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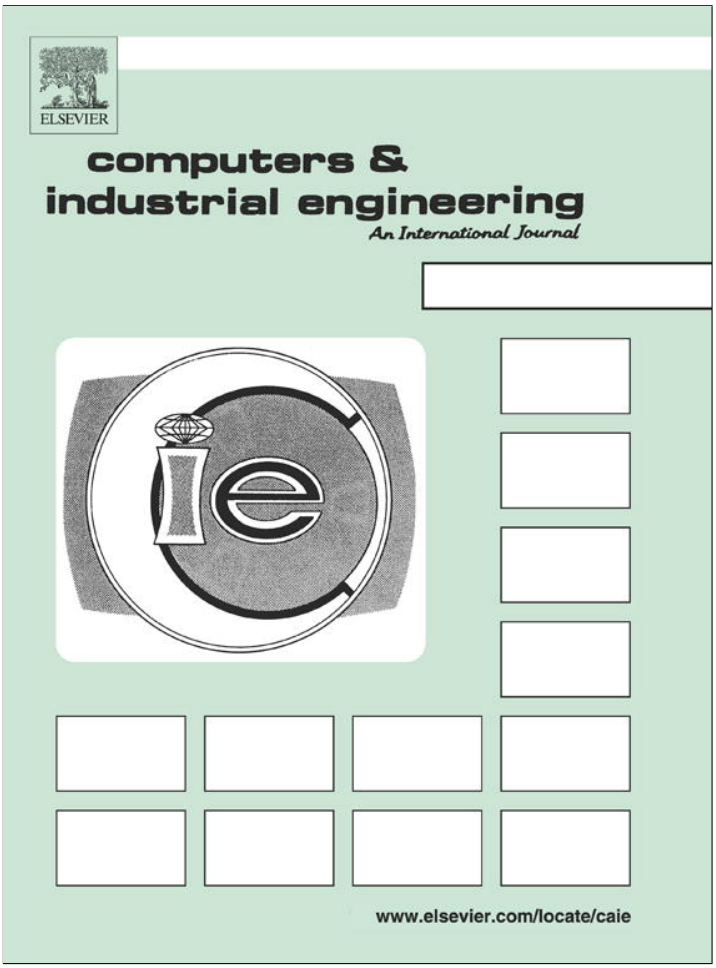

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A priori reformulations for joint rolling-horizon scheduling of materials processing and lot-sizing problem

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ABSTRACT

In many production processes, a key material is prepared and then transformed into different final products. The lot sizing decisions concern not only the production of final products, but also that of material preparation in order to take account of their sequence-dependent setup costs and times. The amount of research in recent years indicates the relevance of this problem in various industrial settings. In this paper, facility location reformulation and strengthening constraints are newly applied to a previous lot-sizing model in order to improve solution quality and computing time. Three alternative metaheuristics are used to fix the setup variables, resulting in much improved performance over previous research, especially regarding the use of the metaheuristics for larger instances.

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1. Introduction

The lot sizing problem is frequently encountered in industrial production planning. How does a planner decide the lot size of each product produced on one or more machines in each demand period over a planning horizon? This problem has been extensively researched, as discussed in the reviews by Bahl, Ritzman, and Gupta (1987), Drexl and Kimms (1997), Brahimi, Dauzere-Peres, Najid, and Nordli (2006), Karimi, Fatemi Ghomia, and Wilson (2003) and Jans and Degraeve (2007), Jans and Degraeve (2008).

In some industrial sectors, production setup costs and times are sequence-dependent, so that the decisions about the production of lots concern their sequence as well as their size. The two sets of decisions are mutually dependent and so should be modelled and decided simultaneously rather than separately. For example, Araujo, Arenales, and Clark (2007) consider the processing of key materials at an initial stage as well as the products that use the materials. In such a two-stage system, the sequencing decisions at the end stage must jointly consider products that use the same materials, so that material changeovers are minimised at the prior stage. In other words, the sequencing at both stages must be integrated. The model in Araujo et al. (2007) is based on a classical formulation of the lot sizing problem, but its complexity meant that an established industrial-grade optimisation solver was unable to find an optimal solution within acceptable computing time. As a result, the paper developed heuristic solution methods based on *relax-and-fix* within a rolling horizon approach and incorporating metaheuristics.

Since then, many other authors have also researched the integrated lot-sizing and sequencing problem with material preparation at a prior stage in a wide range of industrial settings. Examples include soft drink production (Ferreira, Clark, Alamada-Lobo, & Morabito, 2012; Ferreira, Morabito, & Rangel, 2009, 2010; Toledo, França, Morabito, & Kimms, 2009), animal feed (Clark, Morabito, & Toso, 2010; Toso, Clark, & Morabito, 2009), electrofused grains (Luche, Morabito, & Pureza, 2009), glass bottles (Almada Lobo, Klabjan, Carravilla, & Oliveira, 2007; Almada Lobo, Oliveira, & Carravilla, 2008), foundries (Araujo, Arenales, & Clark, 2008; Camargo, Toledo, & Mattiollo, 2012; Tonaki & Toledo, 2010), yogurt packaging company (Marinelli, Nenni, & Sforza, 2007), pharmaceutical company (Stadtler, 2011) and sand casting operations (Hans & Van de Velde, 2011).

Most of this recent research, including Araujo et al. (2007), is based on the *General Lot Sizing and Scheduling Problem* (GLSP) model (Fleischmann & Meyr, 1997) in which the planning horizon is subdivided into macro-periods, in each of which multiple products can be produced. To model the sequence of lots, each macro-period is in turn subdivided into micro-periods in which at most one product can be produced. This special structure involving subperiods within macro time periods is similar to a small-bucket framework (Koçlar, 2005).

However, some papers, such as Clark et al. (2010) and Ferreira et al. (2012) take a different approach, using an asymmetric travelling salesman problem (ATSP) representation for sequencing lots

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rather than a GLSP-type model. The results presented in Ferreira et al. (2012) shown the superiority of their ATSP-type model over a GLSP-type model. One possible reason is the poor quality of the GLSP linear relaxation as a lower bound on the optimal solution.

Taking forward research initiated in Bernardes, Araujo, and Rangel (2010), the first contribution of this current paper is to demonstrate that certain reformulations applied to the GLSP-type model in Araujo et al. (2007) can provide improved solutions using established optimisation solvers, due mainly to their better quality of linear-relaxation lower-bounds, contributing to the growing research in this area. The second contribution is to show computationally that the use of metaheuristic methods can help solve the reformulations more quickly and better than established solvers, such as, Cplex 12.0.

This paper is structured as follows. In Section 2, the original model in Araujo et al. (2007) is presented. Section 3 develops extended formulations and proposes new constraints. In Section 4, a reformulated rolling horizon-based model is proposed while Section 5 presents the metaheuristics. Section 6 computationally compares the quality of the reformulations using the *Branch-and-Cut* search within the solver Cplex 12.0 and using the metaheuristics. Section 7 concludes and poses challenges for future research.

2. Original Formulation (OF)

In Araujo et al. (2007), a material may be used in multiple products, but a product is made from just one material. A product must be manufactured in the same time period in which its material is processed. Thus processed materials cannot be held over from one period to the next. In each time period only one material can be processed on a given machine. A setup changeover from one material to another is sequence-dependent, i.e., it consumes capacity time in a manner that is dependent on the sequence in which the materials are processed. The triangle inequality holds for setup costs and times so that it is optimal to produce at most one lot per product per period. The model allows backlogs as well as inventory.

The sequencing decisions are made by dividing a period into smaller subperiods, as in the *General Lot Sizing and Scheduling Problem* (GLSP) model (Drexl & Kimms, 1997; Fleischmann & Meyr, 1997; Meyr, 2000, 2002). Let K be the total number of materials, P the total number of products, T the total number of periods and η the total number of subperiods. Consider the following indices and data:

Indices	
	$j, k = 1,\ldots,K$ processed materials $p = 1,\ldots,P$ products $t = 1,\ldots,T$ periods $n = 1,\ldots,\eta$ subperiods
Data	
C	Capacity available on the machine in each subperiod
ρ_p	Capacity required to produce one unit of product p
d_{pt}	Demand for product p at the end of period t
$S(k)$	Set of products p that use material k. Each product uses just one material, i.e., $\{1,\ldots,P\} = S(1) \cup,\ldots,\cup S(K)$, and $S(k) \cap S(j) = \emptyset$, for all materials $k \neq j$, implying $\sum_k \lvert S(k)\rvert = P$
h^-_{pt}	Backlog penalty for delaying delivery of a unit of product p at the end of period t
h^+_{pt}	Inventory penalty for holding a unit of product p at the end of period t
s_{jk}	Setup penalty (or cost) for changing over from material j to material k, where $s_{jj} = 0$
st_{jk}	Setup time (loss of machine capacity) for changing over from material j to material k, where $st_{jj} = 0$
Variables	
x_{pn}	Quantity (lot-size) of product p to be produced in subperiod n
I^+_{pt}	Inventory of product p at the end of period t, where I^+_{p0} is the initial inventory at the start of period 1.
I^-_{pt}	Backlog of product p at the end of period t, where I^-_{p0} is the initial backlog at the start of period 1.
y^k_n	Binary variable: $y^k_n = 1$ if the machine is configured for production of material k in subperiod n, otherwise $y^k_n = 0$. Note that the initial setup state y^k_0 is set to zero and so not a variable
z^{jk}_n	Binary setup variable: $z^{jk}_n = 1$ if there is a machine changeover from material j to material k at the start of subperiod n, otherwise $z^{jk}_n = 0$. Thus $z^{jk}_n = 1$ if $y^j_{n-1} = 1$ & $y^k_n = 1$, and $z^{jk}_n = 0$ if $y^j_{n-1} = 0$ or $y^k_n = 0$. It is relaxed to be continuous for reasons explained below
Consider also the following definitions from the GLSP model	
η_t	Maximum number of subperiods in period t
First	$Ft = 1 + \sum_{\tau=1}^{t-1} \eta_t$ subperiod in period t
$L_t = F_t + \eta_t - 1$	Last subperiod in period t
Total	$\eta = \sum_{t=1}^{T} \eta_t$ number of subperiods over periods $1,\ldots,T$

The *Original Formulation* (OF) (Araujo et al., 2007) of the mathematical model is:

$$\text{Minimise} \quad \sum_p \sum_t (h^-_{pt} I^-_{pt} + h^+_{pt} I^+_{pt}) + \sum_j \sum_k \sum_{n=F_1}^{L_T} s_{jk} z^{jk}_n \tag{1}$$

subject to

$$I^+_{p,t-1} - I^-_{p,t-1} + \sum_{n=F_t}^{L_t} x_{pn} - I^+_{pt} + I^-_{pt} = d_{pt} \quad \forall p,t \tag{2}$$

$$\sum_{p \in S(k)} \rho_p x_{pn} + st_{jk} z^{jk}_n \leqslant C y^k_n \quad \forall j,k,n = F_1,\ldots,L_T \tag{3}$$

$$z^{jk}_n \geqslant y^j_{n-1} + y^k_n - 1 \quad \forall j,k,n = F_1,\ldots,L_T \tag{4}$$

$$\sum_k y^k_n = 1 \quad \forall n = F_1,\ldots,L_T \tag{5}$$

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$$\sum_{j=1}^{K}\sum_{k=1}^{K} z_{F_t}^{jk} = 1 \quad \forall t \tag{6}$$

$$y_n^k \in \{0,1\} \quad \text{with} \quad y_0^k = 0 \quad \forall k, n = F_1, \ldots, L_T \tag{7}$$

$$z_n^{jk} \geqslant 0 \quad \forall j,k,n = F_1, \ldots, L_T \tag{8}$$

$$x_{pn} \geqslant 0 \quad \text{and integer} \quad \forall p, n = \mathrm{F}_1, \ldots, L_T \tag{9}$$

$$I_{pt}^{+} \quad \text{and} \quad I_{pt}^{-} \geqslant 0 \quad \forall p,t \tag{10}$$

The objective function (1) minimises a weighted sum of inventory & backlog penalties and sequence-dependent setup penalties. Constraints (2) balance inventories, backlogs, demand and production in each period. Constraints (3) not only keep production within the machine's capacity, but also ensure that only products of the same material are produced in the same machine loading. As y_n^k and y_{n-1}^j are both binary variables, constraints (4) and the objective function (1) force the continuous variable z_n^{jk} to have value 1 if there is a changeover from material j to material k and, along with constraints (7), to have value 0 otherwise. Constraints (5) and (7) ensure that there is only a single machine loading in each subperiod. Constraints (6) ensure one changeover in the first subperiod of each period.

The model assumes that the system is not initially setup for any material, and that every setup must be completed in a subperiod. This means that the capacity time st_{jk} needed to setup a material k must not be greater than C, the capacity available on the machine in each subperiod, as enforced in constraints (3).

Most models generally consider the lot-size variables x_{pn} to be continuous. If an x_{pn} value is large, then simply rounding it to the nearby integer value is usually feasible and near-optimal. However, for problems where capacity or demand is low then the integer conditions (9) on x_{pn} must hold. Constraints (10) measure inventory I_{pt}^{+} and backlogs I_{pt}^{-} as non-negative variables, but for a given pair (p,t), I_{pt}^{+} and I_{pt}^{-} will not both be strictly positive in an optimal solution, given their positive coefficients in the objective function (1).

3. *A priori* reformulations

Recently, many authors (Alfieri, Brandimarte, & D'Orazio, 2002; Denizel, Altekin, Süral, & Stadtler, 2008; Denizel & Süral, 2006; Jans, 2009; Jans & Degraeve, 2004a; Pochet & Van Vyve, 2004; Süral, Denizel, & Van Wassenhove, 2009) have used alternative formulations with stronger linear relaxations that can provide better lower bounds or improved integer rounding of fractional solutions.

Two such reformulations stand out in particular. The first is based on the *Shortest Path* (SP) problem and makes use of variable redefinition (Eppen & Martin, 1987). The second reformulation is based on the *Facility Location* (*FL*) problem (Krarup & Bilde, 1977).

Several reformulation results, both theoretical and computational, have been published. Nemhauser and Wolsey (1988) show that for the single item capacitated problem, the SP and FL reformulations are equivalent and that their linear relaxations coincide with the integer convex hull. In fact, their reformulation matrices are unimodular. Denizel et al. (2008) consider the capacitated multi-item problem with setup times, and also show that the SP and FL reformulations are equivalent. Alfieri, Brandimarte, and D'Orazio (2002) apply these reformulations to the same problem without setup times. The computational tests indicated that the SP and FL linear relaxations' objective value is 60% larger than those of the classical formulation, but 30% larger if setup times are considered (Denizel & Süral, 2006). Diaby, Bahl, Karwan, and e Zionts (1992) use Lagrangean relaxation together with reformulation as a transport problem. Süral et al. (2009) consider an alternative formulation and also use Lagrangean relaxation to solve the problem. Jans and Degraeve (2004b) present a method to obtain lower bounds on the problem with setups using an SP-based formulation.

In addition, several researchers have proposed similar formulations for extentions of the classical lot sizing problem. Wu, Shi, Geunes, and Akartunalı (2011) propose two new mixed integer programming models for capacitated multi-level lot-sizing problems with backlogging, whose linear programming relaxations provide good lower bounds on the optimal solution value. Gramani and França (2006) consider the lot sizing problem integrated with the bidimensional cutting stock problem applied to the furniture industry, proposing a method based on an analogy with the SP problem.

In the same spirit, in Section 3.1 an Extended Formulation (EF) is now developed from the Original Formulation (OF) using the facility location concept studied in Krarup and Bilde (1977) and Denizel et al. (2008). Additional EF constraints will also be proposed. The EF model is then strengthened in Section 3.2 using setup considerations (Belvaux & Wolsey, 2001; Koçlar, 2005; Wolsey, 2002).

3.1. Extended reformulation (EF)

The Extended Formulation (EF) can be obtained by interpreting the Original Formulation (OF) in a manner analogous to the *Facility Location Problem* (Krarup & Bilde, 1977). Let $N = \{1, \ldots, \eta\}$ be the set of facility candidates and $M = \{1, \ldots, T\}$ a set of clients. The Facility Location Problem (FLP) consists of deciding which facilities to install so that demand is satisfied with minimum cost. To interpret the original formulation as an FLP, the production in subperiod n to satisfy demand in period t is taken as equivalent to using installed facility n to meet the demand of client t. Thus subperiods represent facilities and periods represent clients.

The FLP-based extended formulation uses the same indices, parameters and variables as the original formulation in Section 2, and an additional variable, Q_{pnt}, the quantity of product p produced in subperíod n to meet the demand at the end of period t. The production in a subperiod can have multiple destinations.

Note that if the production of subperiod n within period t meets demand for period $t' > t$ after period t, then it is held in inventory until the end of period $(t'-1)$, with consequent costs over periods $t, \ldots, t'-1$. Conversely, if the production of subperiod n within period t meets demand for period $t'' < t$ before period t, then the backlog costs over periods $t'', \ldots, t-1$ must be charged.

Using λ_n to indicate the period of subperiod n, the inventory and backlog costs CQ_{pnt}^{+} and CQ_{pnt}^{-} must be redefined as follows:

CQ_{pnt}^{+} is the unit cost of producing product p in subperiod n to meet demand at the end of period t after λ_n. Note that $CQ_{pnt}^{+} = \sum_{l=\lambda_n}^{t-1} h_{pl}^{+} (\lambda_n \leqslant t)$.

CQ_{pnt}^{-} is the unit cost of producing product p in subperiod n to meet backlogged demand at the end of period t before λ_n. Note that $CQ_{pnt}^{-} = \sum_{l=t}^{\lambda_n - 1} h_{pl}^{-} (\lambda_n \geqslant t)$.

The Extended Formulation (EF) is then given by the objective function (11) and constraints (12)-(15), as now explained.

3.1.1. Extended formulation (EF)

$$Min \quad \sum_{p=1}^{P}\left(\sum_{n=1}^{L_T-1}\sum_{t=\lambda_n+1}^{T} CQ_{p\lambda_n t}^{+} Q_{pnt} + \sum_{n=F_2}^{L_T+1}\sum_{t=1}^{\lambda_n-1} CQ_{p\lambda_n t}^{-} Q_{pnt}\right) + \sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{n=F_1}^{L_T} s_{jk} z_n^{jk} \tag{11}$$

Subject to :

$$\sum_{n=F_1}^{L_T+1} Q_{pnt} = d_{pt} \quad \forall p,t \tag{12}$$

$$\sum_{p \in S(k)} \rho_p \left(\sum_{t=1}^{T} Q_{pnt}\right) + st_{jk} z_n^{jk} \leqslant C y_n^k \quad \forall j,k,n = F_1, \ldots, L_T \tag{13}$$

$$Q_{pnt} \geqslant 0 \text{ and integer} \quad \forall\, p,t,n = F_1, \ldots, L_T \tag{14}$$

and constraints (4)—(9) from the original formulation. (15)

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The objective function (11) again minimizes a weighted sum of inventory & backlog penalties and sequence-dependent setup penalties. Constraints (12) ensure that the demand in period t is met by the production from all subperiods. The variable Q is defined for $\eta+1$ subperiods so as to permit backlogs in the last period. Note that the initial stock is zero. Constraints (13) are identical to (3) in the original formulation.

Although the number of variables has increased, previous research indicates such a reformulation generally improves the lower bounds obtained via linear relaxation (Chen & Thizy, 1990; Denizel et al., 2008; Nemhauser & Wolsey, 1988).

3.2. Strengthening constraints for an extended formulation

This section now presents strengthening constraints based on setup considerations that can be included in an extended formulation (EF).

3.2.1. Flow equations (V1)

The setup variables y and changeover variables z are weakly linked in the linear relaxation of the extended formulation. The setup variables y can take positive values, thus allowing production, without the corresponding changeover variables z also being positive. For example: if $y^j_{n-1}=0.5$ and $y^k_n=0.5$ then constraint (4) means that $z^{jk}_n \geqslant 0.5+0.5-1=0$. This kind of solution causes a loosening of the lower bounds provided by the linear relaxation (as used in the *Branch-and-Cut* method).

The relationship between setup variables y and changeover variables z can be represented via a network in which the value associated with a node (j,n) indicates if there is a setup to material j in subperiod n. The flow along an arc represents a changeover between two consecutive subperiods. The setup variable y^j_{n-1} indicates the node capacity of node $(j,n-1)$ and the changeover variable z^{jk}_n indicates the flow from node $(j,n-1)$ to node (k,n), as illustrated in Fig. 1.

The constraint family V1 presented below, based on Wolsey (2002), establishes a stronger link between the setup and changeover variables. When there is a setup to material k in a subperiod, then a changeover from another material to k must be enforced (constraint (16), V1a) along with a changeover from j to some other material (constraint (17), V1b), thus preserving the flow in the network.

$$\text{V1a}: \quad \sum_{j=1}^{K} z^{jk}_n = y^k_n, k=1,\ldots,K, n=F_1,\ldots,L_T. \tag{16}$$

$$\text{V1b}: \quad \sum_{k=1}^{K} z^{jk}_n = y^j_{n-1}, j=1,\ldots,K, n=F_1+1,\ldots,L_T. \tag{17}$$

3.2.2. Elimination constraints (E1)

Suppose that a solution has a sequence in a period in which the same material is processed in 2 lots in 2 non-consecutive subperiods. Given triangular setups, a no-worse and possibly better solution can be obtained by resequencing so that the subperiods are consecutive, thus eliminating a setup. In other words, there exists an optimal solution such that no material is setup more than once in a period. Constraints E1 impose this:

$$\textbf{E1}: \quad \sum_{j=1|j\neq k}^{K} \sum_{n=F_t}^{L_t} z^{jk}_n \leqslant 1 \quad \forall k=1,\ldots,K, t=1,\ldots,T. \tag{18}$$

4. Reformulated rolling horizon model

The Original Formulation OF and the strengthened extended formulation EF + V1 + E1 (i.e., EF with the new constraints V1a, V1b and E1) were solved and tested with the Cplex 12.0 mixed integer programming (MIP) optimizer. As we will show in the computational results in Section 6.3, the performance of Cplex 12.0 for solving the formulations is unsatisfactory for medium-sized problems. As a result, this section proposes three metaheuristics to solve the EF model on a rolling-horizon basis.

As in Araujo et al. (2007), suppose each period t is a workday. Consider, for example, a planning horizon of $T=5$ workdays of which only the first day ($t=1$) will be scheduled in detail. This is achieved by dividing the first day into $L=\eta_1=10$ subperiods, as up to L material loadings can be processed each day. The remaining days $t=2,\ldots,5$ have just one subperiod each ($\eta_2=\eta_3=\eta_4=\eta_5=1$). Thus $F_1=1$; $L_1=10$; $F_2=L_2=11$; $F_3=L_3=12$; $F_4=L_4=13$; $F_5=L_5=14$, i.e., there are $\eta=14$ subperiods n. The variables y^k_n for the larger subperiods $n=F_2,\ldots,F_5$ are then redefined as "the number of loadings using material k produced in subperiod n".

Only the scheduled decisions relative to the $\eta_1=10$ subperiods of day 1 are actually implemented. The decisions for the remaining 4 days are used only to evaluate the impact of future available capacity, i.e., to identify a provisional production plan in order to have advance warning of possible production backlogs and be able to act accordingly. The extended formulation for rolling horizon use, denominated EF–RH, is:

4.1. Model EF–RH

$$\begin{aligned} Min \quad & \sum_{p=1}^{P}\left(\sum_{n=1}^{L_T-1}\sum_{t=\lambda_n+1}^{T} CQ^{+}_{p\lambda_n t}Q_{pnt} + \sum_{n=F_2}^{L_T+1}\sum_{t=1}^{\lambda_n-1} CQ^{-}_{p\lambda_n t}Q_{pnt}\right) \\ & + \sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{n=F_1}^{L_1} s_{jk}z^{jk}_n \end{aligned} \tag{19}$$

Subject to :

$$\sum_{n=F_1}^{L_T+1} Q_{pnt} = d_{pt} \quad \forall p,t \tag{20}$$

$$\sum_{p\in S(k)} \rho_p\left(\sum_{t=1}^{T} Q_{pnt}\right) + st_{jk}z^{jk}_n \leqslant Cy^k_n \quad \forall j,k,n=F_1,\ldots,L_1 \tag{21}$$

$$\sum_{p\in S(k)} \rho_p\left(\sum_{t=1}^{T} Q_{pnt}\right) \leqslant Cy^k_n \quad \forall j,k,n=F_2,\ldots,L_T \tag{22}$$

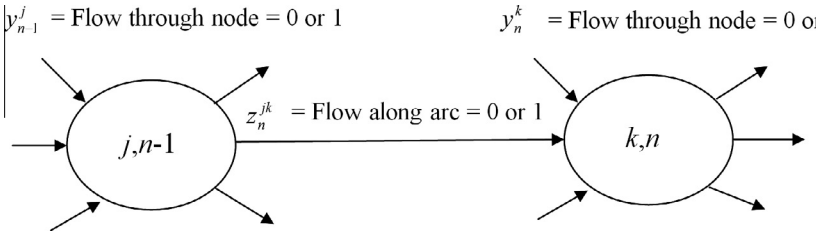

Fig. 1. Flow conservation through nodes and along arcs.

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$$\sum_k y_n^k = \frac{L}{\eta_t} \quad \forall t, n = F_t, \ldots, L_t \tag{23}$$

$$z_n^{jk} \geqslant y_{n-1}^j + y_n^k - 1 \quad \forall j,k,n = F_1, \ldots, L_1 \tag{24}$$

$$\sum_{j=1}^{K} z_n^{jk} = y_n^k \quad \forall k, n = F_1, \ldots, L_1 \tag{25}$$

$$\sum_{k=1}^{K} z_n^{jk} = y_{n-1}^j \quad \forall j, n = F_1 + 1, \ldots, L_1 \tag{26}$$

$$\sum_{j=1|j\neq k}^{K} \sum_{n=F_1}^{L_1} z_n^{jk} \leqslant 1 \quad \forall k \tag{27}$$

$$y_n^k \in \{0,1\} \quad \text{with} \quad y_0^k = 0 \quad \forall k, n = F_1, \ldots, L_1 \tag{28}$$

$$y_n^k \geqslant 0 \quad \text{and integer} \quad \forall k, n = F_2, \ldots, L_T \tag{29}$$

$$z_n^{jk} \geqslant 0 \quad \forall j,k,n = F_1, \ldots, L_1 \tag{30}$$

$$Q_{pnt} \geqslant 0 \quad \text{and integer} \quad \forall p,t,n = \mathrm{F}_1, \ldots, L_1 \tag{31}$$

$$Q_{pnt} \geqslant 0 \quad \forall p,t,n = \mathrm{F}_2, \ldots, L_T \tag{32}$$

$$I_{pt}^+ \text{and} I_{pt}^- \geqslant 0 \quad \forall p,t \tag{33}$$

Constraints (13) are now replaced by (21) for the first period and (22) for the remaining periods. As in Araujo et al. (2007), constraints (5) are replaced by (23) which impose exactly L/η_t setups in subperiod n, i.e., the number of loads in period t divided by the number of subperiods in period t. For example, period 1 has 10 subperiods and can handle 10 loads, so $L/\eta_1 = 10/10 = 1$, whereas period 2 has one subperiod and can handle 10 loads, so $L/\eta_2 = 10/1 = 10$. Constraints (24), as well as the strengthening constraints (25)–(27) are now restricted to the first period.

5. Metaheuristics for the Extended Formulation (EF)

The models were solved on a rolling horizon basis in two steps using the *relax-and-fix* method, as follows:

Relax all integer variables, except the first day's binary variables y_n^k for $n = F_1, \ldots, L_1$, these being the most important decisions in the rolling horizon method. Solve this relaxed problem.
Fix the first day's binary variables y_n^k at their solution values in step 1. Retighten to be integer the y_n^k variables for the subsequent days ($n = F_2, \ldots, L_T$) and the x_{pn} variables for the first day ($n = F_1, \ldots, L_1$). Solve this partially fixed problem.

Note that solving the model on a rolling horizon basis means that the initial setup state is zeroed as we proceed through the rolling horizon. This can result in slightly different solutions to the Cplex solution of the OF and EF formulations.

Step 1 is still time-consuming to solve optimally, so a solution is obtained using one of the three metaheuristics methods described in this section.
Step 2 can be optimally solved in a few seconds with Cplex, since a binary variable y_n^k which is fixed to 1 implies that $Q_{pnt} = 0$ for all $p \notin S(k)$, i.e., products that do not use material k are not manufactured in subperiod n, thus eliminating many integer variables and constraints.

Consequently, the three solution methods developed in the rest of this paper focus on step 1 and are now briefly described. Details can be found in Araujo et al. (2007).

5.1. Descent Heuristic (DH)

Step 1 of the *relax-and-fix* method can be solved using a local search *descent heuristic* (Araujo et al., 2007) to find the values of the first day's binary variables y_n^k for $n = F_1, \ldots, L_1$. The search starts with a random initial solution whose objective function value is obtained by solving the linear programme (LP) that results from fixing the values of the binary variables. The y_n^k solution values for day 1 are then randomly modified to provide a neighbouring solution whose objective function value is then calculated via the dual simplex method to efficiently resolve the modified LP. If it is an improvement, the neighbouring solution is adopted as the current one. Experiments showed that 1000 iterations are sufficient to obtain a good solution within an acceptable running time.

5.2. Diminishing Neighbourhood (DN) Search

This method adapts the local search descent heuristic described above by beginning with a large neighbourhood to encourage diversity, and then gradually diminishing its size so as to increasingly intensify the search. The search starts with the largest possible neighbourhood and after a certain number of iterations the neighbourhood size is reduced. The search ends with the smallest neighbourhood where just one position is changed. 1000 iterations in total were made.

5.3. Simulated Annealing (SA)

Simulated annealing (Eglese, 1990) is a variant of the local search descent heuristic that tries to avoid getting trapped at a local optimum by permitting worsening moves away with probability:

$$p(\Delta of\, v) = e^{-\left(\frac{\Delta of\, v}{Temp}\right)} \tag{34}$$

where *Temp* is a gradually-cooling "temperature" and Δofv the amount by which the new move worsens the objective value of the solution. As the search progresses, the best solution encountered is recorded. 1000 iterations in total were made. The parameters used in the tests below were the same as in Araujo et al. (2007).

6. Computational results

This section first describes how the test data was generated. It then goes onto analyse the use of the Cplex 12.0 solver against the metaheuristics from Araujo et al. (2007) on the Original Formulation (OF). Then Cplex 12.0 is compared against the metaheuristics from Araujo et al. (2007) under the extended formulation incorporating the *a priori* constraints (EF + V1 + E1).

The models were implemented in the AMPL modelling language (Fourer, Gay, & Kernighan, 2002). The optimizer Cplex 12.0 used its *default* parameters. The tests were carried out on an Intel Core 2 Duo CPU P8400 at 2.26 GHz 2.27 GHz with 3.0 GB of RAM, under Window Vista Business.

6.1. Data generation

The data set is presented in Table 1. It is the same as that generated by Araujo et al. (2007). The instances have 10, 50 and 100 products (small, medium and large, respectively). The capacity can be (loose, tight or too tight) and setup costs are high or low, making a total of $3 \times 3 \times 2 = 18$ combinations. Ten replications were run for each of the 18 combinations. The planning horizon extends over 5 periods, each divided into 10 subperiods.

6.2. Comparing Cplex on the OF formulation against the metaheuristics on the OF–RH formulation

Table 2 compares the objective function value resulting from an hour of Cplex 12.0 solving time applied to the Original Formulation

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Table 1
Parameters used for generating uniformly-distributed test data.

Parameters	Values
Number of products and materials: (P,K) pairs	Small problem: (10,2) Medium problem: (50,10) Large problem: (100,20)
Number of periods (T)	5
Inventory penalty (h^+_{pt})	$U[2,10]$
Backlog penalty (h^-_{pt})	$U[20,100]$
Necessary capacity to produce one unit of product p(ρ_p)	$U[0.1,3]$
Setup time (st_{jk})	$U[5,10]$
Setup penalty (s_{jk})	Low: $50 \times st_{jk}$ High: $500 \times st_{jk}$
Demand (d_{pt})	$U[40,60]$
Tightness of capacity (C)	Loose: $LFLC/0.6$
Note: $LFLC = \sum_{p=1}^{P}\sum_{t=1}^{T} d_{pt}\rho_p / L_T$	Tight: $LFLC/0.8$ Too tight: $LFLC/1.0$

(OF) over the values obtained by the Descent Heuristic (DH), Diminishing Neighbourhood Search (DN) and Simulated Annealing (SA).

The heuristic procedures are applied on the rolling horizon-based model RH proposed in Araujo et al. (2007). In this paper this model will be called OF–RH. A solution to model OF–RH is not a solution to model OF (1)–(9), as just the first loadings are scheduled and actually implemented, while the other days are planned only approximately. However, the application T times of model OF–RH, starting consecutively at periods 1, 2,...,5, with an always-shortening horizon ($T = 5,4,3,2,1$), will provide a feasible solution to model (1)–(9), enabling a comparison of results.

The table shows the percentage mean difference between the Cplex solution and the indicated metaheuristic, calculated as:

$$\text{Difference} = \left(\frac{\text{Heuristic solution value} - \text{Cplex solution value}}{\text{Cplex solution value}}\right) \times 100\%.$$

Thus a positive difference value provides a % measure of how much better the Cplex solution is compared to the heuristic solution. Conversely, a negative difference value indicates how much worse the Cplex solution is compared to the heuristic one.

Note from Table 2 that Cplex generally outperforms the metaheuristics for small and medium sizes problems, but certainly not for large problems where the metaheuristics perform far better. This is consistent with the generally acknowledged advantage that metaheuristics show over mathematical programming for large instances.

6.3. Using Cplex to compare the Original Formulations (OF) and extended formulations (EF + V1 + E1)

Tables 3–5 show the results using Cplex on both the Original Formulation (OF) and the extended one (EF + V1 + E1). The values shown are the means from 10 runs for each of the 18 combinations in Table 1 using Cplex for up to an hour, along with upper (UB) and lower (LB) bounds. The gap shown is the percentage mean difference between the upper and lower bounds UB and LB. The nodes

Table 2
Difference (%) between Cplex and the metaheuristics, both on OF.

		Setup cost					
Methods		DH		DN		SA	
Problem	Capacity	Small	Large	Small	Large	Small	Large
10 Products	Loose	25.82	66.46	6.34	59.99	6.34	59.97
	Tight	90.73	60.25	12.89	44.80	6.34	44.82
	Too tight	63.47	32.37	−19.12	4.58	−19.99	7.95
50 Products	Loose	30.01	32.35	8.43	12.55	10.73	12.90
	Tight	28.77	15.05	17.65	−2.69	16.52	−1.55
	Too tight	27.84	6.73	16.79	−1.89	23.29	−3.27
100 Products	Loose	−94.04	−91.69	−94.09	−92.18	−94.14	−92.16
	Tight	−86.14	−88.03	−86.43	−88.66	−86.66	−88.68
	Too tight	−75.31	−82.72	−76.32	−83.91	−76.60	−83.32

Table 3
Tests with 10-product instances.

Problem		OF formulation				
Capacity	Setup cost	UB	LB	Gap (%)	Nodes	Cuts
Loose	Small	1961	1961	0.00	2910	133
	Large	13,040	13,034	0.05	6,845,657	145
Tight	Small	1961	1961	0.00	1278	146
	Large	14,405	14,382	0.16	11,696,057	150
Too tight	Small	6262	6015	3.94	11,149,917	282
	Large	22,575	22,359	0.96	12,292,987	197
Mean		10,034	9952	0.82	6,998,134	175
		EF + V1 + E1 formulation				
Loose	Small	1961	1961	0.00	2198	477
	Large	13,040	12,401	4.90	1,806,223	567
Tight	Small	1961	1961	0.00	776	229
	Large	14,408	14,383	0.17	4,493,654	323
Too tight	Small	6327	6027	4.74	3,186,267	377
	Large	22,816	22,368	1.96	3,459,528	340
Mean		10,085	9850	2.33	2,158,107	386

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Table 4
Tests with the 50-product instances.

Problem		OF formulation				
Capacity	Setup cost	UB	LB	Gap (%)	Nodes	Cuts
Loose	Small	21,154	1900	91.01	13,314	1901
	Large	122,272	2882	97.64	11,253	2109
Tight	Small	33,563	8002	76.16	8311	2077
	Large	147,141	8255	94.39	7270	2272
Too tight	Small	93,839	42,425	54.79	14,329	1381
	Large	212,540	43,799	79.39	11,578	1396
Average		105,085	17,877	82.99	11,009	1856
		EF + V1 + E1 Formulation				
Loose	Small	19,974	4912	75.41	12,193	2938
	Large	117,433	27,807	76.32	9591	3244
Tight	Small	30,515	11,400	62.64	9252	3067
	Large	132,776	33,846	74.51	6673	3406
Too tight	Small	90,254	46,183	48.83	7524	3318
	Large	193,250	68,028	64.80	5228	3428
Average		97,367	32,029	67.10	8410	3233

Table 5
Tests with the 100-product instances.

Problem		OF formulation				
Capacity	Setup cost	UB	LB	Gap (%)	Nodes	Cuts
Loose	Small	3,994,738	35,373	99.11	124	3085
	Large	3,566,167	102,836	97.12	185	39,101
Tight	Small	2,561,219	61,540	97.60	225	2916
	Large	4,135,672	48,497	98.83	127	2916
Too Tight	Small	2,316,719	220,323	90.49	290	220,060
	Large	3,869,352	220,060	94.31	227	2454
Average		3,407,311	114,771	96.63	2041	38,923
		FE + V1 + E1 Formulation				
Loose	Small	140,947	97,523	30.81	723	5324
	Large	295,012	117,271	60.25	675	5742
Tight	Small	718,120	131,557	81.68	442	5074
	Large	326,927	151,379	53.60	559	5647
Too Tight	Small	612,921	293,273	52.15	149	4087
	Large	983,102	321,127	67.33	386	5035
Average		512,838	185,355	63.86	1975	4735

Table 6
Difference (%) between Cplex on EF + V1 + E1 and the metaheuristics on OF–RH.

		Setup cost					
Methods		DH		DN		SA	
Problem	Capacity	Small	Large	Small	Large	Small	Large
10 Products	Loose	25.82	66.46	6.34	59.99	6.34	59.97
	Tight	90.73	60.21	12.89	44.77	6.34	44.78
	Too tight	61.80	30.97	−19.95	3.47	−20.80	6.80
50 Products	Loose	37.69	37.80	14.83	17.19	17.27	17.56
	Tight	41.63	27.50	29.40	7.83	28.16	9.10
	Too tight	32.92	17.38	21.43	7.90	28.19	6.38
100 Products	Loose	68.96	27.27	67.53	19.86	66.13	20.09
	Tight	−50.57	51.43	−51.60	43.46	−52.43	43.19
	Too tight	−6.67	−31.98	−10.51	−36.67	−11.54	−34.34

and cuts columns show respectively the number of nodes and cuts used by Cplex up to the time limit.

Note that for small instances (Table 3) the results are practically identical, while for medium-sized instances (Table 4), the EF always provided slightly better solutions. For large instances (Table 5), the EF always provided much better solutions. These test results clearly illustrate that the extended formulation is more efficient and thus more effective than the original formulation. However, note that, for medium and large problems, the gaps obtained by Cplex are still large, being more than 60% on average, even for the tighter EF + V1 + E1 reformulation.

6.4. Comparing Cplex on the EF+V1+E1 formulation against the metaheuristics on the OF–RH formulation

Table 6 shows the differences between Cplex 12.0 on the EF + V1 + E1 formulation and the metaheuristics on the OF–RH formulation. Note that the EF + V1 + E1 formulation generally

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Table 7
Difference (%) between Cplex on EF + V1 + E1 and the metaheuristics on EF – RH + V1 + E1.

Methods		Setup cost					
		DH		DN		SA	
Problem	Capacity	Small	Large	Small	Large	Small	Large
10 Products	Loose	8.96	58.59	−4.44	7.09	−5.47	2.75
	Tight	6.93	20.69	−16.18	−12.08	−22.37	−27.11
	Too tight	162.86	8.33	−66.92	−53.32	−54.24	−63.10
50 Products	Loose	39.31	−43.27	10.56	−51.24	9.90	−62.65
	Tight	79.48	−29.62	1.92	−46.08	−20.89	−54.11
	Too tight	77.94	17.71	78.47	−41.40	−73.24	−66.34
100 Products	Loose	−27.60	−58.15	−32.38	−55.16	−37.32	−63.56
	Tight	−67.26	2.03	−79.87	−49.70	−87.11	−56.30
	Too tight	−31.83	−37.64	−50.71	−60.93	−67.53	−82.01

outperforms the metaheuristics in small and medium problems, but not for large instances, an observation very similar to that in Section 6.2.

6.5. Comparing Cplex on the EF + V1 + E1 formulation against the metaheuristics on the EF – RH + V1 + E1 formulation

Considering the good results obtained after the resolution of the reformulation with the Cplex 12.0, the metaheuristics from Araujo et al. (2007) were adapted for solving the EF + V1 + E1 reformulation. The results are shown in Table 7. Observe that now the metaheuristics generally outperforms the EF + V1 + E1 formulation in small and medium problems, and very clearly so for large instances. These results are the really interesting ones, showing that metaheuristics are very competitive even against a strengthened mathematical programming reformulation. Comparing the three metaheurstics, note that Simulated Annealing (SA) performs best, followed by Diminishing Neighbourhood (DN).

7. Conclusions

This paper considered a lot sizing model that sequences the preparation of a key material and decides the production lot sizes of the final products. This class of problem is very prevalent in the literature, using GLSP formulations. However, recent ATSP-inspired formulations have shown better performance. Aiming to tighten the linear relaxation of the GLSP-type model in Araujo et al. (2007), a reformulation as a facility location problem was extended with setup elimination constraints and flow equations, strengthening the link between setup and changeover variables. Three different metaheuristics and an industry-standard optimizer were then successfully applied to accelerate and improve the solution of both the original and extended formulations, particularly applying simulated annealing to the latter.

Computational tests showed that on the larger instances the metaheuristics outperformed Cplex and the reformulated model was more effective than the original model. Jointly applied, the metaheuristics and the extended formulation were effective on both small and large sized instances, particularly the latter where metaheuristics really show their advantage.

Future work will extend and adapt the metaheuristics to the case where multiple machines process the raw materials, the predominant case in many industrial processes. Considering the prevalence of this type of problem, future research should analyse and compare the GLSP and ATSP formulations, both theoretically and computationally.

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