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A HEURISTIC METHOD FOR LOT-SIZING IN MULTI-STAGE SYSTEMS

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Scope and Purpose—Material Requirements Planning (MRP) is a computer-based methodology used widely in industry for planning the production of the end items and their components over the medium term. However, most MRP systems present two major drawbacks, namely production and inventory costs and capacity constraints are not taken into account. As a result, uneconomical or capacity infeasible production plans can be generated. In this article we propose a heuristic method for resource capacitated MRP systems which attempts to find a feasible solution and minimize total costs.

Abstract—This article considers the lot-sizing problem in multi-stage production settings with capacity-constrained resources. This problem deals with the determination of a production plan for the end item and its components in order to meet the forecast demand in each period of a planning horizon. The production plan should minimize the sum of production, setup and inventory costs. A heuristic method build upon a formulation of the problem in terms of echelon stock is developed. Computational results are reported and the solutions' quality is evaluated through Lagrangean lower bounds. © 1997 Elsevier Science Ltd

1. INTRODUCTION

Material Requirements Planning (MRP) is a computer-based methodology which is widely used in industry for the medium term planning of production in a multi-stage environment [1]. The input for the MRP is a Master Production Schedule (MPS) that specifies a production plan for each end item over the time period of a planning horizon. An end item production plan generates a dependent demand for its constituent components. Taking into account each component's production or purchasing lead time and its current and planned stock, the MRP method provides a synchronized production plan for all components which states how much to produce of each component in each time period.

In a real multi-stage production setting there are two important aspects that should be considered [2,3]. First, the production of components share one or more capacity-constrained resources. Second, setup and inventory costs are associated with the production plan of each component. However, most MRP systems disregard these aspects; capacity constraints are ignored and the production plans are calculated using

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very simple heuristic methods or even the lot-for-lot method which reduces inventory costs but fails to consider the setup cost economies that could result if some production lots were bunched into a single lot. The production plans generated by such systems can be capacity infeasible and are unlikely to be economical.

This article deals with the multi-stage capacitated lot-sizing (MSCLS) problem which is concerned with the determination of production lot sizes in resource-constrained multi-stage MRP systems so as to minimize the sum of production, setup and inventory costs. Florian *et al.* [4] have shown that the single-item capacitated lot-sizing problem without setup time is NP-hard which implies that the MSCLS problem is also NP-hard. Moreover, when setup times are considered the decision problem associated with the existence of a feasible solution for the MSCLS problem is NP-complete [5]. These results imply that it is unlikely that any optimal algorithm can solve large problems.

Most of the research so far has focused on the uncapacitated version of the problem and optimal solutions were obtained for special cases and small problems [6–13]. As a consequence, several heuristic methods were developed [14–19]. The literature on the MSCLS problem is rather scarce because of its greater complexity. Billington *et al.* [20] present a model for a general product structure with production lead times, overtime and capacity of the work centers. A compression procedure on the product structure is suggested in order to reduce the size of the problem but the authors do not propose a solution method. Optimal algorithms were developed for special cases [21–24]. Clark and Armentano [25] consider a model similar to Billington *et al.* [20] and suggest a Branch-and-Bound solution procedure which was applied to small problems. Some heuristics were proposed for the most general model considered in Billington *et al.* (1983) and Clark and Armentano (1995) [20,25]. Billington *et al.* [26] consider only one capacitated (bottleneck) work center and use smoothing procedures which shift production in order to remove capacity infeasibility. Maes *et al.* [5] assume that setup times are zero and propose heuristics based on Linear Programming relaxation and rounding procedures of the binary variables which represent the setup decisions. Roll and Karni [27] also assume zero setup times and suggest a heuristic approach which starts from a solution generated by the sequential application of the optimal single-item uncapacitated Wagner–Whitin [28] algorithm. Next, eight types of production shifts are used with the aim of eliminating capacity infeasibility and reducing costs. Kuik *et al.* [29] consider assembly systems and assume that only one work center is capacitated and that setup times are zero. They propose heuristics based on simulated annealing and tabu search techniques. Billington *et al.* [30] consider serial systems with multiple end items with zero setup time and capacity constraints in each level of the product structure. They study the effectiveness of several single level heuristics applied sequentially from the end items. This article considers a model for the MSCLS problem which includes setup costs, setup times and multiple constrained resources. Moreover, the product structure is general, i.e., a component can be used in the assembly of more than one component. Component lead times are assumed to be zero. As in Clark and Armentano [31], the heuristic proposed in this article is based on an echelon stock formulation of the problem. Starting from a solution to the corresponding uncapacitated problem the heuristic tries to obtain a capacity feasible solution by moving production backwards and forwards in time. The heuristic in Clark and Armentano [31] stops when a feasible solution is found whereas the heuristic proposed here has additional procedures which increase the likehood of providing feasible solutions and also lower cost solutions.

2. PROBLEM FORMULATION

In this section the MSCLS problem is first formulated in terms of conventional stock and then reformulated in terms of echelon stock.

The component structure of a product can be represented by a directed graph whose nodes correspond to the components. The nodes are numbered from 1 to N such that if component i is a sub-component of component j then j is called a successor of i and $i>j$. The end item is node 1. Define

$S(i)$=the set of immediate successor components to component i;
$P(i)$=the set of immediate predecessors of component i.

An assembly product structure is such that the set $S(i)$ has a single element for all $i=2,\ldots,N$. Otherwise, we have a general product structure. Examples of 4-component product structures are shown in Fig. 1.

In order to formulate the problem, let:

r_{ij}=number of units of component i needed by one unity of component $j\in S(i)$;

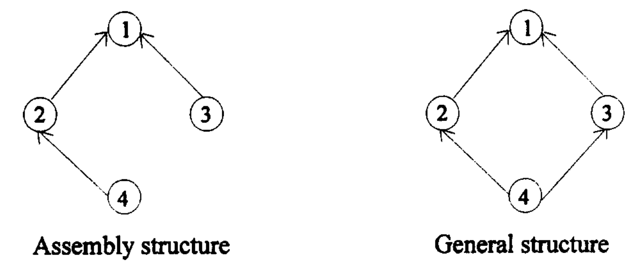

Fig. 1. Examples of product structures.

T=number of periods in the planning horizon;
d_{it}=independent demand for component i in period t;
c_{it}=unit production cost of component i in period t;
s_{it}=setup cost incurred if component i is produced in period t;
h_{it}=unit holding cost of component i at the end of period t;
K=number of capacity-constrained resources used to produce all components;
b_{kt}=amount of resource k available in period t;
f_{ikt}=fixed amount of resource k necessary to produce component i in period t (for example, setup time on a machine);
v_{ikt}=unit amount of resource k necessary to produce component i in period t;
x_{it}=lot-size of component i in period t (a decision variable);
M_{it}=an upper bound on x_{it};
y_{it}=a binary variable which assumes value 1 if component i is produced in period t and 0, otherwise (a decision variable);
I_{it}=inventory stock of component i at the end of period t (a decision variable).

The MSCLS problem can now be formulated as the following mixed integer programming model.

$$(MI)\text{: minimize} \sum_{i=1}^{N} \sum_{t=1}^{T} [s_{it}y_{it}+c_{it}x_{it}+h_{it}I_{it}],$$

subject to;

$$I_{i,t-1}+x_{it}-I_{it}=d_{it}+\sum_{j\in S(i)} r_{ij}x_{jt} \quad i=1,\ldots,N;\ t=1,\ldots,T \tag{1}$$

$$\sum_{i=1}^{N} [f_{ikt}y_{it}+v_{ikt}x_{it}]\le b_{kt} \quad k=1,\ldots,K;\ t=1,\ldots,T \tag{2}$$

$$x_{it}\le M_{it}y_{it} \quad i=1,\ldots,N;t=1,\ldots,T \tag{3}$$

$$x_{it}\ge 0, I_{it}\ge 0, y_{it}=0 \textit{ or } 1 \quad i=1,\ldots,N;t=1,\ldots,T \tag{4}$$

Constraints (1) represent the inventory balance equations and the term $\sum_{j\in S(i)} r_{ij}x_{jt}$ is called the dependent demand of component i in period t. Constraints (2) state that the amount of each resource used for production is limited and constraints (3) ensure that the setup cost s_{it} is charged when x_{it} is positive. The nonnegative constraint on the inventory stock I_{it} in (4) precludes backlogging.A useful equivalent model for the proposed heuristic method is now given in terms of echelon stock [10–12,16,17,25,32]. Define the echelon demand D_{it} and the echelon stock E_{it} of component i at the end of period t in a recursive manner as follows:

$$D_{it}=d_{it}+\sum_{j\in S(i)} r_{ij}D_{jt},$$

and,

$$E_{it}=I_{it}+\sum_{j\in S(i)} r_{ij}E_{jt}.$$

Note that D_{it} is the total system demand for component i in period t and it is composed of the independent demand d_{it} and the demand of its successor components $\sum_{j\in S(i)} r_{ij}D_{jt}$. Similarly, E_{it} is the total system stock of component i at the end of period t made up of its own stock I_{it} and the echelon stock of its successor components $\sum_{j \in S(i)} r_{ij}E_{jt}$. A unit echelon stock holding cost is charged to E_{it} and is defined as;

$$e_{it}=h_{it}-\sum_{j \in P(i)} r_{ji}h_{jt}.$$

It has been shown in [33] that the model (MI) is equivalent to the following model in terms of echelon stock;

$$(ME)\text{: minimize} \sum_{i=1}^{N}\sum_{t=1}^{T}[s_{it}y_{it}+c_{it}x_{it}+e_{it}E_{it}],$$

subject to,

$$E_{i,t-1}+x_{it}-E_{it}=D_{it} \quad i=1,\ldots,N;\ t=1,\ldots,T \tag{5}$$

$$\sum_{j \in S(i)} r_{ij}E_{jt}-E_{it}\leq 0 \quad i=1,\ldots,N;\ t=1,\ldots,T \tag{6}$$

$$\sum_{i=1}^{N}[f_{ikt}y_{it}+v_{ikt}x_{it}]\leq b_{kt} \quad k=1,\ldots,K;\ t=1,\ldots,T \tag{7}$$

$$x_{it}\leq M_{it}y_{it} \quad i=1,\ldots,N;\ t=1,\ldots,T \tag{8}$$

$$x_{it}\geq 0, E_{it}\geq 0, y_{it}=0 \text{ or } 1 \quad i=1,\ldots,N;\ t=1,\ldots,T. \tag{9}$$

Constraints (5) represent the echelon inventory balance equations and the inter-echelon constraints (6) follow from the definition of echelon stock and the non-negativity of the conventional stock I_{it}. It can be shown easily [33] that constraints (6) imply $E_{it}\geq 0$. Therefore, the non-negativity constraint of the echelon stock variable E_{it} in (9) is redundant.

Echelon stock is a useful analytically simplifying concept because, whereas in (1) the amount of conventional stock I_{it} depends not only upon the lot-size of component i but also upon the components $j \in S(i)$, the amount of echelon stock in (5) depends only on the lot-size of component i. As a consequence, the Lagrangean relaxation [34,35] of (ME) with respect to constraints (6) and (7) results in a Lagrangean problem which decomposes into a set of N independent, uncapacitated, single-item lot-sizing problems solvable by the Wagner–Whitin algorithm [28]. The modified subgradient method [36] is then used to optimize the dual function and to obtain a lower bound on the value of the optimal solution for (ME).

3. THE HEURISTIC METHOD

The heuristic proposed for solving problem (ME) consists of four procedures: a starting solution (P1), smoothing (P2), improvement (P3) and merging (P4). In procedure P1 an initial solution is obtained by disregarding the capacity constraints. If the initial solution is capacity infeasible, the smoothing procedure P2 shifts production in an attempt to obtain a feasible solution. The improvement procedure P3 starts from a capacity feasible solution and tries to obtain a lower cost feasible solution. From the solution obtained in P3, production lots are merged in P4 with the aim of reducing setup costs. The resulting solution is in general capacity infeasible and it becomes a new starting solution. Procedures P2, P3 and P4 are executed successively until a stopping criterion is satisfied. The pseudo-code of heuristic method is given next.

Let;

r=iteration counter
$S[r]$=solution obtained at iteration r
$f(S[r])$=cost value of $S[r]$
S^*=incumbent solution
ITMAX=maximum number of iterations of the heuristic.

The pseudo-code of the heuristic is as follows:

```
r=0
S[r]=starting solution                    {procedure P1}
for r=1 to ITMAX do
    if S[r] is infeasible then
        S[r]=smoothing                    {procedure P2}
```

end
if $S[r]$ is feasible then;
    $S[r]$=improvement {procedure P3}
    if $f(S[r])<f(S^*)$ then $S^*=S[r]$
end
$S[r]$=Merging {procedure P4}
end {for r}

If S^* is feasible then we have a solution for (ME) else we do not have a feasible solution for (ME). In the following the four procedures are detailed.

3.1. *Starting solution*

Consider the uncapacitated version of the problem (MI) which is obtained by ignoring the capacity constraints (2). The starting solution is a solution for the uncapacitated problem which is obtained by the sequential application of the single-item Wagner–Whitin algorithm to each component of the product structure. First, the algorithm is applied to the end item (component 1) which has only the independent demand d_{lt}. Next, the algorithm is applied to each component $i \in \{2,..., N\}$ which has both independent and dependent demands, i.e., $d_{it} + \sum_{j \in S(i)} r_{ij} x_{jt}$. If such a solution is capacity feasible in (MI), and consequently in (ME), then procedure P3 is called. Otherwise, the smoothing procedure P2 is applied.

3.2. *Smoothing procedure*

If procedure P1 yields a capacity infeasible solution for (ME) then the smoothing procedure P2 tries to find a feasible solution by moving production from infeasible periods to other periods. A period is said to be infeasible if a constraint in (7) is violated in such a period.

Given an infeasible period t, an attempt is made to transfer a production quantity q_{it} of the production x_{it} of component i in period t to another period $t\ell$. For each component i that is produced in an infeasible period t, two quantities are considered for moving to period $t\ell$:

- $W_{i,t\ell}$=the maximum quantity of the production x_{it} that ensures that the inter-echelon constraints (6) are still satisfied. This amount depends on whether $t\ell > t$ or $t\ell < t$ and will be explained later;
- Q_{itk}=the exact quantity of the production x_{it} which eliminates the overload of resource k in period t. Let $a^+=\max\{0,a\}$. It is easy to see that;

$$Q_{itk} = \sum_{j=1}^{N} [f_{jkt} y_{jt} + v_{jkt} x_{jt} - b_{kt}]^+ / v_{jkt} \text{ if } Q_{itk} < W_{i,tl}$$

Note that the amount Q_{itk} indicates if there is a quantity less than $W_{i,t\ell}$ which can reduce the overuse of resource k in the period t to zero.

The smoothing procedure contains two steps:

3.2.1. Backward shifts. Production shifts from periods $t=T$, $T-1$,..., 2 are analysed in that order. Portions of the production in an infeasible period t are moved to earlier periods until period t becomes feasible. If after these moves period 1 is feasible then we have a feasible solution for (ME).

For a given infeasible period t we consider moving a quantity q_{it} of the production x_{it} of each component i to earlier target periods $t\ell$. These periods are such that $\tau \le t\ell \le t-1$, where τ=max{1, the last period in which there is production of component i prior to period t}. The constraints (5) show that if q_{it} is moved from period t to an earlier period $t\ell$, then the echelon stock levels $E_{i\sigma}$, $\sigma = t\ell,..., t-1$ will increase by amount q_{it}. We must ensure that constraints (6) are satisfied after such a move, i.e., we must have;

$$\sum_{m \in S(j)} r_{jm} E_{m\sigma} + r_{ji} q_{it} \le E_{j\sigma}\, j \in P(i),\ \sigma = t\ell, \ldots, t-1$$

Thus,

$$q_{it} \le W_{i,t\ell} = \min \left\{ \min_{\substack{j \in P(i) \\ \sigma = t\ell,\ldots,t-1}} \left\{ \left(E_{j\sigma} - \sum_{m \in S(j)} r_{jm} E_{m\sigma} \right) / r_{ji} \right\}, x_{it} \right\} \tag{10}$$

Note that there always exists a component r whose entire production x_{rt} can be moved to an earlier period, namely, $r=\max\{i|x_{it}>0\}$. Since $x_{jt}=0$, for all $j\in P(r)$, then the production x_{rt} can be moved from period t to period $t-1$.

The choice of the quantity, component and target period $(q,i,t\ell)$ is based on a criterion which takes into account the cost variation and the use of resources if the quantity q_{it} were moved to $t\ell$. Such a criterion is called the Ratio Test and will be explained later.

If after these moves period 1 is infeasible, then we apply a step which consists of forward shifts. Otherwise, procedure P2 terminates.

3.2.2. Forward shifts. Production shifts from periods $t=1,2,\ldots, T-1$ are examined in that order. Portions of the production in an infeasible period t are moved to a later period $t\ell$. For a given infeasible period t we consider moving a quantity q_{it} of the production x_{it} of each component i to later target periods $t\ell$. These periods are such that $t+1\le t\ell\le\tau$, where $\tau=\min\{T$, the first period in which there is production of component i after period $t\}$. The choice of the quantity, component and target period $(q,i,t\ell)$ is also based on the Ratio Test.

The constraints (5) show that if q_{it} is moved from period t to a later period $t\ell$, then the echelon stock levels $E_{i\sigma}$, $\sigma=t,\ldots, t\ell-1$ will decrease by amount q_{it}. In order to ensure that constraints (6) are satisfied after such a move, we must have;

$$\sum_{j\in S(i)} r_{ij}E_{j\sigma}\le E_{it}-q_{it} \qquad \sigma=t,\ldots,t\ell-1$$

Thus,

$$q_{it}\le W_{i,t\ell}=\min_{\sigma=t,\ldots,t\ell-1}\left\{E_{i\sigma}-\sum_{j\in S(i)} r_{ij}E_{j\sigma}\right\} \tag{11}$$

Note that it is possible that for any component i no production quantity in period t can be moved to a later period. For example, in a lot-for-lot solution we have $I_{it}=E_{it}=0$ for all i and t, which implies $W_{i,t\ell}=0$.

If after these shifts a feasible solution is found then procedure P2 terminates. Otherwise, we apply again the backward shifts step.

3.2.3. Ratio test. This test is used to choose the quantity, component and target period $(q, i, t\ell)$, i.e., the quantity q_{it} of the production x_{it} of component i in period t to be moved to an earlier or later target period $t\ell$. The triple $(q, i, t\ell)$ chosen is that which minimizes the Ratio which is calculated as;

$$\text{Ratio}=\frac{\text{Extra cost }+\beta\text{ Penalty}}{\text{Excess Decrease}} \tag{12}$$

The Extra cost is the ratio additional cost/total cost, where the additional cost is the cost variation caused by the shifting of the quantity q_{it} from period t to period $t\ell$ and the total cost is the sum of the current production and inventory costs for all components.

The additional cost can be expressed as;

$$\text{Additional cost}=q.[(c_{it\ell}-c_{it})+\sum_k e_{ik}Z]+SU1-SU2$$

where;

$$k=\begin{cases} t\ell,\ldots,t-1 & \text{for } \ t\ell<t \text{ (backward step)}\\ t,\ldots,t\ell-1 & \text{for } \ t\ell>t \text{ (forward step)}\end{cases}$$

$$Z=\begin{cases} 1, & \text{for the backward step}\\ -1, & \text{for the forward step}\end{cases}$$

$$\text{SU1}=\begin{cases} s_{it\ell}, & \text{if } x_{it\ell}=0\\ 0, & \text{otherwise}\end{cases}$$

$$\text{SU2}=\begin{cases} s_{it}, & \text{if } q=x_{it}\\ 0, & \text{otherwise}\end{cases}$$

When the quantity q_{it} is moved from period t to period $t\ell$ there is a variation in the use of the resources in such periods. Let;

$$\mathrm{Excess(t)} = \sum_{k=1}^{K} \left\{ \frac{\sum_{i=1}^{N} [f_{ikt} y_{it} + v_{ikt} x_{it}] - b_{kt}}{b_{kt}} \right\}^{+} \tag{13}$$

denote the proportion of overuse of the resources in period t.The Penalty term in the Ratio represents the variation of the use of resources caused by the quantity q_{it} being moved from period t to period $t\ell$. It is defined as;

$$\text{Penalty} = \text{Excess after (t)} + [\text{ Excess after}(t\ell) - \text{ Excess before }(t\ell)]$$

where

$$\text{Excess after(t)} = \text{Excess(t) after the move,}$$

$$\text{Excess before(t)} = \text{Excess(t) before the move.}$$

Note that Penalty is non-negative and it can be interpreted as a cost for overuse of resources in periods t and $t\ell$.

The denominator Excess Decrease in (12) is the difference between Excess after (t) and Excess before (t).

Let a cycle denote a sequence of a backward step and a forward step in the smoothing procedure. In the first cycle we set $\beta=1$. If a feasible solution is not found in the first cycle we start the second cycle with $\beta=2$. In general, in cycle n, we set $\beta=n$. The increase of the factor β at each cycle reflects a greater importance to the overuse of resources. The number of cycles executed in this procedure is prespecified. If after such a number a feasible solution is not found then the smoothing procedure fails.

It is worth pointing out at this point that the heuristic method proposed in Clark and Armentano [31] consists of the first two procedures: starting solution and smoothing. However the smoothing procedure in Clark and Armentano [31] has only backward moves and no penalty is used to choose moves as in (12). Computational results show that such a penalty is essential to increase the likelihood of finding a feasible solution.

3.3. Improvement procedure

Starting from a feasible solution this procedures tries to find a lower cost feasible solution by moving production to earlier or later periods. The procedure is similar to procedure P2 and also has a backward step and a forward step. However, we only allow feasible and improving moves, i.e., moves which do not cause capacity infeasibility and which reduce the total cost.

For each period t we consider moving a quantity q_{it} of the production x_{it} of each component i to a target period $t\ell$, such that $t\ell<t$ in the backward step and $t\ell>t$ in the forward step. For each item i and each target period $t\ell$ we examine two quantities in period t: $q=W_{i,t\ell}$ and q sampled from the uniform distribution $U[0,W_{i,t\ell}]$. One reason for the second option is that we may reach different solutions with the same starting point. Another reason is that the costs are variant over time and a shift of a quantity $q<W_{i,t\ell}$ may result in a lower cost. Among all candidates $(q,i,t\ell)$ in period t we choose the one which minimizes the *Extra cost* as defined in procedure P2. The procedure ends when no feasible and improving moves exist.

3.4. Merging procedure

In this procedure we move the entire production x_{it} of component i in period t to an earlier or later period $t\ell$ where there is also production of component i. The objective is to merge entire production lots in order to save setup costs. Evidently, the production x_{it} can be moved to period $t\ell$ if $x_{it}=W_{i,t\ell}$, where $W_{i,t\ell}$ is given by (10) for $t\ell<t$, and (11) for $t\ell>t$. If the solution obtained at the end of this procedure is capacity feasible we proceed to procedure P3. Otherwise, a new starting point is obtained for procedure P2.

The origin periods t and the target periods $t\ell$ are determined from a sorting method which takes into account the setup costs and the proportion of capacity slack in period t which is given by;

$$\text{Slack(t)} = \sum_{k=1}^{K} \left\{ \frac{b_{kt} - \sum_{i=1}^{N} [f_{ikt} y_{it} + v_{ikt} x_{it}]}{b_{kt}} \right\}$$

Note that Slack(t) can assume positive and negative values while Excess(t) takes on only non-negative values.

The sorting method is better understood with the aid of an example. Suppose that $T=4$ and component 2 is produced in periods $t=1,2,3,4$. The setup costs for component 2 are given by $s_{21}=50$, $s_{22}=100$, $s_{23}=10$, and $s_{24}=150$. The slack in periods 1,2,3 and 4 is given by 2.0, 1.0, 0.1, and 1.5, respectively.

First, the periods are sorted in non-decreasing order of setup costs as shown in SORT1. Next, the periods are sorted in non-increasing order of slacks as shown in SORT2.

SORT1 3 1 2 4 SORT2 1 4 2 3

Note that from the viewpoint of setup costs it would be interesting to move the entire production of component 2 in period 4 to period 3. However, in terms of slack it is more attractive to move the entire production of component 2 in period 3 to period 1.

A single sorting of the periods is obtained from weights which are calculated as the sum of the positions of the periods in the arrays SORT1 and SORT2 as shown below.

Period	Position in SORT1	Position in SORT2	Weight
1	2	1	3
2	3	3	6
3	1	4	5
4	4	2	6

The periods are now sorted in non-decreasing order of weights. If two periods have the same weight then the period with lower setup cost precedes the other one. Finally, we obtain an array ORD in which the periods are sorted according to the setup costs and the slacks.

ORD 1 3 2 4

From this array, the periods considered as origin periods are $t=4$, 2, and 3 in that order, and the periods considered as target periods are $t\ell=1$, 3, and 2, in that order. We then examine moving the entire production of component 2 in period t to period $t\ell$ in the following order:4 to 1, 4 to 3, 4 to 2, 2 to 1, 2 to 3, and 3 to 1.

The first move allowed is carried out and such a merging procedure is repeated for all items.

3.5. An example

Consider the general structure in Fig. 1 ($N=4$) constrained by one resource ($K=1$) over a 4-period planning horizon ($T=4$) with the parameters in Table 1. The initial solution is shown in Table 2, which was obtained by sequentially applying the Wagner–Whitin algorithm (P1). The steps of procedures P2,

Table 1. Parameters for the example

	Period	1	2	3	4
Costs	c_{it}	1	1	1	1
	s_{it}	500	1000	100	1500
	e_{it}	2	2	2	2
Demand	d_{1t}	15	50	40	80
	d_{it} $(i\neq 1)$	0	0	0	0
Resources	v_{it}	1	1	1	1
	f_{it}	75	10	100	50
	b_t	600	600	600	600

Table 2. Initial solution for the example (infeasible)

Period	1	2	3	4		1	2	3	4
x_{1t}	65	0	120	0	E_{1t}	50	0	80	0
x_{2t}	65	0	120	0	E_{2t}	50	0	80	0
x_{3t}	65	0	120	0	E_{3t}	50	0	80	0
x_{4t}	130	0	240	0	E_{4t}	100	0	160	0

P3 and P4 are shown in Table 3 and the solutions obtained at the end of such procedures are shown in the Tables 4, 5 and 6, respectively.

Table 3. Heuristic steps for the example

			Smoothing procedure-backward
$t=4$	Excess(t)=0		
$t=3$	Excess(t)=0.67		
	$i=1, 2, 3$	$tl=2$	$W_{i,2}=0$, no production of these items can be transferred to previous periods
	$i=4$	$tl=2$	$W_{4,2}=240$ and $Q_{4,3}=400$ (note that Q is not considered since $Q>W$). $q=240$ ratio=0.70
		$tl=1$	$W_{4,1}=240 \Rightarrow q=240$ ratio=1.21
	Transfer $q=240$ of item $i=4$ in period $t=3$ to period $tl=2 \Rightarrow$ Excess(t)=0.10		
	$i=1$	$tl=2$	$W_{1,2}=0$
	$i=2$	$tl=2$	$W_{2,2}=120$ and $Q_{2,3}=60 \Rightarrow q=120$ ratio=1.90; $q=60$ ratio=1.87
		$tl=1$	$W_{2,1}=0$
	$i=3$	$tl=2$	$W_{3,2}=120$ and $Q_{3,3}=60 \Rightarrow q=120$ ratio=1.90; $q=60$ ratio=1.87
	$tl=1$		$W_{3,1}=0$
	Transfer $q=60$ of item $i=2$ in period $t=3$ to period $tl=2 \Rightarrow$ Excess(t)=0		
$t=2$	Excess(t)=0		
			Smoothing procedure-forward
$t=1$	Excess(t)=0.04		
	$i=1$	$tl=2$	$W_{1,1}=50$ and $Q_{1,1}=25 \Rightarrow q=50$ ratio=3.03; $q=25$ ratio=3.2
		$tl=3$	$W_{1,3}=0$
	$i=2$, 3 and 4		$W_{i,2}=0$
	Transfer $q=50$ of item $i=1$ in period $t=1$ to period $tl=2 \Rightarrow$ Excess(t)=0		
$t=2, 3$	Excess(t)=0		
Feasible solution, improvement procedure			
			Improvement Procedure-Backward
$t=4$	no production		
$t=3$	$i=1$	$tl=2$	$W_{1,2}=0$
	$i=2$	$tl=2$	$W_{2,2}=60 \Rightarrow q=60$ ratio>0; $q=1$ ratio>0 ($q=1$ was sampled from [1, 60])
		$tl=1$	$W_{2,1}=0$
	$i=3$	no transfer with ratio<0	
$t=2$	no transfer with ratio<0		
			Improvement procedure-forward
$t=1$	$i=2$	$tl=2$	$W_{2,2}=50 \Rightarrow q=50$ ratio=-1.25; $q=2$ ratio=-0.05
	Transfer $q=50$ of item $i=2$ in period $t=1$ to period $tl=2$		
$t=2, 3$	no transfers with ratio<0		
			Improvement procedure-backward
	no transfers with ratio<0		
			Improvement procedure-forward
$t=1$	$i=4$	$tl=2$	$W_{4,1}=50 \Rightarrow q=50$ ratio=-1.26; $q=9$ ratio=-0.23
	Transfer $q=50$ of item $i=4$ in period $t=1$ to period $tl=2$		
$t=2, 3$	no transfers with ratio<0		
Next backward and forward steps in improvement procedure yield no transfers with ratio <0, thus the merging procedure is carried out			
			Merging procedure
$i=1, 2, 3$	no transfers		
$i=4$	setup cost(t)=[500 1000 100 1500]		
	slack(t)=[-20.8-20.0 0.0-100.0]		
	ord=[1 3 4 2]		
	$t=2$,	$tl=1$	$W_{1,1}=290$ ($W_{1,1}=x_{1,2}$, thus the entire production can be transfered)
	Transfer $q=290$ of component $i=1$ in period $t=2$ to period $tl=1$		

Table 4. The solution at the end of the smoothing procedure (feasible with cost=8025)

Period	1	2	3	4		1	2	3	4
x_{1t}	15	50	120	0	E_{1t}	0	0	80	0
x_{2t}	65	60	60	0	E_{2t}	50	60	80	0
x_{3t}	65	0	120	0	E_{3t}	50	0	80	0
x_{4t}	130	240	0	0	E_{4t}	100	240	160	0

Table 5. The solution at the end of the improvement procedure (cost=7825)

Period	1	2	3	4		1	2	3	4
x_{1t}	15	50	120	0	E_{1t}	0	0	80	0
x_{2t}	15	110	60	0	E_{2t}	0	60	80	0
x_{3t}	65	0	120	0	E_{3t}	50	0	80	0
x_{4t}	80	290	0	0	E_{4t}	50	240	160	0

Table 6. The solution at the end of the merging procedure (infeasible)

Period	1	2	3	4		1	2	3	4
x_{1t}	15	50	120	0	E_{1t}	0	0	80	0
x_{2t}	15	110	60	0	E_{2t}	0	60	80	0
x_{3t}	65	0	120	0	E_{3t}	50	0	80	0
x_{4t}	370	0	0	0	E_{4t}	340	240	160	0

4. COMPUTATIONAL RESULTS

The heuristic was programmed in C and run on a SUN SPARC station 20 with clock of 50 MHz. Three configurations of number of items (N), number of periods (T) and number of resources (K) were chosen as (10, 12, 1), (10, 12, 2) and (17, 10, 1). For each configuration we considered three different product structures: flat, serial and general. A flat structure is an assembly structure such that $S(i)=1$, for $i=2,..., N$. A serial structure is also an assembly structure such that $S(i)=i-1$, for $i=2,..., N$. The general structures considered are shown in Figs 2 and 3.

Table 7 shows the parameters that were generated from a uniform distribution defined on an interval [a,b] and represented by U[a,b].

The parameter r_{ij} was set to 1 for all $i=1,..., N$ and $j \in S(i)$. Capacity was constant over time, i.e., $b_{kt}=b_k$ for $t=1,..., T$ and was generated from the lot-for-lot solution. The amount of resource k consumed by this

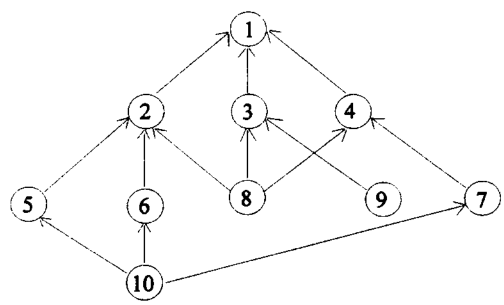

Fig. 2. General structure for $N=10$[25].

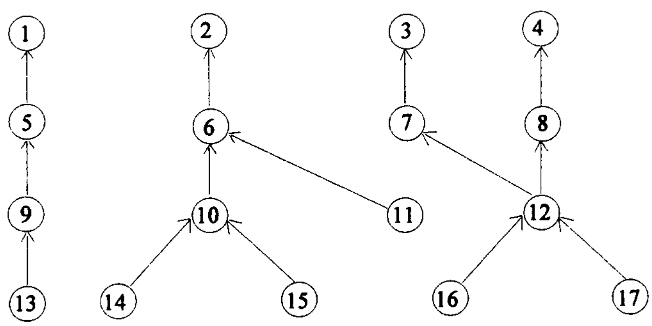

Fig. 3. General structure for $N=17$[5].

Table 7. Parameters generated from a uniform distribution U[a,b]

Parameter		Interval	Observation
Unit production cost	c_{it}	U[1.5, 2]	-
Setup cost	s_{it}	U[5, 95]	Low setup cost
		U[50, 950]	High setup cost
Unity echelon holding cost	e_{it}	U[0.2, 0.4]	-
Unity amount of resource	v_{ikt}	U[1.5, 2]	For k=1
		U[2, 3]	For k=2
Fixed amount of resource	f_{ikt}	U[150, 250]	For k=1
		U[200, 300]	For k=2
Demand	d_{it}	U[0, 180]	For the end items
		U[0, 18]	For the components

solution is calculated for each period t and the average (b_k) was taken over all periods. We then considered three types of capacity:

- Tight capacity: cap $1=0.9b_k$
- Normal capacity: cap $2=b_k$
- Loose capacity: cap $3=1.1b_k$

Our objective was to study how the heuristic's performance is affected by the following factors: configuration (N, T, K), product structure, setup cost and production capacity. For each combination of these factors we have a type of problem which was repeated 20 times with different random seeds. Thus, 1080 problems were generated.

The tests were performed in three phases. In phase 1 we tested the heuristic with procedures P1 and P2 only. In phase 2 we analysed the effect of the inclusion of procedure P3. Finally, in phase 3 we tested the complete heuristic (procedures P1, P2, P3 and P4). The maximum number of iterations (ITMAX) was set to 100 and the maximum number of cycles in procedure P2 was 6. This number was chosen for it was noted that in 95% of the problems no more than 3 cycles were required to find a feasible solution.

For a given infeasible solution, the average percentage of overuse of the resources is given by;

$$\mathrm{EXC}=\left(\sum_{t=1}^{T}\mathrm{Excess}(t)\right)/T\cdot 100, \tag{13}$$

where Excess(t) is given by (13). Let EXCi and EXCf denote the mean of EXC over the infeasible initial solutions obtained in P1 and the infeasible solutions provided by the heuristic, respectively.

Let FEA denote the percentage of feasible solutions that the heuristic was able to find for nontrivial infeasible problems. A problem is said to be trivial if it is infeasible in the first period. Hence, FEA is defined as;

$$\mathrm{FEA}=\frac{\mathrm{NF}}{\mathrm{NG}-\mathrm{NT}}\cdot 100,$$

where,

NF=number of problems where a feasible solution was found,
NG=number of problems generated, and
NT=number of trivial problems generated.

For a given problem the optimality guarantee of the best heuristic solution found for such a problem is given by;

$$\frac{Z_0-Z_1}{Z_1}\cdot 100$$

where,

Z_0=value of the heuristic solution, and
Z_l=Lagrangean lower bound for (ME).

The average quality of the heuristic solutions for a class of problems is the mean of the optimality guarantees and is denoted by GAP.

Table 8 shows how phase 1 is affected by the penalty parameter β in terms of feasible solutions that

Table 8. Effect of the penalty parameter β.

Structure	Setupcost	%EXCi	CA %FEA	CA %EXCf	$\beta=0$ %FEA	$\beta=0$ %EXCf	$\beta=1$ %FEA	$\beta=1$ %EXCf	$\beta=n$ %FEA	$\beta=n$ %EXCf
Flat	Low	7.7	33.5	1.6	80.2	5.8	88.0	5.1	88.6	5.4
	High	13.6	0.0	10.5	77.2	5.1	87.2	4.8	87.4	4.9
	Mean	10.6	16.8	6.9	78.7	5.4	87.6	4.9	88.0	5.2
Serial	Low	12.1	55.2	2.6	69.9	8.1	71.0	6.4	70.6	6.2
	High	20.1	1.2	9.3	61.3	6.9	71.8	6.0	72.4	6.1
	Mean	16.	28.2	7.2	65.6	7.4	71.4	6.2	71.5	6.1
General	Low	12.2	42.8	2.9	64.5	8.3	74.5	6.1	75.3	6.2
	High	19.1	0.0	11.9	59.0	6.4	75.2	5.9	75.9	5.9
	Mean	15.6	21.4	8.6	61.7	7.3	74.8	6.0	75.6	6.0
Overall mean		14.1	22.1	7.6	68.8	6.9	77.9	5.9	78.4	5.9

were found. We examined three cases: $\beta=0$ (no penalty); $\beta=1$ during all cycles; $\beta=n$, which means that $\beta=1$ in the first cycle and increases by one in subsequent cycles. Each cell in Table 8 is the average of 60 problems (20 problems for each type of capacity as described before). The column CA shows the percentage of feasible solutions found by the heuristic proposed in [31]. The overall mean in this table shows that the smoothing procedure with backward and forward shifts increase significantly the number of feasible solutions found (78.4% for $\beta=n$) when compared to the corresponding number (22.1%) provided by the heuristic Clark and Armentano [31] which has only backward moves. Moreover, increasing values of β yield more feasible solutions. For this reason we adopted this penalty strategy in all phases. Note also that the smoothing procedure was able to reduce the degree of capacity infeasibility from 14.1% to 5.9% as indicated by EXCi and EXCf for $\beta=n$.

Table 9 shows the results for phases 1 and 2 in terms of FEA and GAP. The objective of Table 9 is to show the degree the success of the improvement procedure P3. Note that GAP was reduced for all classes of problems.

Finally, Table 10 shows the results for the complete heuristic (phase 3). The last column indicates the CPU-time taken as the mean of the CPU-times for those problems that the heuristic found at least one feasible solution.

Table 9. Computational results for phases 1 and 2

Structure	Setup cost	Capacity available	Phase 1 % FEA	Phase 1 % GAP	Phase 2 % GAP
Flat	Low	Cap 1	64.7	2.8	1.7
		Cap 2	98.3	1.5	1.0
		Cap 3	100.0	1.1	0.8
	High	Cap 1	60.8	15.9	9.5
		Cap 2	98.3	10.7	6.0
		Cap 3	100.0	6.8	4.9
		Mean	88.0	5.9	3.7
Serial	Low	Cap 1	18.4	9.8	8.5
		Cap 2	85.7	7.6	6.8
		Cap 3	100.0	6.0	5.5
	High	Cap 1	26.5	41.5	34.8
		Cap 2	83.9	34.8	30.0
		Cap 3	100.0	25.1	21.6
		Mean	71.5	19.0	16.5
General	Low	Cap 1	34.0	8.4	5.3
		Cap 2	87.7	4.8	3.4
		Cap 3	98.3	3.5	2.8
	High	Cap 1	34.0	32.9	23.8
		Cap 2	89.5	23.5	16.8
		Cap 3	98.3	14.4	11.5
		Mean	75.6	12.7	9.6
	Overall mean		78.4	12.1	9.5

Table 10. Computational results for phase 3

Structure	Setup cost	Capacity available	%EXCi	Phase 3 %FEA	Phase 3 %EXCf	Phase 3 %GAP	Time (s)
Flat	Low	Cap 1	12.0	72.5	5.9	1.7	15.1
		Cap 2	7.3	98.3	1.1	1.0	12.1
		Cap 3	4.3	100.0	-	0.8	10.9
	High	Cap 1	17.4	72.5	5.2	11.5	13.6
		Cap 2	13.5	98.3	1.5	6.0	8.2
		Cap 3	10.5	100.0	-	4.9	6.2
		Mean	10.6	91.0	4.9	4.0	10.6
Serial	Low	Cap 1	18.7	30.6	7.8	9.0	5.7
		Cap 2	11.8	91.1	2.3	6.8	5.0
		Cap 3	6.9	100.0	-	5.5	4.1
	High	Cap 1	25.7	34.7	7.9	41.9	5.9
		Cap 2	19.8	89.3	1.8	32.1	4.3
		Cap 3	15.6	100.0	-	21.6	3.4
		Mean	16.1	76.4	7.0	17.5	4.4
General	Low	Cap 1	18.6	48.0	7.0	5.1	16.9
		Cap 2	11.7	93.0	0.4	3.4	8.6
		Cap 3	7.2	98.3	0.3	2.8	7.5
	High	Cap 1	24.6	48.0	7.0	28.0	16.0
		Cap 2	18.8	94.7	0.3	17.3	8.0
		Cap 3	14.8	98.3	0.3	11.5	5.4
		Mean	15.6	81.6	6.0	10.1	8.9
	Overall mean		14.1	83.1	6.3	10.1	8.2

Let us analyze the results in Table 10. Observe that for the flat structure, the interechelon constraints (6) show that the echelon stock of all components $i \geq 2$ are related only to the echelon stock of component 1. In contrast, serial and general structures exhibit a greater interdependence of the components echelon stock. As a consequence, the flat structure allows more moves which do not violate constraints (6) and, therefore, it is easier to find feasible solutions for such a structure as shown by the mean values. It was also observed that the number of moves allowed in the heuristic was smaller for the serial structure than for the general structures considered. As a result, the quality of the solutions for the serial structure is lower and the heuristic stops earlier.

With regard to the degree of capacity feasibility the heuristic has a good performance. The overall mean in this table shows that the heuristic was able to find a feasible solution for 83.1% of the problems. There is no guarantee that the remaining problems (about 17%) are infeasible. Nevertheless the mean excess of the consumption of resources for such problems is low (6.3%) as the column EXCf shows. Note that this excess is less than half of the mean excess of the initial solutions (14.1%) as shown in column EXCi.

A point that deserves a comment in concerned with cycling in the heuristic caused by procedure P4. In other words the feasible solution obtained in procedure P2 coincides with that obtained in the previous iteration. In the computational experiments, cycling was not noticed in feasible problems and a possible reason is that a move considered in the Improvement Procedure comes from a uniform distribution. In any case the Merging Procedure is effective for it causes an increase in feasible solutions found by the heuristic, from 78.4% (Table 9) to 83.1% (Table 10).

In general, Table 10 shows that the heuristic performs better for loose capacity and low setup cost. A possible reason for higher gaps associated with high setup costs for serial and general structures is that the solution of the Lagrangean problem tends to have few setups and therefore the cost of a feasible solution is considerably underestimated. More research is required on the heuristic in order to have a greater number of moves allowed, specially for serial and general structures. This would probably lead to new regions of feasible solutions.

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