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Article *in* Optics and Spectroscopy · June 2009

DOI: 10.1134/S0030400X09060204

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ISSN 0030-400X, Optics and Spectroscopy, 2009, Vol. 106, No. 6, pp. 910–912. © Pleiades Publishing, Ltd., 2009.
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Simple Formula for a Gaussian Beam with General Astigmatism in a Homogeneous Medium

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Received November 13, 2008

Abstract—A simple analytical formula that describes the evolution of a Gaussian beam with general astigmatism in a homogeneous medium is obtained. The quadratic form matrix, which determines the transverse distribution of the field in the beam for an arbitrary value of the longitudinal coordinate, is represented in the form of a linear combination of such matrix in a certain initial cross section and a unit matrix. The coefficients of the linear combination are complex-valued and are expressed via the longitudinal coordinate of the considered cross section, the refractive index of the medium, and the determinant and the trace of the initial matrix.

PACS numbers: 42.25.Bs

DOI: 10.1134/S0030400X09060204

INTRODUCTION

Gaussian beams with general astigmatism have repeatedly been considered in the literature [1–8]. In particular, the fundamental mode of a ring cavity with a nonplanar axial contour represents such a beam. In scalar form, the expression for the field of the beam up to terms on the order of $k^{-1/2}$ has the form

$$u(z, r, k) = U(z)\exp\{ik[\tau(z) + r^t\Gamma r/2]\}.$$

Here, k is the wavenumber; z is the longitudinal coordinate counted along the optical axis; $\tau(z)$ is the eikonal; the transverse coordinates are combined into the two-dimensional vector r; the superscript t denotes transposition; and Γ is the symmetric 2×2 matrix with the positive definite imaginary part, which depends on z. The matrices $\Gamma(z)$ and $\Gamma(0) = \Gamma_0$ are related by the equality

$$\Gamma(z) = (C(z) + D(z)\Gamma_0)(A(z) + B(z)\Gamma_0)^{-1}, \qquad (1)$$

where $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are the real-valued (in the absence of amplification and absorption) 2×2 matrices, which, in the aggregate, form the 4×4 symplectic matrix $ABCD$, which relates the characteristics of the wave field in the initial cross section and in the cross section with the longitudinal coordinate z. Here and below, the operations of matrix inversion and division by the matrix determinant do not restrict the generality of formulas because matrices, whose imaginary parts are positive or negative definite, are nondegenerate.

The $ABCD$ matrix, which describes the propagation of the beam in free space, has the form $\begin{pmatrix} E & zE \\ O & E \end{pmatrix}$, where E and O are the unit and zero 2×2 matrices, respectively. In this case, $\tau(z) = \tau_0 + z$, and formula (1) changes to

$$\Gamma(z) = \Gamma_0(E + z\Gamma_0)^{-1}, \qquad (2)$$

and the preexponential factor $U(z)$ takes the form

$$U(z) = U(0)\sqrt{\det(\Gamma(z)\Gamma_0^{-1})} = U_0/\sqrt{\det(E + z\Gamma_0)}. \qquad (3)$$

The objective of this note is to simplify formula (2) by representing $\Gamma(z)$ as a linear combination of Γ_0 and unit matrix.

DERIVATION OF THE FORMULA

Recall that, for an arbitrary number λ and a 2×2 matrix M, the following relations hold:

$$\det(\lambda M) = \lambda^2 \det M,$$

$$\det(M - \lambda E) = \det M - \lambda \operatorname{Tr} M + \lambda^2.$$

In particular, these relations yield the formula for the radicand in (3),

$$\det(E + z\Gamma_0) = 1 + z\operatorname{Tr}\Gamma_0 + z^2\det\Gamma_0. \qquad (4)$$

In addition, M satisfies its characteristic equation

$$E\det M - M\operatorname{Tr} M + M^2 = O, \tag{5}$$

which yields the following equality for the nondegenerate matrix M:

$$M^{-1} = (E\operatorname{Tr} M - M)/\det M. \tag{6}$$

We apply (6) to the matrix $E + z\Gamma_0$; its trace is

$$\operatorname{Tr}(E + z\Gamma_0) = 2 + z\operatorname{Tr}\Gamma_0.$$

Then,

$$(E + z\Gamma_0)^{-1} = \frac{(1 + z\operatorname{Tr}\Gamma_0)E - z\Gamma_0}{\det(E + z\Gamma_0)}, \tag{7}$$

where the expression for the denominator is given by formula (4). To find $\Gamma(z)$, we multiply (7) by Γ_0

$$\Gamma(z) = \frac{(1 + z\operatorname{Tr}\Gamma_0)\Gamma_0 - z\Gamma_0^2}{\det(E + z\Gamma_0)} \tag{8}$$

and, using (5), express Γ_0^2 in (8) via Γ_0 and E. Then, reducing similar terms, we obtain the final result

$$\Gamma(z) = \frac{\Gamma_0 + z\det\Gamma_0 E}{1 + z\operatorname{Tr}\Gamma_0 + z^2\det\Gamma_0}. \tag{9}$$

We make two evident remarks. First, our constructions can be easily generalized to the case of a homogeneous medium with the refractive index n. Since, in this case the ray matrix has the form $\begin{pmatrix} E & n^{-1}zE \\ O & E \end{pmatrix}$, it is necessary to replace z by z/n in all the formulas pertaining to the matrix Γ. In this case, $\tau(z) = \tau_0 + nz$. Second, if the coordinate of the initial cross section is nonzero, $z = z_0 \neq 0$, the considered formulas will be valid after the replacement of z by $z - z_0$.

ALTERNATIVE APPROACHES

It is surprising that such an elementary and easy-to-derive formula has not been obtained before (in any case, it is unavailable from the literature known to us). In all likelihood, Eq. (2) and, especially, the formula

$$\Gamma^{-1}(z) = \Gamma_0^{-1} + zE \tag{10}$$

which is equivalent to this equation, look so simple that it did not occur to anyone to try to simplify them any further. At the same time, there are other methods for describing Gaussian beams with general astigmatism. The first of these methods, which belongs to Arnaud and Kogelnik [1] (see also [8]), is based on the diagonalization of Γ upon changing to the eigenbasis. As is seen from (10), the eigenvectors of the matrix Γ^{-1} (and Γ) remain constant, while the eigenvalues of Γ^{-1} (the complex-valued parameters of the beam) linearly depend on the longitudinal coordinate. In the case where the principal axes of the ellipses of phase and intensity coincide (simple astigmatism), the eigenvectors of Γ are directed along these axes and the diagonalization is reduced to the rotation of the coordinate system. Otherwise (general astigmatism), the eigenvectors prove to be not real-valued, and the changing to the eigenbasis is interpreted by the authors of that study as a rotation by a complex-valued angle. This approach makes it possible to describe and analyze changes in the shape of the beam in the course of its propagation (in particular, rotation of the ellipses of intensity and phase) using the natural internal terminology in terms of eigenvalues and eigenvectors of the matrix Γ. However, this merit changes to a drawback in the case where Γ_0 is specified by rather complicated analytical expressions, since, here, the formulas for the complex-valued parameters of the beam and complex-valued rotation angle may prove to be very cumbersome, not to mention the resultant formulas for $\Gamma(z)$. Clearly, after the simplification, the result will be equivalent to expression (9); however, the process of its obtaining may prove to be very tedious. In addition, this approach is not universal because matrices of the form $\alpha E + \beta\begin{pmatrix} \pm 1 & i \\ i & \mp 1 \end{pmatrix}$, whose both eigenvalues coincide with α cannot be diagonalized at $\beta \neq 0$ (the imaginary part of such matrices is positive definite at $\operatorname{Im}\alpha > |\beta|$). At the same time, formula (9) remains valid for the matrices of this form as well.

The second approach was developed by Belousova and was reported in [6]. It is based on solving the set of ordinary differential equations for the elements of the matrix $\Gamma(z)$, which follows from the parabolic equation of the system. As a result of rather complicated constructions, the author arrives at a set of very cumbersome formulas for matrix elements. In fact, matrix equation (9) represents this entire set in laconic form (unfortunately, both editions of monograph [6] contain misprints in some of the formulas considered here).

CONCLUSIONS

We obtained a simple analytical formula that describes the evolution of a Gaussian beam with general astigmatism in a homogeneous medium. The matrix $\Gamma(z)$, which determines the transverse distribution of the field in the cross section of the beam at an arbitrary point z, is represented as a linear combination of the same matrix in a certain initial cross section and a unit matrix. The coefficients of the linear combination are complex-valued and are expressed via the longitudinal coordinate of the considered cross section, the refractive index of the medium, and the determinant and the trace of the initial matrix Γ_0.

Because they are equivalent to the well-known methods of description of Gaussian beams in the domain of their applicability, our formula is substantially simpler in form and does not require any auxiliary operations (such as the diagonalization of the matrix and its inversion or the introduction of auxiliary variables) and makes it possible to directly relate the expressions for the field in two arbitrary cross sections. We believe that formula (9) may prove to be a very convenient and useful tool for studying and describing Gaussian beams in homogeneous media.

REFERENCES

1. J. A. Arnaud and H. Kogelnik, Appl. Opt. **8** (8), 1687 (1969).
2. M. M. Popov, Vestn. Leningr. Gos. Univ. Phys. Chem., No. 22, 42 (1969).
3. V. M. Babich and V. S. Buldyrev, *Asymptotic Methods in Short-Wavelength Diffraction Theory.*
4. T. F. Pankratova, Opt. Spektrosk. **36** (5), 969 (1974).
5. I. I. Savel'ev and A. M. Khromykh, Kvantovaya Élektron. (Moscow) **3** (7), 1517 (1976) [Sov. J. Quantum Electron. (Moscow) **6** (7), 821 (1976)].
6. A. M. Goncharenko, *Gaussian Light Beams* (Nauka i Tekhnika, Minsk, 1977) [in Russian].
7. I. V. Golovnin, A. I. Kovrigin, A. N. Konovalov, and G. D. Laptev, Kvantovaya Élektron. (Moscow) **22** (5), 461 (1995) [Quantum Electron. (Moscow) **25** (5), 436 (1995)].
8. V. P. Bykov and O. O. Silichev, *Laser Cavities* (Fizmatlit, Moscow, 2004) [in Russian].

Translated by V. Rogovoĭ