CSE4203: Computer Graphics

Lecture – 8 (Part - A)

Fractal Geometry

Outline

– Fractal
– Fractal Dimension
– The Cantor Set
– Koch Snowflake
– Sierpinski Triangle
– Mandelbrot Set

Fractal

- A **Fractal** is a geometric shape generated using set of recursive rules
- Fractal share a **self-similarity** property where part of fractal resembles the whole fractal
- The property is often seen in nature, e.g clouds, plants and landscapes etc.
- We will look at methods used to generate some popular fractals in this lecture

Credit:https://www.youtube.com/@mathematicsofcomputergraph6990

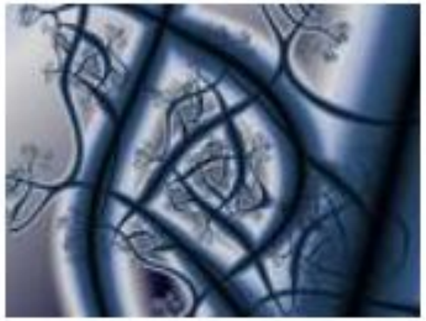
Example of Fractals

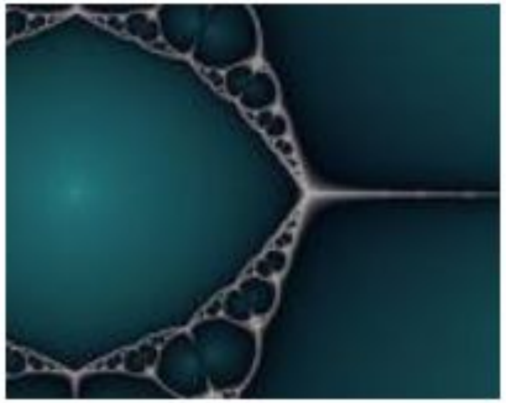


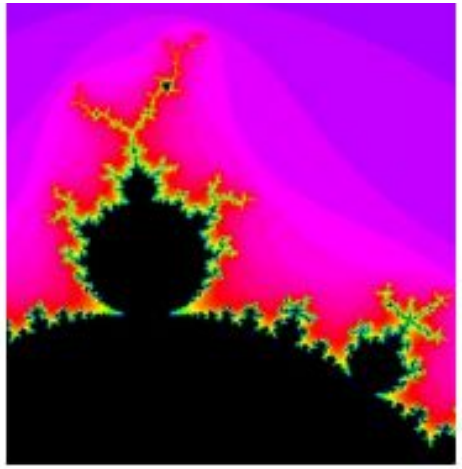



Credit:Internet

Artificial Fractals







Credit:Internet

The Cantor Set (1/6)

- A simplest example of a fractal is a Cantor Set devised by George Cantor
- Starting with a straight line at $n = 0$, we remove middle third of each line at each iteration
- As n gets larger, line segments gets smaller

The Cantor Set (2/6)

- A simplest example of a fractal is a Cantor Set devised by
- n = 0 

The Cantor Set (3/6)

- A simplest example of a fractal is a Cantor Set devised by
- n = 0
- n = 1

The Cantor Set (4/6)

- A simplest example of a fractal is a Cantor Set devised by

- n = 0
- n = 1
- n = 2

The Cantor Set (5/6)

- A simplest example of a fractal is a Cantor Set devised by

- n = 0
- n = 1
- n = 2
- n = 3

The Cantor Set (6/6)

- A simplest example of a fractal is a Cantor Set devised by

- n = 0
- n = 1
- n = 2
- n = 3
- n = 4

Fractal Dimension

- The fractal dimension is a measure of the complexity of a fractal
- Let N_n be the number of new elements added at iteration n and ϵ be the scaling factor of the size of new elements then the fractal dimension is

$$D = -\lim_{n\to\infty} \frac{\log(N_n)}{\log(\epsilon_n)}$$

- For a 2D fractal D ∈ [1, 2] and for a 3D fractal D ∈ [2, 3]

Dimension of Cantor Set (1/2)

- n = 0
- n = 1
- n = 2
- n = 3
- n = 4

Line segments is related to iterations

iterations	no. line segments	length of segment
0	1	1
1	2	$\frac{1}{3}$
2	4	$\frac{1}{9}$
3	8	$\frac{1}{27}$
$\vdots$	$\vdots$	$\vdots$
n	2^n	3^{-n}

Dimension of Cantor Set (2/2)

Using

$$D = -\lim_{n\to\infty} \frac{\log(N_n)}{\log\,(\epsilon_{\mathrm{n}})}$$

then the fractal dimension of Cantor set is

$$D = -\lim_{n\to\infty} \frac{\log(2^n)}{\log \frac{1}{3^n}} = -\lim_{n\to\infty} \frac{\mathrm{n}\log\ 2}{n\log \frac{1}{3}} = -\lim_{n\to\infty} \frac{\log\ 2}{\log 3^{-1}}$$

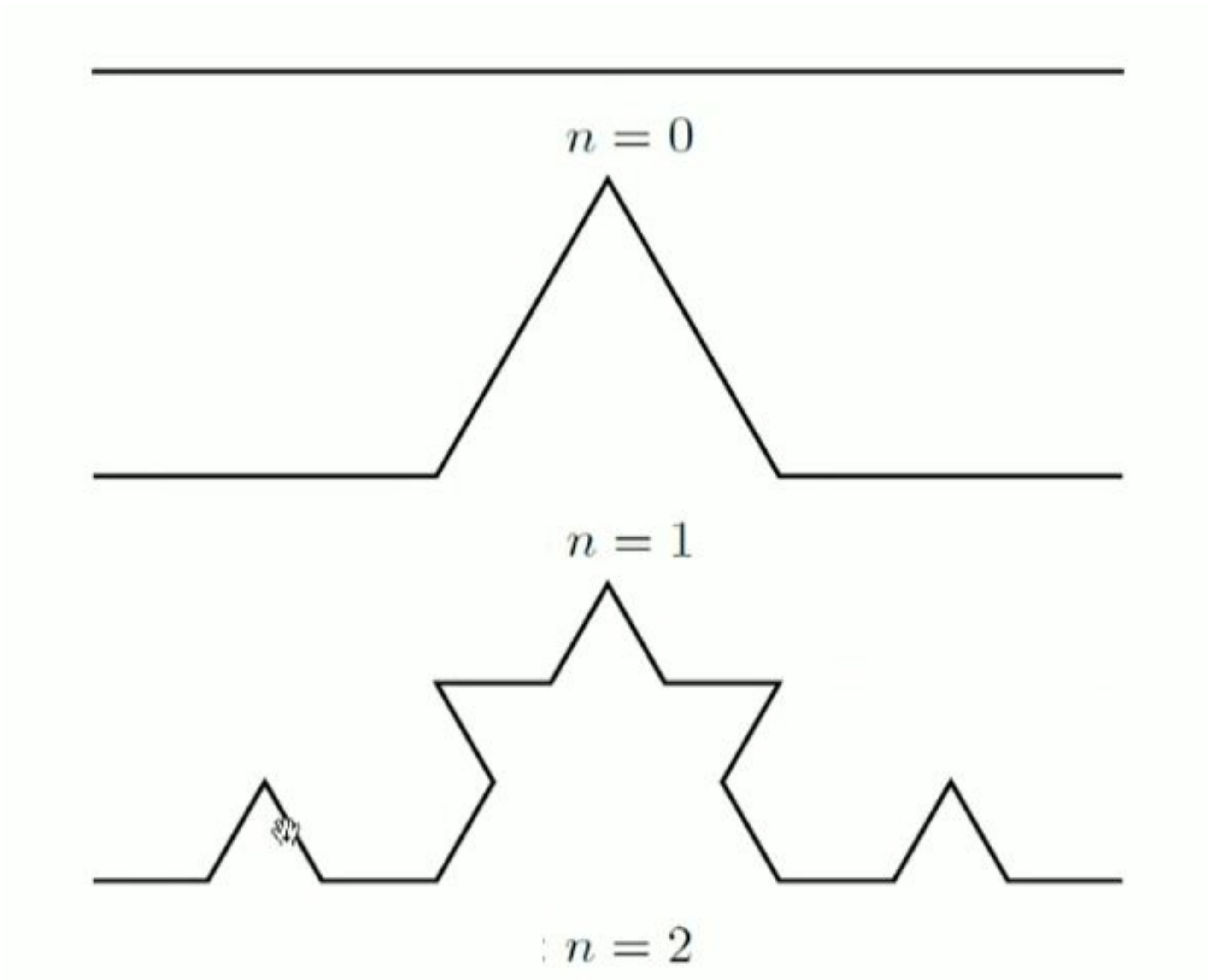
$$= \lim_{n\to\infty} \frac{\log\ 2}{\log 3}$$

$$= 0.6309\ \ldots$$

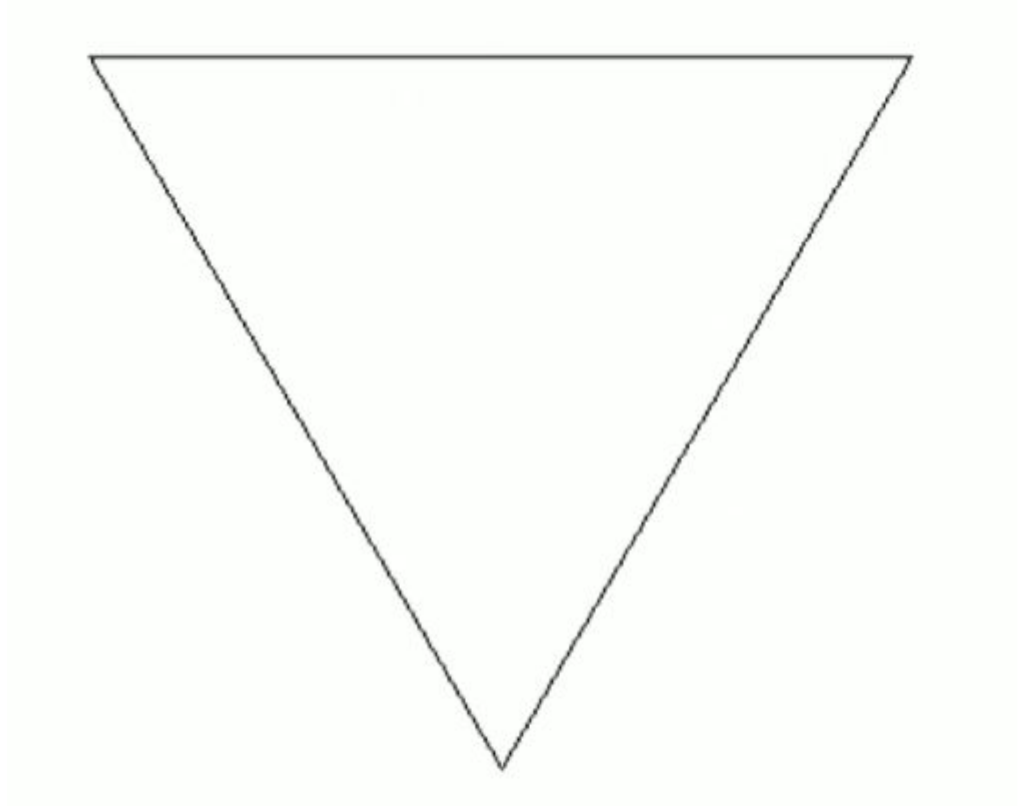
The Koch Snowflake (1/7)

- The Koch Snowflake named after Helge von Koch is a shape with a finite area bounded by infinite circumference
- The rules for generating a Koch Snowflake start with a equilateral triangle at stage = 0
  - Divide each line into three lines of equal length
  - Place an equilateral triangle so that the base fits along the middle length
  - Remove the middle length that is base of the triangle

The Koch Snowflake (2/7)

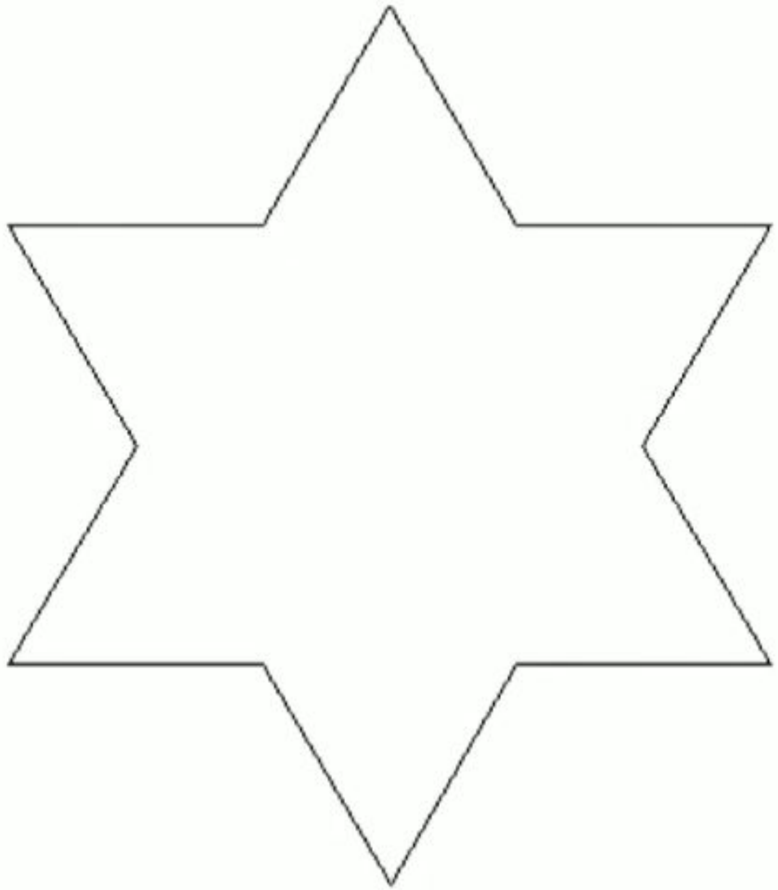


The Koch Snowflake (3/7)



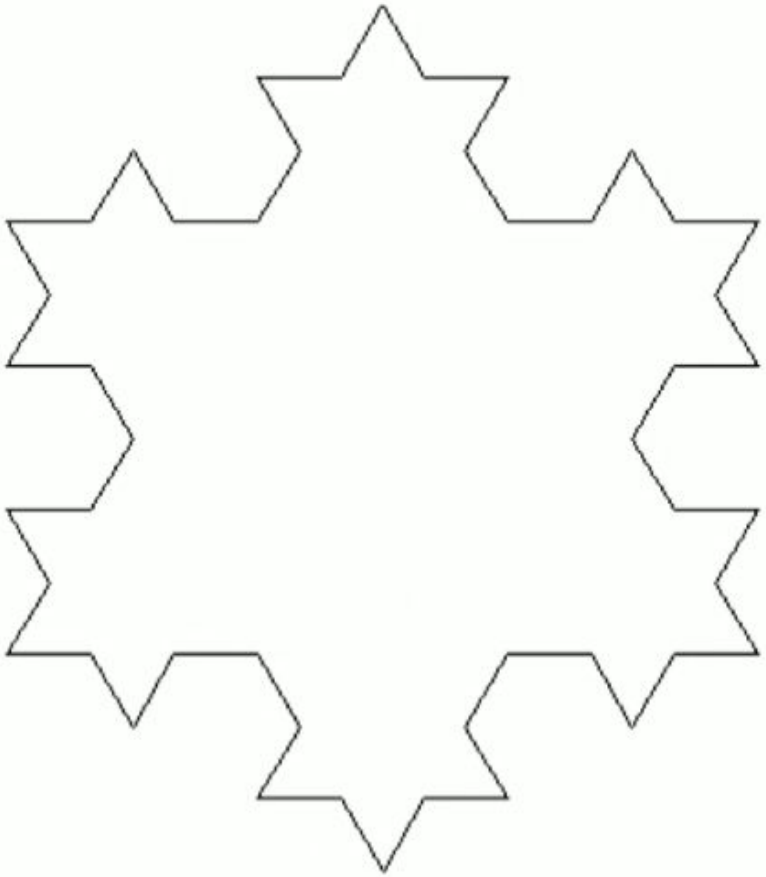
stage	no of sides	length of sides
0	3	1

The Koch Snowflake (4/7)



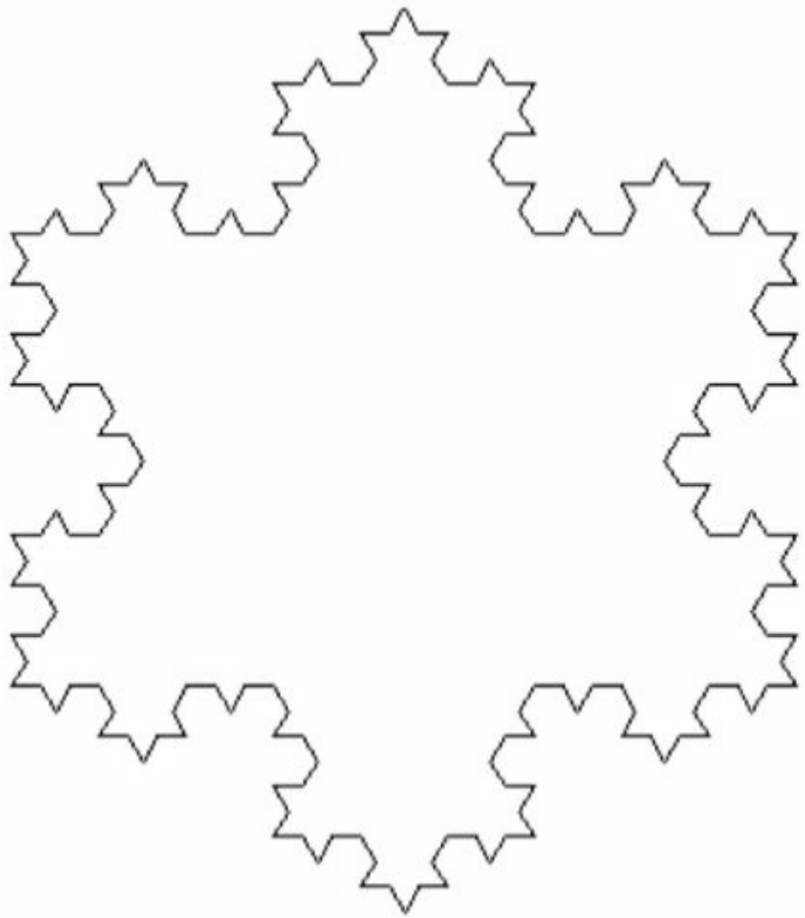
stage	no of sides	length of sides
1	3 x 4	1/3

The Koch Snowflake (5/7)



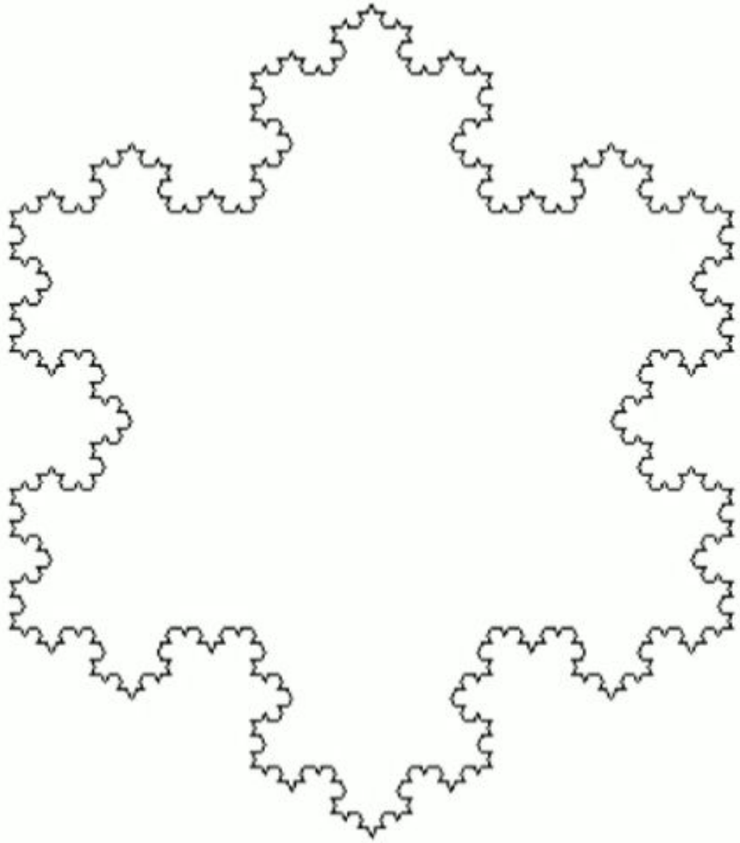
stage	no of sides	length of sides
2	3 x 4 x 4	$1/3^2$

The Koch Snowflake (6/7)



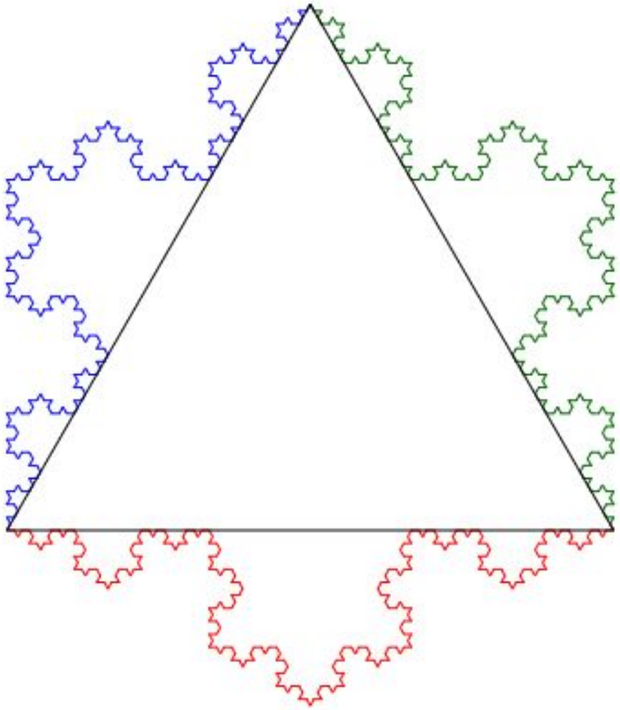
stage	no of sides	length of sides
3	3 x 4 x 4 x 4	$1/3^3$

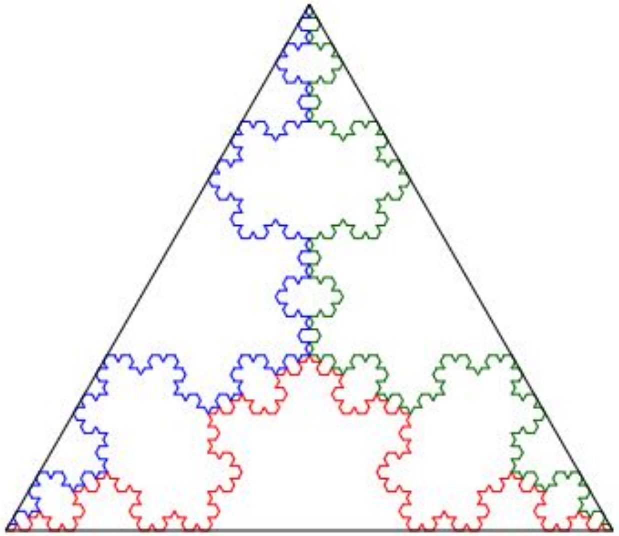
The Koch Snowflake (7/7)

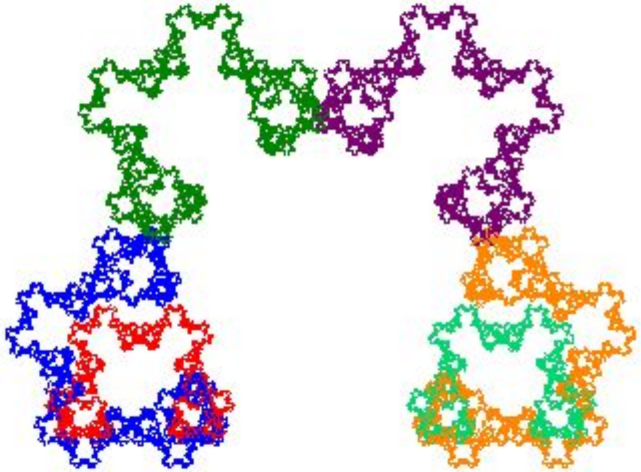


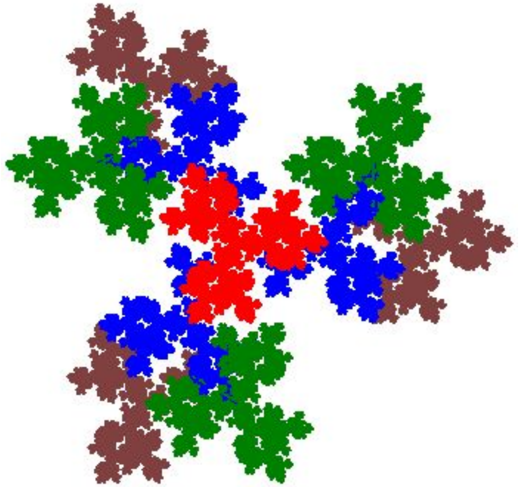
stage	no of sides	length of sides
4	3 x 4 x 4 x 4 x 4	$1/3^4$

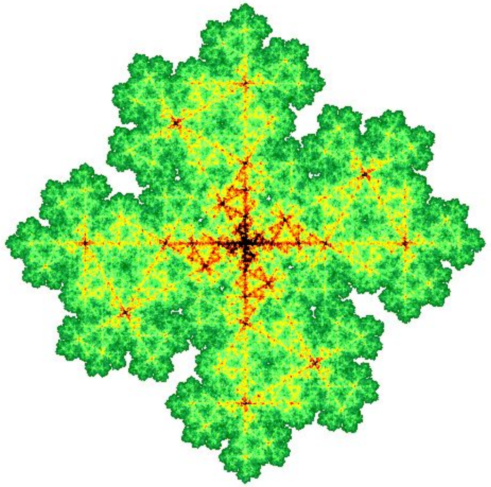
Colorful Koch Snowflakes

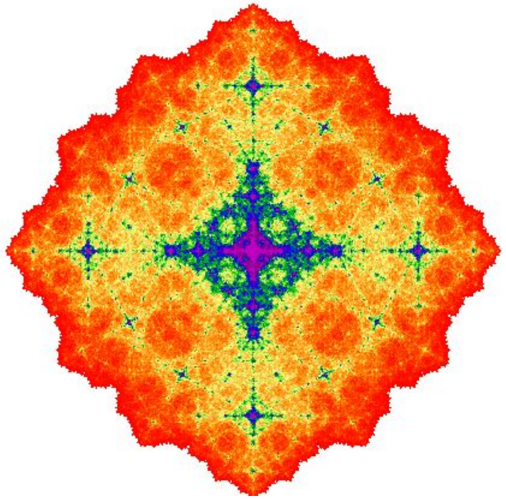






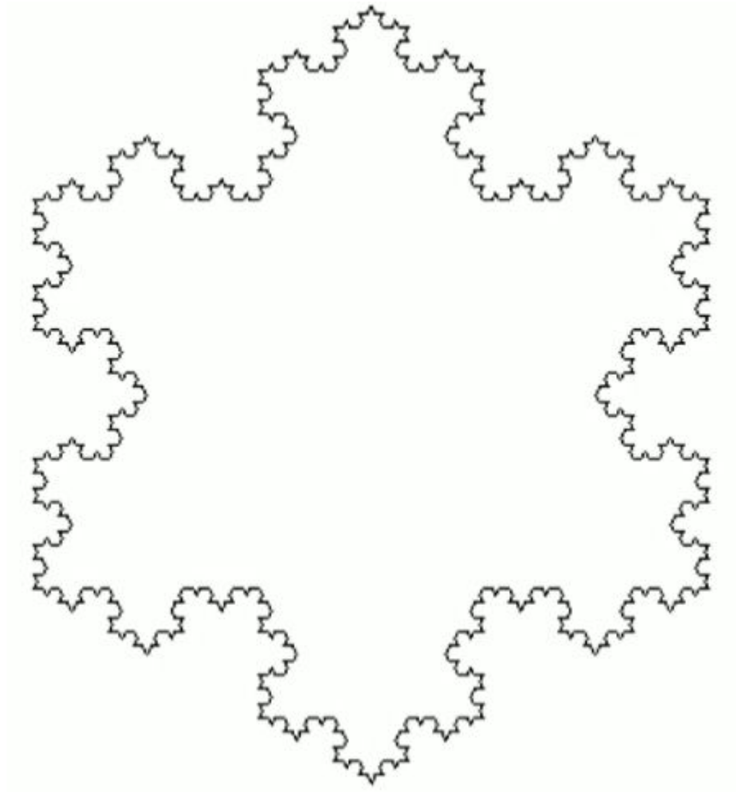






Koch Curve (agnesscott.org)

Fractal Dimension of Koch Snowflake (1/2)



stage	no of sides	length of sides
0	3	1
1	3 x 4	1/3
2	3 x 4 x 4	$1/3^2$
3	3 x 4 x 4 x 4	$1/3^3$
4	3 x 4 x 4 x 4 x 4	$1/3^4$
…	…	…
n	3×4^n	$1/3^n$

Fractal Dimension of Koch Snowflake (2/2)

Using

$$D = -\lim_{n\to\infty} \frac{\log(N_n)}{\log(\epsilon_n)}$$

then the fractal dimension of Koch Snowflake is

$$D = -\lim_{n\to\infty} \frac{\log\ [3(4^n)]}{\log\left[\frac{1}{3^n}\right]} \quad = -\lim_{n\to\infty} \frac{\log(3) + n\log 4}{-n\log(3)}$$

$$= \lim_{n\to\infty} \frac{\log(3)}{n\log(3)} + \frac{n\ \log 4}{n\log(3)} \quad = \frac{\log(4)}{\log(3)}$$

$$= 1.2619 \ldots$$

Fractal Dimension of Koch Snowflake (2/2)

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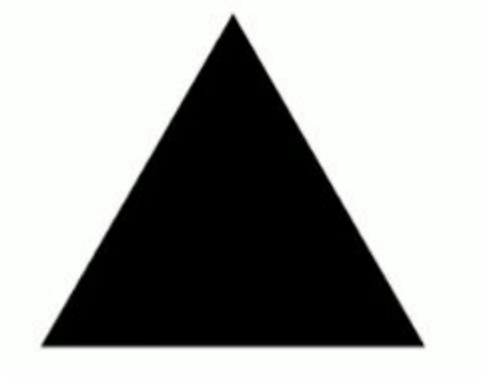
$$= 1.2619\ \ldots$$

Sierpinski Triangle (1/5)

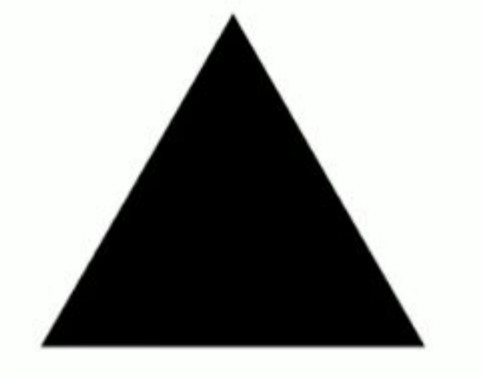
- Attributed to Waclaw Sierpinski, the Sierpinski has also been seen in the medieval mosaics
- It starts with a filled equilateral triangle at stage 0 and the rules are:
  - For each filled triangle, make an unfilled copy
  - Scale the copy by half and invert it
  - Place the copy in the center of the filled triangle

Sierpinski Triangle (2/5)



stage 0

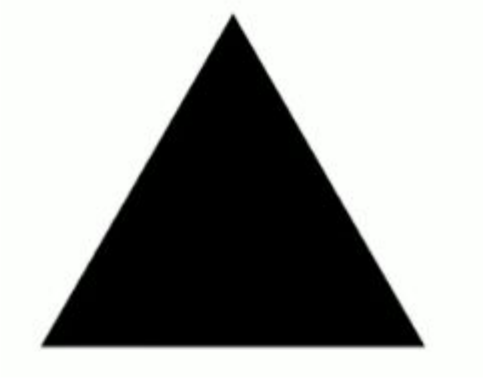
Sierpinski Triangle (3/5)



stage 0

stage 1

Sierpinski Triangle (4/5)

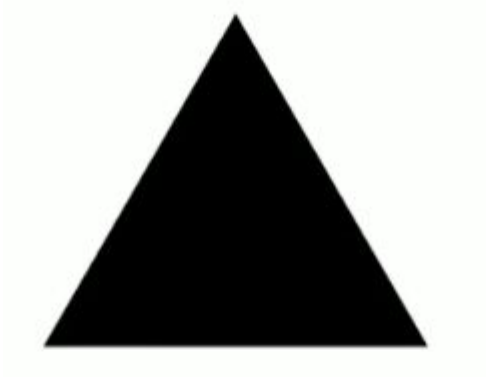


stage 0

stage 1

stage 2

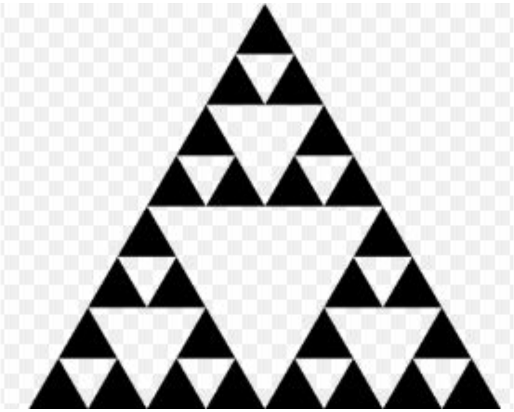
Sierpinski Triangle (5/5)



stage 0

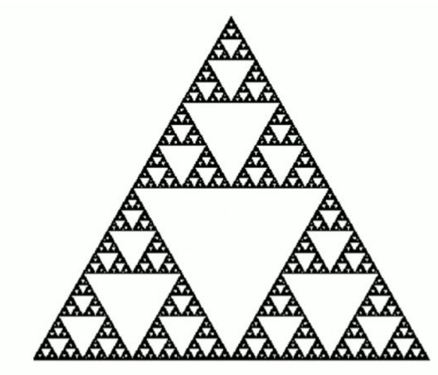
stage 1

stage 2



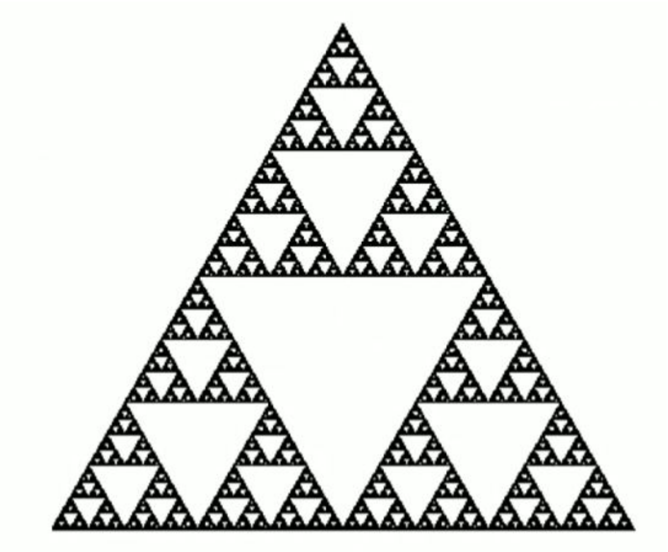
stage 3

stage 5



stage 6

Fractal Dimension of Sierpinski Triangle



$$D = \frac{\log(3)}{\log(2)}$$

$$= 1.585 \ldots$$

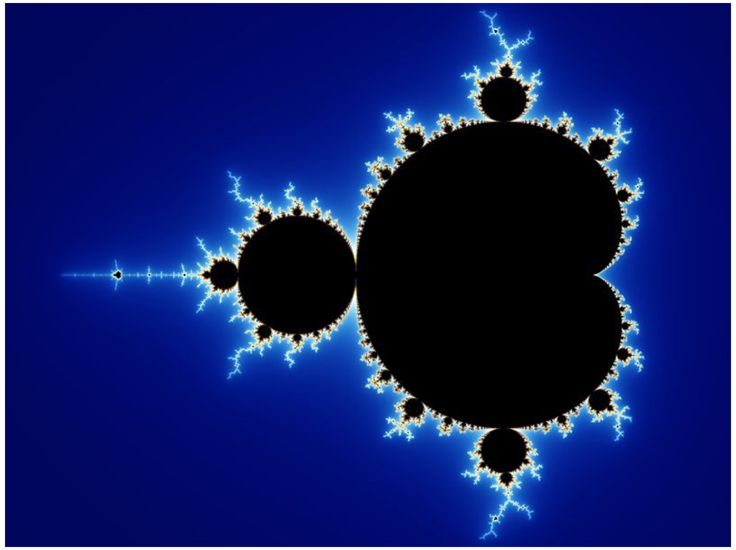
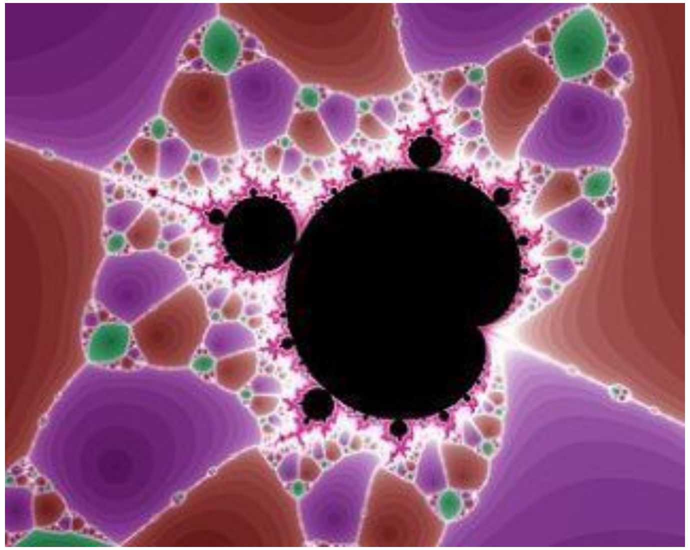
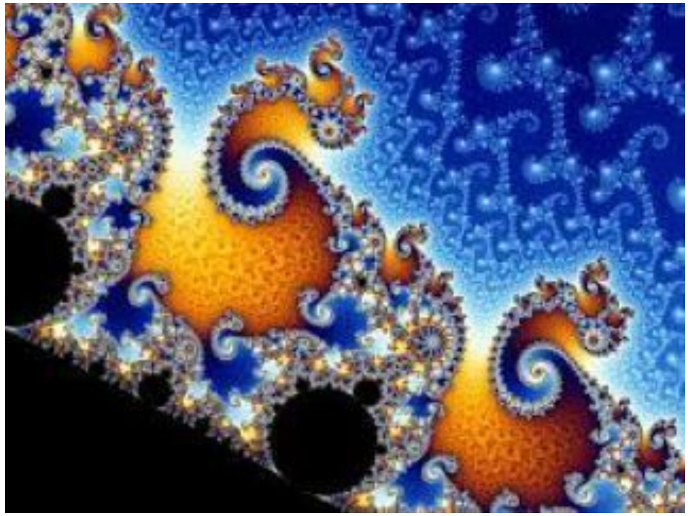
How?

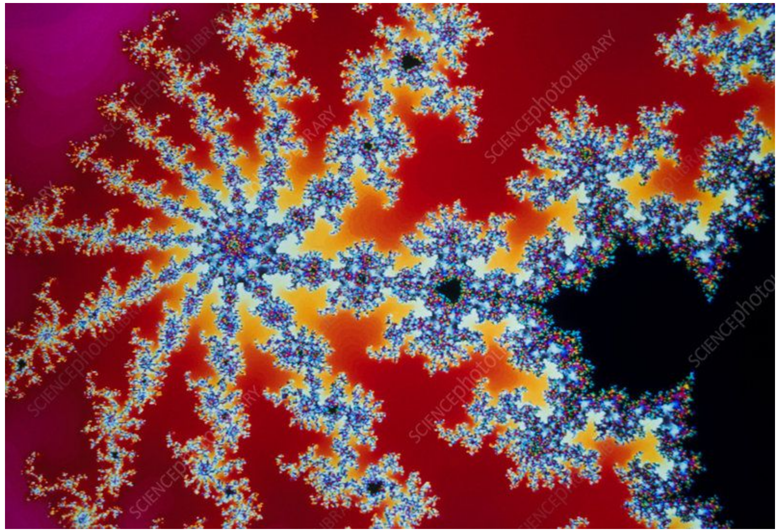
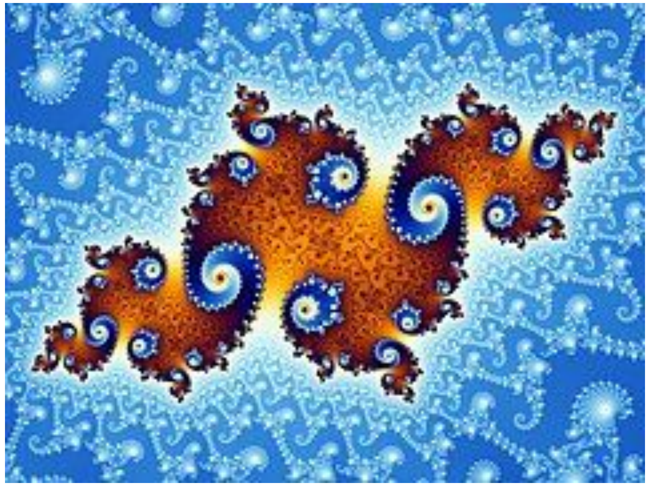
[Zoom into Sierpoinski Triangle]

The Mandelbrot Set

- The most famous fractal is the **Mandelbrot set** by Benoit Mandelbrot
- It is a very very beautiful and complex shape that is generated from a iterative scheme
- The Mandelbrot set can be generated by a simple iterative formula, called quadratic recurrence relation

The Mandelbrot Set

Complex Numbers

Complex Numbers:

- The Mandelbrot set is defined using the complex numbers
- A complex number z is defined using the number of the form:

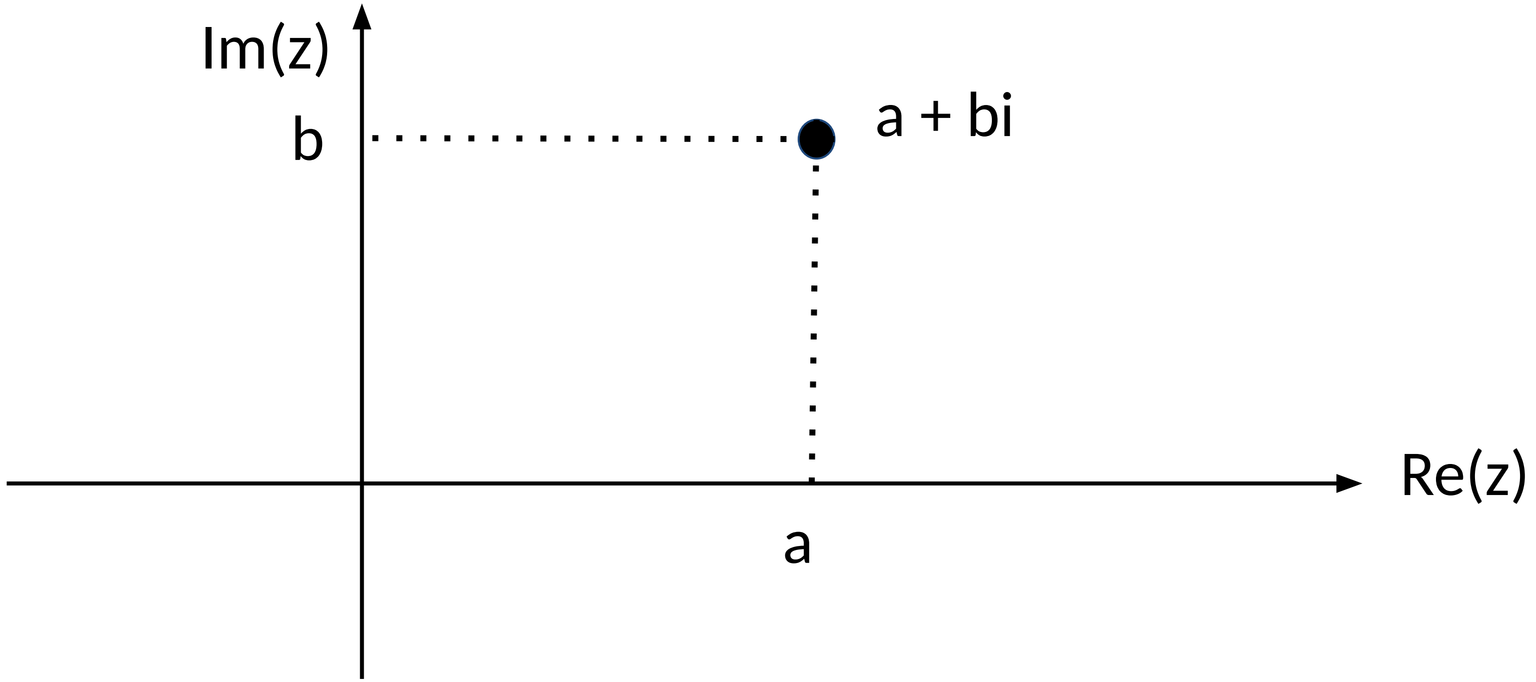
$$z = a + bi$$

where $a, b \in R$ and $i^2 = -1$

- a is known as the real part of z and is denoted by Re(z)
- b is known as the imaginary part of z and is denoted by Im(z)

Complex Numbers

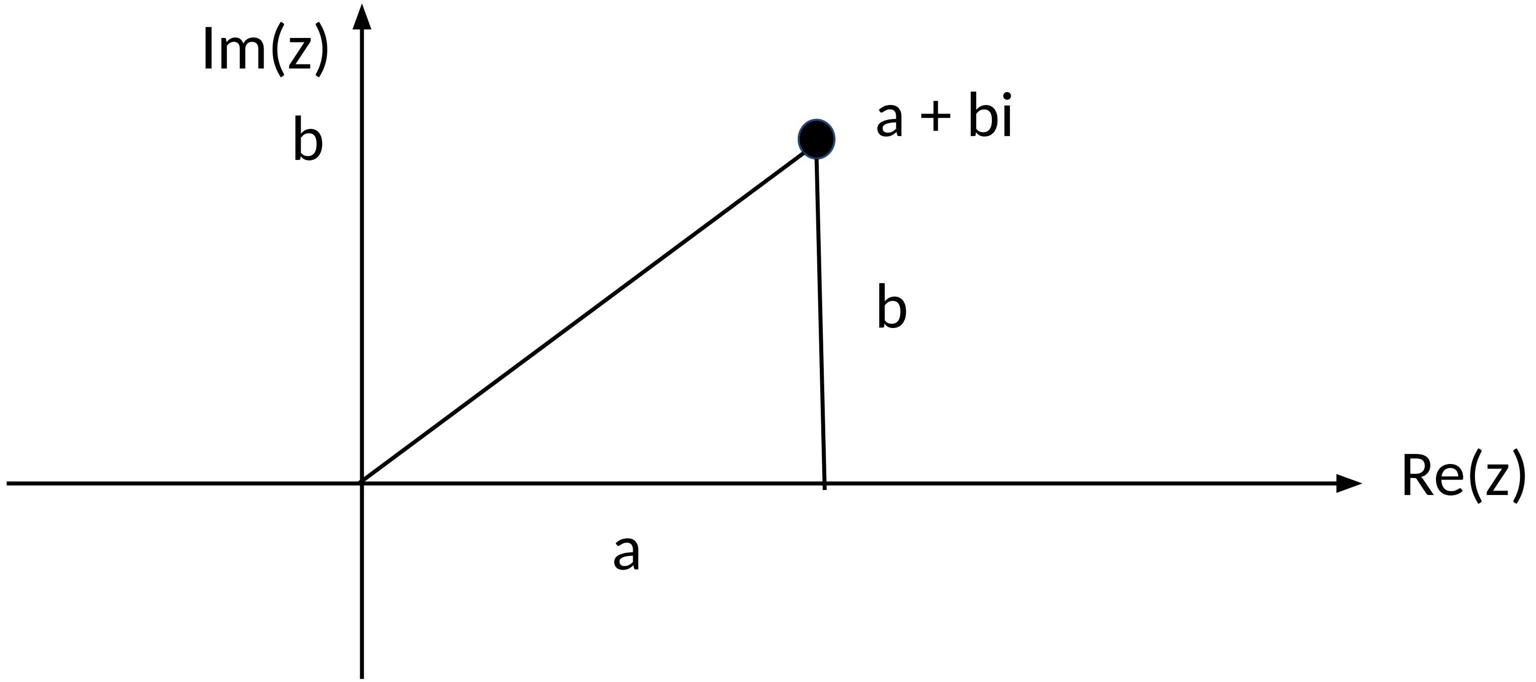
- Complex number can be plotted on a set of axes where the horizontal axes represent real part and the vertical axes represent imaginary part
- The axes are known as **complex plane**



Complex Numbers

- The modulus of a complex number, |a + bi|, is the distance between the complex number and the origin

$$|a + bi| = \sqrt{a^2 + b^2}$$



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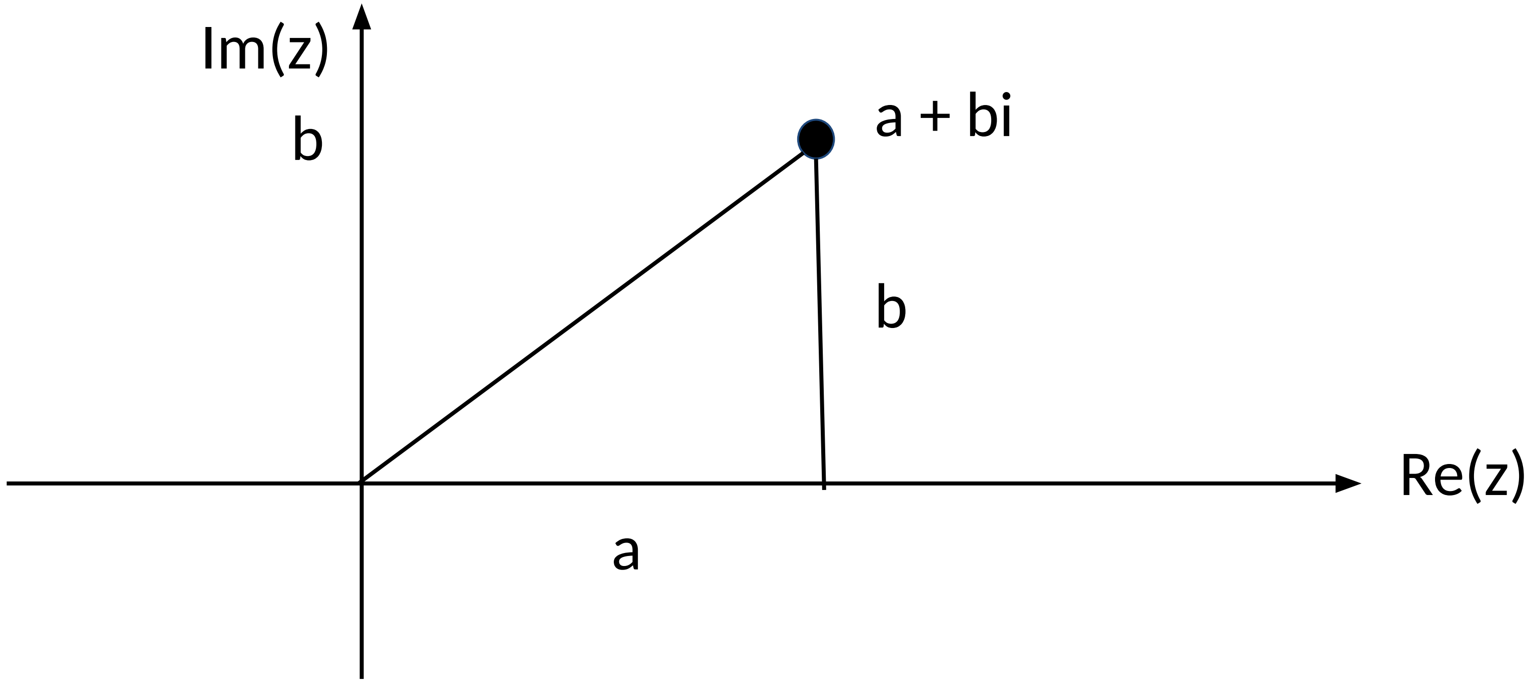


Generating Mandelbrot Set (1/8)

- The Mandelbrot set is defined as a set of points in the complex plane for which the iterative scheme:

$$z_{n+1} = {z_n}^2 + c$$

remains bounded, i.e $|z_n|$ does not tends to infinity

- c is a complex number
- $z_0 = 0$ is used for starting value
- A point c is said to have escaped (therefore not the member of Mandelbrot set) if z_n is larger than some escape radius (usually 2)

Generating Mandelbrot Set (2/8)

iterative scheme:

$$z_{n+1} = z_n{}^2 + c$$

- Consider the iterations for c = 1 + 0i

$z_0 = 1 + 0i$	$\lvert z_0 \rvert = 1$
$z_1 = 1^2 + 1 + 0i = 1 + 1 = 2$	$\lvert z_1 \rvert = 2$
$z_2 = 2^2 + 1 + 0i = 4 + 1 = 5$	$\lvert z_2 \rvert = 5$
$z_3 = 5^2 + 1 + 0i = 25 + 1 = 26$	$\lvert z_3 \rvert = 26$
$z_4 = 26^2 + 1 + 0i = 1 + 1 = 677$	$\lvert z_4 \rvert = 677$

- Therefore c = 1 + 0i is not in the Mandelbrot set

Generating Mandelbrot Set (3/8)

iterative scheme:

$$z_{n+1} = {z_n}^2 + c$$

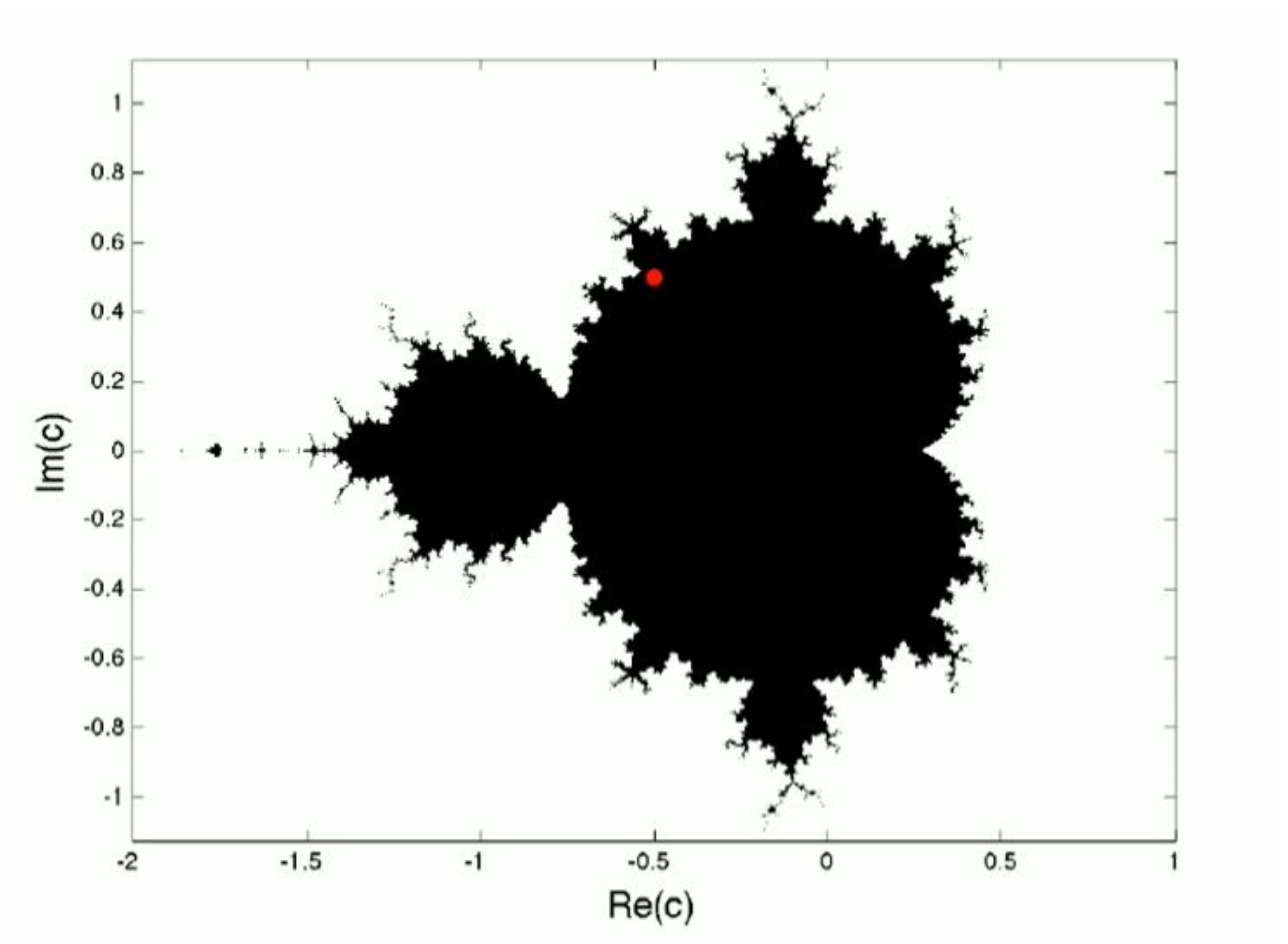
- Consider the iterations for c = 0 + i

$z_0 = 0 + i$	$\|z_0\| = 1$
$z_1 = (0 + i)^2 + 0 + i = -1 + i$	$\|z_1\| = \sqrt{2}$
$z_2 = (-1 + i)^2 + 0 + i = -i$	$\|z_2\| = 1$
$z_3 = (0 - i)^2 + 0 + i = -1 + i$	$\|z_3\| = \sqrt{2}$
$z_4 = (-1 + i)^2 + 0 + i = -i$	$\|z_4\| = 1$

- Therefore c = 0 + i is in the Mandelbrot set

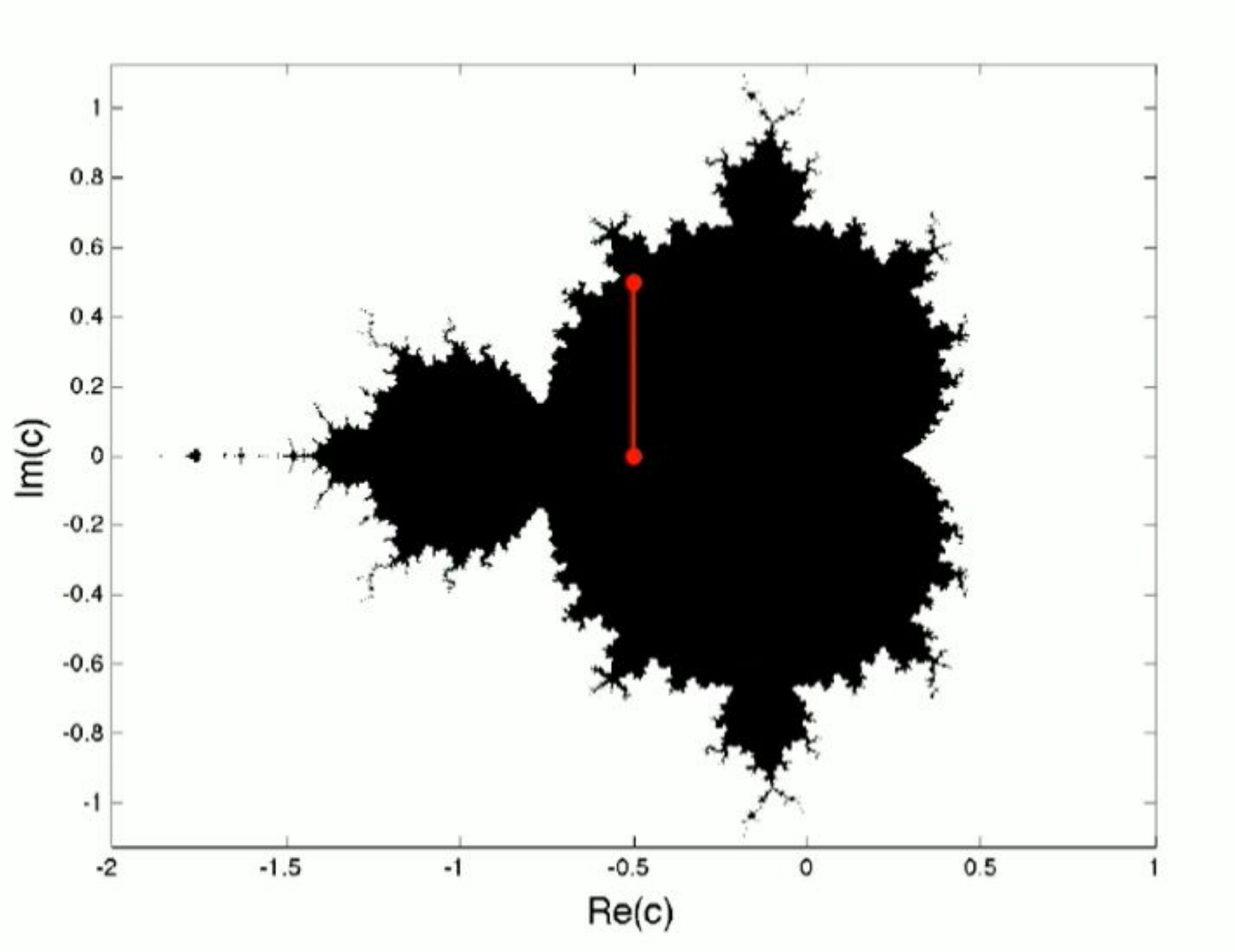
Generating Mandelbrot Set (4/8)

Convergence behaviour of c = 0.5 + 0.5i



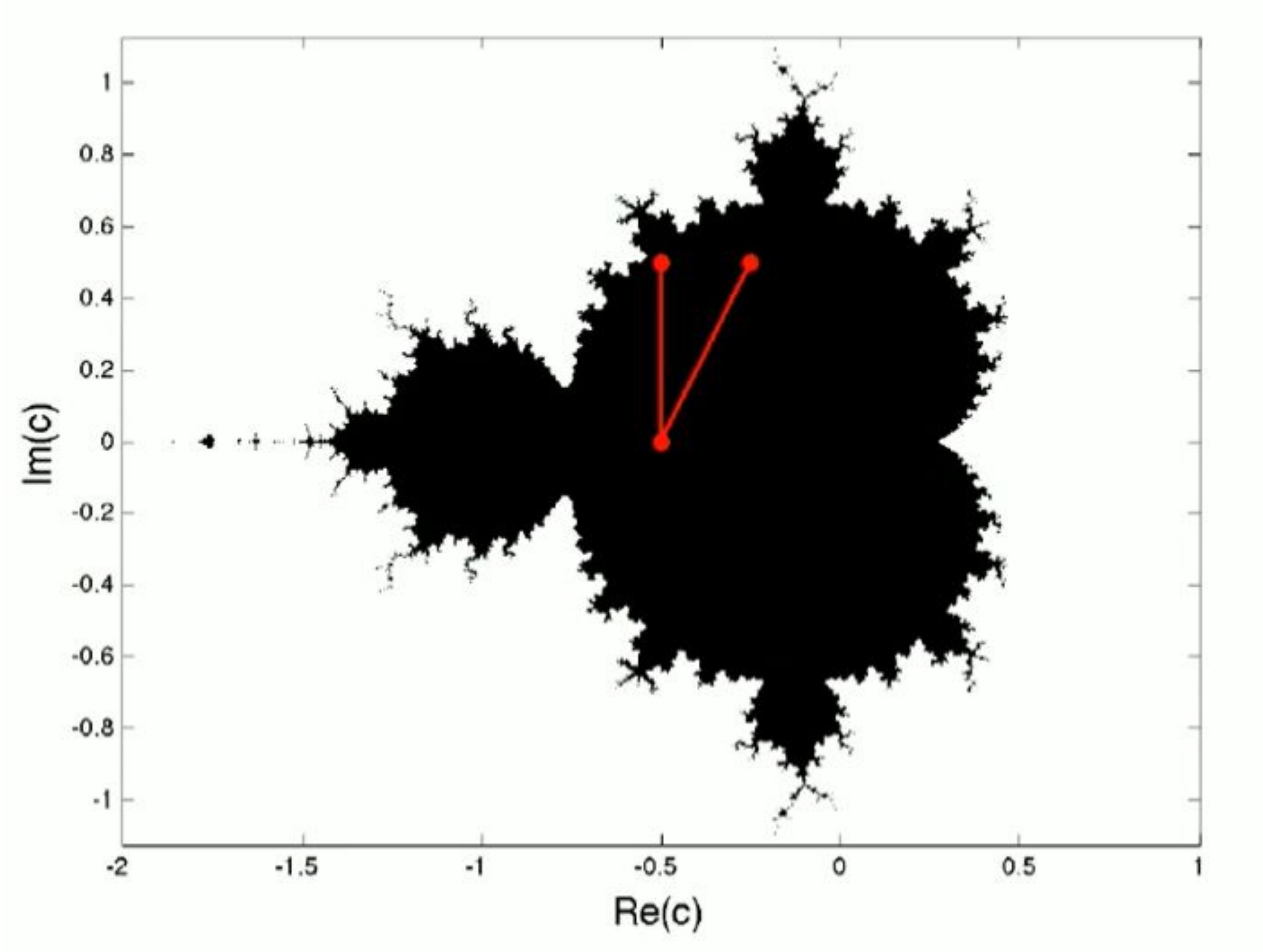
Generating Mandelbrot Set (5/8)

Convergence behaviour of c = 0.5 + 0.5i



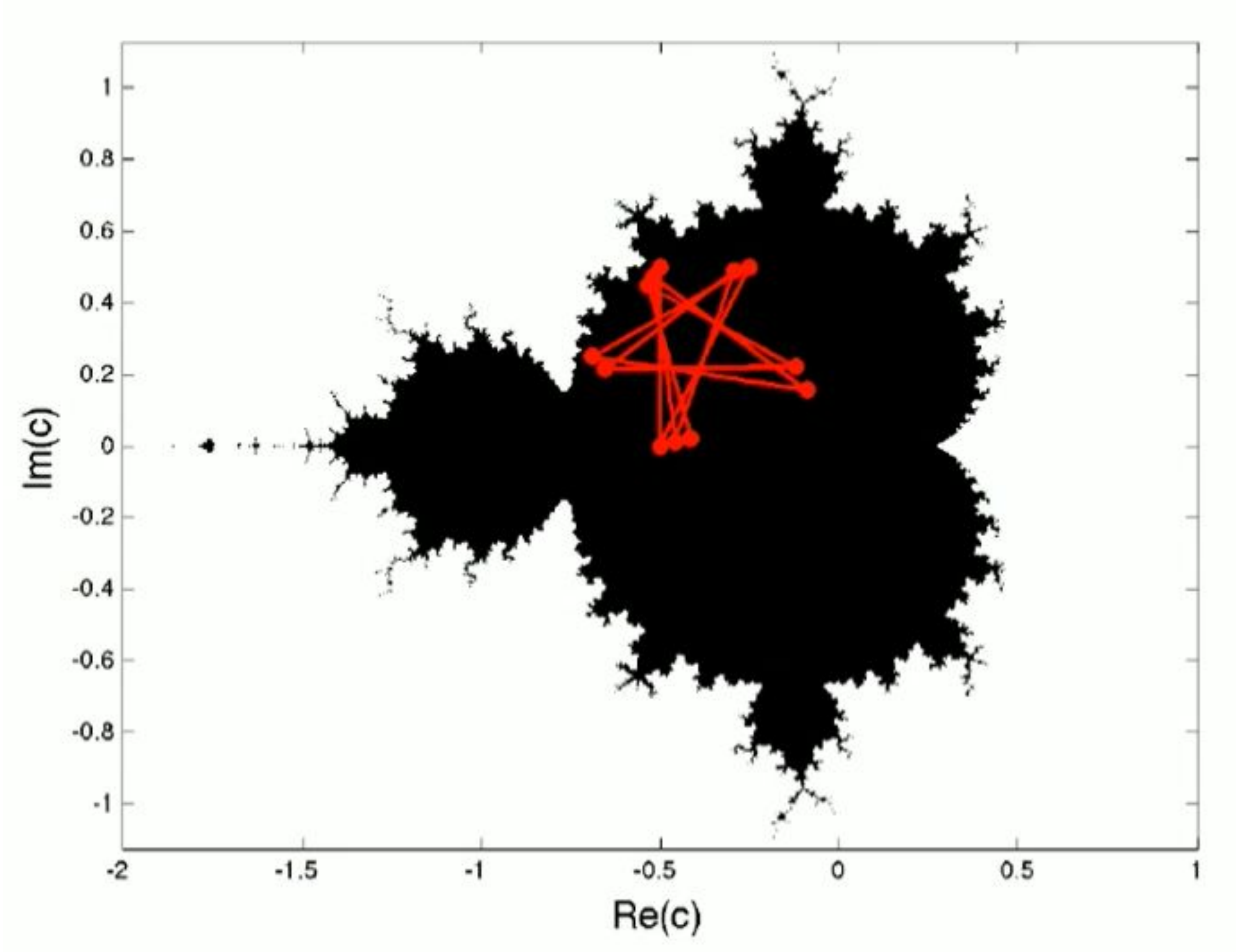
Generating Mandelbrot Set (6/8)

Convergence behaviour of c = 0.5 + 0.5i



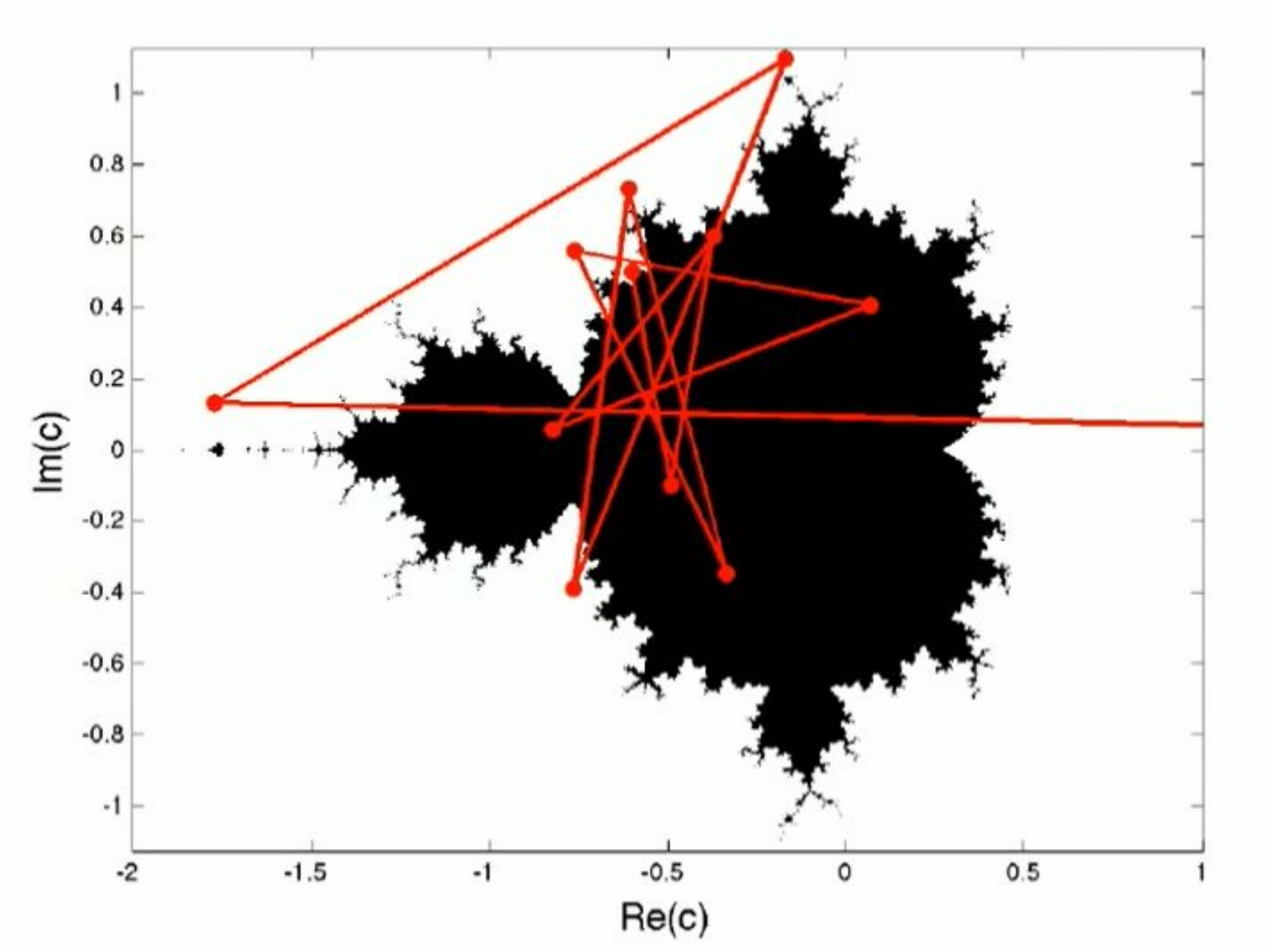
Generating Mandelbrot Set (7/8)

Convergence behaviour of c = 0.5 + 0.5i



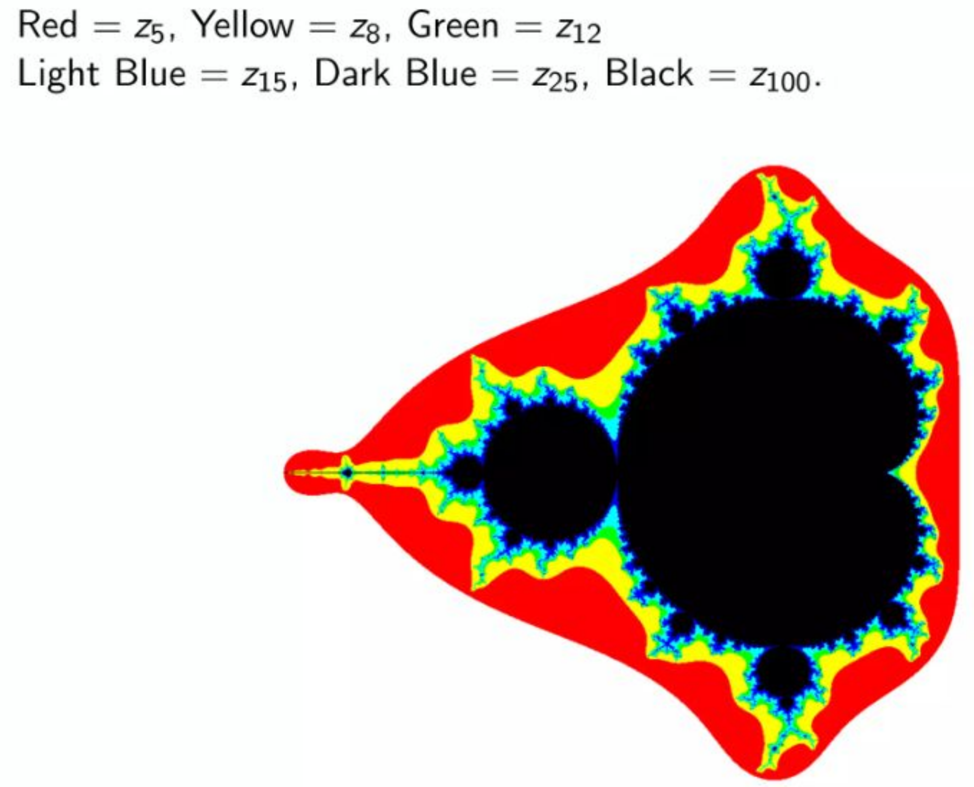
Generating Mandelbrot Set (8/8)

Convergence behaviour of $c = -0.6 + 0.5i$



Mandelbrot Set using Color

- Our Mandelbrot set do not plot points that aren't in the set
- We can color these points depending on the number of iterations to divergence



[zoom into Mandelbrot Set]

Mandelbrot Set using Color

- Python code for implementing Mandelbrot Set link

Further Study

- https://www.youtube.com/watch?v=bEXW7V9mATU
- https://www.youtube.com/watch?v=NGMRB4O922I&ab_channel=Numberphile