CSE4203: Computer Graphics
Lecture - 3

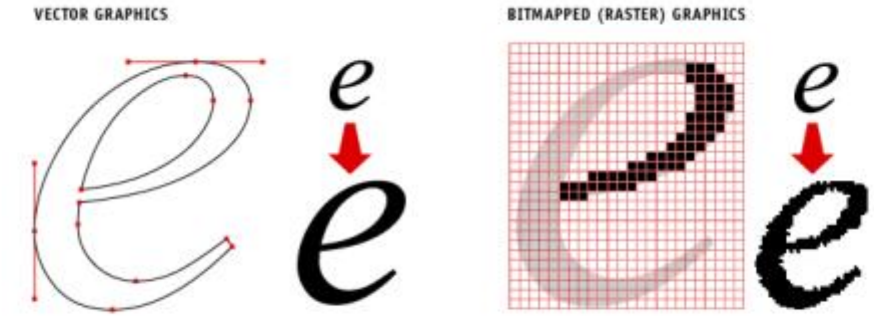
Vector Graphics

Outline

- Vector Image
- Curves
- Polynomial Curves
- Bézier Curves

Vector Image (1/3)

- storing descriptions of shapes
- areas of color bounded by lines or curves
- no reference to any pixel grid.



- Need to store *instructions for displaying the image rather* than the pixels needed to display it.

Vector Image (2/3)

- Resolution independent – scaling the image will not affect the quality
- Typically small in size compared to raster image
- Generates smooth curves and edges
- Limited photorealism
- Must be rasterized before they can be displayed
- File formats: SVG, AI, PDF

Vector Image (3/3)

- Differences between raster and vector images?

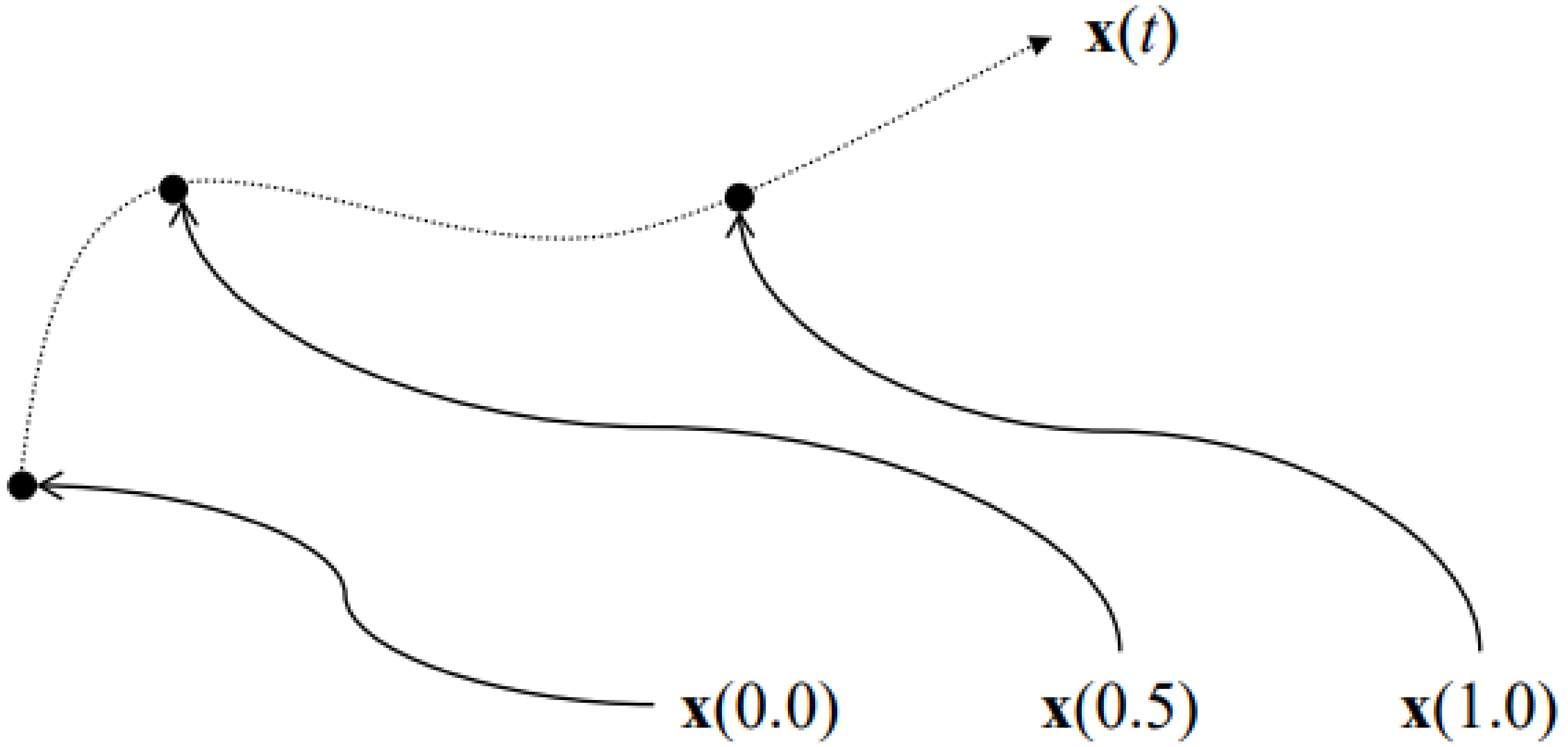
Curves (1/2)

- Curve is the continuous image of some interval in an n-dimensional space
- a continuous map from a one-dimensional space to an n-dimensional space.

Curves (2/2)

How many dimensions in the curve?

- It lies on 2D plane, but actually it's 1D



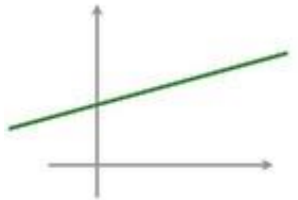
Credit: https://umd.instructure.com/courses/1286262/modules

Polynomial Curve (1/1)

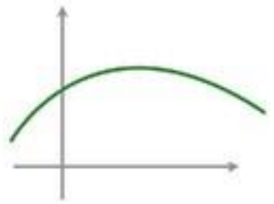
- A polynomial is a sum of variables raised to powers and multiplied by coefficients

$$\sum(a_i x^i) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

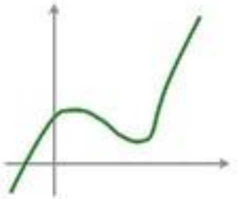
- The degree (or order) of a polynomial is the highest power of variables



1st degree (linear) polynomial: $y = ax^0 + bx^1$



2nd degree (quadratic) polynomial: $y = ax^0 + bx^1 + cx^2$



3rd degree (cubic) polynomial: $y = ax^0 + bx^1 + cx^2 + dx^3$

- Higher the degree, more change of directions

Bézier Curves (1/3)

- One of the most common representations for free-form curves in computer graphics
- First developed in 1959 by Paul de Casteljau
- Formalized and popularized by engineer Pierre Bézier
- Pierre Bézier used them for designing cars at Renault
- Common applications: 3D Modeling, CAD, typeface etc.

Pierre Étienne Bézier

Bézier Curves (2/3)

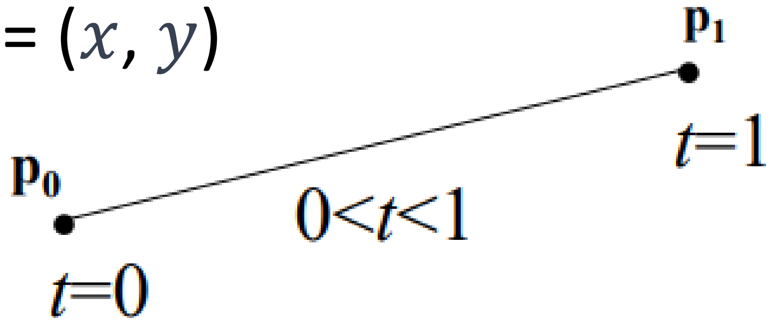
Parametric Equation

- Bézier Curves are expressed as parametric equations
- A parameter t is used to determine the value

$$x(t) = (1 - t)x_0 + tx_1$$
$$y(t) = (1 - t)y_0 + ty_1$$

where $0 \le t \le 1$. Let $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$ and P = (x, y)

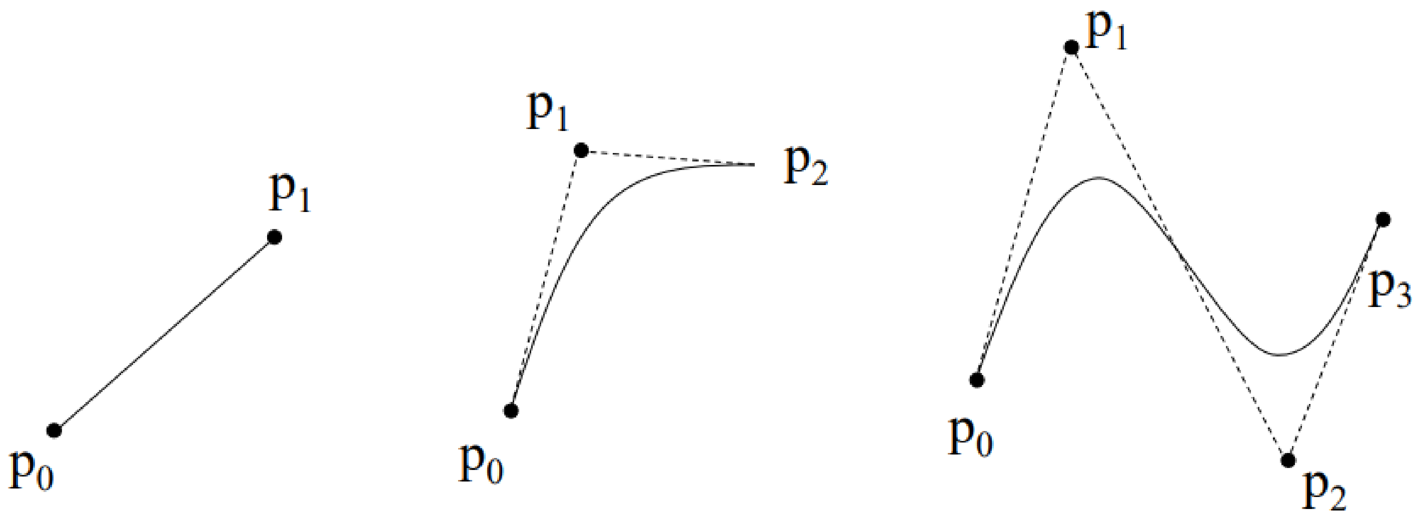
$$P(t) = (1 - t)P_0 + tP_1$$
or $P(t) = P_0 + t(P_1 - P_0)$



Bézier Curves (3/3)

Control Points

- A Bézier curve is defined by a set of control points
- An N degree Bézier curve is define by (N+1) control points
- For N control points, $\boxed{\text{degree } d = N - 1}$



Linear
N = 2

Quadratic
N = 3

Cubic
N = 3

Bézier Curves Derivation (1/13)

Derivation of a quadratic Bézier curve

- A quadratic (d=2) Bézier curve has 3 control points (P_0, P_1, P_2)
- Assume Q_0 and Q_1 lies on the line $P_0 \rightarrow P_1$ and $P_1 \rightarrow P_2$

$$Q_0 = P_0 + u(P_1 - P_0)$$

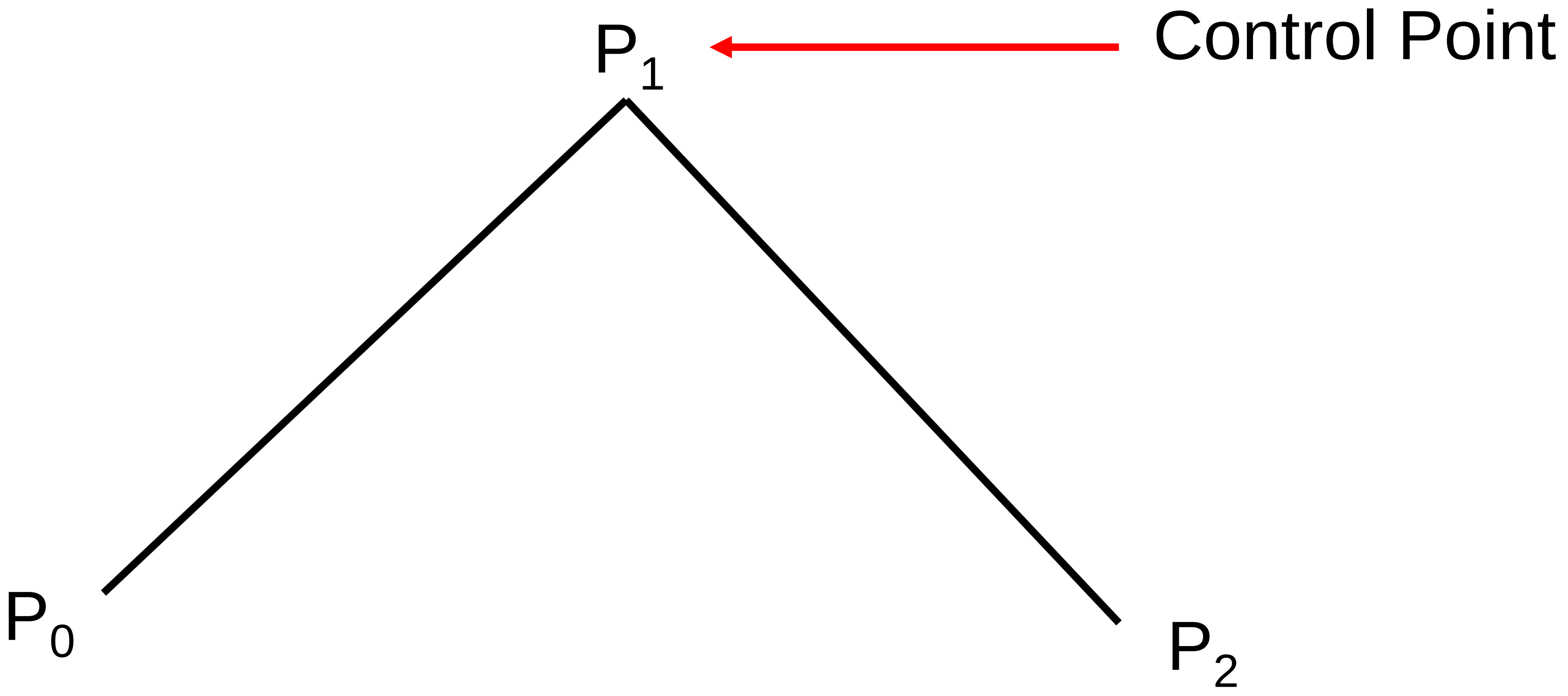
$$Q_1 = P_1 + u(P_2 - P_1)$$

- Point on the Bezier curve lies on the line $Q_0 \rightarrow Q_1$

$$Q(u) = Q_0 + u(Q_1 - Q_0)$$

Where u is a parameter ranges between 0 to 1

Bézier Curves Derivation (2/13)

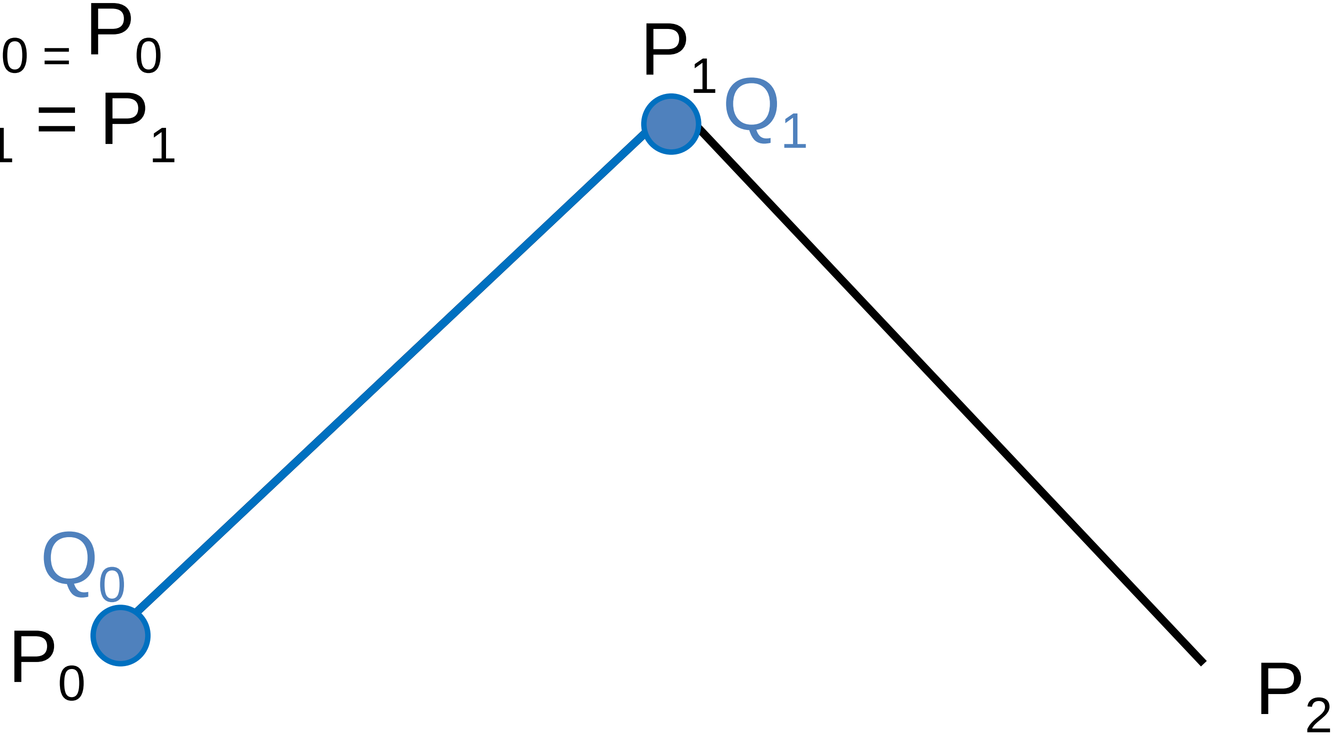


Bézier Curves Derivation (3/13)

at $u = 0$,

$Q_0 = P_0$

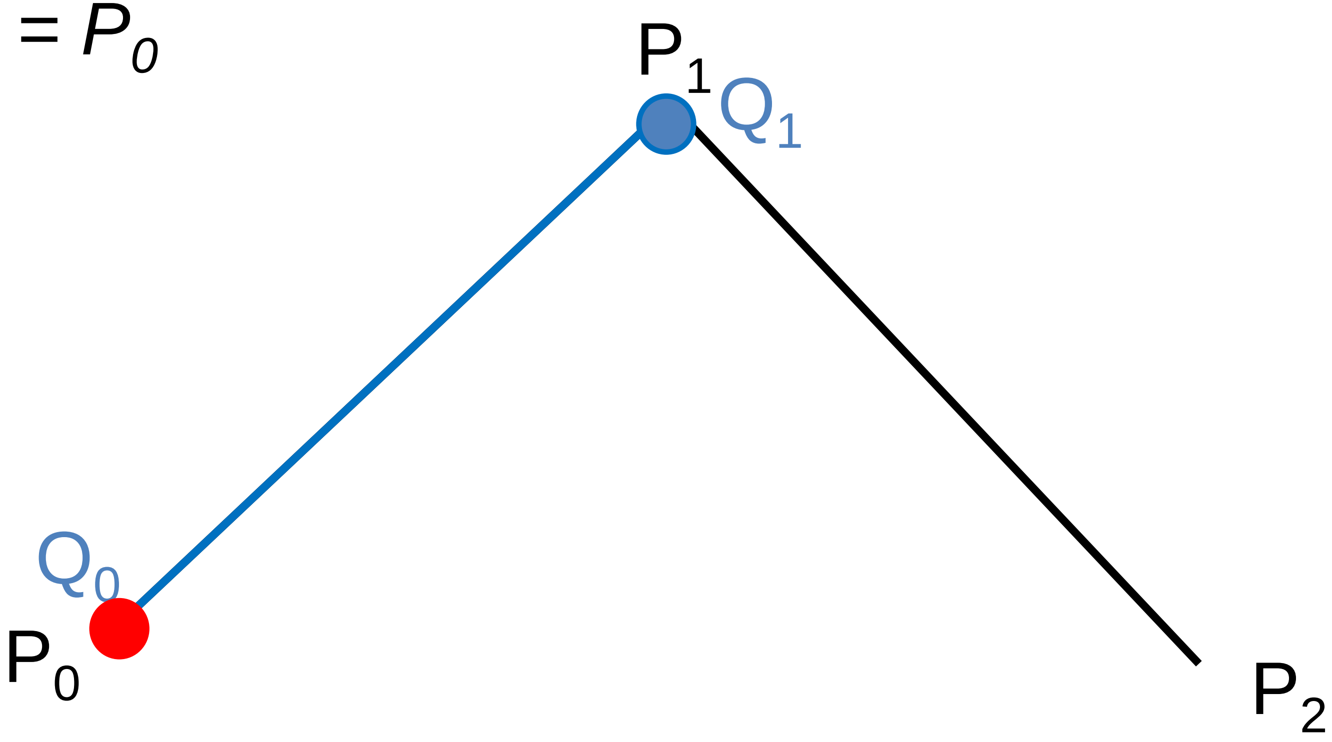
$Q_1 = P_1$



Bézier Curves Derivation (4/13)

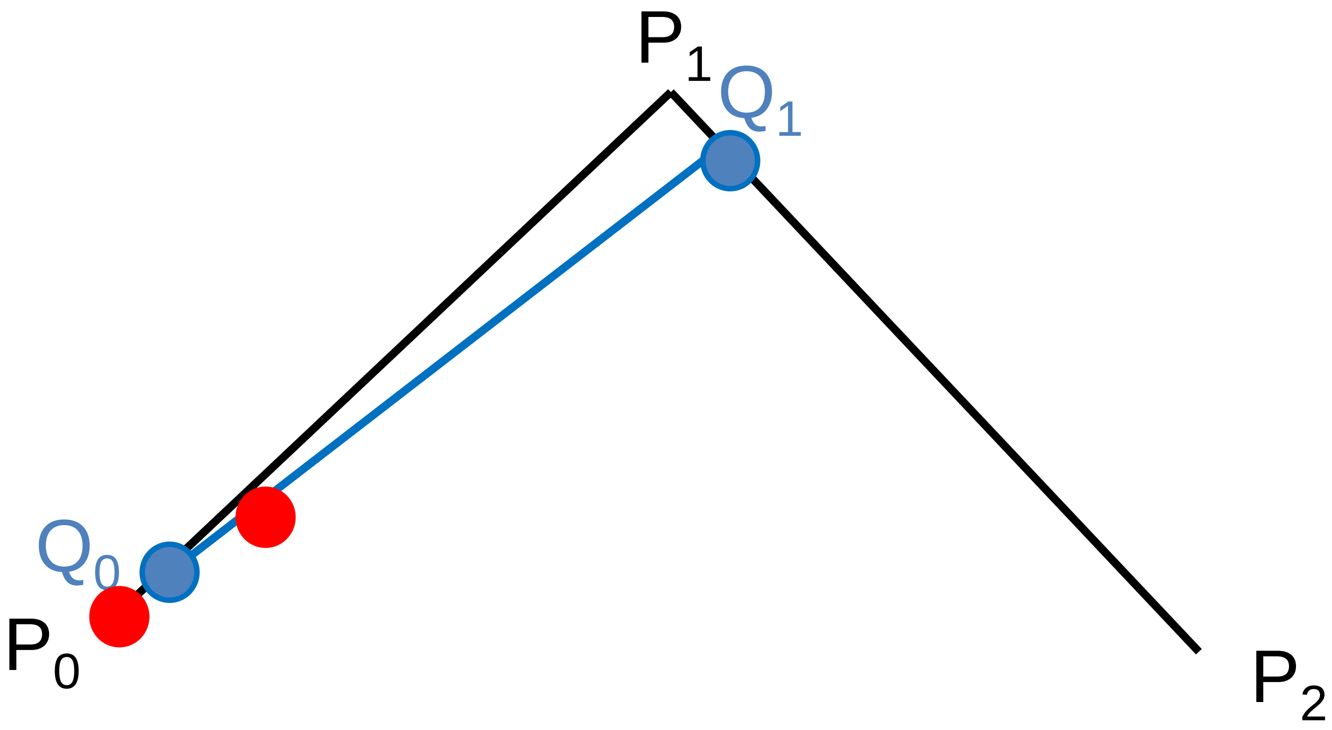
at $u = 0$,

$Q = P_0$



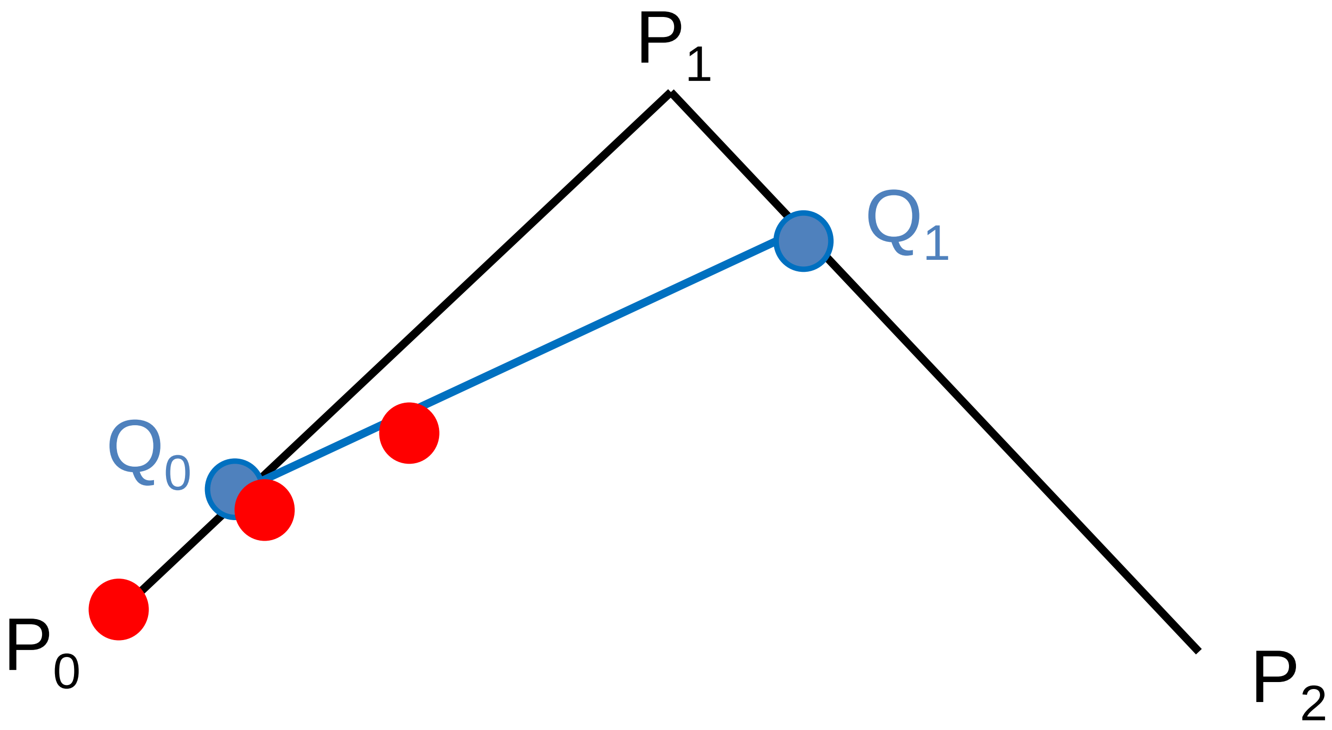
Bézier Curves Derivation (5/13)

at $u = 0.1$,



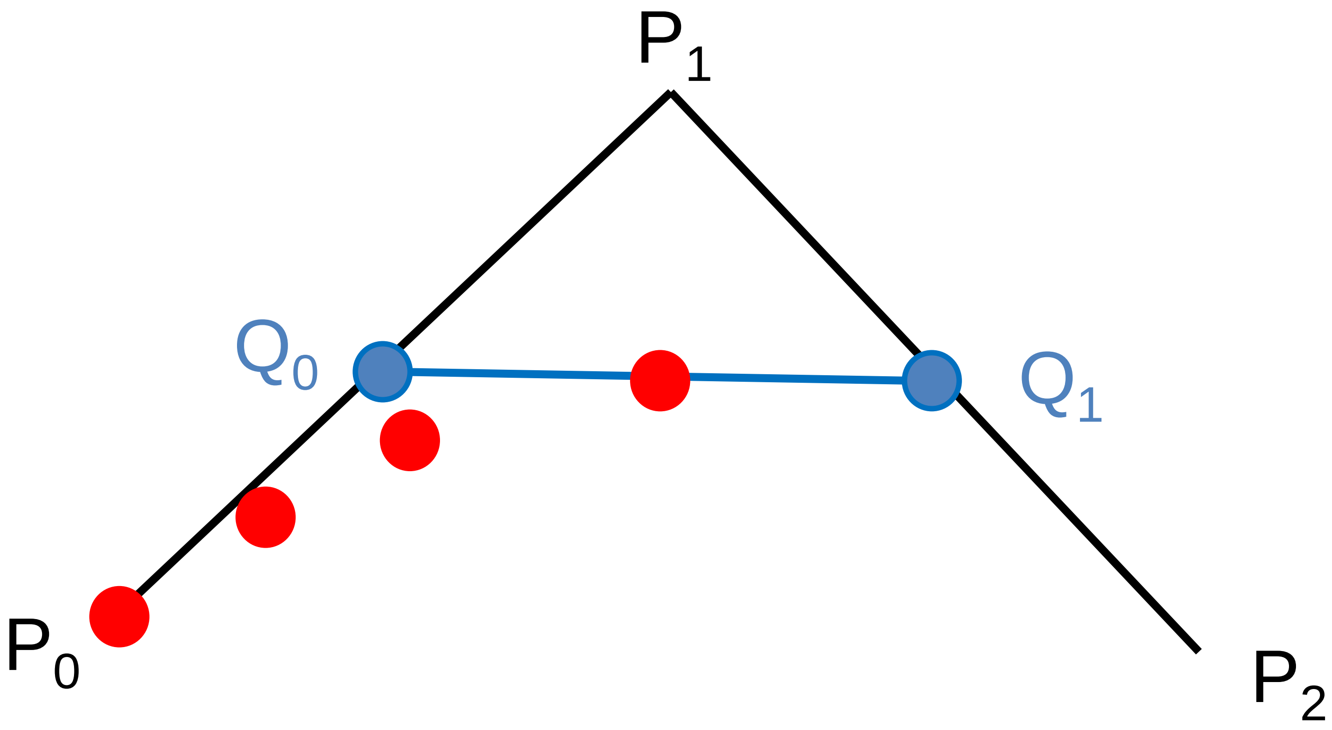
Bézier Curves Derivation (6/13)

at $u = 0.2$,



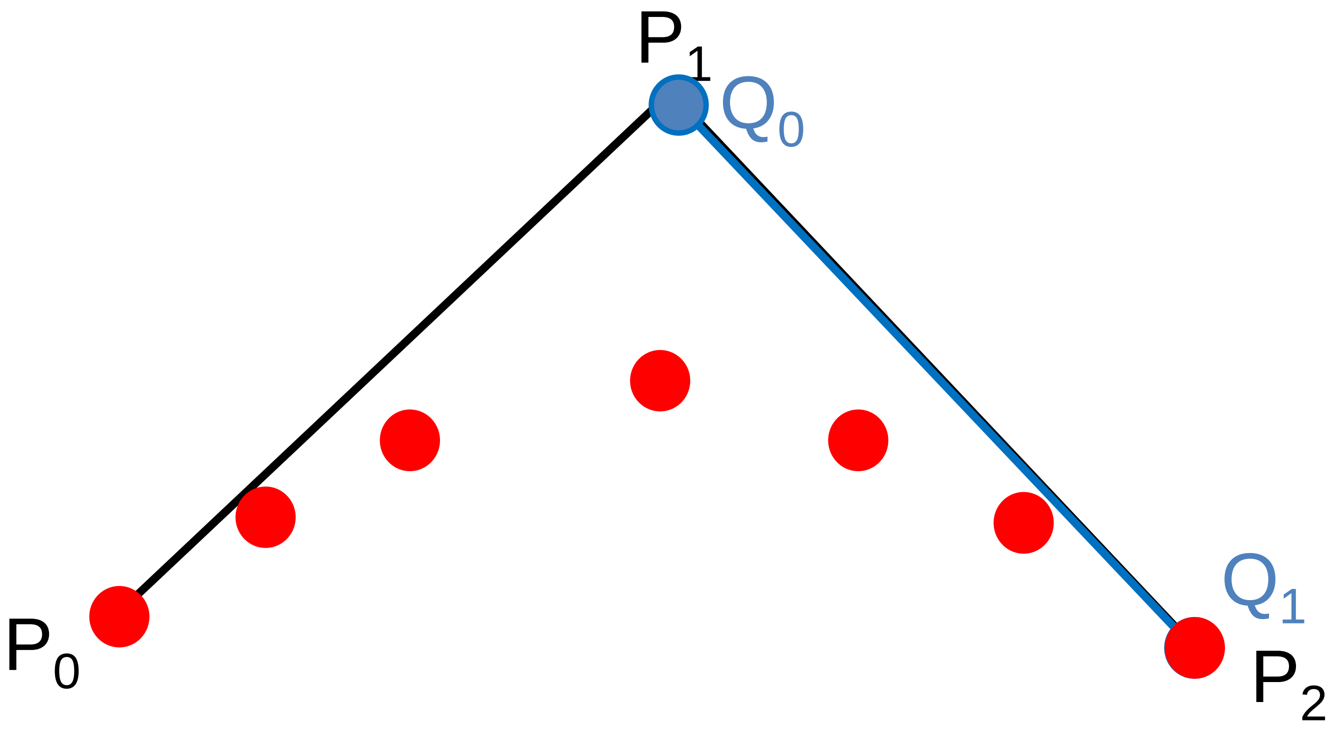
Bézier Curves Derivation (7/13)

at $u = 0.5$,

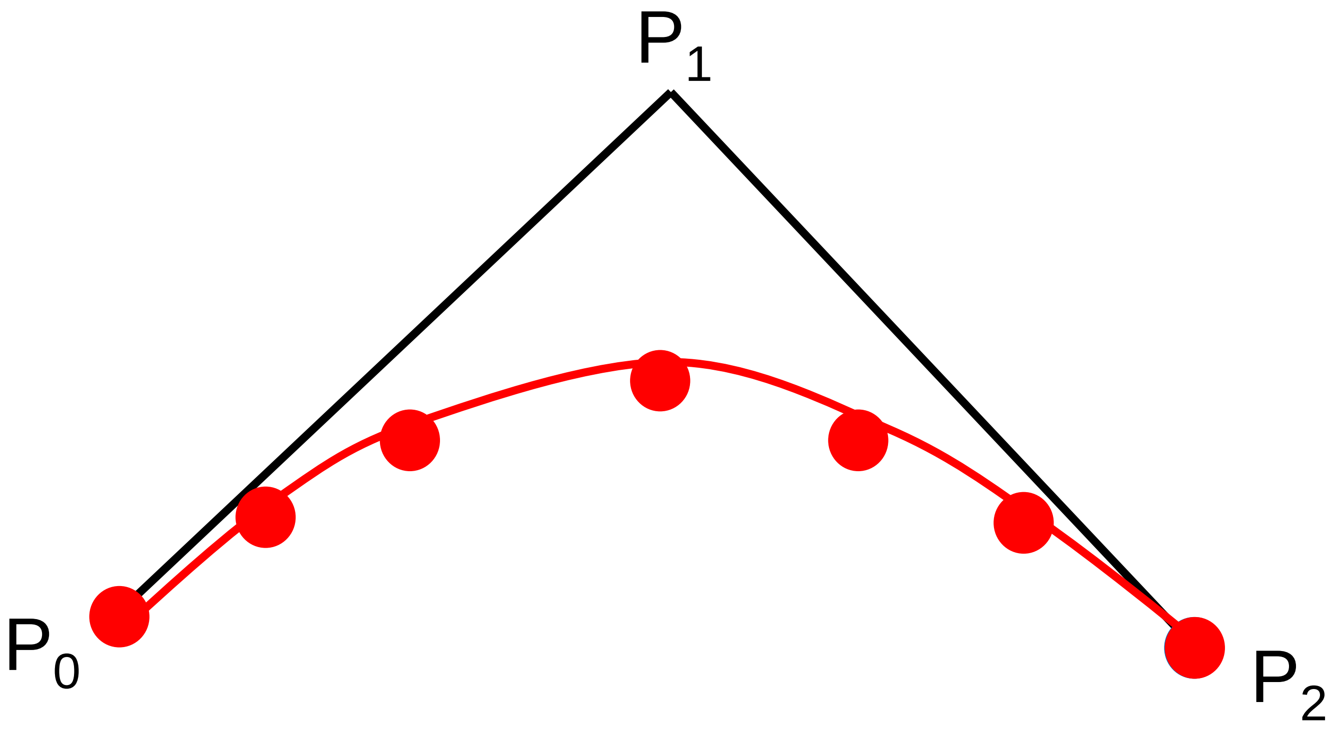


Bézier Curves Derivation (8/13)

at $u = 1$,



Bézier Curves Derivation (9/13)



Bézier Curves Derivation (10/13)

- Assume Q_0 and Q_1 lies on the line $P_0 \rightarrow P_1$ and $P_1 \rightarrow P_2$

$$Q_0 = P_0 + u(P_1 - P_0)$$

$$Q_1 = P_1 + u(P_2 - P_1)$$

- $Q(u)$ is the point on the Bezier curve on the line $Q_0 \rightarrow Q_1$

$$Q(u) = Q_0 + u(Q_1 - Q_0)$$

- Combining them gives

$$Q(u) = (1-u)^2 P_0 + 2u\,(1-u)P_1 + u^2 P_2$$

Bézier Curves Derivation (11/13)

de Casteljau's Algorithm

- This derivation is known is de Casteljau's Algorithm
- Let $P_{i,j}$ denote the control points
- where $P_{i,0}$ are the original control points P_0 to P_2

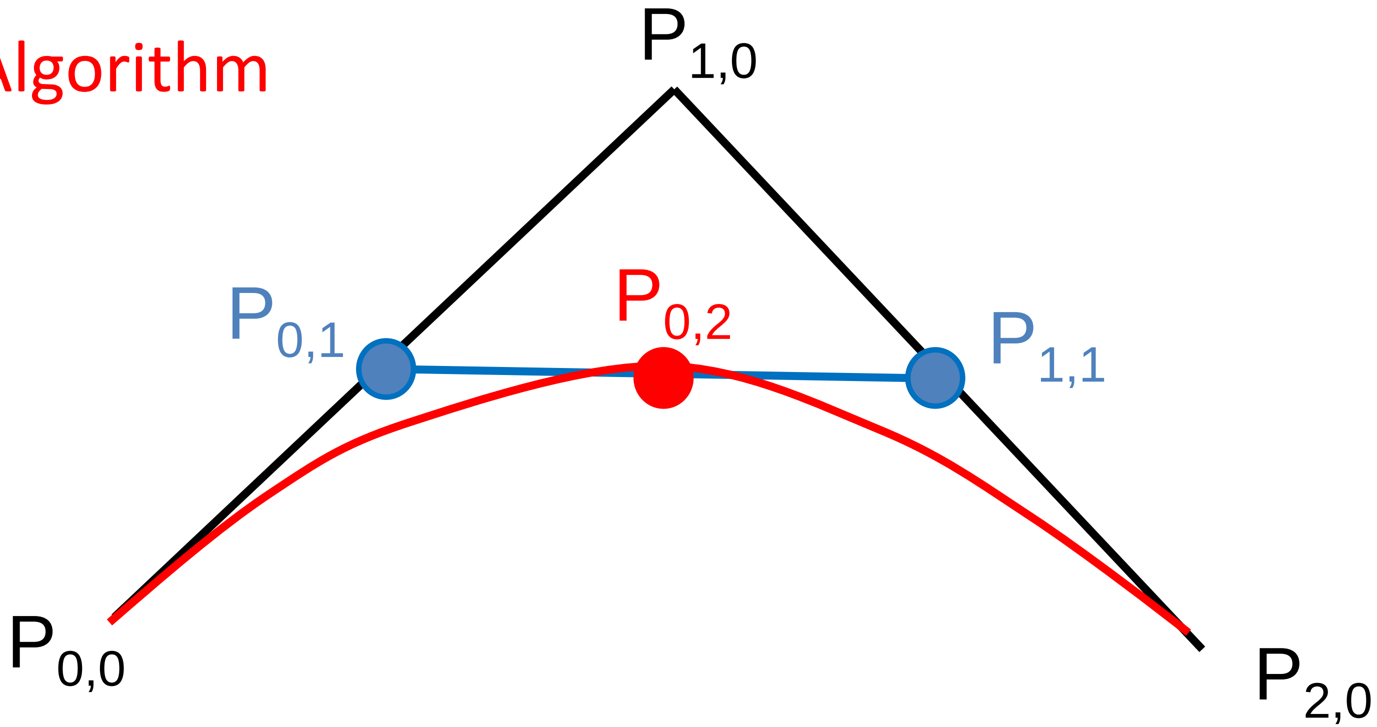
  $P_{i,1}$ are the points Q_1 to Q_2

  $P_{0,2}$ is the Q(u) then,

$$P_{i,j} = (1 - u)\, P_{i,j-1} + u\, P_{i+1,j-1}$$

Bézier Curves Derivation (12/13)

de Casteljau's Algorithm



$$P_{0,2} = (1-u)\, P_{0,1} + u\, P_{1,1}$$

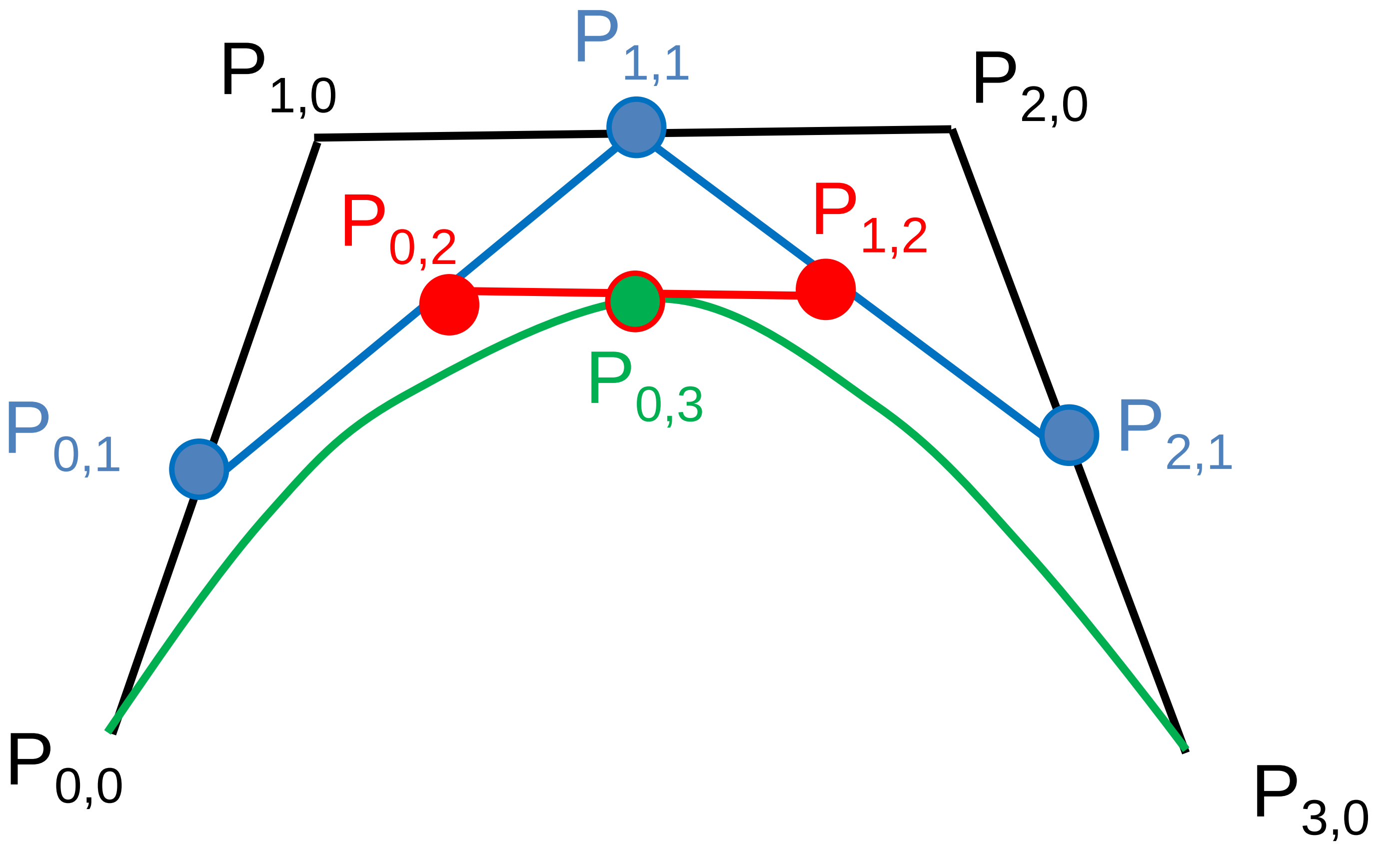
$$P_{0,2} = (1-u)\, [(1-u)\, P_{0,0} + u\, P_{1,0}] + u\, [(1-u)\, P_{1,0} + u\, P_{2,0}]$$

$$P_{0,2} = (1-u)^2\, P_{0,0} + 2u\, (1-u)\, P_{1,0} + u^2\, P_{2,0}$$

Bézier Curves Derivation (13/13)

For cubic Bézier Curves with 4 control points

$$P_{0,3} = (1-u)^3\, P_{0,0} + 3u\,(1-u)2\, P_{1,0} + 3u^2\,(1-u)\, P_{2,0} + u^3\, P_{3,0}$$



General form of Bézier Curves

General form of Bézier Curves:

$$Q(u) = \sum_{\mathrm{i=0}}^{\mathrm{d}} B_{i,d}(u)\mathbf{P}_i \qquad 0 \le u \le 1$$

$B_{i,d}(u)$ is called Bernstein polynomials

$$B_{\mathrm{i,d}}(u) = \begin{pmatrix} d \\ i \end{pmatrix} u^i (1-u)^{d-i}$$

$$where \begin{pmatrix} d \\ i \end{pmatrix} = \frac{d!}{i!(d-i)!} u^i (1-u)^{d-i}$$

Bézier curves are weighted sum of control points using n’th-order Bernstein polynomials

General form of Bézier Curves

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$$B_{i,d}(u) = \binom{d}{i} u^i (1-u)^{d-i}$$

$$where \binom{d}{i} = \frac{d!}{i!(d-i)!} u^i (1-u)^{d-i}$$

$$B_{0,2}(u) = (1-u)^2 \qquad B_{0,3}(u) = (1-u)^3$$
$$B_{1,2}(u) = 2u(1-u) \qquad B_{1,3}(u) = 3u(1-u)^2$$
$$B_{2,2}(u) = u^2 \qquad B_{2,3}(u) = 3u^2(1-u)$$
$$B_{3,3}(u) = u^3$$

$$Q_2(u) = (1-u)^2 P_{0,} + 2u\,(1-u)\,P_1 + u^2\,P_2$$

General form of Bézier Curves

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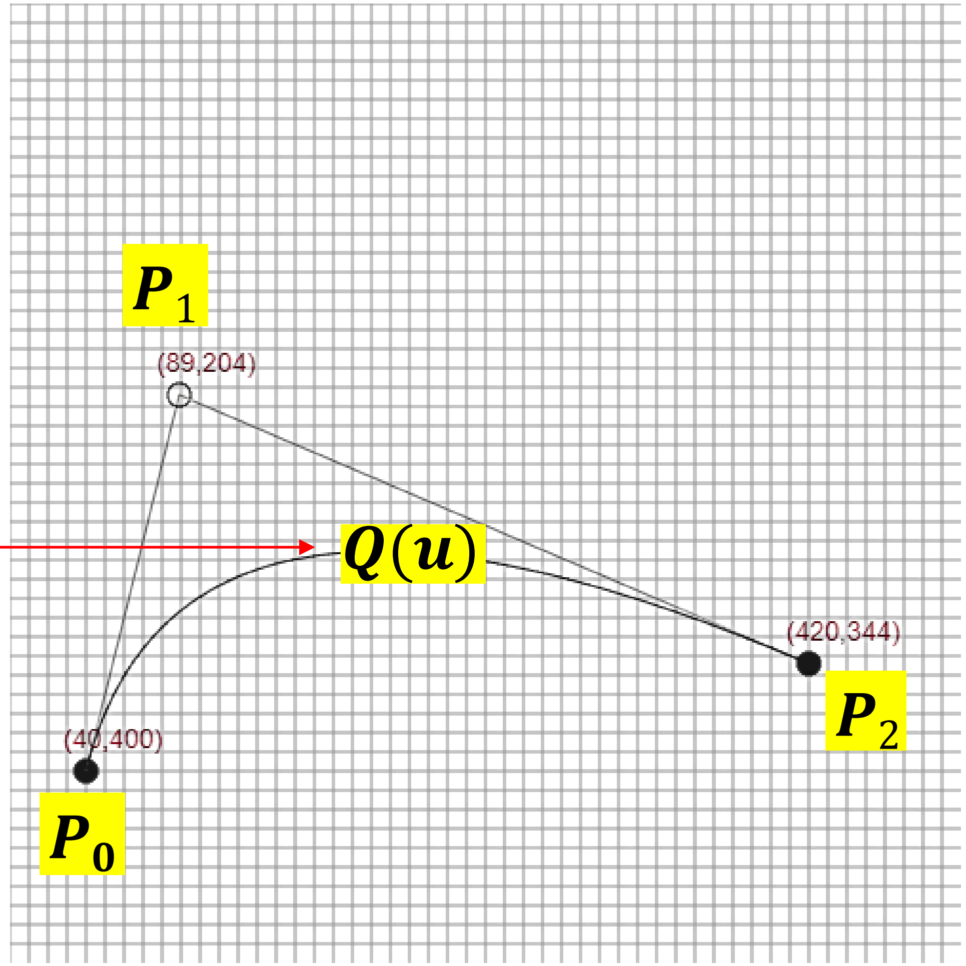
A point on the curve



General form of Bézier Curves

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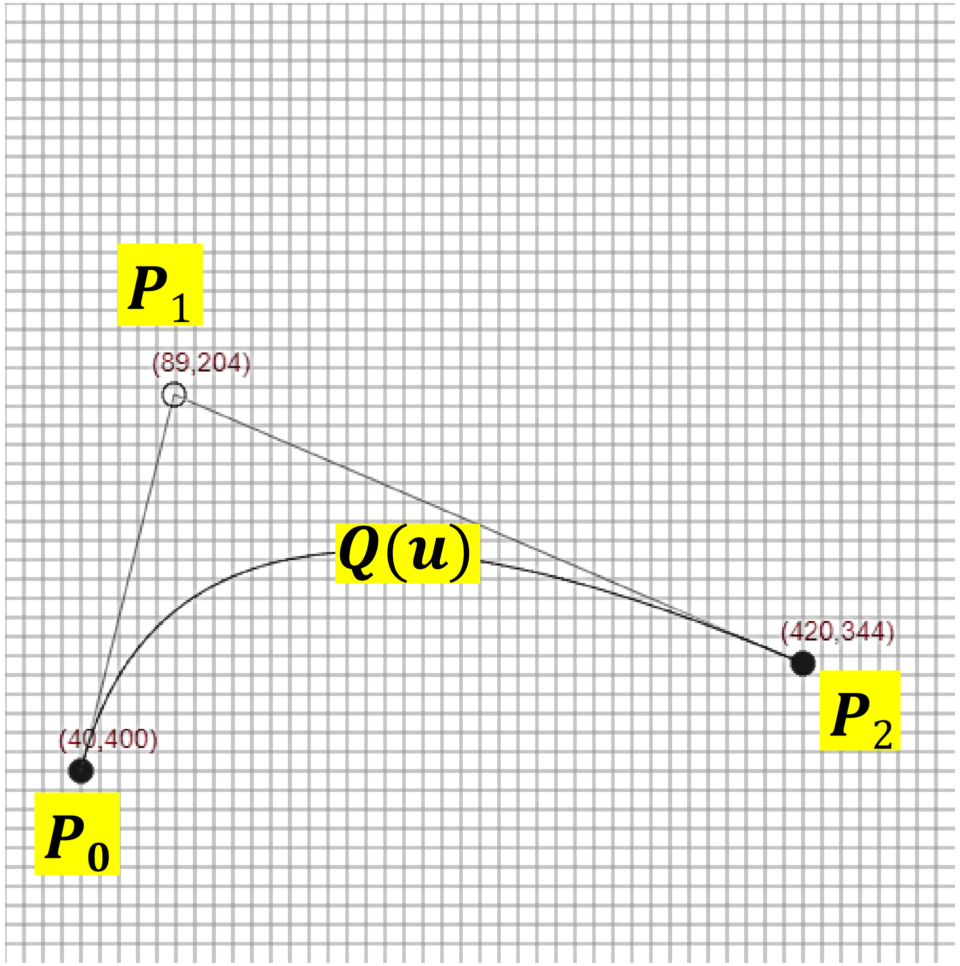
$$where \ \binom{d}{i} = \frac{d!}{i!(d-i)!} u^i (1-u)^{d-i}$$

Denoted with $Q_d(u)$



Credit: CPSC 589/689 Course Notes, University of Calgary, Faramarz Samavati

General form of Bézier Curves

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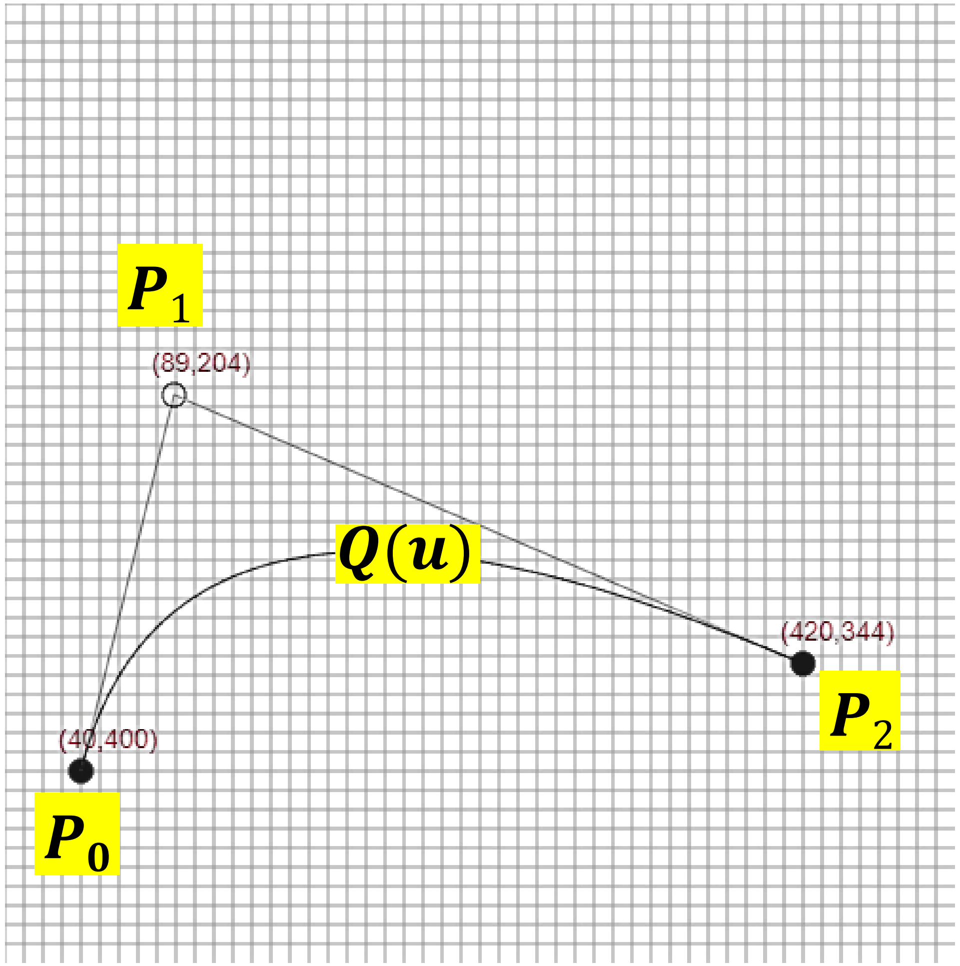
$$Q(u) = \sum_{\mathrm{i}=0}^{\mathrm{d}} B_{i,d}(u)\mathbf{P}_i$$

$$B_{\mathrm{i,d}}(u) = \begin{pmatrix} d \\ i \end{pmatrix} u^i (1-u)^{d-i}$$

$$where \begin{pmatrix} d \\ i \end{pmatrix} = \frac{d!}{i!(d-i)!} u^i (1-u)^{d-i}$$



Where is $Q_d(0.5)$ situated?

Where is $Q_d(0)$ situated?

Where is $Q_d(1)$ situated?

Practice Problem 1

Given control points $P_0 = (0,0), P_1 = (4,2), P_2 = (8,0)$, find the Bézier curve values $Q_2(0)$, $Q_2(\frac{1}{2})$ and $Q_2(1)$.

Why subscript 2 for $Q_2(u)$?

Practice Problem 1

Given control points $P_0 = (0,0), P_1 = (4,2), P_2 = (8,0)$, find the Bézier curve values $Q_2(0)$, $Q_2(\frac{1}{2})$ and $Q_2(1)$.

$$Q_2(u) = \sum_{i=0}^{n} B_{i,2}(u) P_i \quad 0 \le u \le 1$$

$$B_{i,d}(u) = \binom{d}{i} u^i (1-u)^{d-i} \qquad \binom{d}{i} = \frac{d!}{i!(d-i)!}$$

$$Q_2(u) = B_{0,2}(u)P_0 + B_{1,2}(u)P_1 + B_{2,2}(u)P_2$$

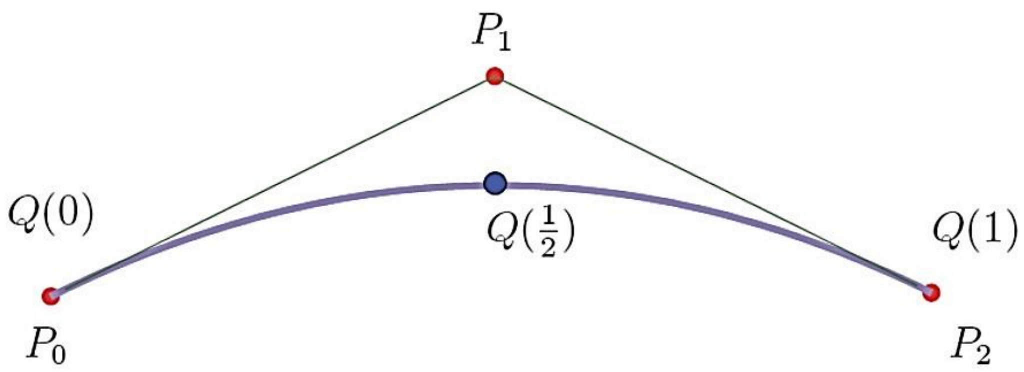
$$Q_2(u) = (1-u)^2 P_0 + 2(1-u)uP_1 + u^2 P_2$$

Practice Problem 1

Given control points $P_0 = (0,0), P_1 = (4,2), P_2 = (8,0)$, find the Bézier curve values $Q_2(0)$, $Q_2(\frac{1}{2})$ and $Q_2(1)$.

$$Q_2(u) = (1-u)^2 P_0 + 2(1-u)uP_1 + u^2 P_2$$

- $Q_2(0) = (1-0)^2 P_0 + 2(1-0)0P_1 + 0^2 P_2 = P_0 = (0,0)$
- $Q_2(\frac{1}{2}) =$ ..... *Do calculations* .... $= (4,1)$
- $Q(1) =$ ... *Do calculations* ... $= (8,0)$

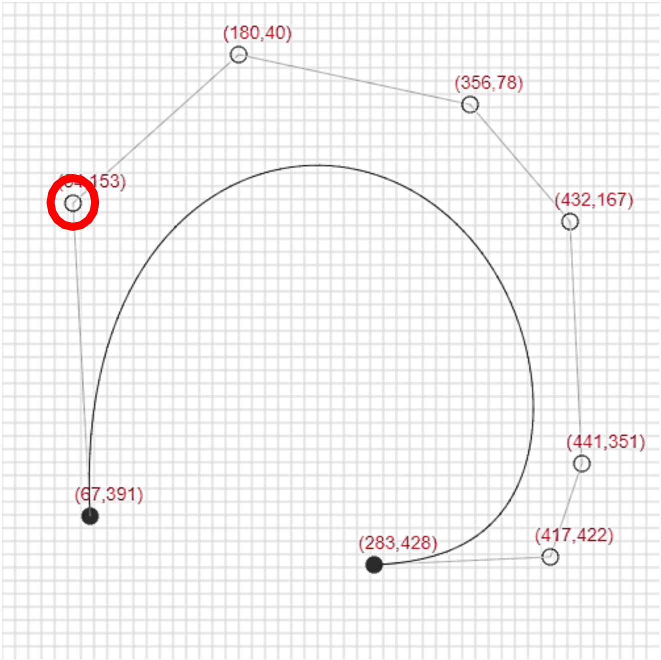


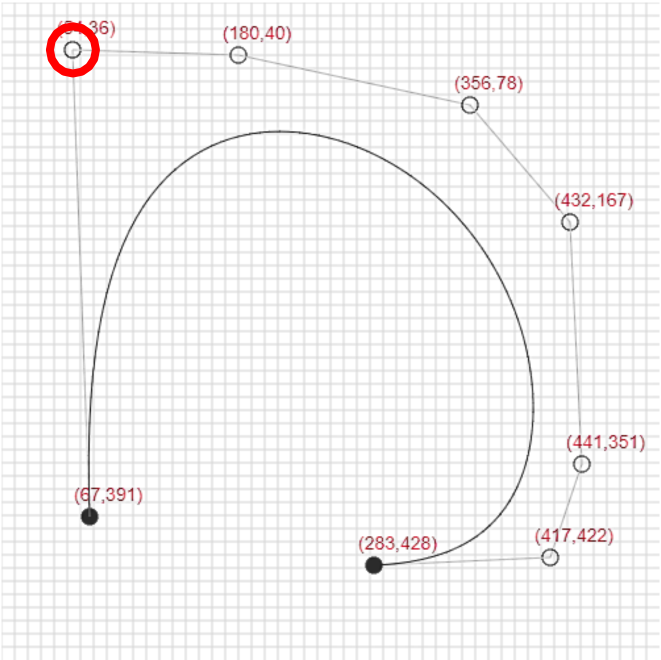
Properties of Bezier Curves

- They generally follow the shape of the control polygon, which consists of the segments joining the control points
- They always pass through the first and last control points
- They are contained in the convex hull of their defining control points
- The degree of the polynomial defining the curve segment (d) is one less than that the number of defining polygon point (n) i.e. n = d+1

Disadvantages

- A change to any of the control point alters the entire curve.
- Having a large number of control points requires high polynomials to be evaluated. This is expensive to compute.

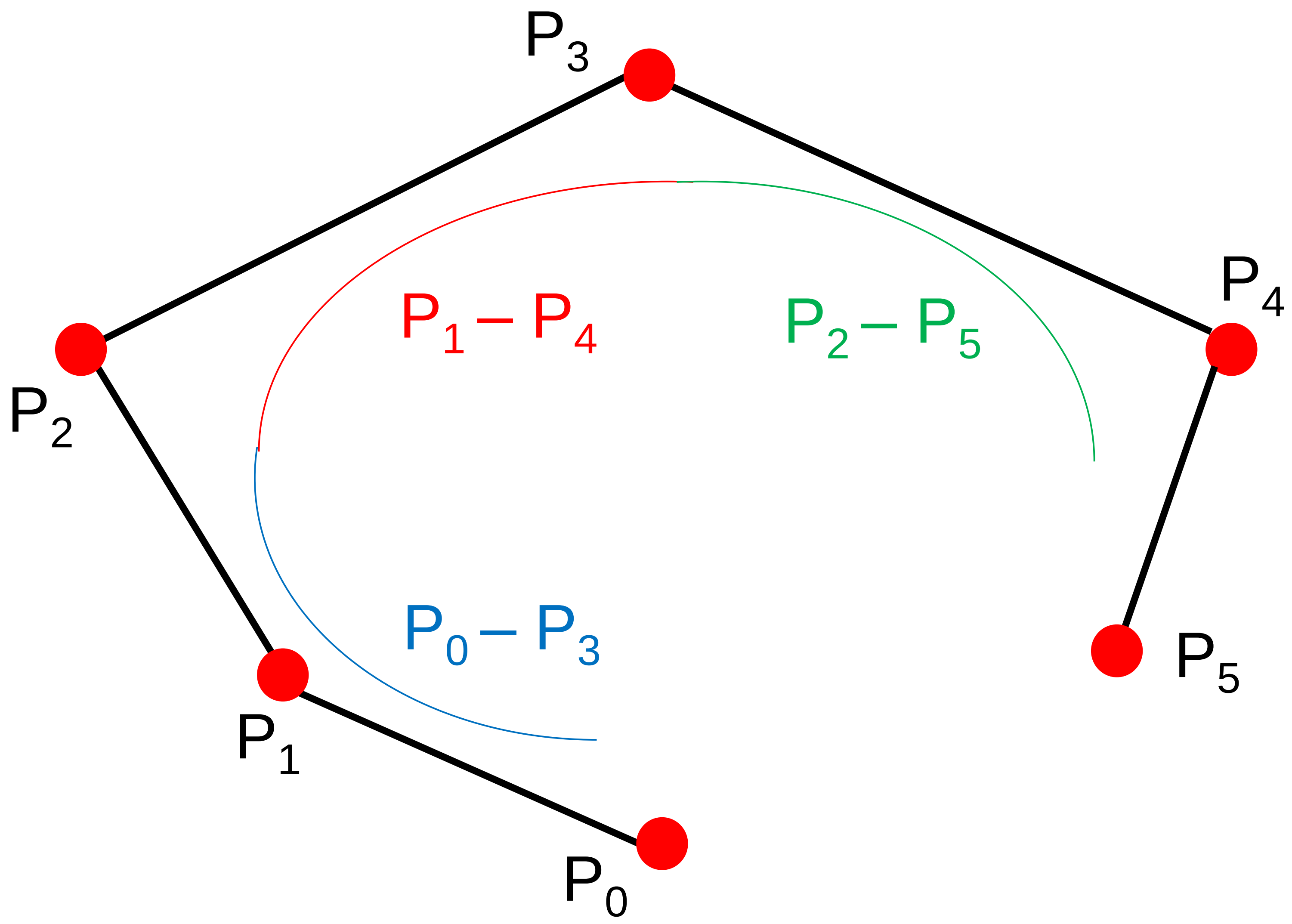




B-Spline Curve (1/4)

- B-splines (or Basis Splines) use several Bezier Curves joined end on end
- a k degree B-spline curve defined by $n+1$ control points will consists of $n - k + 1$ Bezier Curves
- For example, a cubic B-Spline Curve defined by 6 control points P_0 P_1 P_2 P_3 P_4 P_5 consists of $n - k + 1 = 5 - 3 + 1 = 3$ Bezier Curves

B-Spline Curve (2/4)



Cubic B-Spline Curve with 6 control Points

B-Spline Curve (3/4)

- Degree is independent of the number of control points
- The final point of the first Bezier curve has the same co-ordinate as the first point on the second Bezier curve (C^0 continuity)
- The first derivative at the end of the first Bezier curve is the same as the first derivative at the start of the second Bezier curve (C^1 continuity)
- The second derivative at the end of the first Bezier curve is the same as the second derivative at the start of the second Bezier curve (known as C2 continuity).

B-Spline Curve (4/4)

- A B-spline curve S(t), is defined by,

$$S(t) = \sum_{\mathrm{i}=0}^{\mathrm{n}} N_{i,k}(t)\mathbf{P}_i$$

- where (P_0, P_1,…,P_n) are the control points
- k is the order of the polynomial segments of the B-spline curve
- $N_{i,k}$(t) are the “normalized B-spline blending functions”.

Cox-de Bour formula (1/1)

- The blending function $N_{i,k}(t)$ is defined by Cox-de Bour recursion formula,

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases},$$

$$N_{i,j}(t) = \frac{t - t_i}{t_{i+j} - t_i} N_{i,j-1}(t) + \frac{t_{i+j+1} - t}{t_{i+j+1} - t_{i+1}} N_{i+1,j-1}(t).$$

- The values of t_i is taken from non-decreasing sequence of real numbers called knot vector, $T = \{t_0, t_1, t_2, \ldots\ldots, t_m\}$
- The number of knots in a knot vector, $m = k + n + 1$

Credit: Mathematics of Computer Graphics and Virtual Environments – Dr. Jon Siach

Uniform Quadratic B-spline (1/4)

- If the knots are equidistant then we have a uniform B-spline
- If a uniform quadratic B-spline is defined by the control points (P_0, P_1, P_2), hence k = 2, n = 2
- then m = k + n + 1= 2 + 2 + 1 = 5
- knot vector, T = $\{t_0, t_1, t_2, t_3, t_4, t_5\}$ = {0, 1, 2, 3, 4, 5}
- The B-Spline curve is defined by,

$$P(t) = \sum_{\mathrm{i}=0}^{2} N_{i,2}(t)\mathbf{P}_i$$

Uniform Quadratic B-spline (2/4)

- Equation of a quadratic B-spline curve is

$$P(t) = \sum_{i=0}^{n} N_{i,k}(t)\mathbf{P}_i$$

- when $t_2 \le t \le t_3$

$$N_{0,2}(t)P_0 = \frac{1}{2}[(t-t_2)^2 - 2(t-t_2) + 1]P_0,$$
$$N_{1,2}(t)P_1 = \frac{1}{2}[-2(t-t_2)^2 + 2(t-t_2) + 1]P_1,$$
$$N_{2,2}(t)P_2 = \frac{1}{2}(t-t_2)^2 P_2,$$

Uniform Quadratic B-spline (3/4)

- which can be written in matrix form as:

$$S(t) = \begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$$

- In general terms a uniform quadratic B-spline curve is written as:

$$\boxed{S_i(t) = \begin{pmatrix} P_i & P_{i+1} & P_{i+2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix},}$$

where i = 0, 1,...,n − k + 1

Uniform Quadratic B-spline (4/4)

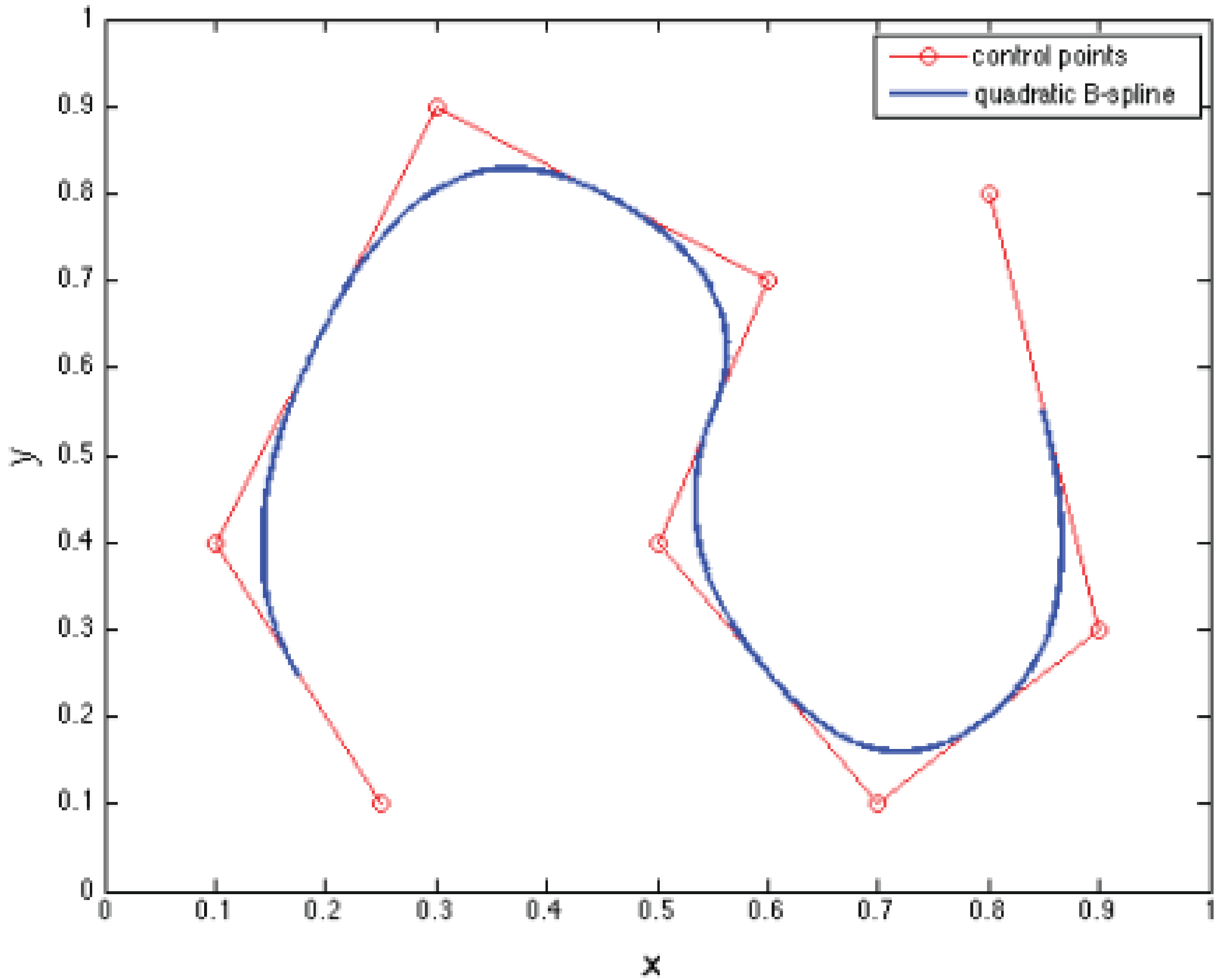


Fig: A uniform quadratic B-spline curve

Credit: Mathematics of Computer Graphics and Virtual Environments – Dr. Jon Siach

Open Uniform B-spline curves (1/3)

- Uniform B-splines aren't defined for all the range $[t_{0,} t_m]$.
- As a result, B-spline curve is undefined at the start and end control points.
- Open uniform B-spline curves overcome this by setting the first k knots the same value and the last k knots have the same value
- For example, consider an open uniform quadratic B-spline curve defined by the control points P_0, …., $P_{4.}$ Here $m = k + n + 1 = 2 + 4 + 1 = 7$
- So the knot vector is $T = \{t_0, t_1, t_2, t_3, t_4, t_{5,} t_6, t_7\}$

$$= \{0, 0, \quad 0, 1, 2, 3, \quad 3, 3\}$$

Open Uniform B-spline curves (2/3)

- Matrix form of an open uniform B-spline curve is

$$S_0(t) = \begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix},$$

$$S_i(t) = \begin{pmatrix} P_i & P_{i+1} & P_{i+2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}, \qquad 1 \leq i \leq n-3,$$

$$S_{n-2}(t) = \begin{pmatrix} P_{n-2} & P_{n-1} & P_n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix},$$

Open Uniform B-spline curves (3/3)

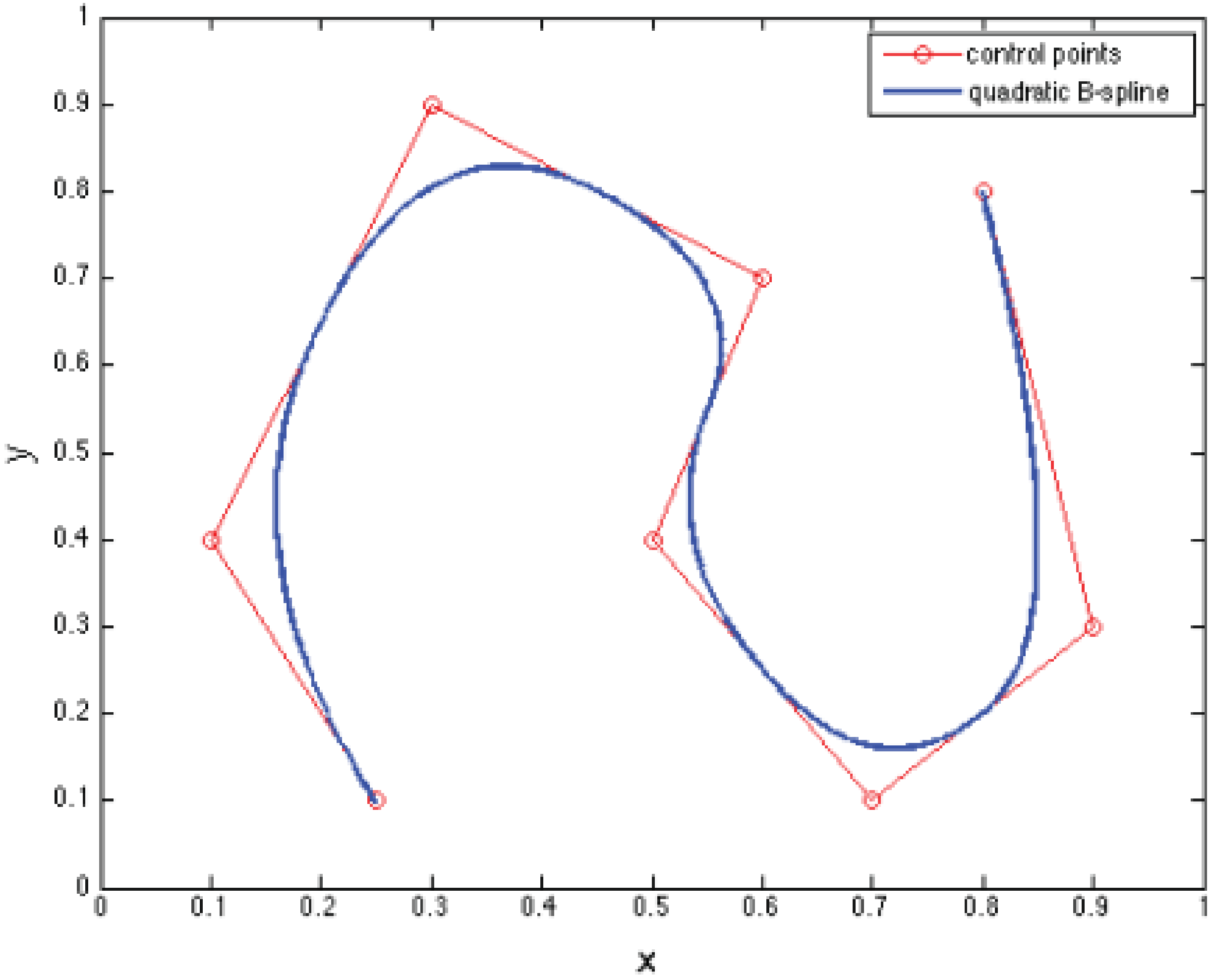


Fig: A Open Uniform B-spline curve

Credit: Mathematics of Computer Graphics and Virtual Environments – Dr. Jon Siach

Practice Problem 2

An uniform quadratic B-Spline curve is defined by 5 control points $P_0(1, 2)$, $P_1(3, 8)$, $P_2(5, 2)$, $P_3(7, 1)$ and $P_4(8, 0)$. Find the points on the curve segments for t = 0

Practice Problem 2

An uniform quadratic B-Spline curve is defined by 5 control points $P_0(1, 2)$, $P_1(3, 8)$, $P_2(5, 2)$, $P_3(7, 1)$ and $P_4(8, 0)$. Find the points on the curve segments for t = 0

solution:

k = 2, n = 4

Number of curve segments = n – k + 1 = 4 – 2 + 1 = 3

$$S_i(0) = \begin{bmatrix} P_i & P_{i+1} & P_{i+2} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

Practice Problem 2

$$S_0(0) = \begin{bmatrix} P_0 & P_1 & P_2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$S_1(0) = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} = ?$$

$$S_2(0) = \begin{bmatrix} P_2 & P_3 & P_4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} = ?$$

Credit: Mathematics of Computer Graphics and Virtual Environments – Dr. Jon Siach

Practice Problem 3

An open uniform quadratic B-Spline curve is defined by 5 control points $P_0(1, 2)$, $P_1(3, 8)$, $P_2(5, 2)$, $P_3(7, 1)$ and $P_4(8, 0)$. Find the points on the curve segments for $t = 0$

solution:

$k = 2, n = 4$

Number of curve segments = $n - k + 1 = 4 - 2 + 1 = 3$

Practice Problem 3

$$S_0(0) = \begin{bmatrix} P_0 & P_1 & P_2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$S_1(0) = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} = ?$$

$$S_2(0) = \begin{bmatrix} P_2 & P_3 & P_4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} = ?$$

Further Reading

- Fundamentals of Computer Graphics, 4th Edition - Chapter 15
- https://www.youtube.com/watch?v=2HvH9cmHbG4
- https://www.youtube.com/watch?v=qhQrRCJ-mVg

End