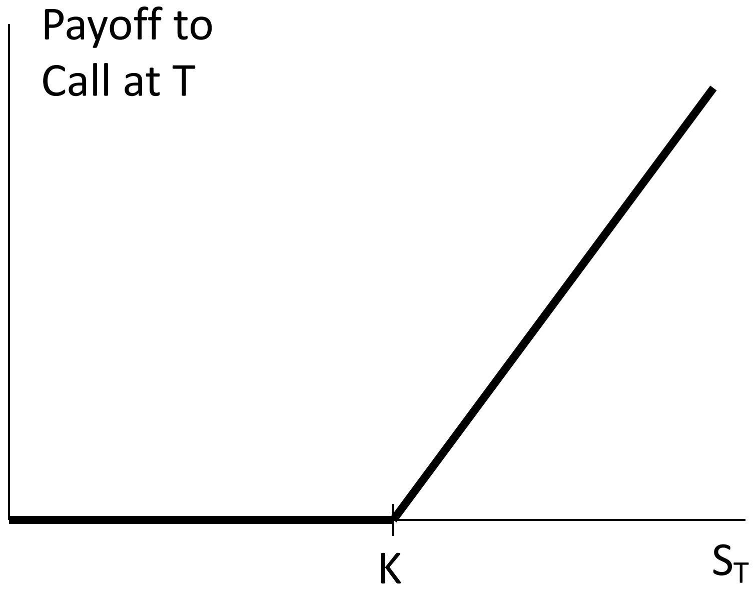
• Current date	0
• Expiration date	T
• Price of underlying asset	S_0 or S_T
• Exercise price	K (or X)
• Price of a Call	C_0 or C_T
• Price of a Put	P_0 or P_T

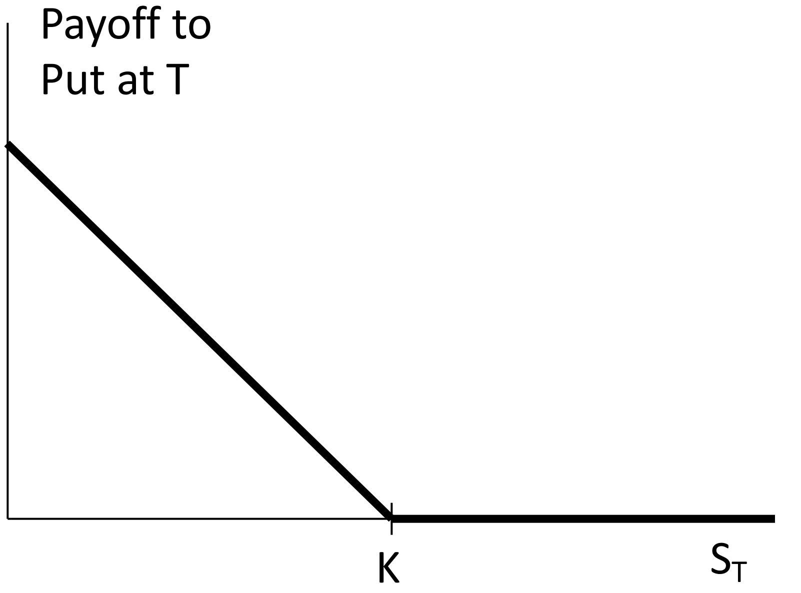
Call and Put Option payoffs at expiration

- $C_T = \max(0, S_T - K)$ $\quad \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T < K \end{cases}$

- $P_T = \max(0, K - S_T)$ $\quad \begin{cases} 0 & \text{if } S_T > K \\ K - S_T & \text{if } S_T < K \end{cases}$

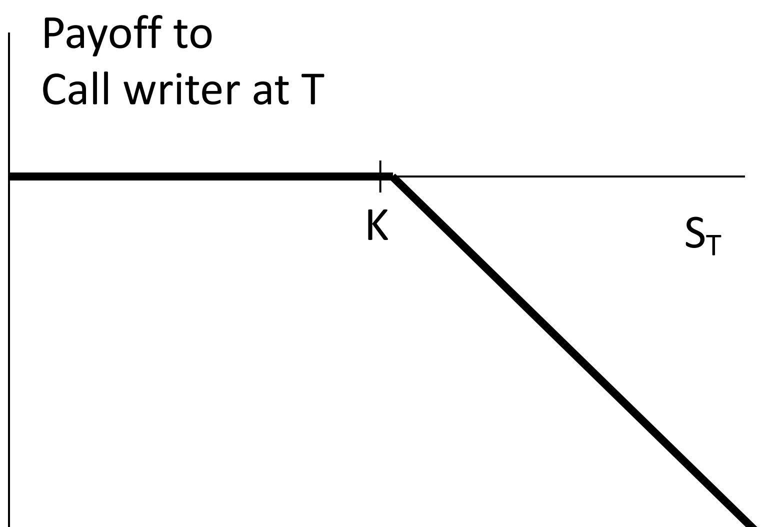
Option payoff diagrams

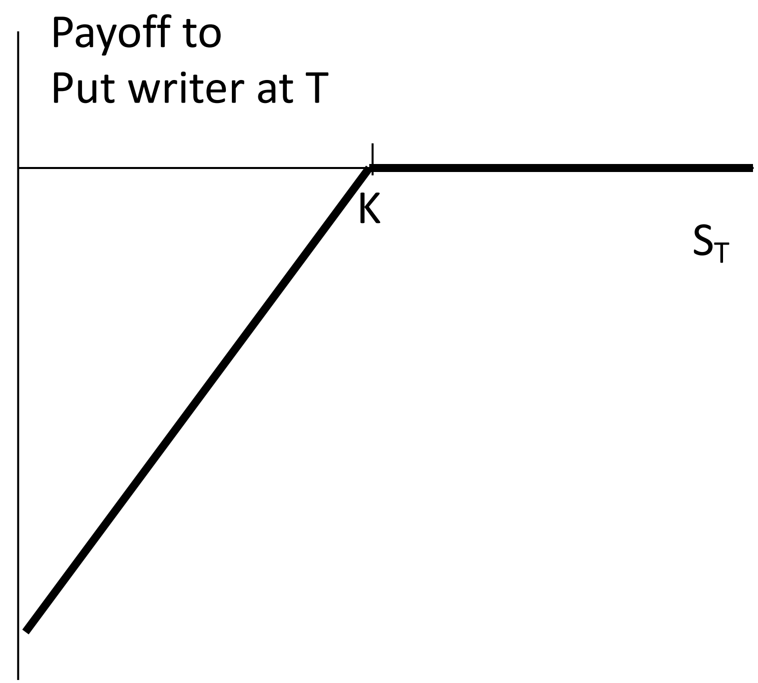




Long position in the option

Payoffs to option writers





Short position in the option

Option prices

- Payoff of an option buyer is never negative, since exercise is a right not an obligation
- But, option contracts require initial cash outlay called **option price** or **premium**
- Payoff of an option seller in never positive, but he/she gets initial premium

Put-call parity

- European call and put **prices** are related to each other by the equation
  - Same maturity
  - Same strike price

$$C_0 - P_0 = S_0 - K/(1+r_f)^T$$

$$\text{or} \qquad C_0 + K/(1+r_f)^T = S_0 + P_0$$

Put-call parity proof

- Compare these two portfolios:
  - A: Buying a European call option with strike price K and investing (lending) $K/(1+r_f)^T$ in T-Bills
  - B: Buying the underlying stock at its current price S_0 and buying a put option on the stock with strike price K

$$C_0 + K/(1+r_f)^T = S_0 + P_0$$

- Hold both portfolios until expiration

Put-call parity proof (contd..)

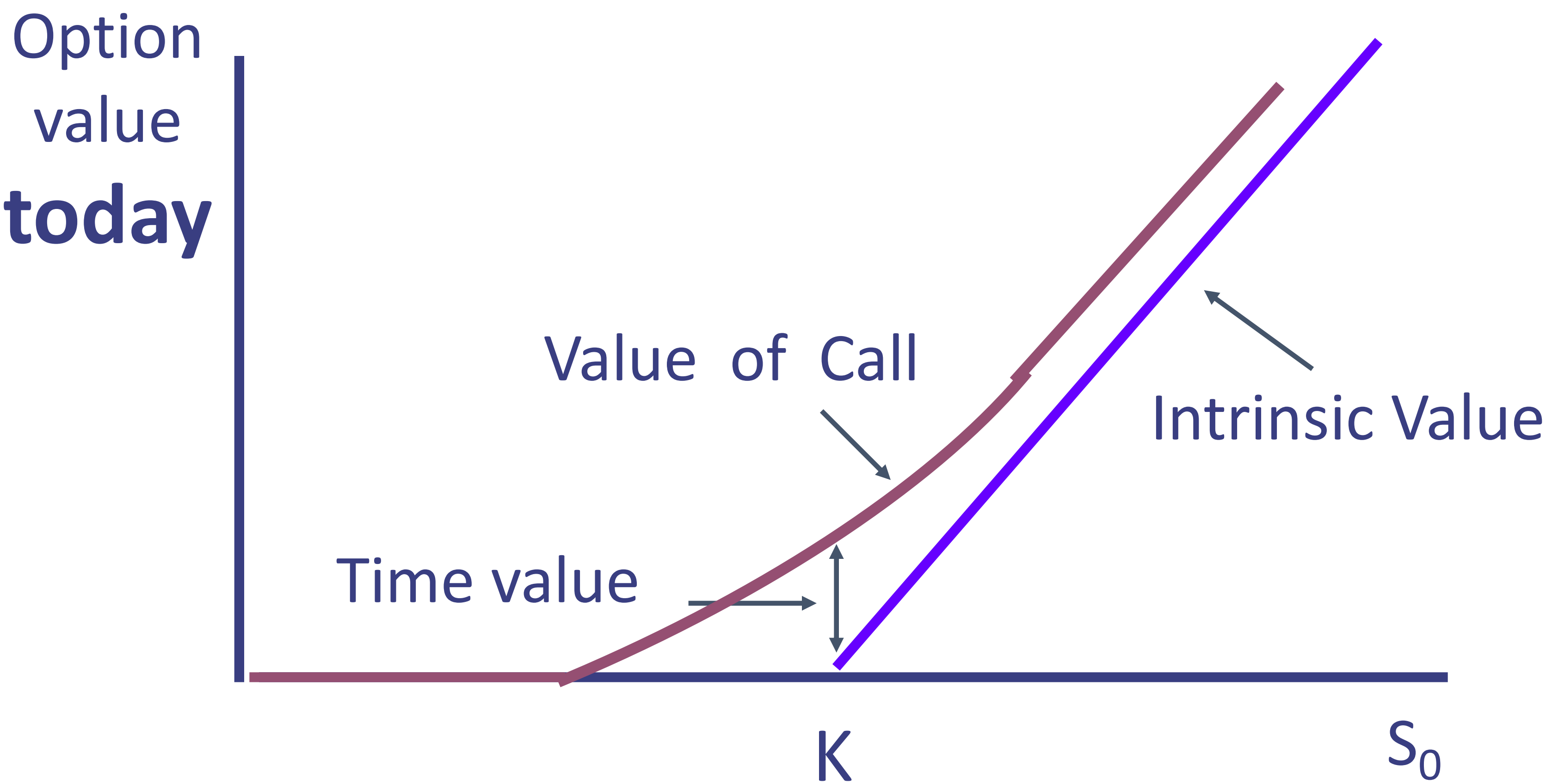
- The two strategies have identical payoffs at expiration

Portfolio	Initial Cost	Payoff $S_T > K$	Payoff $S_T = K$	Payoff $S_T < K$
A	$C_0 + K/(1+r_f)^T$	$S_T - K + K$	$0+K$	$0 + K$
B	$S_0 + P_0$	$S_T + 0$	$K+0$	$S_T + K - S_T$
A-B	??	0	0	0

- Their values today must be identical, else arbitrage

$$C_0 + K/(1+r_f)^T = S_0 + P_0$$

Current value of call option



Exact Option Pricing Formula?

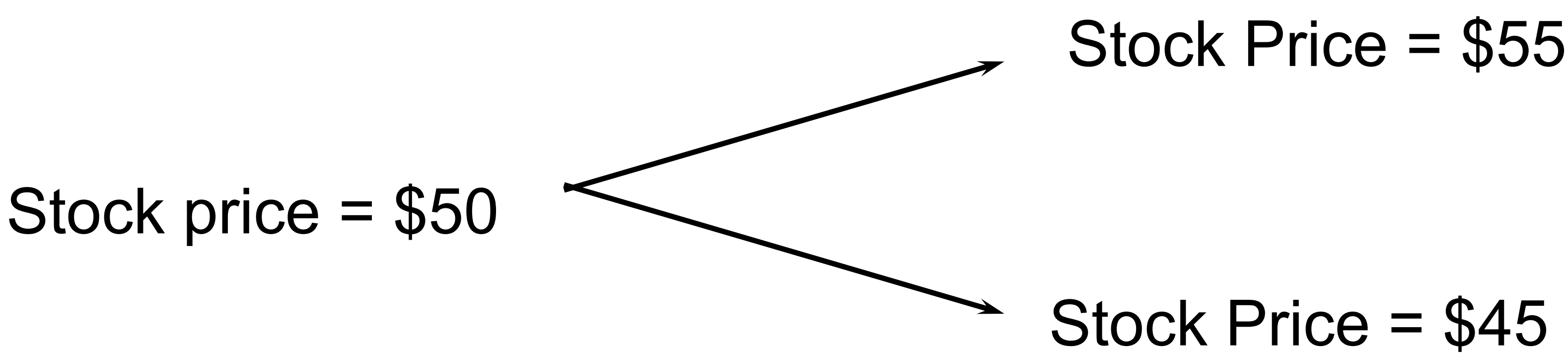
- We know the value of the option at maturity.
- The question is: What is the exact functional form for the dependence of the option price on underlying asset price prior to maturity?
- Put Call Parity and other arbitrage relationships did not need assumptions of asset price behaviour
- For an exact option pricing formula we need assumptions about how asset prices evolve
- What assumptions appear reasonable?

How Asset Prices Evolve

- Random walk?
  - ➔ Past history is irrelevant
    - Successive returns are independent
    - Variances are additive
  - ➔ If returns are identically distributed across time
    - Drift or Expected Return is proportional to the time interval
    - Variance is proportional to the time interval
  - ➔ Normally distributed continuously compounded returns
  - ➔ Jargon: Geometric Brownian motion
  - ➔ Limit of a Binomial Model

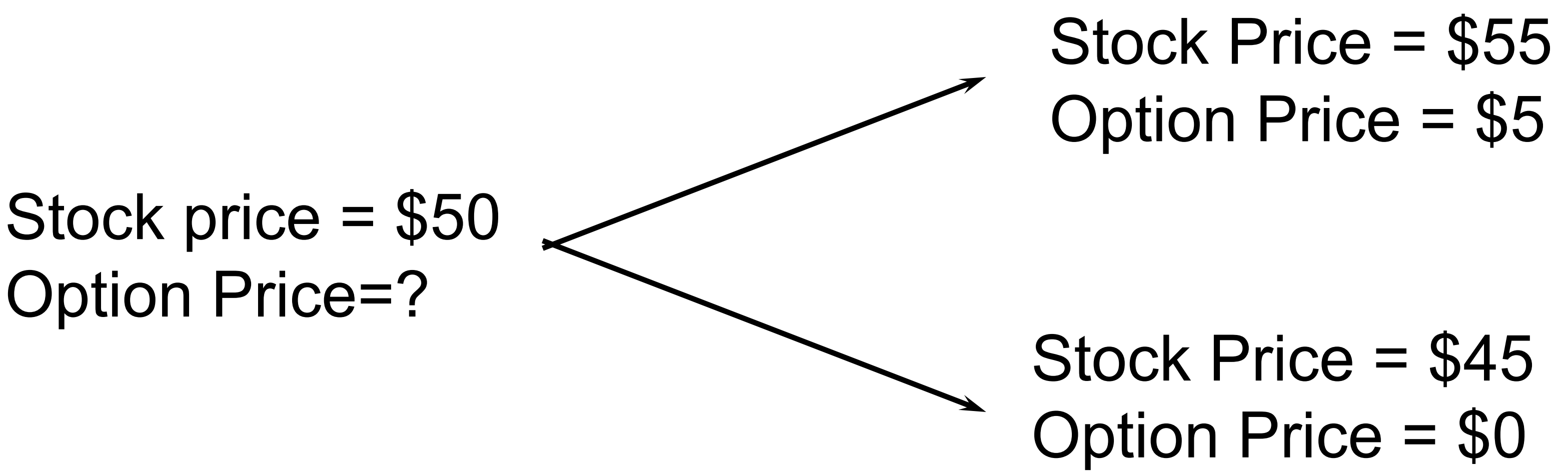
A Simple Binomial Model

- A stock price is currently \$50
- In three months it will be either \$55 with probability q or \$45 with probability (1-q)



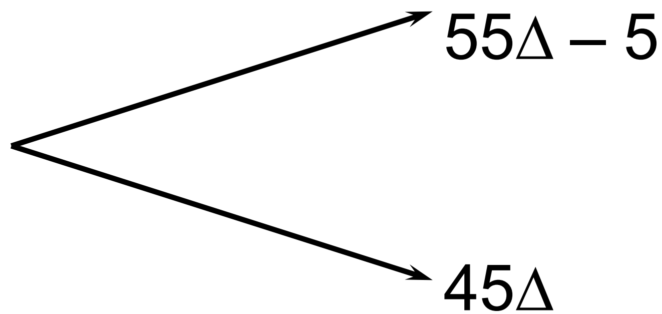
A Call Option

A 3-month call option on the stock has a strike price of $50.



Setting Up a Riskfree Portfolio

- Consider the Portfolio: long Δ shares
  short 1 call option



- Portfolio is riskless when $55\Delta - 5 = 45\Delta$
- or $\Delta = 0.5$

Valuing the Portfolio

- The riskless portfolio is:

  long 0.5 shares
  short 1 call option

- The value of the portfolio in 3 months is

  $$55 \times 0.5 - 5 = 22.50$$

- Suppose risk-free rate is 12%
- The value of the portfolio today is

  $$22.5e^{-0.12 \times 0.25} = 21.835$$

Valuing the Option

- The portfolio that is

  long 0.5 shares
  short 1 option

  is worth 21.835
- The value of the shares component is
  25.000 (= 0.5×50)
- The value of the option is therefore
  = 25.000 – 21.835 = 3.165

Binomial Model

- Stock price follows a binomial distribution
- It can go up (uS) or down (dS) with probability q and 1-q
- Given that we know K we can price the Call option at maturity
- We can form a portfolio between the Stock and the Call that is riskless
- The return on that portfolio has to be the risk-free rate
- This give the binomial formula for the Call

Replicating Portfolio

- The basic idea behind option pricing theory is that it is possible to duplicate the payoff of an option using a *dynamic* replicating portfolio investing only in the underlying asset and cash
  - The equivalent replicating portfolio for forwards is *static*
  - Real world dynamic replication has limits due to transaction costs, liquidity, price jumps and unexpected changes

- "Delta" is the number of units of the underlying asset in the replicating portfolio for one option.

- **Option Value = (Asset Price × Delta) + Cash**

Binomial Call Pricing : Notation

Let us use the following notation:

R = 1 + riskless return (over known but arbitrary time interval)

u = 1 + asset return (if asset price increases)

d = 1 + asset return (if asset price decreases)

q = subjective probability of increase

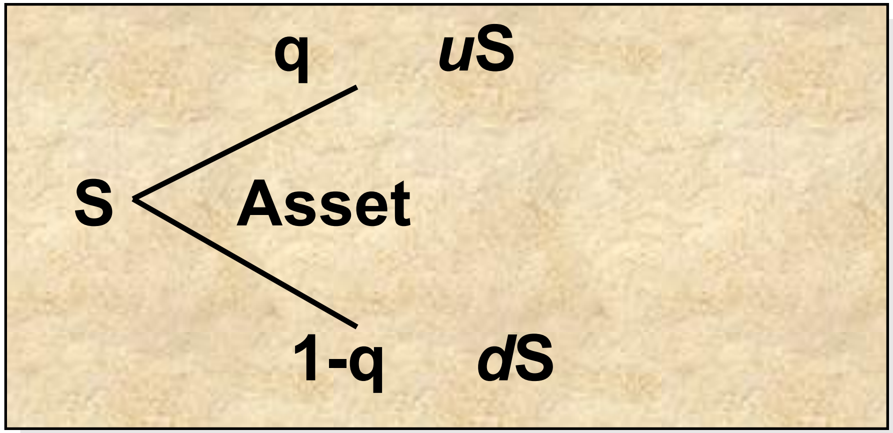
$$\boxed{u > R > d}$$

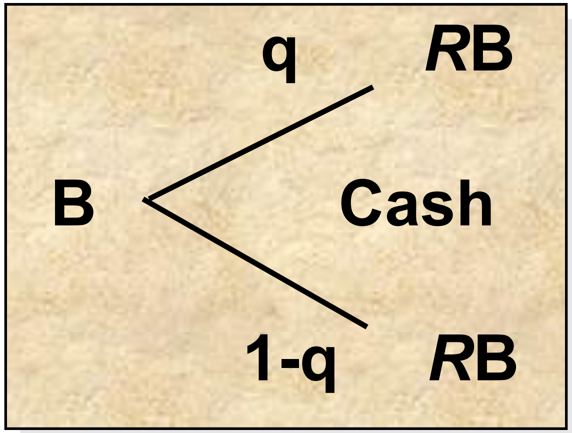
Both u and d cannot be greater than R, since that means that the underlying asset would for sure have a return higher than cash.

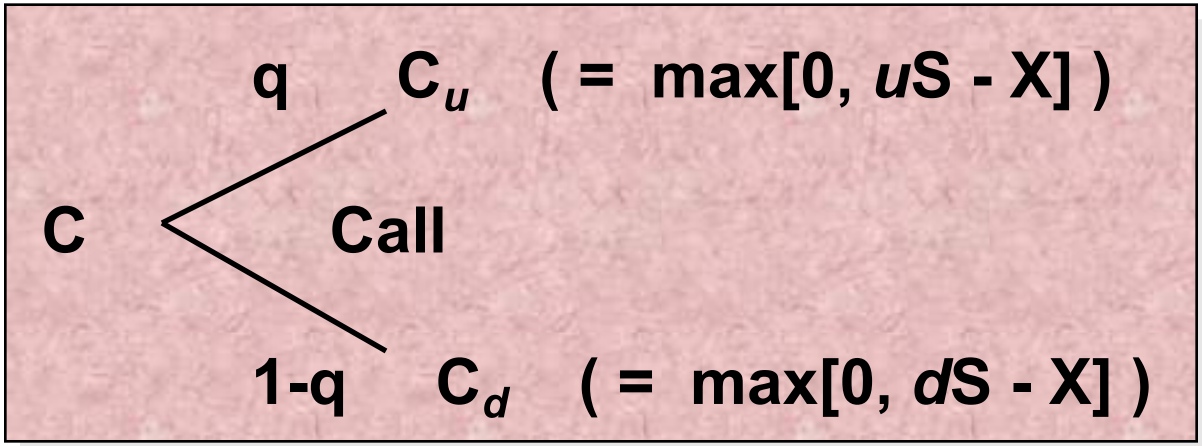
Similarly, both u and d cannot be less than R since that means that cash would for sure have a higher return than the underlying asset.

This leaves as the only possibility: $u > R > d$.

Binomial Call Pricing : Notation







Binomial Call Pricing

- Consider the portfolio that is long Δ shares and short one call

$$Su\Delta - C_u$$

$$Sd\Delta - C_d$$

- The portfolio is riskless when $Su\Delta - C_u = Sd\,\Delta - C_d$ or

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

Binomial Call Pricing

- Value of the portfolio at time T is $Su\,\Delta - C_u$
- Value of portfolio today has to be $(Su\,\Delta - C_u)/R$ or $(Su\,\Delta - C_u)e^{-rT}$
- The cost of setting up the portfolio today is $S\,\Delta - C$
- Hence $C = S\,\Delta - (Su\,\Delta - C_u)/R$

$$C = \cancel{S}\,\frac{C_u - C_d}{\cancel{S}(u-d)} - \left(\cancel{S}\,\frac{u(C_u - C_d)}{\cancel{S}(u-d)} - C_u\right)\Big/$$

$$= \frac{C_u - C_d}{u-d} - \left(\frac{u C_u - u C_d}{u-d} - \frac{u C_u - \cdots}{u-d}\right.$$

$$+\left(\frac{u C_d - d C_u}{u-d}\right)\Big/ R$$

$$= \frac{R C_u - R C_d + u C_d - d C_u}{R(u-d)}$$

$$= \frac{1}{R}\left[\frac{C_u(R-d) + C_d(u-R)}{u-d}\right]$$

$$= \frac{1}{R}\left[\underbrace{\frac{R-d}{u-d}}_{p}\, C_u + \underbrace{\frac{u-R}{u-d}}_{1-d}\, C_d\right]$$

$$1-p = \frac{u-d}{u-d} - \frac{R-d}{u-d} = \frac{u-R}{u-d}$$

Binomial Call Pricing

- Substituting for Δ we obtain

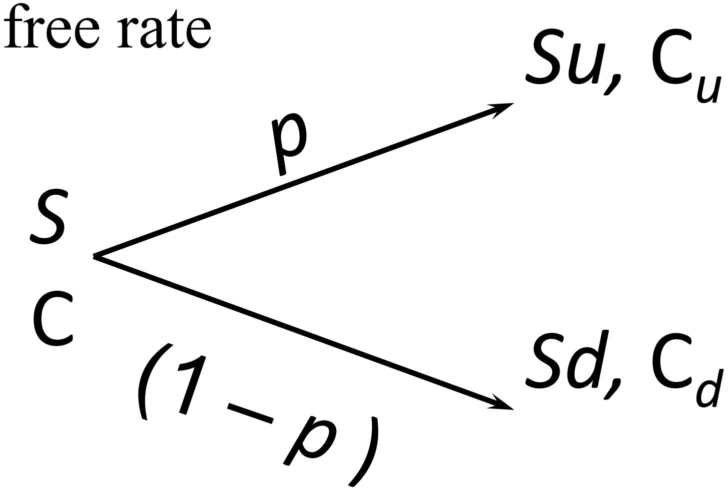
$$C = [\, p\, C_u + (1 - p\,)C_d \;]/R$$

where

$$p = \frac{R - d}{u - d}$$

Binomial Call Pricing

- $C = [\,p\,C_u + (1-p)C_d\,]/R$
- The variables p and $(1-p)$ can be interpreted as the *risk-neutral* probabilities of up and down movements. p is a probability since $0 < p < 1$, and since p satisfies $pu + (1-p)d = R$, p qualifies as a "risk-neutral probability"
- The value of the call is its expected payoff in a *risk-neutral* world discounted at the risk-free rate



Option value depends only upon S, X, *u, d, and R and NOT on q and hence NOT on the expected return*

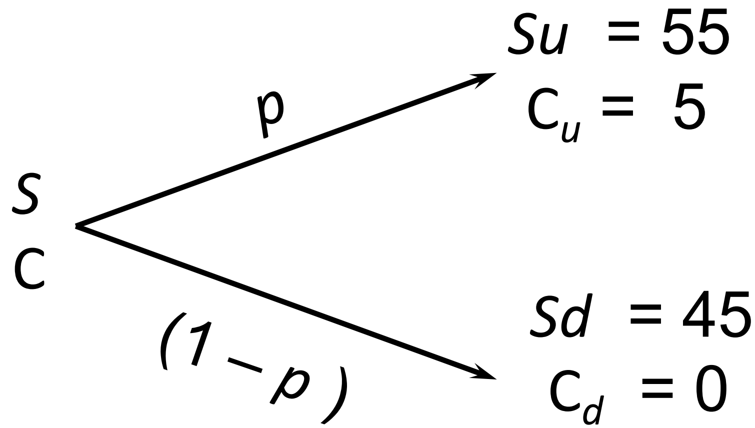
$$pu + (1-p)\,d$$

$$= \frac{R-d}{u-d}\,u + \left(1 - \frac{R-d}{u-d}\right) d$$

$$= \frac{Ru - \cancel{du} + \cancel{ud} - \cancel{d^2} - Rd + \cancel{d^2}}{u-d}$$

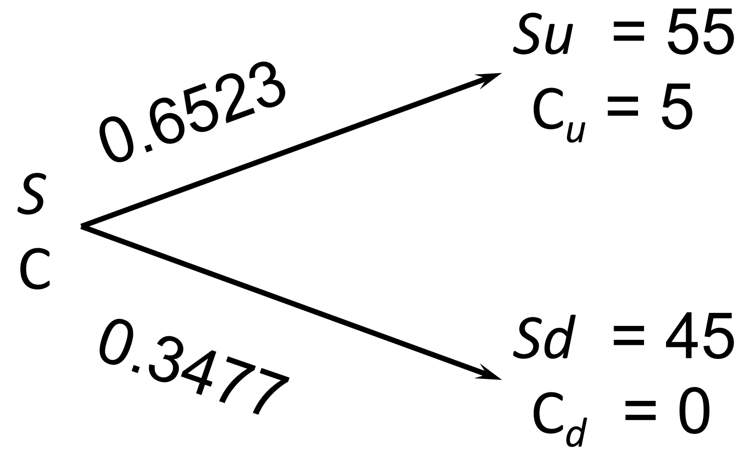
$$= \frac{R\,\cancel{(u-d)}}{\cancel{(u-d)}} = R$$

Earlier Example Revisited



$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option: Earlier Example

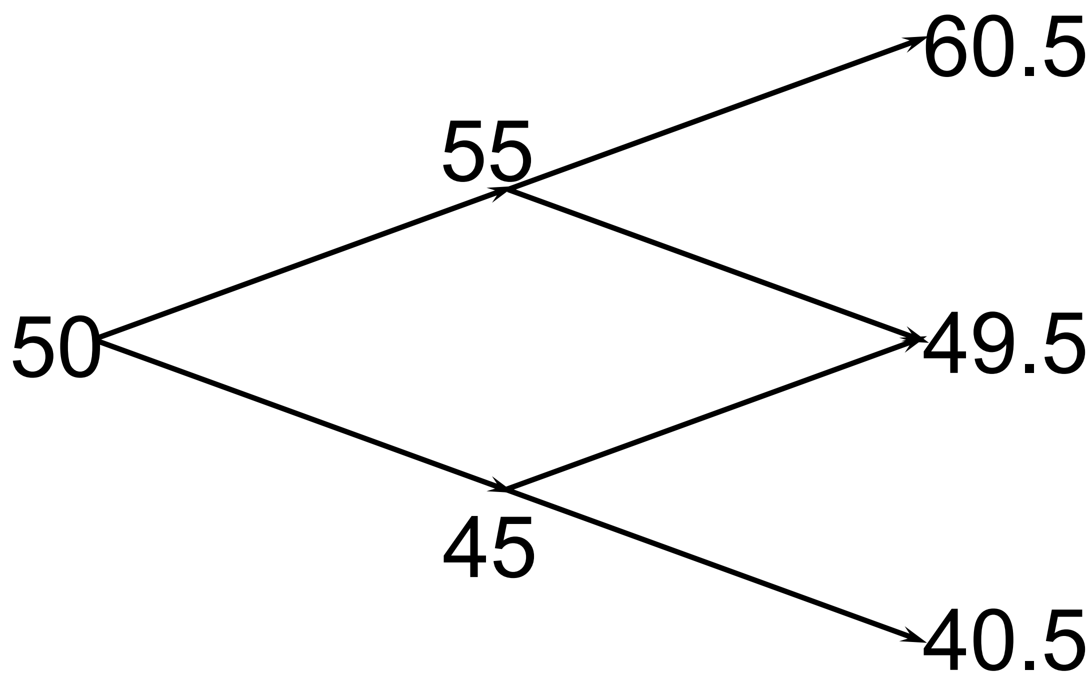


The value of the option is

$e^{-0.12\times0.25}\,[0.6523\times5 + 0.3477\times0] = 3.165$

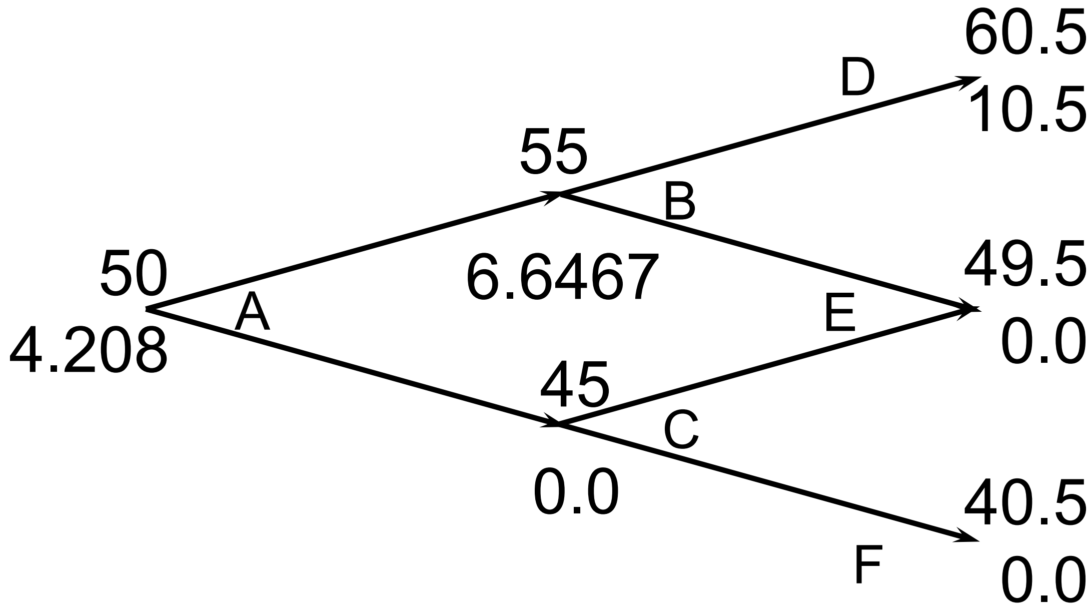
Same as before

A Two-Step Example



- Each time step is 3 months

Valuing a Call Option



- Value at node B
  $= e^{-0.12\times0.25}(0.6523\times10.5 + 0.3477\times0) = 6.6467$
- Value at Node C = 0
- Value at node A
  $= e^{-0.12\times0.25}(0.6523\times6.6467 + 0.3477\times0)$
  $= 4.208$

Key problem: Choosing *u* and *d*

- Suppose σ is the volatility and Δt is the length of the time step.
- To matching the true volatility one can set:

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

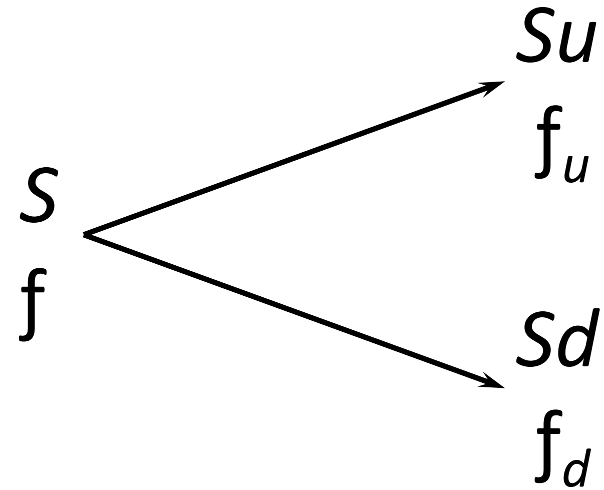
- Suppose σ is 20%, the volatility and Δt is 3 months
- To matching the true volatility one can set:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{0.25}} = e^{0.1} = 1.1052$$

$$d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.2\sqrt{0.25}} = e^{-0.1} = 0.9048$$

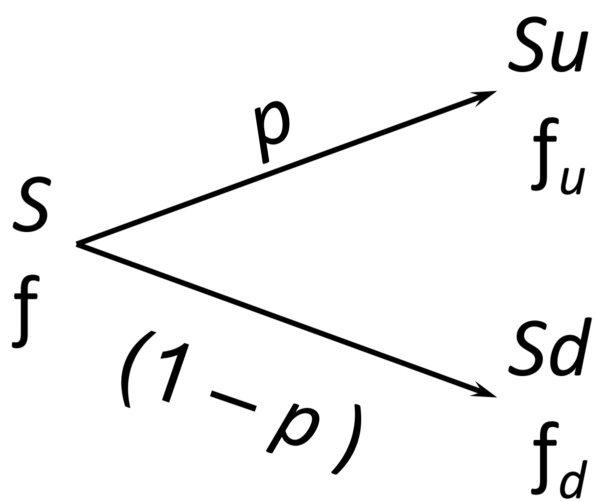
Generalization for any Option

- Consider a derivative with a payoff of f_u in up-state and f_d in down-state
- Suppose the price of the option is f

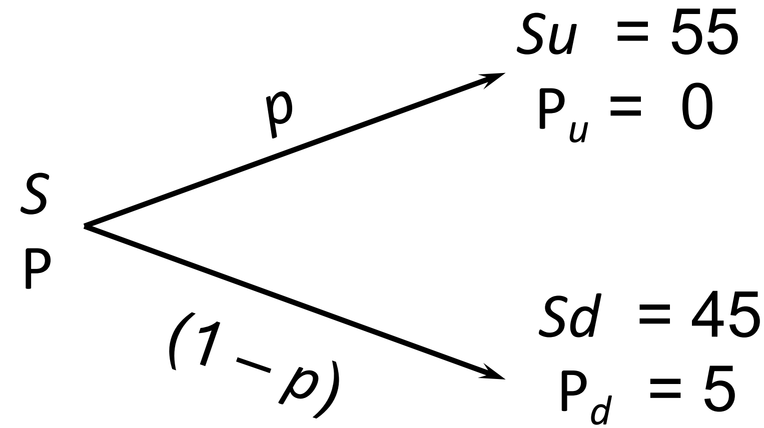


Generalization for any Option

- Then $f = [\, p\, f_u + (1 - p\,)f_d\ ]e^{-rT}$ where $$p = \frac{e^{rT} - d}{u - d}$$
- The variables p and $(1 - p\,)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



Value of Put: Earlier Example



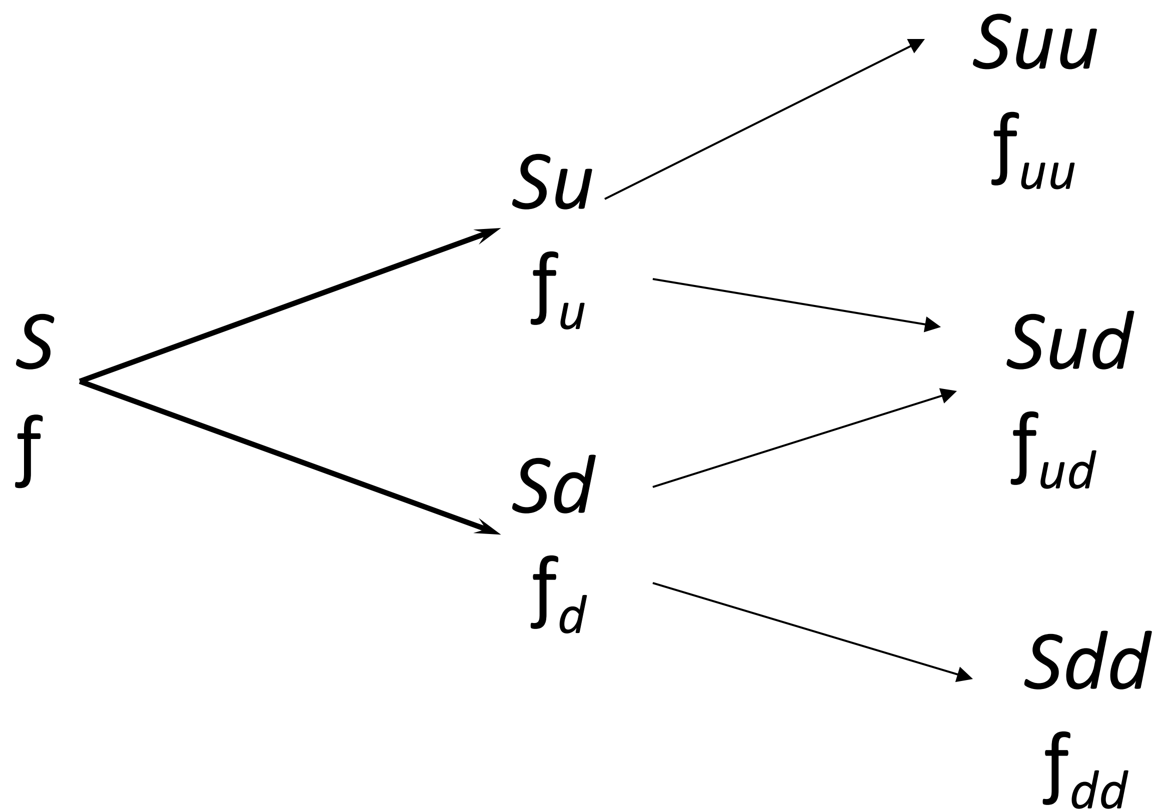
$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

The value of the option is

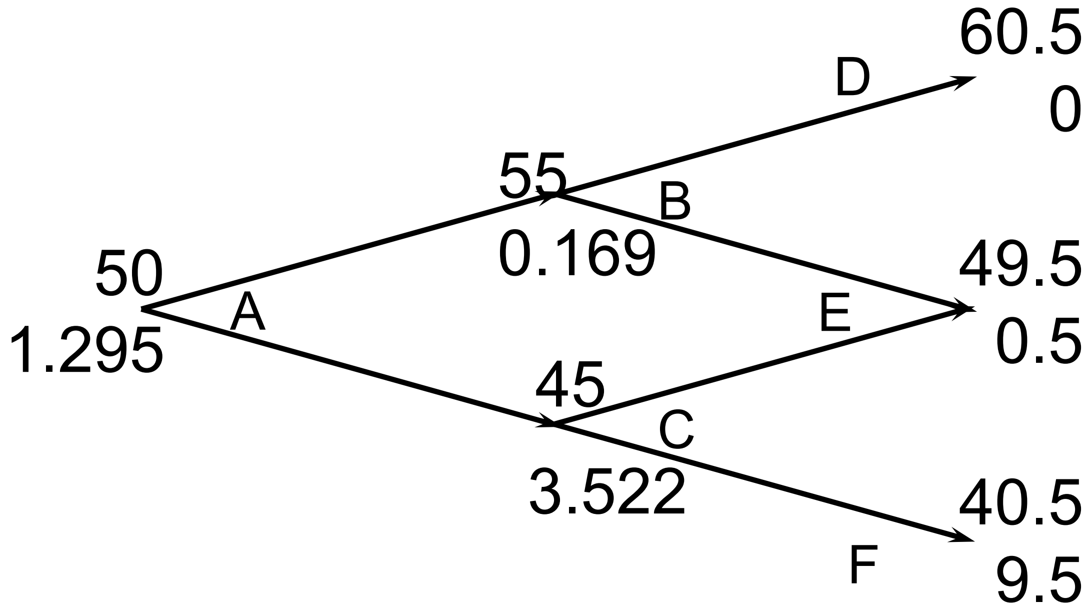
$e^{-0.12 \times 0.25}\ [0.6523 \times 0 + 0.3477 \times 5] = 1.687$

Does Put-Call Parity hold? Check!!

Generalization for any Option



Valuing a European Put Option



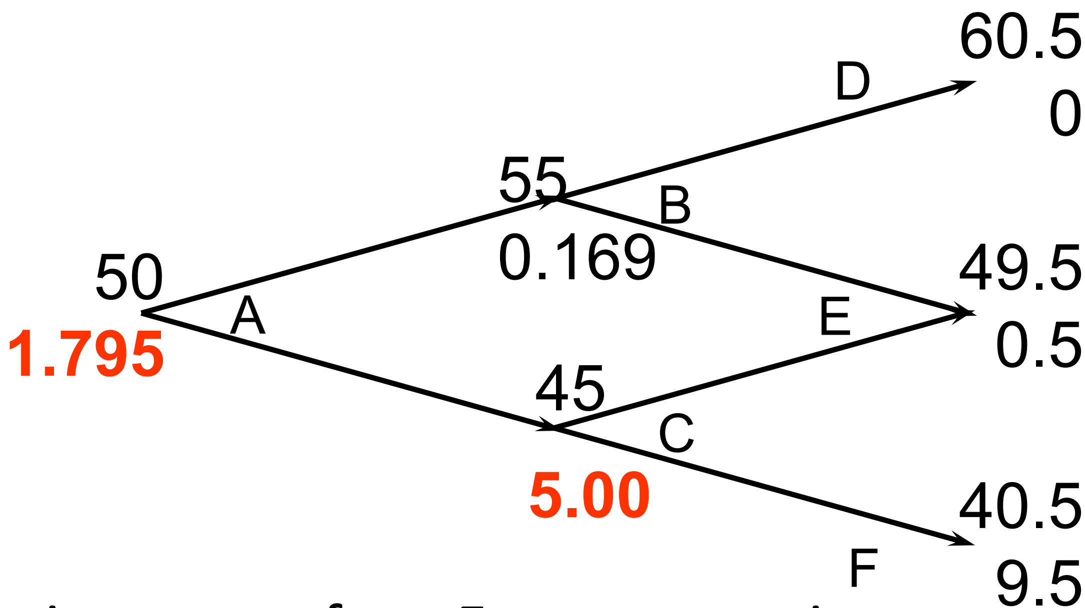
Value at node B = $(0.6523\times0.0 + 0.3477\times0.5)\, e^{-0.12\times0.25} = 0.169$

Value at node C = $(0.6523\times0.5 + 0.3477\times9.5)\, e^{-0.12\times0.25} = 3.522$

Value at node A = $(0.6523\times0.169 + 0.3477\times3.522)\, e^{-0.12\times0.25} = 1.295$

Again, does Put-Call Parity hold?

Valuing a American Put Option



Value at final nodes is same as for a European option

Test at each earlier node if early exercise is optimal by comparing payoff from early exercise with the value from the European tree.

Note it is optimal to exercise early at Node C above

Value at node A = $(0.6523 \times 0.169 + 0.3477 \times 5.00)\, e^{-0.12 \times 0.25}$ = 1.795

The Black-Scholes Formula

Let the number of periods in Binomial model go to infinity.
Replace discrete interest rates by continuously compounded rates.
Let r be the interest rate, and let the option mature at *T*.
Note: ln( ) is the natural log

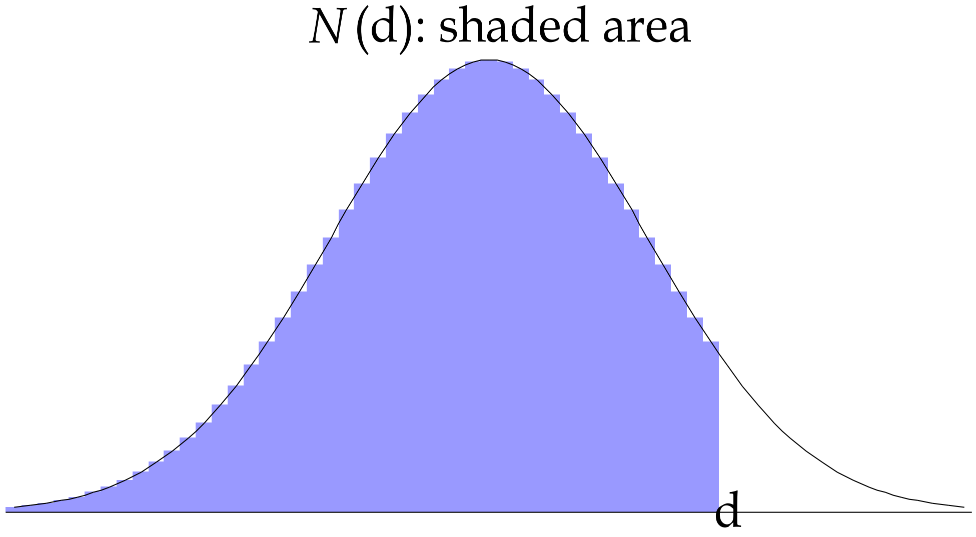
$$C = S_0\, N(d_1) - X\, e^{-rT} N(d_2)$$

$$\text{where} \quad d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / X) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Black-Scholes formula

- N(d) = probability that a random draw from a normal distribution with mean 0 and variance 1 will be less than d
  - normsdist() in Excel
- N(d) is the area under the unit normal curve from minus infinity to d
  - Always between zero and one



$N(d_1)$ is also the delta or the hedge ratio

Black-Scholes formula

- Call price depends on
  - Current price of stock S_0
  - Exercise price X
  - Time to maturity T
  - Volatility of stock σ^2
  - Interest rate r

- Does not depend on
  - Expected return on the stock
  - Why?

Black-Scholes formula

- Formula is derived for European calls
- Also gives us price of European puts through put-call parity

$$P_0 = C_0 + \mathrm{PV}(X) - S_0$$

$$P_0 = Xe^{-rT}N\left(-d_2\right) - S_0N\left(-d_1\right)$$

- Also gives us price of an American call, as long as the underlying pays no dividends during life of option
- Does *not* give us the price of an American Put

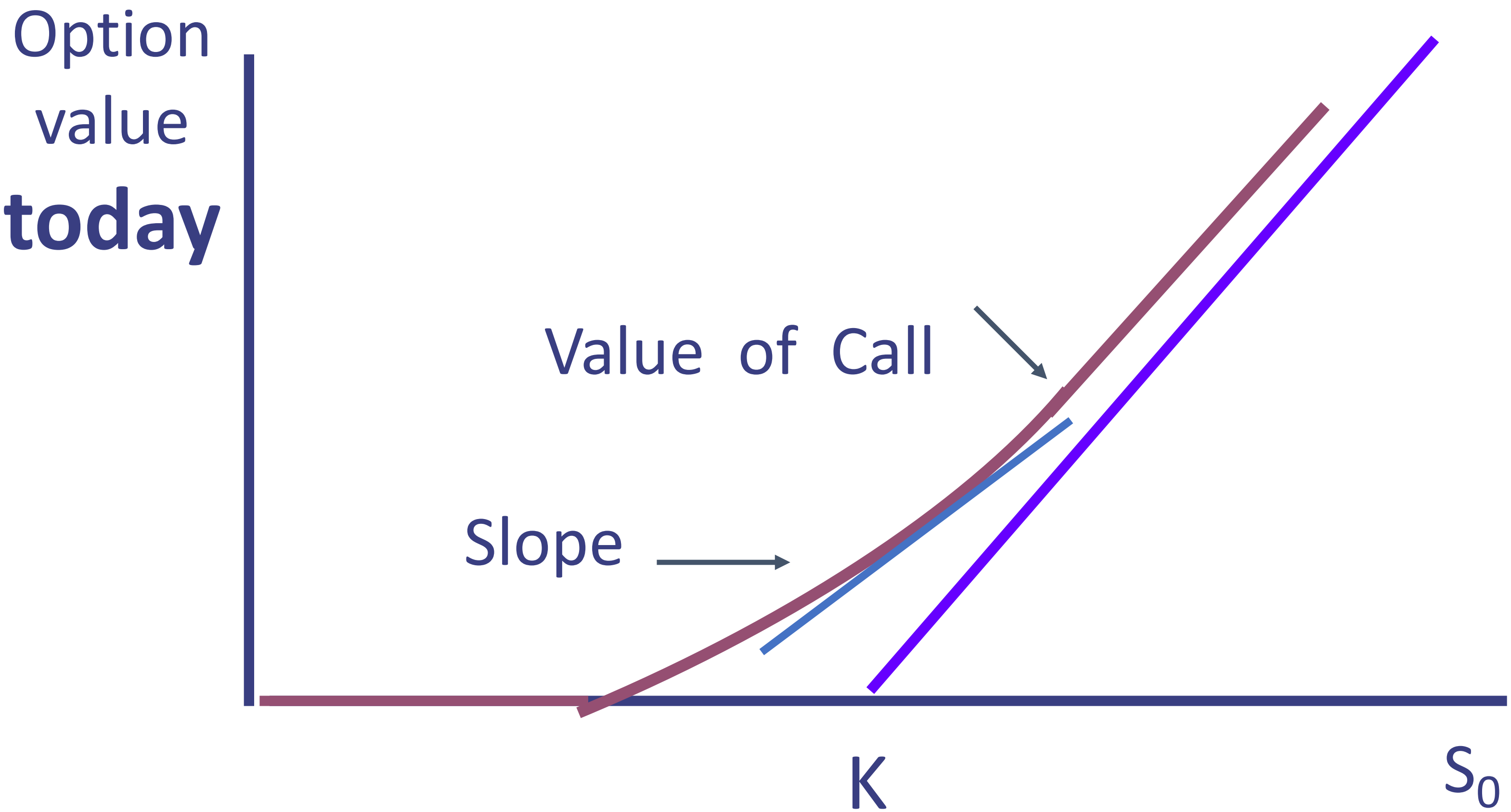
Delta

- Measures the dollar change in the value of the option for a $1 change in the value of the stock
- Using Black-Scholes formula, we can get exact expressions for Delta
- For a call

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

  - Always lies between 0 and 1

Delta: Slope of the curve $\Delta = \frac{\partial C}{\partial S} = N(d_1)$



Implied volatility

- BS formula shows that option price depends on five parameters:
  - S, X, r, T, and σ
- The first four of these are directly observable
- Can use the market price of the option to back out the volatility σ^2 that the market is using to price the option
  - Called *implied volatility*
- Implied volatility is the value of σ that solves:

Market price = Black-Scholes price

Volatility of the Call as a function of the Volatility of the Stock

$$\sigma_C = \frac{\partial C}{\partial S}\frac{S_0}{C_0}\sigma \quad = \frac{\%change\ in\ C}{\%change\ in\ S}\sigma$$

$$\sigma_C = N(d_1)\frac{S_0}{C_0}\sigma$$

What have we Discussed?

- Assumed normal distributions for continuously compounded returns.
- Basic insight is that you can combine an option with the asset to construct a risk-free hedge portfolio that must earn the risk-free rate of return.
- Used the binomial approximation to calculate option prices.
- In the limit we get the Black-Scholes formula.