# Modelling the emergence of replicator selection from interactor selection

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In these notes we review a model of language behaviour in a speech community. This model incorporates two systematic biases. The first corresponds to listeners paying attention to some of the speakers more than others: this we call *interactor selection* (for reasons outlined below). The second corresponds to a preference towards one particular way of expressing a particular meaning out of multiple alternatives: this we call *replicator selection*. Previous research has shown that certain empirical features of language change (for example: the rapid rate of change of new-dialect formation described in Ref. [1]; that new conventions spread through the community following an S-curve trajectory [2]; and that the timescales over which conventions remain stable are highly diverse [3]) require replicator selection to be present. More specifically, they rely on a consensus among the community of speakers as to which variant of a linguistic variable they prefer. This opens the question: how did this consensus arise in the first place? The aim of this project is to explore mechanisms by which shared preferences can emerge from interactions between speakers, and in particular, the pre-existing variation in the influence of different speakers in the speech community.

We begin with a brief overview of the conceptual theory that underpins this work, and then define the Utterance Selection Model that is based on this theory. We then propose an extension to this model that may allow the question posed above to be studied. We further provide a number of references to the literature which will form the basis of further background reading.

## 1 Croft's utterance selection theory

Croft [4] sets out a detailed evolutionary theory for language change. The key dynamical process in this theory is *replication*, specifically replication of linguistic structures at all levels (e.g., words, idioms, grammatical constructions, phonemes, ...). In a generalised analysis of selection due to Hull [5], the name given to a unit of replication is the *replicator*. Different replicators can be replicated at different rates: this is a consequence of a process called *selection*. For concreteness, we tend to consider a single linguistic variable (that is, a meaning or grammatical function to convey) that exists in multiple variants (e.g., two or more different words with the same meaning). These variants then correspond to different replicators.

The theory of utterance selection distinguishes between two sources of selection. The first comes from the replicator itself: one variant may be preferred by members of the speech community, for example, because it has become associated with a group of people that a speaker wants to be associated with. This is thus an intrinsic property of a replicator, and where one replicator reproduces faster than another we say that *replicator selection* is operating. On the other hand, a speaker may simply be exposed to more uses of one variant than another; or be strongly influenced by a speaker who happens to use a particular variant more than another. In this case, there is no intrinsic property of the replicator that contributes to its rate of reproduction: rather, any differences stem from the properties of the speakers (or, in the theory of Hull, the *interactors*). Selection of this type is thus referred to as *interactor selection*.

## 2 The Utterance Selection Model

A simple 'physics-type' model of language change based on the utterance selection was introduced in Ref. [6], and dubbed the 'Utterance Selection Model' (USM). The version of the model set out in [6] incorporates interactor selection, but not replicator selection. We therefore describe a later refinement of the model that appeared in [2] that does allow replicator selection to be included. (However, one should refer to [6] for a full discussion of how to analyse a model of this type mathematically).

The model comprises $N$ speakers, each of whom has knowledge of a set of $V$ variants of a linguistic variable (vowel sound, sentence structure, ...). Speakers are indexed with the letter $i$, $i = 1, 2, \ldots, N$, and variants with $v = 1, 2, \ldots, V$. A speaker's knowledge of the language (her *grammar*) is specified by a set of variables $x_{iv}$ which give the frequency with which speaker $i$ perceives variant $v$ to be used in her speech community. These variables evolve by repeatedly iterating the following steps:

1. A pair of individuals is chosen to interact; pair $(i, j)$, $i < j$ is chosen with probability $G_{ij}$; this must be normalised $\sum_{i<j} G_{ij} = 1$.

2. Both speakers produce a string of $T$ tokens (i.e., instances of a variant). The number of tokens of variant $v$ produced by speaker $i$ is $n_{iv}$; analogously for speaker $j$. The probability that a given token produced by speaker $i$ is of variant $v$ is $x_{iv}$. Likewise for speaker $j$. The distribution of token numbers $n_{iv}$ and $n_{jv}$ is multinomial [7].

3. Based on these token productions, the speakers construct a *perceived* frequency of each variant, $y_{iv}$ and $y_{jv}$. We take the prescription

$$y_{iv} = f_{iv}\left(\frac{n_{iv}}{T}\right) + H_{ij} f_{iv}\left(\frac{n_{jv}}{T}\right) \tag{1}$$

   for speaker $i$, and the same expression with $i$ and $j$ exchanged to get the corresponding frequency for speaker $j$. The parameter $H_{ij}$ allows speaker $i$ to give a weight to speaker $j$'s utterances that can vary with $j$: this is the mechanism by which interactor selection is incorporated into the model. In the original USM [6], the function $f_{iv}(u) = u$: the perceived frequency is a linear combination of the token frequencies encountered in the interaction. In models incorporating replicator selection $f_{iv}(u)$ can vary between speakers and the variant in question (hence different variants can be favoured by different speakers), see below.

4. Both speakers modify their grammars via

$$x'_{iv} = \frac{x_{iv} + \lambda y_{iv}}{Z} \tag{2}$$

   with $i$ and $j$ exchanged to get the update for speaker $j$. The normalisation $Z$ is chosen such that

$$\sum_v x'_{iv} = 1 \tag{3}$$

   after each interaction.

This model, although simple, incorporates a variety of effects. Speakers monitor their own language behaviour, as well as that of the people they interact with. The parameters $H_{ij}$ control the weight of other speaker's behaviour relative to their own. As noted above, variation in $H_{ij}$ allows interactor selection to be incorporated, that is, some speakers to be paid more attention than others. This is a distinct effect from the interaction frequency $G_{ij}$ that can also vary between speakers, but is necessarily symmetric. If speaker $i$ is interacting with speaker $j$, speaker $j$ is interacting with speaker $i$; however, speaker $i$ doesn't have to pay the same amount of attention to speaker $j$ as vice versa: this would lead to weights $H_{ij}$ that are asymmetric: $H_{ij} \neq H_{ji}$.

Meanwhile, a wide variety of effects can be incorporated into the function $f_{iv}(u)$. We will be interested in cases where speaker $i$ is biased towards (or against) specific variants. As such, we define a quantity $S_{iv}$, which is the intrinsic replicator weight speaker $i$ ascribes to variant $v$. Variants with higher $S_{iv}$ will be preferred if we take, for example,

$$f_{iv}(u) = (1 + S_{iv})u \quad \text{or} \quad f_{iv}(u) = u + S_{iv}u(1 - u) \ . \tag{4}$$

The second expression is common in models of population dynamics where one species has more offspring in each generation than another [8]; however we have found that the first (simpler) expression leads to qualitatively similar dynamics [2].

Finally, the parameter $\lambda$ controls how much each interaction affects a speaker's grammar. Typically this parameter is assumed to be small, and in simulations it is often set to a value of around 0.01.

## 3 A brief survey of results for the Utterance Selection Model

We summarise briefly some of the behaviour that has been established for the Utterance Selection Model. References are provided for further details.

- *Single-speaker model.* Even with a single speaker, the USM is interesting because the speaker monitors their own production. This causes fluctuations in variant frequencies that eventually causes one of them to go to fixation (i.e., be the only one remaining). This case is described in detail in [6], and turns out to be key in understanding more complicated versions of the model.

- *Multi-speaker model in a flat society.* A flat society is one in which all pairs of speakers interact with each other equally often, and ascribes the same weight $H_{ij} = H$ to all other speakers' utterances. As in the single-speaker model, fluctuations in variant frequencies take place until such time as the all but one of them has gone to fixation. The mean time to fixation turns out to be proportional to $N^2$ interactions; we tend to view the interactions as occurring in parallel, and this would correspond to a real time that increases linearly with $N$. Mathematical analysis of this model (see [6] and Section 5 below) shows that to obtain a reasonable continuous-time limit, one should take $H \propto \lambda$. For larger $H$, the model becomes essentially deterministic, and fixation is not expected to occur. (The crossover to the large $H$ region has however been studied in [9], and shown to be nontrivial).

- *Multi-speaker model on a network and interactor selection.* More generally, speakers can occupy positions on complex social network structures. Much work has been done on the USM and related models (such as the voter model [10]) on networks. It turns out the structure have have a dramatic effect on the probability a low-frequency variant goes to fixation, and the time it does to do so [11, 12, 13]. In particular, on scale-free networks [14] where the variance in the degree (number of neighbours of each node) diverges in the limit of an infinite network, fixation can be very fast ($O(N)$ interactions, rather than $O(N^2)$ interactions as is typically the case). These results rely on large asymmetries in the interactor selection weights, i.e., where $H_{ij}$ and $H_{ji}$ can be very different. Typically these asymmetries arise from large differences in the number of neighbours on the network that speakers $i$ and $j$ have; however one can also set up more uniform network structures and vary the interaction weights $H_{ij}$ and $H_{ji}$ independently to similar effect [15].

  A further—quite striking—result is that if the interactor weights are symmetric, $H_{ij} = H_{ji}$ for all $i$ and $j$, fixation probabilities and times are independent of the network structure, as long as they are sufficiently well connected [12]. This result was used to exclude interactor-symmetric models as a possible explanation for the rapid rate of new-dialect formation [1]: in all these models, the fixation time was found to take a time proportional to $N^2$ interactions, which was found to be untenable for realistically large speech communities. Finally, for models that invoke only interactor selection (and no replicator selection), new linguistic variants have been found typically to follow a trajectory given by an exponential function ($x(t) = 1 - \mathrm{e}^{-kt}$ for some $k$), which does not have the S-shape characteristic of historical language changes [2].

- *Effects of replicator selection.* Replicator selection was also briefly studied in [2], and found to be the only reasonable way to obtain an S-curve trajectory for a historical language change. Specifically, what was needed was for the majority of the speakers to apply a boost $S_{iv}$ to an initially low-frequency variant $v$ that is larger than that for all other variants $w$, i.e., we need $S_{iv} > S_{iw}$ for all $w \neq v$, for a majority of the speakers $i$. This was the case where $S_{iv}$ takes one of two values; more generally one might expect a criterion like $\bar{s}_v > \bar{s}_w$, where the bar denotes an average over speakers; this could be something to investigate as part of this project.

These earlier studies (along with others in progress) point towards replicator selection as being an essential component of historical language change. However, whenever replicator selection has been included, it is been pre-specified from the outset. What is unclear is how these shared preferences might emerge from interactions between speakers (and in particular how these are established in advance of the language behaviour itself spreading through the community). Our hypothesis is that the preexisting variation in the interactor weights ($H_{ij}$), which are local quantities, can be exploited as a means to assign one variant a higher replicator weight than another.

## 4 Model for the emergence of replicator weights from interactor weights

One possible model for the emergence of replicator weights from interactor weights equips speakers with a more fine-grained memory of their interactions, and in particular one that records the behaviour of specific individuals. To that end, we define the set of variables $u_{ijv}$ that specify the frequency that speaker $i$ believes speaker $j$ to use variant $v$. These variables will change over time as a result of interactions between speakers $i$ and $j$, as specified below.

We assume now that the replicator weight $S_{iv}$ given by speaker $i$ to variant $v$ is given by

$$S_{iv} = \sigma \sum_j H_{ij} u_{ijv} \; . \tag{5}$$

Since the $u_{ijv}$ vary with time, so will $S_{iv}$. Also, $S_{iv}$ will vary between speakers, as each speaker will experience a different set of interactions (and may interact in different ways with the other members of the speech community). The rationale behind this replicator weight is that if there is a speaker $j$ who typically uses a variant with a high frequency (so that $u_{ijv}$ is large for one of the variants $v$), and this speaker has a high interactor weight $H_{ij}$, the replicator weight $S_{iv}$ will likely be larger than the other weights $S_{iw}$, where $w \neq v$. By incorporating the replicator weight $S_{iv}$ in the function $f_{iv}(u)$ as in Eq. (4), this behaviour will be more likely to be replicated in later interactions. As such this provides a mechanism for interactor selection to be translated into replicator selection. There may be other (better?) ways to do it, which could be investigated as part of this project.

More precisely, the update rule for model could be implemented as follows:

1. As in the original USM (see above), a pair of speakers $i$ and $j$ is chosen to interact, and they both generate $T$ tokens according to their stored frequencies $x_{iv}$ and $x_{jv}$.

2. The new step (relative to the original USM) is that both speakers update their per-speaker store variables $u_{ijv}$ and $u_{jiv}$ according to the prescription

$$u'_{ijv} = \frac{u_{ijv} + \gamma \left(\frac{n_{jv}}{T}\right)}{1+\gamma} \quad \text{and} \quad u'_{jiv} = \frac{u_{jiv} + \gamma \left(\frac{n_{iv}}{T}\right)}{1+\gamma} \; . \tag{6}$$

   The denominator $1 + \gamma$ ensures that the store variables are normalised: $\sum_v u_{ijv} = \sum_v u_{jiv} = 1$.

3. The replicator weights $S_{iv}$ are then computed using (5) and the most-recently updated values of $u_{ijv}$. Likewise $S_{jv}$.

4. The usage frequencies $x_{iv}$ and $x_{jv}$ are then updated as in the USM, using a function $f(u)$ that incorporates the replicator weights, e.g., one of those given by (4).

**NB** – After discussions with Gareth Baxter and Alan McKane, it became clear that we do not—in fact—need to keep track of all the per-interlocutor stores $u_{ijv}$. Instead, each speaker can keep track of its $S_{iv}$ value; one can find an updated $S'_{iv}$ using (5) and inserting the values for $u'_{ijv}$ from the above expressions to obtain something that depends only on $S_{iv}$ and the $n_{jv}$ values from the most recent interaction. This helps both mathematically and computationally: it reduces the number of variables we need to store by a potentially considerable factor.

We have previously investigated this model in a model where all pairs of speakers $i, j$ interact equally often, but are divided into two groups of equal size. The interactor weights $H_{ij}$ take one of two values: $H_{ij} = H_s$ if speakers $i$ and $j$ are in the same group, and $H_{ij} = H_d$ if the two speakers are in different groups. We have $H_s > H_d$ (typically by a factor of about 10). As in the original USM, we also take the $H$ values to be of similar magnitude to $\lambda \approx 0.01$, so for example, $H_d = \lambda = 0.01$ and $H_s = 10H_d$ would be reasonable values to start with, with $\gamma$ and $\sigma$ also of the same order as $\lambda$.

Within this setup, we have found that when initially all members of one group use one variable (e.g., $x_{i1} = 1, x_{i2} = 0$) and all members of the other group use another ($x_{i1} = 0, x_{i2} = 1$), this difference can be maintained between the two groups for sufficiently large $H_s/H_d$. This contrasts with the behaviour of the standard USM which always goes to fixation on one of the variants. As such, it is the first variant of the USM what allows for stable variation between speech communities (e.g., dialects). A first step in this project would be to reproduce this behaviour.

A number of questions, however, remain:

- Over what range of $H$ are separate dialects maintained? It is known that for very small $H$, speakers tend to use only one variant, as opposed to a mixture of variants. Is it necessary to be in this regime?
- Is the state with multiple dialects stable, or only metastable? If eventually everyone ends up using the same variant, what is the waiting time to this event, and how does the change go through?
- Can innovations propagate in this model? We have considered the case where two variants are initially dominant in separate subpopulations. Many changes start from very low frequency: can these invade the speech community? And if so, how?
- Is it possible to maintain separate dialects without the initial condition matching the sub-group structure? And what happens on networks? (See below for more on this).
- Are the per-speaker stores a necessary component of this model, or does the model in fact have a simpler formulation? (The mathematical analysis below is suggestive this may be the case). **NB** — we have already answered this question, see above!
- What other ways of exploiting variation between speakers in the speech community to construct replicator weights exist? Are they equivalent to this model?

## 5 Coda: Steady state of the two-group model within a mean-field approximation

To analyse a model like the USM, what one needs to do is establish the expected changes in the stochastic dynamical variables in one update. More precisely, if one has a stochastic variable that is known to be $X$ at the start of a time step, and whose distribution of values $X'$ at the end of the timestep is specified by the update rule, the quantities one must calculate are the *jump moments*

$$a_k(X) = \langle (X' - X)^k \rangle \tag{7}$$

where the angle brackets denote an average over all possible realisations of the random numbers that enter into the update rule. Once one has these jump moments, one can write down a Fokker-Planck equation for the stochastic process [16]. This is a variant of the diffusion equation familiar from physics.

For a stochastic process with a single random variable, the Fokker-Planck equation reads

$$\frac{\partial P(X,t)}{\partial t} = -\frac{\partial}{\partial X}\left[a_1(X)P(X,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial X^2}\left[a_2(X)P(X,t)\right] \ . \tag{8}$$

The first term in this equation gives the deterministic contribution to the dynamics, and the second term gives the leading stochastic contribution. Here we have stopped at two terms: quite often this yields an exact description of the process in an appropriate limit; however sometimes it is necessary to include higher-order stochastic terms in the equation. These more general equations, along with extensions to the case with more than variable, can be found in [16].

We now consider the model of Section 4, specialised to the case where there are two groups of speakers and only two variants. Each speaker's linguistic behaviour is then specified by the single variable $x_i$ that determines how frequently they use variant 1. (The frequency of variant 2 is given by $1 - x_i$). They also have $N-1$ store variables $u_{ij}$, $j \neq i$, at their disposal.

The probability that a particular pair of speakers $i, j$ is chosen is $G = \frac{2}{N(N-1)}$. The per-speaker store variable $u_{ij}$ thus remains unchanged with probability $1-G$. Otherwise, it is updated to

$$u'_{ij} = \frac{u_{ij} + \gamma\frac{n_j}{T}}{1+\gamma} \ . \tag{9}$$

The random variable in this equation is $n_j$, the number of tokens of variant 1 produced by speaker $j$ in the interaction. This is a binomial random variable with mean

$$\langle n_j \rangle = T x_j \tag{10}$$

and variance

$$\langle n_j^2 \rangle - \langle n_j \rangle^2 = T x_j (1 - x_j) \ . \tag{11}$$

We can therefore work out the first two jump moments of $u_{ij}$. First it is convenient to write

$$u'_{ij} - u_{ij} = \gamma\left[\frac{n_j}{T} - u_{ij}\right] + O(\gamma^2) \ . \tag{12}$$

Under the assumption that $\gamma$ is a small parameter, we will keep only the first term in this expansion. Then

$$a_1(u_{ij}) = \langle (u'_{ij} - u_{ij}) \rangle = G\gamma\left\langle \frac{n_j}{T} - u_{ij}\right\rangle = G\gamma(x_j - u_{ij}) \tag{13}$$

$$a_2(u_{ij}) = \langle (u'_{ij} - u_{ij})^2 \rangle = G\gamma^2\left\langle \left(\frac{n_j}{T} - u_{ij}\right)^2\right\rangle = G\gamma^2\left[\frac{x_j(1-x_j)}{T} + (x_j - u_{ij})^2\right] \ . \tag{14}$$

The behaviour of the speaker's usage frequency $x_i$ is more complicated. Again, this will be unchanged with probability $1 - G$, and will be updated with probability $G$. To write the update rule (2) in a convenient form, we need first to work out the normalisation $Z$. To this end, we introduce the 'twiddled' quantities $\tilde{x}_i$, $\tilde{y}_i$ etc that correspond to frequencies of variant 2. The normalisation is always such that $x_i + \tilde{x}_i = 1$. Hence, from (2),

$$Z = x_i + \lambda y_i + \tilde{x}_i + \lambda \tilde{y}_i = 1 + \lambda(y_i + \tilde{y}_i) \ . \tag{15}$$

Anticipating that $\lambda$ is small, we now have from (2) that

$$x_i' - x_i = \lambda \left[ y_i - (y_i + \tilde{y}_i) x_i \right] + O(\lambda^2) \ . \tag{16}$$

We now choose a specific form for $f_i(u)$ appearing in (1). We will use the linear form, which reads $f_i(u) = (1 + S_i)u$ and implies $\tilde{f}_i(u) = (1 + \tilde{S}_i)u$. Then

$$y_i + \tilde{y}_i = f_i\left(\frac{n_i}{T}\right) + H_{ij} f_i \left(\frac{n_j}{T}\right) + \tilde{f}_i\left(1 - \frac{n_i}{T}\right) + H_{ij} \tilde{f}_i \left(1 - \frac{n_j}{T}\right) \tag{17}$$

$$= 1 + H_{ij} + S_i \left(\frac{n_i}{T} + H_{ij}\frac{n_j}{T}\right) + \tilde{S}_i \left( \left[1 - \frac{n_i}{T}\right] + H_{ij} \left[1 - \frac{n_j}{T}\right] \right) \ . \tag{18}$$

Putting this all together we get

$$\begin{aligned} x_i' - x_i = \lambda \Bigg[ & \frac{n_i}{T} - x_i + H_{ij}\left(\frac{n_j}{T} - x_i\right) \\ & + S_i(1 - x_i)\left(\frac{n_i}{T} + H_{ij}\frac{n_j}{T}\right) - \tilde{S}_i x_i \left( \left[1 - \frac{n_i}{T}\right] + H_{ij}\left[1 - \frac{n_j}{T}\right] \right) \Bigg] \end{aligned} \tag{19}$$

to leading order in $\lambda$.

At this point it is worth recalling the definition of the replicator weight $S_i$ from (5). We have that

$$S_i = \sigma \sum_j H_{ij} u_{ij} \ . \tag{20}$$

Now, we have that $u_{ij} + \tilde{u}_{ij} = 1$, and so

$$\tilde{S}_i = \sigma \sum_j H_{ij}(1 - u_{ij}) = S_i^* - S_i \quad \text{where} \quad S_i^* = \sigma \sum_j H_{ij} \ . \tag{21}$$

Note that $S_i^*$ is the maximal value that $S_i$ may attain.

Using this result, and the statistics of the binomial distribution, we find that

$$\begin{aligned} & a_1(x_i) = \langle x_i' - x_i \rangle = \\ & G\lambda \left[ H_{ij}(x_j - x_i) + (2S_i - S_i^*) x_i (1 - x_i) + S_i H_{ij}(1 - x_i) x_j - (S_i^* - S_i) H_{ij} x_i (1 - x_j) \right] \ . \end{aligned} \tag{22}$$

We could go on to calculate $a_2(x_i)$; however we will first take a brief pause to save ourselves some work. In the original USM, it was customary to take $H_{ij} = \lambda h_{ij}$, where $h_{ij}$ is typically a number of order unity. The reason for this is that $a_1(x_i) \propto \lambda H_{ij}$, in the absence of the replicator selection terms (i.e., with $\sigma = 0 \implies S_i = S_i^* = 0$). Meanwhile, the second moment will be proportional to $\lambda^2$. Hence, for both terms to be of the same order of magnitude, we take $H_{ij} \sim \lambda$. Since the $H_{ij}$ also enter into the definition of the replicator weights, these will also be of order $\lambda$. Hence, terms in (22) that have $S_i$ multiplied by $H_{ij}$ will be of order $\lambda^3$, and so we can neglect these in the small-$\lambda$ limit. Then, we find

$$a_1(x_i) = G\lambda^2 \left[h_{ij}(x_j - x_i) + (2s_i - s_i^*)x_i(1-x_i)\right] + O(\lambda^3) \tag{23}$$

$$a_2(x_i) = G\lambda^2 \frac{x_i(1-x_i)}{T} + O(\lambda^3) \tag{24}$$

where

$$s_i = \sigma \sum_j h_{ij} u_{ij} \quad \text{and} \quad s_i^* = \sigma \sum_j h_{ij} \ . \tag{25}$$

Calculating the second jump moment to leading order is left as an exercise (the trick is to identify that only one term in (19) will contribution to leading order). It is interesting to note that Eq. (23) corresponds to models of population dynamics with migration between islands (speakers) and selection

Rather than pursue the full Fokker-Planck equation for this process, we shall assume (perhaps unjustly) that fluctuations can be neglected. In this regime, we can use the jump moments to derive equations of motion for the mean values of $u_{ij}$ and $x_i$. We refer to this as a *mean-field theory*. The recipe is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_{ij}\rangle = \langle a_1(u_{ij})\rangle = G\gamma(\langle x_j\rangle - \langle u_{ij}\rangle) \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_i\rangle = \sum_{j\neq i}\langle a_2(x_i)\rangle = G\lambda^2 \sum_{j\neq i}\left[h_{ij}(\langle x_j\rangle - \langle x_i\rangle) + (2\langle s_i\rangle - s_i^*)\langle x_i\rangle(1-\langle x_i\rangle)\right] \ . \tag{27}$$

Three comments: (i) The sum over $j$ appears in the second equation because the jump moment was conditioned on an interaction with a specific speaker $j$; we need to take into account all the other speakers that speaker $i$ could have interacted with. (ii) In assuming that fluctuations around the mean values can be neglected, it follows that quantities like $\langle x_i^2\rangle$ are to be approximated as $\langle x_i\rangle^2$. (iii) The mean replicator selection weight is given by

$$\langle s_i\rangle = \sigma \sum_{j\neq i} h_{ij}\langle u_{ij}\rangle \ . \tag{28}$$

We are interested in knowing what stable steady states exist in this model. We thus first set all time derivatives to zero. Eq. (26) then implies that $\langle u_{ij}\rangle = \langle x_j\rangle$: that is, speaker $i$'s impression of speaker $j$'s usage frequency is accurate.

To model the case of two groups, we take speakers $1 \le i \le n = N/2$ to be in group $A$ and the rest, $n < i \le N$, to be in group $B$. We further define two different $h$ values:

$$h_{ij} = \begin{cases} h_s & \text{if } i \text{ and } j \text{ are in the same group} \\ h_d & \text{if } i \text{ and } j \text{ are in different groups} \end{cases} \tag{29}$$

and, since members of a group are interchangeable, we have that

$$\langle x_i\rangle = \begin{cases} x_A & \text{if } i \text{ in group } A \\ x_B & \text{if } i \text{ in group } B \end{cases} \tag{30}$$

Consequently,

$$\langle s_i \rangle = \sigma \times \begin{cases} (n-1)h_s x_A + n h_d x_B & \text{if } i \text{ in group } A \\ (n-1)h_s x_B + n h_d x_A & \text{if } i \text{ in group } B \end{cases} \tag{31}$$

and

$$s_i^* = \sigma[(n-1)h_s + n h_d] \; . \tag{32}$$

Now, by choosing $i$ from group $A$ and $B$ respectively, we find from (27) that

$$n h_d (x_B - x_A) + (N-1)\sigma[(n-1)h_s(2x_A - 1) + n h_d(2x_B - 1)]x_A(1-x_A) = 0 \tag{33}$$
$$n h_d (x_A - x_B) + (N-1)\sigma[(n-1)h_s(2x_B - 1) + n h_d(2x_A - 1)]x_B(1-x_B) = 0 \; . \tag{34}$$

There are always solutions of these equations if $x_A = x_B = 0, \frac{1}{2}, 1$, that is, if all speakers in both groups have converged on categorical use of variant 1 ($x = 1$), variant 2 ($x = 0$), or an equal mixture of both. Intuition suggests that the latter state will be unstable to fluctuations; on the other hand if both groups reach consensus on the same variant, there is no mechanism to reinstate the other variant, so these states will be stable.

In the special case where $h_d = 0$, new solutions $(x_A, x_B) = (0,1), (1,0)$ become available. These correspond to the two (isolated) groups having reached a different consensus. This suggests that with $0 < h_d \ll h_s$, there may be a stable state with $(x_A, x_B)$ close to $(0,1)$ or $(1,0)$. Putting $x_A = 1 - \epsilon$ and $x_B = \epsilon$, substituting into (33)–(34), and discarding terms of order $\epsilon^2$ or higher, we find

$$\epsilon = \frac{n}{(n-1)(N-1)\sigma} \frac{h_d}{h_s} + O\left( \left[ \frac{h_d}{h_s} \right]^2 \right) \; . \tag{35}$$

We anticipate that these ‘multiple-dialect’ solutions exist only when $h_d/h_s$ is sufficiently small. A more thorough analysis of these equations may allow the critical point to be established, or potentially even for the full solution of Eqs. (33)–(34) to be found. Generalisations of these equations to more complex network structures may admit multiple solutions, with clusters of speakers exhibiting different conventions. Such equations would probably be analytically intractable, but numerical solution may be possible. It may then be possible in these models for different dialects to emerge out of a random initial condition as a consequence of early-time fluctuations within the different clusters. Whether this actually happens—or indeed if this mean-field analysis allows predictions to be made for fluctuating systems—is an open question.

## References

Take note of the way in which journal articles, books and internet URLs are presented in this list of references. The conventions used here are largely standard and are expected to be followed in your report. Note also that the arXiv: prefix relates to preprints that can be downloaded from the site `http://arxiv.org`. Type the paper identifier into the search box at the top of the page. Alternatively, you can jump straight to an abstract by entering the url `http://arxiv.org/abs/paper-id`, e.g., `http://arxiv.org/abs/0710.3256`.

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