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A Heuristic for a Resource-Capacitated Multi-Stage Lot-Sizing Problem with Lead Times

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Abstract. In this paper we propose a heuristic for the resource capacitated multi-stage lot-sizing problem with general product structures, setup costs and resource usage, work-in-process inventory costs and lead times. To facilitate the functioning of the heuristic, we use the formulation of the problem based on Echelon Stock in a rolling horizon scheme. The heuristic first obtains a reasonable solution to the corresponding uncapacitated problem and then tries to attain capacity feasibility by shifting production backwards in time. The concept of echelon stock makes the task of checking the inventory feasibility of proposed shifts easier than what would be the case with conventional installation stock. The heuristic is first tested computationally for problems with a five-component product structure over a twelve-period planning horizon for which optimal solutions were available and for which optimality precision guarantees were also obtained via Lagrangian Relaxation. The heuristic's performance is also explored with two different forty-component product structures, with high and low setup costs and is compared to the Lagrangian precision guarantees.

Key Words: Production Planning, Lot-Sizing, Inventory, Resource-Capacitated, Heuristic

Introduction

Material Requirements Planning (MRP)[1] is a computer-based methodology for planning the batch production of industrial products over the medium term. The user first develops a Master Production Schedule (MPS) that specifies in which time periods and what quantities each end-product will be produced. The MPS is then "exploded" to calculate its implications for the production or purchase of component parts, taking into account each component's production lead time, current and planned stock, and location in the end-product structures. The result is an MRP plan whose detailed production capacity implications are then calculated using the Capacity Requirements Planning (CRP) methodology. If the MRP plan is shown to be capacity infeasible then the production planner must alter the MPS and repeat the MRP-CRP calculations. The MPS-MRP-CRP cycle is repeated until a capacity feasible plan results. In this article we try to avoid the cycling procedure by allowing production shifts in the MRP plan so as to attain, if possible, capacity feasibility.

The question of determining production lot sizes to balance setup and work-in-process inventory costs in a multi-stage MRP system is known as the multi-stage lot-sizing (MSLS) problem. The MSLS problem is NP-hard[2] and optimal solutions have been obtained only for special cases or small problems[3-11]. As a result many heuristic approaches have been developed for the uncapacitated problem[12-16]. However, as is pointed out in two research surveys[2,17], the finite capacity of production facilities is a reality in all manufacturing systems and yet most MSLS research ignores this fact. When tight capacity constraints are included, it may be difficult to find even a feasible solution, let alone an optimal one. Heuristics have been developed and tested for special cases[18-20] including finite capacity at a single work centre[21,22] and small capacitated problems with zero setup and lead times[22,23]. However little research has been carried out into the more complex, but widely occurring, case in which lead times and multiple work centre setups may exist at any component production stage. The contribution of this paper is a heuristic for solving a generalisation of this case, namely a resource-capacitated MSLS problem with component production lead times, the setup use of multiple capacitated resources at all component stages and the sharing of capacitated resources between component stages, a reality in many industrial settings. To the best of our knowledge, such a general multi-stage model and heuristic have not yet been developed by other researchers.

The performance of the heuristic is first compared with optimal solutions for a general product structure with five components over a twelve-period rolling planning horizon, the largest problem for which optimal solutions were available. Optimality precision guarantees were also obtained for the problem via Lagrangian Relaxation. The heuristic's performance is then compared with the precision guarantees on two forty-component products, one with a very simple flat structure and the other with a general structure, with high and low setup costs. Encouraging conclusions are drawn concerning the heuristic's quality in the presence of tight resource constraints.

Problem Description and Model Formulation

In this section, we first formulate the uncapacitated MSLS problem with component lead time in terms of conventional installation stock and show how the component planning periods must be synchronized. The problem is then reformulated in terms of echelon stock in order to facilitate the functioning of the proposed heuristic. Finally resource-capacity constraints are added, leading to the problem tackled by our proposed heuristic. We note here that we study the problem on a rolling horizon basis.

The component structure of a product can be represented by an acyclic graph whose nodes correspond to the components. The nodes are numbered from 1 to N such that if component i is a sub-component of component j then j is called a successor of i and $i > j$. The end-item is node 1. Define

$S(i)$: the set of all immediate successor components to component i,

$P(i)$: the set of immediate predecessors of component i; if $P(i)=\varnothing$, then i is called *primary*,

r_{ij} : the number of units of component i needed by one unit of component $j \in S(i)$,

d_{it} : the independent demand for component i in period t,

c_{it} : the unit production cost of component i in period t,

s_{it} : the fixed setup cost incurred if component i is produced in period t,

h_{it} : the unit holding charge of component i at the end of period t,

x_{it} : the lot-size of component i in period t (a decision variable),

y_{it} : a dummy variable with value 1 if component i is produced in period t and 0 if not (a consequence of x_{it}),

I_{it} : the inventory stock of component i at the end of period t (a consequence of x_{it}),

$L(i)$: the production lead time of component i, measured as an integer multiple of production planning periods, ensuring that the lot x_{it} is available for consumption only at the beginning of period $t+L(i)$.

$T(i)+1$: the period at which, on a rolling horizon of T periods, production planning of component i must start in order to satisfy demand for its immediate successor components $S(i)$ at periods $t \geq T(i)+1+L(i)$ and eventually with that of the end item. The lot x_{it} belongs to one of the following three categories: (1) if $t \leq T(i)$ then x_{it} is known from previous rolling horizon applications of the model and has a fixed value; (2) if $T(i)+1 \leq t \leq T(i)+T$ then x_{it} is a model decision variable; (3) if $t > T(i)+T$ then x_{it} is beyond the planning horizon and will enter the model only in future rolling horizon applications. For any component i with lead time $L(i)$ and successor component $j \in S(i)$, we must have $T(i)+L(i) \leq T(j)$ otherwise lead time synchronization of the production line is not possible. Consequently the end item has the largest planning horizon $T(1)+T$ of any component. Further details about $T(i)$ and how it is determined can be found in Clark and Armentano[24], but it is worth clarifying here that the $T(i)$ concept permits the incorporation of MRP *back schedule* logic into multi-stage lot-sizing models[1],

$H(i)$: the T planning periods $T(i)+1,...,T(i)+T$.

The uncapacitated MSLS problem is formulated as the following mathematical model that minimizes the total of setup, production and inventory costs over synchronized rolling planning horizons:

$$F_I : \quad \text{minimize} \sum_{i=1}^{N} \sum_{t \in H(i)} \left[s_{it} y_{it} + c_{it} x_{it} + h_{i,t+L(i)} I_{i,t+L(i)} \right] \tag{1}$$

subject to

$$I_{i,L(i)+t-1} + x_{it} - I_{i,L(i)+t} = d_{i,L(i)+t} + \sum_{j \in S(i)} r_{ij} x_{j,L(i)+t} \qquad \begin{matrix} i=1,...,N \\ t \in H(i) \end{matrix} \tag{2}$$

$$x_{it} \geq 0; \quad I_{i,L(i)+t} \geq 0; \qquad \begin{matrix} i=1,...,N \\ t \in H(i) \end{matrix} \tag{3}$$

$$y_{it} = 0 \text{ when } x_{it} = 0; \quad y_{it} = 1 \text{ when } x_{it} > 0; \qquad \begin{matrix} i=1,...,N \\ t \in H(i) \end{matrix} \tag{4}$$

Since $T(i)+L(i) \leq T(j)$, it is possible that $t+L(i) \leq T(j)$ for some $t \in H(i)$ indicating that $x_{j,L(i)+t}$ in constraint (2) has a fixed value known from previous rolling horizon applications of the model, as commented in the explanation of $T(i)+1$ above.

Recall that the component lead time value $L(i)$ simply indicates that the production x_{it} becomes available for use in period $t+L(i)$ and as such does not include the time spent in queues between component production stages. Total manufacturing lead time is essentially made up of cumulative production lead times $L(i)$, the queuing times at component production centers and time spent waiting for out-of-stock components. The relationship between these constituent times is intricate as many of them occur in parallel and cannot simply be summed. When production capacity is not a limiting factor, the model F_I helps to cope with this complexity by relating total manufacturing lead time to lot-sizes, demand and stock availability.

Production capacity and the total manufacturing lead time are intimately linked since, with infinite capacity, queues between component production centers would be eliminated and total manufacturing lead time would be substantially reduced[21,25,26]. To model the relationship between production capacity and the total manufacturing lead time of an end-product, define

K : the number of capacitated resources of type k=1,...,K used to produce the components i=1,...N;

b_{kt} : the availability of resource type k in period t, already paid for independently of its utilization by the production of the components i=1,...N,

s_{ikt} as the amount of resource type k necessary to setup the production of component i in period t,

c_{ikt} as the amount of resource type k necessary to produce one unit of component i in period t.

Thus the total amount of resource type k utilized in period t to produce all necessary components is

$$R_{kt} = \sum_{\substack{i=1 \\ i|t \le T(i)+T}}^{N} \left[s_{ikt}\, y_{it} + c_{ikt}\, x_{it} \right] \le b_{kt} \qquad \begin{matrix} k=1,...,K \\ t=1,...,T(1)+T \end{matrix} \qquad (5)$$

These constraints are added to F_I to form the capacitated formulation F_I^C. The condition $i \mid t \le T(i)+T$ is specified in the summation since x_{it} does not exist beyond component i's planning horizon. Note that the summation includes the already known values $x_{it} \mid t \le T(i)$ decided in previous rolling horizon application of the model, given that this prefixed production consumes capacity. The constraint is enforced up to the period $t = T(1)+T$, the longest of the component planning horizons, namely that of the end-item.

The heuristic method proposed in the next section attempts to make a production plan feasible with respect to capacity by shifting some component production backwards to earlier periods. Such transfers can only be carried out if the constraints in (2) and (3) continue to be satisfied, in other words, only if there is sufficient production and inventory of predecessor subcomponents. These preconditions are more easily checked if the model F_I^C is reformulated in terms of echelon stock[3,5,8,11,27]. To do so, first recursively define the echelon demand D_{it} and the echelon stock E_{it} of component *i* at the end of period *t* as

$$D_{it}=d_{i,L(i)+t}+\sum_{j\in S(i)} r_{ij} D_{j,L(i)+t} \qquad \text{and}$$

$$E_{it}=I_{i,L(i)+t}+\sum_{j\in S(i)} r_{ij} E_{j,L(i)+t} \qquad \text{both for } t\le T(i)+T \qquad (6)$$

where $D_{1t}=d_{1,L(1)+t}$ and $E_{1t}=I_{1,L(1)+t}$ fare the initial conditions and where D_{it} and E_{it} are evaluated in the order *i* = 1,...,*N*. In words, D_{it} is the total system demand for component *i* in period *t*, both its own independent demand and that caused by the demand for its successor components *S*(*i*) offset by the lead time *L*(*i*). Similarly E_{it} is the total system stock, offset by lead time, of component *i* at the end of period *t*, both as a stand-alone component, $I_{i,L(i)+t}$, and as part of successor components $\sum_{j\in S(i)} r_{ij} E_{j,L(i)+t}$. The relationship between echelon demand and stock, in (8) below, is as simple as that in the zero lead-time, single-stage case. Further details about the definitions of D_{it} and E_{it} can be found in Clark and Armentano[24].

A unit echelon stock holding cost e_{it} is attached to E_{it}, defined as $e_{it}=h_{i,L(i)+t}-\sum_{j\in P(i)} r_{ji} h_{jt}$ where $h_{i,L(i)+t}=0$ for $i\in\{1,...,N\}$ and $t\notin H(i)$.

The formulation F_I above is equivalent to the following reformulation F_E in terms of echelon stock[24]:

$$F_E:\ \text{minimize} \quad \sum_{i=1}^{N} \sum_{t\in H(i)} \left[s_{it} y_{it}+c_{it} x_{it}+e_{it} E_{it}\right] + \sum_{i|P(i)\neq\varnothing} \sum_{t=1}^{T(i)} e_{it} E_{it} \qquad (7)$$

subject to

$$E_{i,t-1}+x_{it}-E_{it} = D_{it} \qquad \begin{matrix} i=1,...,N \\ t\in H(i)\end{matrix} \qquad (8)$$

$$\sum_{j\in S(i)} r_{ij} E_{j,L(i)+t}-E_{it}\le 0 \qquad \begin{matrix} i=2,...,N \\ t\in H(i)\end{matrix} \qquad (9)$$

$$x_{it}\ge 0;\quad E_{it}\ge 0 \qquad \begin{matrix} i=1,...,N \\ t\in H(i)\end{matrix} \qquad (10)$$

$$y_{it}=0 \text{ when } x_{it}=0;\quad y_{it}=1 \text{ when } x_{it}>0; \qquad \begin{matrix} i=1,...,N \\ t\in H(i)\end{matrix} \qquad (11)$$

where $\{E_{it} \mid i=1,...,N;\ t=1,...T(i)\}$ are predetermined echelon stock levels. Note that the constraint $\sum_{j\in S(i)} r_{ij} E_{j,L(i)+t}-E_{it}\le 0$ implies $I_{i,L(i)+t}\ge 0$. The formulation F_E is also used in the derivation of precision guarantees for the proposed heuristic.

The constraint (5) is added to F_E to form the capacitated echelon stock formulation F_E^C.

The Heuristic

The starting point for the heuristic is a feasible solution for the uncapacitated problem F_I, found by sequentially applying the optimal Wagner-Whitin algorithm[28] on components 1,2,3,...,N. If the capacities b_{kt} in constraints (5) are tight, then it is likely that the feasible solution for F_I, or equivalently F_E, is not feasible for F_E^C. In this case, we try to achieve feasibility by moving production backwards in time from overloaded periods to earlier underloaded ones. Period $t \in \{1,...T(1)+T\}$ is overloaded if $R_{kt} > b_{kt}$ for at least one $k \in \{1,...,K\}$.

When moving production to other periods, we must ensure that the constraints in (9) and (10) are still satisfied. The constraints in (9) and $E_{1t} \geq 0$ for $t \in H(i)$ imply the echelon stock non-negativity constraints in (10). The constraints in (9) are also equivalent to

$$\sum_{m \in S(j)} r_{jm} E_{m,L(j)+\bar{t}} \leq E_{j\bar{t}} \qquad \begin{array}{l} n=1,...,N \\ j \in P(n) \\ \bar{t} \in H(j) \end{array} \tag{12}$$

Now the constraints in (8) show that if a quantity q_{it} of a lot x_{it} of a component i in a period t is moved to an earlier period τ then the echelon stock levels $\{E_{i\sigma} \mid \sigma=\tau,...,t-1\}$ will increase by amount q_{it}. After such a move, however, some constraints in (12) could be violated when $n=i$ and $\bar{t}=\sigma-L(j)$. Specifically, we must ensure that

$$\sum_{m \in S(j)} r_{jm} E_{m\sigma} + r_{ji} q_{it} \leq E_{j,\sigma-L(j)} \qquad \begin{array}{l} j \in P(i) \\ \sigma=\tau,...,t-1 \\ \sigma-L(j) \leq T(j)+T \end{array}. \tag{13}$$

In words, production q_{it} of a component i can be moved to an earlier period τ only if sufficient stocks of its own immediate sub-components $P(i)$ are available. The inequalities in (13), central to the heuristic below, will be valid only if we choose q_{it} to be

$$q_{it} \leq M_{i\tau t} = \min_{\substack{j \in P(i) \\ \sigma=\tau,...t-1 \\ \sigma-L(j) \leq T(j)+T}} \left\{ \left(E_{j,\sigma-L(j)} - \sum_{m \in S(j)} r_{jm} E_{m\sigma} \right) \Big/ r_{ji} \right\} \tag{14}$$

The condition (14) is incorporated into the following heuristic that tries to find a feasible solution to F_E^C by moving production earlier in time. Before formally specifying the heuristic, we will outline the reasoning within it.

The starting point for the heuristic is a feasible solution to the uncapacitated problem F_E. If the production in a certain period t is beyond the capacity of at least one of the resources $k \in \{1,...,K\}$, then portions of this production are moved to previous periods until the constraint in (5) for the particular value of t is satisfied.

The value

$$\text{Excess}(t) \;=\; 100 \sum_{k=1}^{K}\left[\left(R_{kt}-b_{kt}\right)^{+}\Big/ b_{kt}\right] \tag{15}$$

is the percent measure of the total overload on capacity in period t. For $t = T(1)+T$, $T(1)+T$-1, ...,2, portions of the production in period t are moved to previous periods until Excess(t) = 0. If after these moves the constraint in (5) is satisfied for t = 1 then we have a feasible solution to the capacitated problem F_E^C.

Let $i\in\{1,...,N\}$, $t\in\{2,...,T(1)+T\}$, and specifically define τ to be:

$\tau = \max\{T(i)+1$, the last period in which there is production of component i prior to period $t\}$.

Also let $\sigma\in\{\tau,...,t\text{-}1\}$ and suppose that Excess(t) > 0. The condition in (14) implies that the quantity $M_{i\sigma t}$ is the maximum amount of the production x_{it} of component i in period t that the inter-echelon restrictions allow to be moved to period σ and that $M_{i\sigma t} \;\le\; M_{i,\sigma+1,t}$ where $M_{itt} = x_{it}$. Thus the following backward-recursive relationship

$$M_{i\sigma t}=\min\left\{M_{i,\sigma+1,t},\ \min_{\substack{j\in P(i)\\ \sigma-L(j)\le T(j)+T}}\left\{\left(E_{j,\sigma-L(j)}-\sum_{m\in S(j)} r_{jm}E_{m\sigma}\right)\Big/ r_{ji}\right\}\right\} \tag{16}$$

may be used to calculate successive values of $M_{i\sigma t}$ in the heuristic.

The heuristic first considers moving the amount $M_{i\sigma t}$ from period t to period σ. If all of the production x_{it} were moved from period t to the previous period in which component i is produced then we would economize the setup cost s_{it}. Obviously such a move must be considered if τ is the previous production period of component i and the value of $M_{i\tau t}$ permits. In the formal specification of the heuristic below, this move is considered in the first iteration k=0 of loop k of the last iteration $\sigma=\tau$ of loop σ, if τ is the previous production period of component i and $M_{i\tau t} = M_{itt}$ still. If not, then although such a move is unlikely to be economic it is still considered in order to not complicate the heuristic.

Let $k\in\{1,...,K\}$. If the amount

$$Q_{itk} = \left(R_{kt}-b_{kt}\right)^{+}\Big/ c_{ikt} \tag{17}$$

of the production x_{it} of component i in period t were moved, then the overload of resource k in period t would be eliminated. Hence, if $Q_{itk} < M_{i\sigma t}$, moving the amount $q_{itk} = Q_{itk}$ to period σ is also considered.

Thus for $k \in \{0\} \cup \{1,...,K | Q_{itk} < M_{i\sigma t}\}$, the shifting of the amount q_{itk} to period σ is considered. It should be emphasized that the quantity q_{itk} is one of several options being considered for moving to period σ. In turn period σ is just one of the target periods under consideration within the interval {τ,...,t-1}, and, in turn again, component i is simply one of the N components also being considered.

The value Ratio(i,σ,k) = ExtraCost(i,σ,k) / Decrease(i,σ,k) is the extra cost per unit decrease in the value of Excess(t) if the amount q_{itk} of the production x_{it} of component i were moved from period t to period σ. ExtraCost(i,σ,k) and Decrease(i,σ,k) are calculated as follows:

- ExtraCost(i,σ,k) = ExtraSetupCost + $\left(c_{i\sigma} - c_{it} + \sum_{\rho=\sigma}^{t-1} e_{i\rho} \right) q_{itk}$

  where ExtraSetupCost =

s_{i_σ}	if $q_{itk} < x_{it}$ and $x_{i_\sigma} = 0$,
$s_{i_\sigma} - s_{it}$	if $q_{itk} = x_{it}$ and $x_{i_\sigma} = 0$,
$- s_{it}$	if $q_{itk} = x_{it}$ and $x_{i_\sigma} > 0$,
0	if $q_{itk} < x_{it}$ and $x_{i_\sigma} > 0$.

- Decrease(i,σ,k) = Excess(t) – $100 \sum_{\substack{k=1 \\ R_{kt} > b_{kt}}}^{K} \left[\left(R_{kt} - Reduction_{ikt} - b_{kt} \right)^{+} / b_{kt} \right]$

  where $Reduction_{ikt}$ =

$c_{ikt}\, q_{itk}$	if $q_{itk} < x_{it}$
$s_{ikt} + c_{ikt}\, q_{itk}$	if $q_{itk} = x_{it}$.

The heuristic's criterion is to choose the triple (i,σ,k) that minimizes Ratio(i,σ,k). In order to limit calculations, the values of σ are confined to the interval {τ,...,t-1}. In addition, note that the use of the echelon stock formulation F_E^C means that calculation of all the different values of Ratio(i,σ,k) is not burdensome, since the only inputs that change value between the options under consideration are $x_{i\sigma}$, x_{it} and $E_{i\sigma},...,E_{i,t-1}$ (all by the same amount, namely q_{itk}) and since the calculation of ExtraCost(i,σ,k) is facilitated by the the use of the echelon stock holding cost e_{it}. Such calculations would be considerably more complex and time-consuming if the installation formulation F_I^C were used.

The heuristic, specified in indented pseudo-code with line numbers, is formally specified as follows:

1. Apply the Wagner-Whitin algorithm sequentially on components 1,2,3,...,N, in order to obtain a feasible solution to F_I.

2. **for** $t = T(1)+T$ **downto** 2 **do begin**

3. Calculate Excess(t) as in (15);

4. **while** Excess(t) > 0 **do begin**

5. **for** $i = 1$ to N **do begin**

6. **if** $t \in H(i)$ **and** $x_{it} > 0$ **then begin**

7. $\tau = \max\{T(i)+1$, period before t in which component i was last produced$\}$;

8. **for** $\sigma = t$-1 **downto** τ **do begin**

9. Calculate $M_{i\sigma t}$ as in (16)

10. **if** $M_{i\sigma t} = 0$ **then** break to the next iteration of loop i;

11. **for** $k = 0$ to K **do begin**

12. **if** $k = 0$ **then** $q_{itk} = M_{i\sigma t}$;

13. **else begin**

14. Calculate Q_{itk} as in (17);

15. **if** $0 < Q_{itk} < M_{i\sigma t}$ **then** $q_{itk} = Q_{itk}$;

16. **else** break to the next iteration of loop k;

17. **end** { of first else }

18. ExtraCost(i,σ,k) = extra cost if amount q_{itk} of the production x_{it} is moved to period σ;

19. Decrease(i,σ,k) = decrease in the value of Excess(t) if the production x_{it} is reduced by amount q_{itk};

20. Ratio(i,σ,k) = ExtraCost(i,σ,k) / Decrease(i,σ,k);

21. **end** {of loop k}

22. **end** {of loop σ}

23. **end** {of if $t \in H(i)$ and $x_{it} > 0$}

24. **end** {of loop i}

25. **if** (i^*,σ^*,k^*) = arg min{Ratio(i,σ,k)} exists,

26. **then** move the production q_{i^*,t,k^*} from period t to period σ^*;

27. **else** stop - the heuristic is unable to find a feasible solution for F_E^C.

28. **end** {of while Excess(t)>0}

29. **end** {of loop t}

30. **if** $R_{k1} \le b_{k1}$ for $k \in \{1,...,K\}$ **then** we have a feasible solution for F_E^C **else** we do not have a feasible solution for F_E^C.

In order to better understand the heuristic, consider the five-component (N=5) product structure in Figure 1 constrained by one resource (K=1) over a twelve period planning horizon (T=12) with the parameters and data of Table 1. Table 2(a) shows the initial solution obtained by sequentially applying the Wagner-Whitin algorithm. This solution is capacity infeasible and Table 2(b) charts the heuristic's progress to the feasible solution of Table 2(c).

When $L(i)$ = 0 for i = 1,...,N then for t = 2,...,T there always exists a component ι whose entire production $x_{\iota t}$ can be moved from period t to period t-1, namely ι = $\max\{i|x_{it} > 0\}$. Since $x_{jt} = 0$ $\forall$ $j \in P(i)$, all of the production $x_{\iota t}$ can be moved to period t-1. However, in the case of non-zero component lead time $L(i)$, it is possible that none of the production of any component can be moved from period t to period t-1, without first moving some of the production of periods before t to even earlier periods. For example, consider a two-component structure with L(1) = 0, L(2) = 1, $r_{21} = 1$, T = 4, $d_{11} = 0$, $d_{1t} = d_t > 0$ for t = 2,..,5, $d_{2t} = 0$ for t = 1,...,5, and the production schedule in Table 3. Suppose that Excess(4) > 0 and that consequently some of the production of period 4 has to be moved to previous periods. The only component produced in period 4 is component 1 with $x_{14} = d_4$. However note that it is impossible to move any of this production to period 3 without also moving some of the production $x_{23} = d_4 + d_5$ of component 2 from period 3 to period 2.

The impossibility of moving any production from period t to period t-1 without first moving some production from earlier periods is revealed in line 9 of the heurisitc by the deadlock condition $M_{i,t-1,t} = 0 \ \forall \ i \in \{1,..., N|x_{it} > 0\}$ which implies that (i*,σ*,k*) = arg min{Ratio(i,σ,k)} does not exist, stopping the heuristic at line 27.

Even if a feasible solution exists, the heuristic may not find one for either of two reasons:

1. At some stage, the combination of Excess(t) > 0 and $M_{i,t-1,t}$ = 0 $\forall$ $i \in \{1,..., N \,|\, t \in H(i), x_{it} > 0\}$ may cause an impasse, as noted immediately above.
2. The capacity of at least one of the resources k is tight and the heuristic may end up overloading period 1, detected in line 30.

A Precision Guarantee

A precision guarantee for the heuristic solution is obtained by relaxing the constraints (5) and (9) into the objective function, resulting in a Lagrangian problem whose solution is a lower bound on F_E^C [29]. The Lagrangian problem is $Z_D(\lambda,\mu)$ =

$$\min \sum_{i=1}^{N} \sum_{t\in H(i)} \left[s_{it}\,y_{it} + c_{it}\,x_{it} + e_{it}\,E_{it}\right] + \sum_{i|P(i)\neq\varnothing} \sum_{t=1}^{T(i)} e_{it}\,E_{it}$$

$$+\sum_{i=2}^{N} \sum_{t\in H(i)} \lambda_{it}\left[\sum_{j\in S(i)} r_{ij}\,E_{j,L(i)+t} - E_{it}\right] + \sum_{k=1}^{K} \sum_{t=1}^{T(1)+T} \mu_{kt}\left[\sum_{i|t\leq T(i)+T}\left[s_{ikt}\,y_{it} + c_{ikt}\,x_{it}\right] - b_{kt}\right] \qquad (18)$$

subject to the conditions in (8) and (10), where $\lambda_{it} \geq 0$ and $\mu_{kt} \geq 0$ are Lagrangian multipliers. Defining $\lambda_{1t} = 0$ for $t\in H(1)$, noting that

$$\sum_{i=2}^{N} \sum_{t\in H(i)} \lambda_{it} \sum_{j\in S(i)} r_{ij}\,E_{j,L(i)+t} = \sum_{i=1}^{N} \sum_{j\in P(i)} r_{ji} \sum_{t\in H(j)} \lambda_{jt}\,E_{i,L(j)+t}$$

(19)

and recalling that $T(j)+L(j) \leq T(i)$ for $i\in\{1,...,N\}$ and $j\in P(i)$, it can be checked that $Z_D(\lambda,\mu)$ decomposes into the sum of the constant term:

$$C(\lambda,\mu) = \sum_{i=1}^{N} \sum_{j\in P(i)} r_{ji} \sum_{t=T(j)+L(j)+1}^{T(i)} \lambda_{j,t-L(j)}\,E_{it} - \sum_{k=1}^{K} \sum_{t=1}^{T(1)+T} \mu_{kt}\,b_{kt}$$

$$+ \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{t=1}^{T(i)} \mu_{kt}\left[s_{ikt}\,y_{it} + c_{ikt}\,x_{it}\right] + \sum_{i|P(i)\neq\varnothing} \sum_{t=1}^{T(i)} e_{it}\,E_{it} \qquad (20)$$

and the following N single-stage lot-sizing uncapacitated problems:

$$Z_{iD}(\lambda_i,\mu) = \min \sum_{t\in H(i)} \left[\left(s_{it} + \sum_{k=1}^{K} \mu_{kt}\,s_{ikt}\right) y_{it} + \left(c_{it} + \sum_{k=1}^{K} \mu_{kt}\,c_{ikt}\right) x_{it}\right]$$

$$+\left[\sum_{t\in H(i)} \left(e_{it} - \lambda_{it}\right) + \sum_{j\in P(i)} r_{ji} \sum_{t=T(i)+1}^{T(j)+L(j)+T} \lambda_{j,t-L(j)}\right] E_{it} \qquad (21)$$

subject to (8) and (10) for $i = 1,...,N$ where $\lambda_{1t} = 0$ for $t \in H(1)$ in $Z_{iD}(\lambda_i,\mu)$.

The holding cost function of E_{it} is linear and production costs are concave and so for all $i = 1,...,N$ there exists[30] at least one optimal solution to $Z_{iD}(\lambda_i,\mu)$ for which $E_{i,t-1}\,x_{it} = 0$ for all $t = 1,...,T$. Hence $Z_{iD}(\lambda_i,\mu)$ can be solved using the fast Wagner-Whitin algorithm[28] and a rapidly calculated lower bound to F_E^C is given by

$$Z_D(\lambda,\mu) = \sum_{i=1}^{N} Z_{iD}(\lambda_i,\mu) + C(\lambda,\mu). \qquad (22)$$

High values of $Z_D(\lambda, \mu)$ were sought by iterating 1000 times on λ and μ, using subgradient optimization[31].

Experimental Tests

The heuristic was tested computationally first for the largest problems for which optimal solutions were obtainable in reasonable time using IBM's OSL[32] mixed integer programming solver on an IBM RS/6000 32H workstation. For ten problems with the five-component product structure in Figure 1 and two constraining resources over a twelve-period planning horizon, optimal and heuristic solutions were obtained. The Lagrangian precision guarantees in (22) were also calculated with the objective of estimating, for large real-life problems, what proportion of the total gap between the heuristic solution and the Lagrangian guarantee could be attributed to the primal gap (between the optimal and heuristic solutions) and what proportion to the dual gap (between the optimal solution and the Lagrangian guarantee). These results will be reported and commented in the context of the results for further and more extensive tests that were carried out on products with 40 components in the presence of 2 constraining resources over a twelve-period planning horizon. The aim of the further tests was to investigate how the heuristic's performance is affected by the following three factors:

A. The product structure. Two contrasting structures were tested:

1. The flattest possible structure: $S(i) = \{1\}$ for $i = 2,...,40$.
2. A typically large general structure, namely: S(2)={1}, S(3)={1}, S(4)={1}, S(5)={2}, S(6)={4}, S(7)={4}, S(8)={3}, S(9)={2, 3}, S(10)={6}, S(11)={9}, S(12)={5}, S(13)={8}, S(14)={6,7}, S(15)={10}, S(16)={10,14}, S(17)={7}, S(18)={8}, S(19)={8}, S(20)={11, 12}, S(21)={11}, S(22)={13}, S(23)={7, 12}, S(24)={13,18,19}, S(25)={15}, S(26)={20}, S(27)={21}, S(28)={22}, S(29)={15,16}, S(30)={16,17}, S(31)={17,23}, S(32)={24}, S(33)={25}, S(34)={25,29}, S(35)={30}, S(36)={26, 31}, S(37)={26, 27}, S(38)={27}, S(39)={28, 32}, S(40)={28}.

B. The mean size of the setup costs s_{it} relative to the echelon stock costs e_{it}. The costs s_{it} were randomly sampled from the following uniform distributions:

1. U(50, 950) representing high setup costs
2. U(5, 95) representing low setup costs.

C. The production capacity in terms of the resource availabilities b_{kt}. Capacity was constant over time, i.e. $b_{kt} = b_k$ for $k \in \{1,2\}$ and $t \in \{1,...,T(1)+T\}$. Two extremes were tested:

1. Infinite capacity, represented by $b_{kt} = b_k = \infty$.
2. Very tight capacity, represented by values of b_1 and b_2 that during initial tests represented the tightest capacity for which the heuristic could find a feasible solution. The values depend upon the mix of setup costs and product structure, as shown in Table 4.

Note in Table 4 that the tightest capacity for the flat structure is less than half that for the general structure. This almost certainly can be attributed to two reasons: (1) components 2,...,40 in the flat structure have no subcomponents and hence their production is free to move to earlier periods; (2) the sequential application of the Wagner-Whitin algorithm to the general structure tends to concentrate the production of the deeper components in the initial periods, leaving fewer options for moving this production. Note also from Table 4 that the tightest capacity with the high setup costs is about 30% tighter than that with the low setup costs, probably because the high costs cause the Wagner-Whitin algorithm to bunch production in larger lots, thereby economizing on total capacity consumption.

The variable production costs c_{it} were randomly sampled from the uniform distribution U(1.5,2.0), the end item demands d_{1t} from U(0,180), the independent component demands $\{d_{it} \mid i = 2,...,N\}$ from U(0,18), the setup resource requirements s_{i1t} and s_{i2t} from U(150,250) and U(200,300) respectively, the unit resource requirements c_{i1t} and c_{i2t} from U(1.5,2.5) and U(2.0,3.0) respectively, and the lead times $L(i)$ from the set {0,1}. For simplicity, $r_{ij} = 1$ for $i \in \{1,...,N\}$ e $j \in S(i)$. Concerning the stock holding costs, recall that

$$e_{it} = h_{i,L(i)+t} - \sum_{j \in P(i)} r_{ji} h_{jt} \qquad \text{where } h_{i,L(i)+t} = 0 \quad \text{for } \begin{cases} i \in \{1,...,N\} \\ t \notin H(i) \end{cases}. \tag{23}$$

To randomly generate the conventional stock holding costs $\{h_{i,L(i)+t} \mid i = 1,...,N;\ t \in H(i)\}$, 'provisional' values of the echelon stock holding costs e_{it} were first sampled from U(0.2,0.4). The provisional values of e_{it} were then used to calculate the costs $\{h_{i,L(i)+t} \mid i = 1,...,N;\ t \in H(i)\}$ via the identity $h_{i,L(i)+t} = e_{it} + \sum_{j \in P(i)} r_{ji} h_{jt}$. Finally, the 'definitive' values of the costs e_{it} were calculated using the identity (23).

The initial stock levels $\{I_{it} \mid i = 1,...,N;\ t = L(i),...,T(i)+L(i)\}$ and the prefixed production levels $\{x_{it} \mid i = 1,...,N;\ t = 1,...,T(i)\}$ were randomly and realistically determined, and took into account that the pre-fixed production will consume part of the availability b_{kt} of the resource k in period $t \in \{1,...,T(1)\}$.

For each of the four possible combinations of product structure and mean setup cost, problems with the very tight capacity shown in Table 4 were repeatedly and randomly generated, continuing until five problems had been identified for which the heuristic found a feasible solution. For each of the twenty problems thus generated, capacity was progressively loosened, passing through each of the slacker capacities in Table 4, and terminating with infinite capacity. The results are shown in Table 5 where each cell shows the percentage precision guarantee of the heuristic solution, calculated as (Solution × 100% / Lagrangian Lower Bound) - 100%. The heuristic is fast, taking, for the scale of problems tested, less than 0.5 seconds elapsed time on a 50 MHz 486DX2 microprocessor.

Commenting on Table 5, note that the precision guarantees are much better with the low setup costs than with the high ones and, within each cost category, are generally better with the flat structure than with the general one. We have no theoretical explanation for this behaviour. However, Trigeiro, Thomas and McClain[33] also found a larger gap for higher setup costs when testing their single-stage capacitated heuristic. Note also that for a given problem the precision guarantee generally improves substantially as capacity is loosened, particularly between the tightest and next tightest capacity categories. A possible reason for this behaviour is that, when capacity is tight, the minimum value of Ratio(i,σ,k) may be influenced more by capacity overload reduction opportunities and less by cost considerations. The heuristic gives more weight to regaining capacity feasibility than to economic solutions.

Without the optimal solution of a problem, it is impossible to know how much of the precision guarantee can be attributed to the distance between the heuristic and optimal solutions on the one hand and to the duality gap between the Lagrangian lower bound and the optimal solution on the other hand. However, as mentioned at the beginning of this section, optimal and heuristic solutions and Lagrangian guarantees were obtained for the largest problems for which optimal solutions could be computed in reasonable time, ranging from 1½ minutes to 2 hours, namely for twelve problems with the five-component general product structure of Figure 1, with the high setup costs from U(50, 950) and two tightly constrained resources over a twelve-period planning horizon. The results are shown in Table 6 where the percentage values were calculated in an equivalent manner to those in Table 5. Note that, on average, the heuristic was within 4.9% of the optimal solution while the

Lagrangian guarantee was over 25.0% away from the heuristic solution and over 19% away from the optimal solution. Observe also that the size of the dual gap can vary substantially. Comparing these results to the five Lagrangian guarantee percentages {20.7, 34.2, 37.8, 18.1, 44.9} for the general product structure with the high setup costs and very tight capacity in Table 5, we can expect the heuristic's performance to be much better than that indicated by the Lagrangian guarantees.

Conclusions

This paper developed a fast and simple heuristic solution method for a difficult but frequently occurring generalization of the NP-hard capacitated multi-stage lot-sizing problem, involving the setup use of multiple capacitated resources at all component stages and the sharing of capacitated resources between component stages. These conditions are present in many plants, yet industrial-size problems are unlikely to solved by optimal methods in reasonable computing time. For small problems with high setup costs and obtainable optimal solutions experiments showed the mean primal gap of 5% between the heuristic and optimal solutions to be only a quarter of the size of the mean dual gap of 20% between the optimal solution and Lagrangian lower bound. As a result it is not unreasonable to expect similar solution quality and primal-dual gap proportions for large problems with high setup costs, indicating the usefulness of the heuristic in solving what is recognised as a particularly complex problem[2]. Much better solution guarantees were obtained for problems with low setup costs than with high ones.

Further research could focus on:

1. more detailed experimentation of the effect on the heuristic's quality of parameters such as product structure, the size of setup costs and resource use, and capacity tightness;
2. the development of a heuristic for the uncapacitated problem that is better than just the sequential application of the Wagner-Whitin algorithm, for example an extension of Afentakis's heuristic[12];
3. at the end of the heuristic, the moving of production forward in time to later periods in order to reduce inventory holding costs and overcome infeasibility provoked by the bunching of production in early periods, while respecting the inter-echelon and no-stockout constraints (9) and (10);
4. the extension of the heuristic to solve a more general model that would represent the consumption of resource k by the production x_{it} in not just period t, but also in periods t+1,...,T+$L(i)$;

5. the development of a cost effective method of finding a way of moving subcomponent production backwards in time from earlier periods when confronted with the deadlock condition $M_{i,t-1,t} = 0$ $\forall \; i \in \{1,\ldots,N | x_{it} > 0\}$ in period *t*;
6. the theoretical development of tighter lower bounds that would result in better precision guarantees than those supplied by Lagrangian relaxation and subgradient optimization[34].

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Figure and Table Headings

Figure 1 - A Simple General Product Structure

Table 1 - Example Parameters and Data

Table 2 - (a) Initial Capacity Infeasible Solution

(b) Heuristic Progress

(c) Final Capacity Feasible Solution

Table 3 - Two-component Solution

Table 4 - Tightest Feasible Capacities

Table 5 - Lagrangian Solution Precision Guarantees for Forty-Component Products (%)

Table 6 - Solutions and Guarantee for Five-Component Products (%)

Figure 1 - A Simple General Product Structure

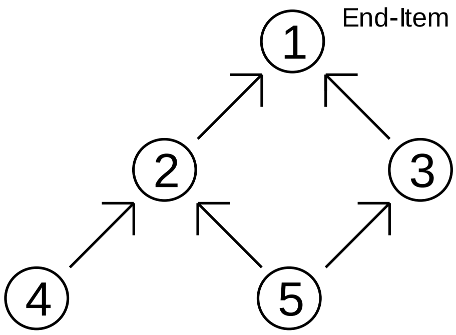

Table 1 - Example Parameters and Data

Parameter / Data	Values
Product Structure	$r_{21} = r_{31} = r_{42} = r_{52} = r_{53} = 1$
Lead Time	$L(1) = L(5) = 0$, $L(2) = L(3) = L(4) = 1$ so that $T(1) = 2$, $T(2) = T(3) = T(5) = 1$, $T(4) = 0$ [24]
Costs	$s_{it} = 1000$, $c_{it} = 0$, $e_{it} = 1 \;\forall\; i, t$;
Constrained Resources	$s_{11t} = 2000$, $s_{21t} = s_{31t} = 1500$, $s_{41t} = s_{51t} = 1000 \;\forall\; t$; $c_{11t} = 2$, $c_{21t} = c_{31t} = 1.5$, $c_{41t} = c_{51t} = 1 \;\forall\; t$; $b_{1t} = 1900 \;\forall\; t$;
Demand	111, 91, 80, 133, 81, 111, 146, 121, 102, 136, 66 and 128 units for the end-item over periods 3 to 14 respectively; Independent demand d_{it} for components 2 to 5 is zero
Prefixed Production Levels	x_{11}, x_{12}, x_{21}, x_{31} and x_{51} are all zero for simplicity's sake. [Their values are not normally zero since they come from previous rolling horizon applications of the model.]
Initial Echelon Stock Levels	E_{12}, E_{21}, E_{31}, E_{40} and E_{51} are all zero

Table 2(a) - Initial Capacity Infeasible Solution

Period t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
D_{1t} (d_{1t})			111	91	80	113	81	111	146	121	102	136	66	128
x_{1t}			111	$\boxed{171}$	-	214	-	257	-	223	-	202	-	128
E_{1t}			-	80	-	81	-	146	-	102	-	66	-	-
D_{2t}		111	91	80	113	81	111	146	121	102	136	66	128	
x_{2t}		282	-	-	214	-	257	-	223	-	330	-	-	
E_{2t}		171	80	-	81	-	146	-	102	-	194	128	-	
D_{3t}		111	91	80	113	81	111	146	121	102	136	66	128	
x_{3t}		282	-	-	214	-	480	-	-	-	330	-	-	
E_{3t}		171	80	-	81	-	369	223	102	-	194	128	-	
D_{4t}	111	91	80	113	81	111	146	121	102	136	66	128		
x_{4t}	496	-	-	-	-	480	-	-	-	330	-	-		
E_{4t}	385	294	214	81	-	369	223	102	-	194	128	-		
D_{5t}		222	182	160	226	162	222	292	242	204	272	132	256	
x_{5t}		564	-	-	428	-	$\boxed{960}$	-	-	-	$\boxed{660}$	-	-	
E_{5t}		342	160	-	162	-	738	446	204	-	388	256	-	

Note: The quantities moved during the heurisitc are in boxes.

Table 2(b) - Heuristic Progress

Line N°.	Heuristic Response
1	The feasible solution to F_E shown in Table 1(a)
2-4	Excess(t) = 0 for t = 14, 13 and 12 and then Excess(11) = 7.895%
5-6	Testing: only components i = 2, 3 and 5 are produced in period t = 11
7	For i = 2, we have τ = 9
9	$M_{2,10,11} = \min\{x_{2,11}, E_{4,9} - E_{2,10}, E_{5,10} - (E_{2,10} + E_{3,10})\} = \min\{330, 0-0, 0-(0+0)\} = 0$. Thus no production of component 2 can be moved to previous periods (without also, it may be confirmed from Table 1(a), moving a part of the production of component 5 in period 11 to earlier periods)
10	Go to the next iteration of loop i
7	For i = 3, τ = 7
8	σ = 10
9	$M_{3,10,11} = \min\{x_{3,11}, E_{5,10} - (E_{2,10} + E_{3,10})\} = \min\{330, 0-(0+0)\} = 0$. Thus no production of component 2 can be moved to previous periods (without also moving a part of the production of component 5 in period 11 to earlier periods).
7	For i = 5, τ = 7
8, 9	σ = 10; $M_{5,10,11} = \min\{x_{5,11}\} = 660$
11	k = 0
12	Moving all $q_{5,11,1}$ = 660 units of the production of component 5 in period 11 to period 10 is an option.
18-20	ExtraCost(5,10,0) = 660, Decrease(5,10,0) = 7.895 %, Ratio(5,10,0) = 83.6
11	k = 1
14-15	$0 < Q_{5,11,1} = 150 < M_{5,10,11} = 660$. Thus moving $q_{5,11,1}$ = 150 units of the production of component 5 in period 11 to period 10 is an option.
18-20	ExtraCost(5,10,1) = 1150, Decrease(5,10,1) = 7.895 %, Ratio(5,10,1) = 145.667
8, 9	σ = 9; $M_{5,9,11}$ = 660

11-20	Ratio(5,9,0) = 167.20 and Ratio(5,9,1) = 183.667
8, 9	σ = 8; $M_{5,8,11}$ = 660
11-20	Ratio(5,8,0) = 334.40 and Ratio(5,8,1) = 221.667
8, 9	σ = 7; $M_{5,7,11}$ = 660
11-20	Ratio(5,7,0) = 374.933, and Ratio(5,7,1) = 133.00
25-26	Thus the most attractive shift is to move all 660 units of the production of component 5 in period 11 to period 10.
4	Excess(11) is now zero.
2-4	Excess(t) = 0 for t = 10, 9 and 8
4, ...	Excess(7) = 29.76% is zeroed by moving all 960 units of the production of component 5 in period 7 to period 5.
2-4	Excess(6) = 0.
4, ...	Excess(5) = 27.89% is zeroed by moving all 1388 units of the production of component 5 in period 5 to period 4.
4, ...	Excess(4) = 6.84% is zeroed by adding all 171 units of the production of component 1 in period 4 to the 111 units already planned in period 3. This latter move produces a negative value of Ratio since the eliminated setup cost in period 4 more than compensates for the additional stock holding cost.
2-4	Excess(t) = 0 for t = 3 and 2.
30	Excess(1) = 0 implies that $R_{k1} \leq b_{k1}$ for $k \in \{1,\ldots,K\}$ and shows that we now have a feasible solution for F_E^C.

Table 2(c) - Final Capacity Feasible Solution

Period *t*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
D_{1t} (d_{1t})			111	91	80	113	81	111	146	121	102	136	66	128
x_{1t}			$\boxed{282}$	$\boxed{-}$	-	214	-	257	-	223	-	202	-	128
E_{1t}			171	80	-	81	-	146	-	102	-	66	-	-
D_{2t}		111	91	80	113	81	111	146	121	102	136	66	128	
x_{2t}		282	-	-	214	-	257	-	223	-	330	-	-	
E_{2t}		171	80	-	81	-	146	-	102	-	194	128	-	
D_{3t}		111	91	80	113	81	111	146	121	102	136	66	128	
x_{3t}		282	-	-	214	-	480	-	-	-	330	-	-	
E_{3t}		171	80	-	81	-	369	223	102	-	194	128	-	
D_{4t}	111	91	80	113	81	111	146	121	102	136	66	128		
x_{4t}	496	-	-	-	-	480	-	-	-	330	-	-		
E_{4t}	385	294	214	81	-	369	223	102	-	194	128	-		
D_{5t}		222	182	160	226	162	222	292	242	204	272	132	256	
x_{5t}		564	-	$\boxed{1388}$	$\boxed{-}$	$\boxed{-}$	$\boxed{-}$	-	-	$\boxed{660}$	$\boxed{-}$	-	-	
E_{5t}		342	160	1388	1122	960	738	446	204	660	388	256	-	

Note: The periods of the production shifts are in boxes.

Table 3 - Two-component Solution

		Period t				
		1	2	3	4	5
Production	x_{1t}	0	d_2	d_3	d_4	d_5
	x_{2t}	d_2	d_3	d_4+d_5	0	0

Table 4 - Tightest Feasible Capacities b_1, b_2

	Low Setup Costs		High Setup Costs	
Flat Product Structure:	10,000	12,000	7,000	9,000
General Product Structure:	20,000	25,000	16,000	20,000

Table 5 - Lagrangian Solution Precision Guarantees (%)

		Capacity Available				
	k = 1 :	7,000	10,000	16,000	20,000	Infinite Capacity
	k = 2 :	9,000	12,000	20,000	25,000	
General Structure and Low Costs	Problem 1	∞	∞	∞	2.57	1.30
	2	∞	∞	∞	1.63	1.38
	3	∞	∞	∞	1.83	1.40
	4	∞	∞	∞	3.28	1.40
	5	∞	∞	∞	8.98	1.58
General Structure and High Costs	6	∞	∞	20.7	20.4	19.9
	7	∞	∞	34.2	21.4	17.5
	8	∞	∞	37.8	17.8	17.0
	9	∞	∞	18.1	15.4	15.4
	10	∞	∞	44.9	24.0	18.3
Flat Structure and Low Costs	11	∞	1.71	0.56	0.51	0.51
	12	∞	3.62	1.28	1.17	1.17
	13	∞	1.38	0.60	0.60	0.60
	14	∞	1.70	0.37	0.37	0.37
	15	∞	1.49	0.34	0.31	0.31
Flat Structure and High Costs	16	9.50	9.37	9.32	9.32	9.32
	17	17.0	13.9	12.7	12.3	12.3
	18	22.8	15.0	13.8	13.8	13.8
	19	16.8	16.4	16.4	16.4	16.4
	20	11.7	9.20	8.74	8.74	8.74

Table 6 - Solutions and Guarantee for Five-Component Products (%)

	Percentage Precision Guarantee (Solution × 100% / Lower Bound) - 100%		
Problem	Heuristic Solution with Lagrangian Lower Bound (*Total Gap*)	Heuristic Solution with Optimal Solution Lower Bound (*Primal Gap*)	Optimal Solution with Lagrangian Lower Bound (*Dual Gap*)
1	24.23	0.82	23.22
2	16.17	10.51	5.11
3	25.57	2.71	22.26
4	19.39	4.12	14.66
5	39.22	11.88	24.44
6	23.35	optimal	23.35
7	13.09	1.24	11.71
8	17.42	1.91	15.23
9	36.56	10.80	23.24
10	43.38	7.36	33.54
11	20.65	7.30	12.44
12	21.90	0.42	21.39
Mean	25.08	4.92	19.22
Standard Deviatio n (n-1)	9.60	4.42	7.59