ENME302 S2
2023

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

Course Handbook

Computational and Applied Mechanical Analysis

Semester Two

Te Tari Pūhanga Pūrere | Mechanical Engineering

ENME302-23S2-HBK

Te Kaupeka Pūhanga | Faculty of Engineering

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ENME302 S2

Computational and Applied Mechanical Analysis

Semester 2

2023

Mechanical Engineering

Finite Element Analysis

Week 1-4

Professor Geoff Rodgers

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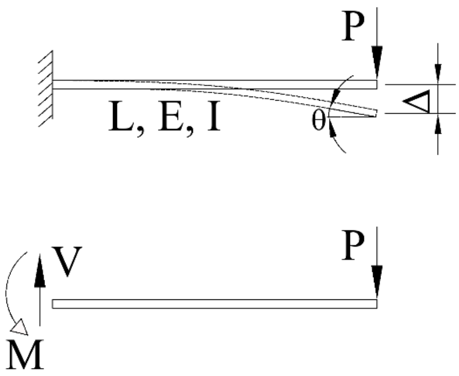
Finite element analysis

Scenario 1

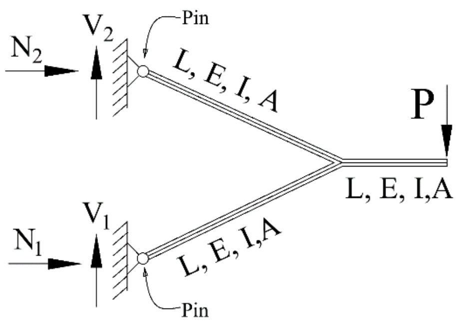


Question:

- What is deflection, Δ?
- What are reaction loads V, M?

From ENME202 we know $\Delta_{tip} = \frac{PL^3}{3EI}$, $\theta_{tip} = \frac{PL^2}{2EI}$, $V = -P, M = -PL$

Scenario 2



Question:

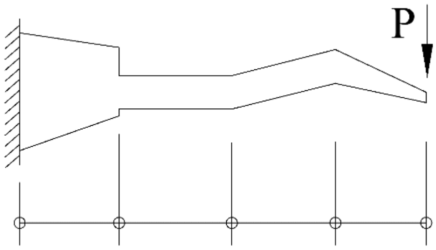
- What is the end deflection at load point, P?
- The are reaction forces, V_1, V_2, N_1, N_2?

…And it gets harder from there, especially when we consider "elements" that carry moment (beams) instead of purely axial load carrying elements (bars).

__The Solution__: use Finite Elements (FE) to break the problem into simpler pieces and simplify solution.

- ⇨ Break a large, complex structure into a number of smaller, simpler pieces, where each piece can be modelled easily, and can together be used to model a range of possible structures.
- ⇨ Develop a framework under which these simple, basic, elements can be combined together to represent the larger, more complicated, structures.

Scenario 3



Question:

- How to calculate the end deflection of this complex element at load point, P?

Solution→ break the element into at least 4 unique finite elements with unique cross-sections.

Almost any structure can be broken down into simpler pieces.

The Finite Element Method consists of the following steps:

A. Choosing the element type	• Axial trusses- carry only axial forces (tension/compression), <u>but do not carry</u> any moments. • Flexural beam- long and slender, carry moments and/or shear forces • Shear beam- short and squat, carry moments and/or shear forces
B. Identifying boundary conditions	• Fixed • Free • Constrained
C. Applying forces	• Point forces • Moments • Continuous (spread) loads
D. Choosing degrees of freedom	• Locations where deflections are to be solved for directly

The “Finite Element” Method

Again, any structure can be broken down (discretized) into simpler elements and be analysed using finite element method. The problem can be formulated and analysed as follows:

Problem formulation:

1) Approximate exact force-displacement behaviour of each piece:

$$f^e = K^e d^e$$

K- stiffness matrix

f^e- a vector of applied loads

d^e- a vector of allowed deflections

2) Reassemble $(f^e = K^e d^e)$ into a larger model ($\hat{F} = \hat{K}\hat{D}$)

3) Apply constraints to remove DOF that are fixed or known (motions) to get a final model $(F = KD)$.

4) Solve the deflections for the given applied loads: $D = K^{-1}F$

Analysis of results:

5) Using this solved deflection vector, x, to calculate reaction forces (V,N,M)
6) From D as we will see you can also estimate/approximate all motion <u>within</u> an element (not just at the ‘nodal’ or ‘end’ points).

→ Overall, you can solve very difficult problems using FEM.

We will cover:

1) Bar (or truss) elements:

- carry axial loads & motion only
- carry no moment

2) Beam elements:

- carry moment, shear forces, lateral displacements and rotations
- can have shear motion out of but not along length of element

For each case we will:

1) Demonstrate element formulation $f^e = K^e d^e$.

2) Discuss rotation of the coordinate system

3) Demonstrate assembly of many elements into a single model for solution

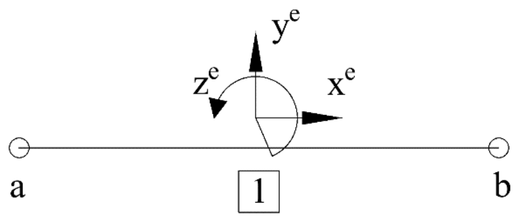
4) Work through different examples and develop Codes to solve.

⇨ Then, if you are using a commercial finite element package, such as ANSYS, ComSol, Abaqus, etc, it's not just a "black box" where you turn a handle and numbers come out. Instead, you have an idea of what's going on "under the bonnet" and can therefore better understand when something goes wrong, and a better understand of limitations to the analysis.

Nomenclature

This course will use the following nomenclature to describe analysed structures:

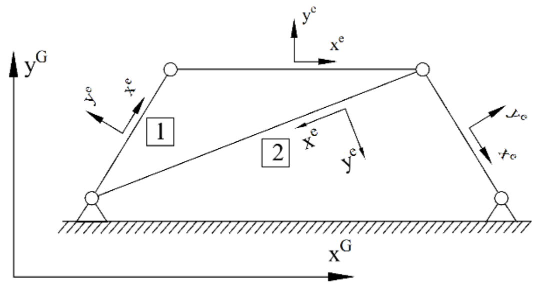
1. Element:



Element #1

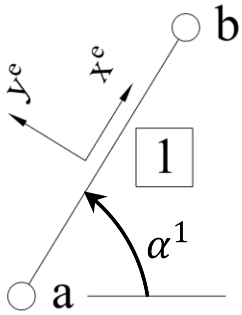
- (x^e, y^e) – element coordinate system
- x^e is along a→b the length of element
- (x^e, y^e) are always set so z^e is out of page

2. Global Model



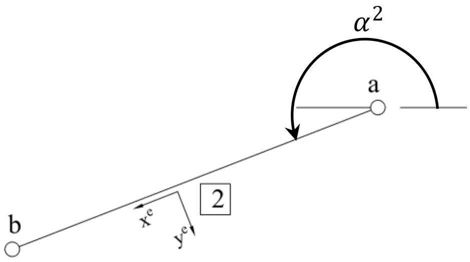
- You can pick any end as "a" as long as you align x^e along that direction
- (x^G, y^G)- global coordinate system. Set for convenience. Usually x^G is set along the ground (horizontal) as this is usually the most convenient.

<u>Element #1</u>

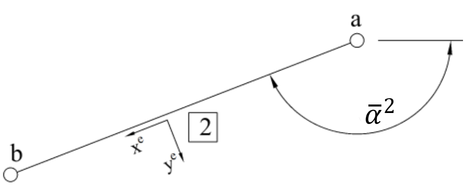


- α^1 corresponds to angle from x^G→ x^e for element #1 (it will need to be transformed)

<u>Element #2</u>



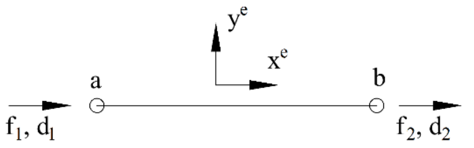
- α^2 corresponds to angle from x^G→ x^e for element #2
- angle can be measured in either direction, $\bar{\alpha}^2 = \alpha^2 - 360°$
- In this case, you could use a transformation angle of α^2 =210° or $\bar{\alpha}^2$ = -150° and the result will be the same.



Overview: What the finite element method tries to accomplish

(1) A simple structure

- Assume one element



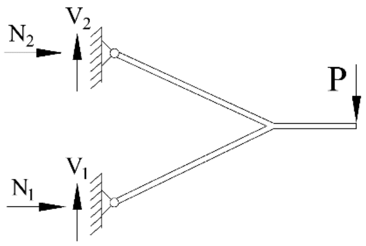
- Analytical solution is easy enough

- Can create an element stiffness matrix K^e relating f_1 & f_2 to d_1 & d_2

$$f^e = K^e d^e$$

- Using $u(x)$ some approximation to the easy analytical solution knowing only $u(x = 0) = d_1$ and $u(x = L) = d_2$

(2) A (reasonably) complex structure



Given 3 elements that comprise the structure, analytical solution may no longer be possible. Hence, FE formulation becomes convenient:

1. Break the structure into separate simple elements and define stiffness matrix for all elements, K^e.

2. Assemble elements into a global model

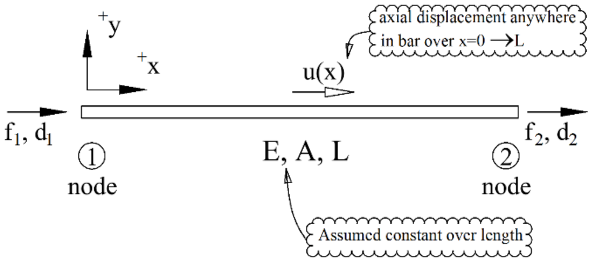
3. Solve for motion at element ends or "nodes", d^e.

4. $u(x)$ approximation gives motion within each element

5. Motion in each element → σ, ε as functions of x along each element ($\sigma(x), \varepsilon(x)$).

Goal is to use "simple" elements to create solutions to complex problems where analytical methods fail (or become too complex to solve easily).

Bar elements



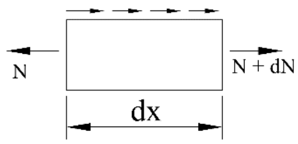
f_1, f_2- nodal forces at nodes 1 and 2, respectively
d_1, d_2- nodal displacements at nodes 1 and 2, respectively

(1) Note: 2 DOF (1 at each end of element)

(2) Goal: be able to define/find $u(x)|_{x=0\to L}$ based on d_1 and d_2.

(3) Consider an infinitesimally small section of bar:

p.dx = axial force internal to the bar



$$\sum F_x = 0 = -N + (N + dN) + p\,dx$$
$$-dN = p\,dx$$
$$-\frac{dN}{dx} = p$$

(4) We know from ENME202 → $N = \sigma A = (E\varepsilon)A = EA\frac{du}{dx}$

(5) Hence, substituting Equation (4) to Equation (3)

$$\boxed{-\frac{dN}{dx} = -\frac{d}{dx}\left(EA\frac{du}{dx}\right) = p(x)}$$

(6) **Governing Boundary Value Problem (BVP):**

<u>Given</u>: A, E, f_1 and f_2 over element x = 0→L
<u>Find</u>: u(x) such that:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

$$u(x = 0) = d_1$$

$$u(x = L) = d_2$$

Equilibrium of a "small" element, based upon the homogeneous form, with $p(x)$ = 0

Called the "strong form" of problem, where the exact solution is hard. This form requires the conditions above to be true for all the element, from = 0 to x = L.

(7) Homogenous strong form in (6) assumes the distributed axial load acting between nodes is equal to zero, $p(x) = 0$. This means that the bar carries a constant axial force along its length.

(8) Therefore, to solve (6) assuming, E, A are constant, we define a solution:

$$u(x) = a_0 + a_1 x$$

Where a_0 and a_1 are determined from boundary conditions (BC's)

$$\left.\begin{array}{l} u(x = 0) = d_1 \\ u(x = L) = d_2 \end{array}\right\}$$

$$a_0 + a_1\,0 = d_1 \rightarrow \boldsymbol{a_0 = d_1}$$

$$d_2 = d_1 + a_1\,L \rightarrow \boldsymbol{a_1} = \frac{\boldsymbol{d_2 - d_1}}{\boldsymbol{L}}$$

Hence, the function describing displacement of element anywhere along 0→L can be written:

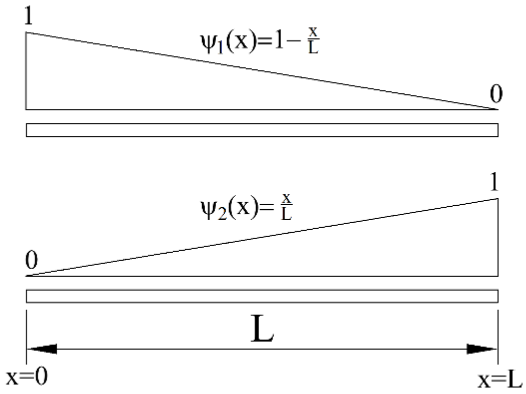
$$u(x) = d_1 + \frac{d_2 - d_1}{L}\,x$$

$$u(x) = \left(1 - \frac{x}{L}\right)d_1 + \frac{x}{L}d_2$$

$$u(x) = \psi_1(x)d_1 + \psi_2(x)d_2$$

(9) Where ψ_1, ψ_2 are shape functions describing change in u(x) between nodes. In other words, these shape functions interpolate between nodal displacements d_1 and d_2.

Shape functions ψ_1 and ψ_2 can be defined:



$$u(x) = \left(1 - \frac{x}{L}\right) d_1 + \frac{x}{L} d_2$$

$$u(x) = \psi_1(x) d_1 + \psi_2(x) d_2$$

Example:

Distance, x	$x = 0.5L$	$x = 0.75L$
Displacement within the bar, $u(x)$	$u(x = 0.5L) = 0.5d_1 + 0.5d_2$	$u(x = 0.75L) = 0.25d_1 + 0.75d_2$
Strain, $\varepsilon(x)$	$\varepsilon(x) = \frac{du}{dx} = \frac{(-d_1 + d_2)}{L}$ Strain $\varepsilon(x)$ is independent of x, hence $\varepsilon(x) = const = \frac{(-d_1+d_2)}{L}$	
Stress, $\sigma(x)$	$\sigma(x) = E\varepsilon = \frac{E(d_2 - d_1)}{L} = E\frac{\Delta L}{L}$	

Equation (9) defines displacement for $x = 0 \rightarrow L$. To get a stiffness matrix relating f and d (where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$) we need forces along $x = 0 \rightarrow L$.

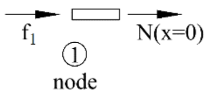
Recall, Equation (4) $N = EA\frac{du}{dx}$ using (9):

$$N = EA\frac{d}{dx}\left(\left(1 - \frac{x}{L}\right)d_1 + \left(\frac{x}{L}\right)d_2\right)$$

$$N = \frac{EA}{L}(d_2 - d_1)$$

Hence axial force, N, is constant along $x = 0 \rightarrow L$ as expected with distributed axial load intensity, $p(x) = 0$ for the homogenous form.

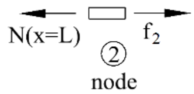
Now relate axial force, N, with nodal forces f_1 and f_2



$$\sum F_x = 0 = f_1 + N(0)$$

$$f_1 = -N_1(0) = -\frac{EA}{L}(d_2 - d_1)$$

$$f_1 = \frac{EA}{L}(d_1 - d_2)$$



$$\sum F_x = 0 = f_2 - N(L)$$

$$f_2 = N(L)$$

$$f_2 = \frac{EA}{L}(d_2 - d_1)$$

(13) Combine into matrix form:

Element formulation for an axial bar element

$$\begin{pmatrix} -N(0) \\ N(L) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$f^e = K^e d^e$$

(14) **Note**: K^e is singular $\det(K^e) = 0$

This means that $f^e = K^e d^e$ has no unique solution b/c it is not constrained

Hence: $d_1 = d_2 = 10 \rightarrow d^e = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

$$f^e = K^e d^e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} 10 \\ 10 \end{pmatrix} = \frac{EA}{L}\begin{pmatrix} 10 - 10 \\ -10 + 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

You get motion without force when a structure is not constrained. This holds for any nodal displacement values where $d_1 = d_2$.

(15) Questions

a) What happens if E and A varies over $x = 0 \rightarrow L$
   - Need a new element
   - Or to use many elements

b) What happens if $p(x) \neq 0$?
   - You get
     $$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = p(x) \neq 0$$
   - Get a new element form
   - Or you see later in the notes

Shape Functions:

Shape functions have two key purposes:

1. To enable a system of equations to be developed so that solving for displacements calculated at only a few selected discrete points within the system (the nodal points) can represent the full, continuous structural system.
2. Once these displacements are solved, we can then use the shape functions and the solved nodal deflections (and element geometric and material properties) to calculate internal deflections, forces, stresses and strains <u>within</u> the element (not just at the nodal points).

➔ Shape functions are used to both "build-up" and "break down" or interpret the structure.

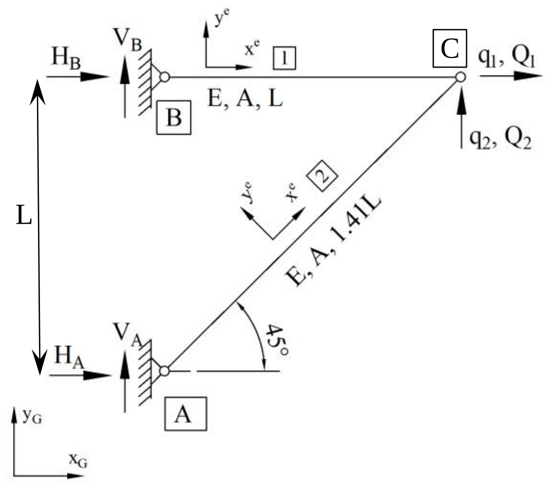
A brief review of energy methods and virtual displacements

You will (hopefully) recall the use of energy methods to solve displacements in structures. We will briefly re-cap Energy Methods, the Work-Energy Method for Single Loads, and the Method of Virtual Work.

First up, we'll consider a simple system of axial bars. We will solve this first through a manual method as we might have done last year in ENME202. We will then solve it through the Method of Virtual Work, but do this manually. Observing this process in detail for a given example will set the scene for the more general system that we develop to solve a range of problems.

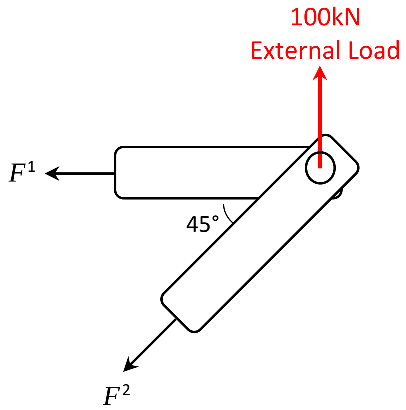
Simple Axial Bar Problem

External forces $Q_1 = 0$, $Q_2 = 100kN$ act on the structure below. Find the structural deflections at pin-joint C. Assume $L_1 = 10m$ and $L_2 = 14.1m$, and both members are made of 100mm diameter solid steel bars with elastic modulus, $E = 200$ GPa.



Solution – Using simple hand calculation for statically determinate systems

We will first consider force equilibrium at the top right pin joint:



Vertical Force Equilibrium: $\sum F_y = 0: \quad 100kN - F^2\cos(45°) = 0$

$$F^2 = \frac{100kN}{\cos(45°)} = 141.42kN$$

Horizontal Force Equilibrium: $\sum F_x = 0: \ -F^1 - F^2\cos(45°) = 0$

$$F^1 = -F^2\cos(45°) = -100kN$$

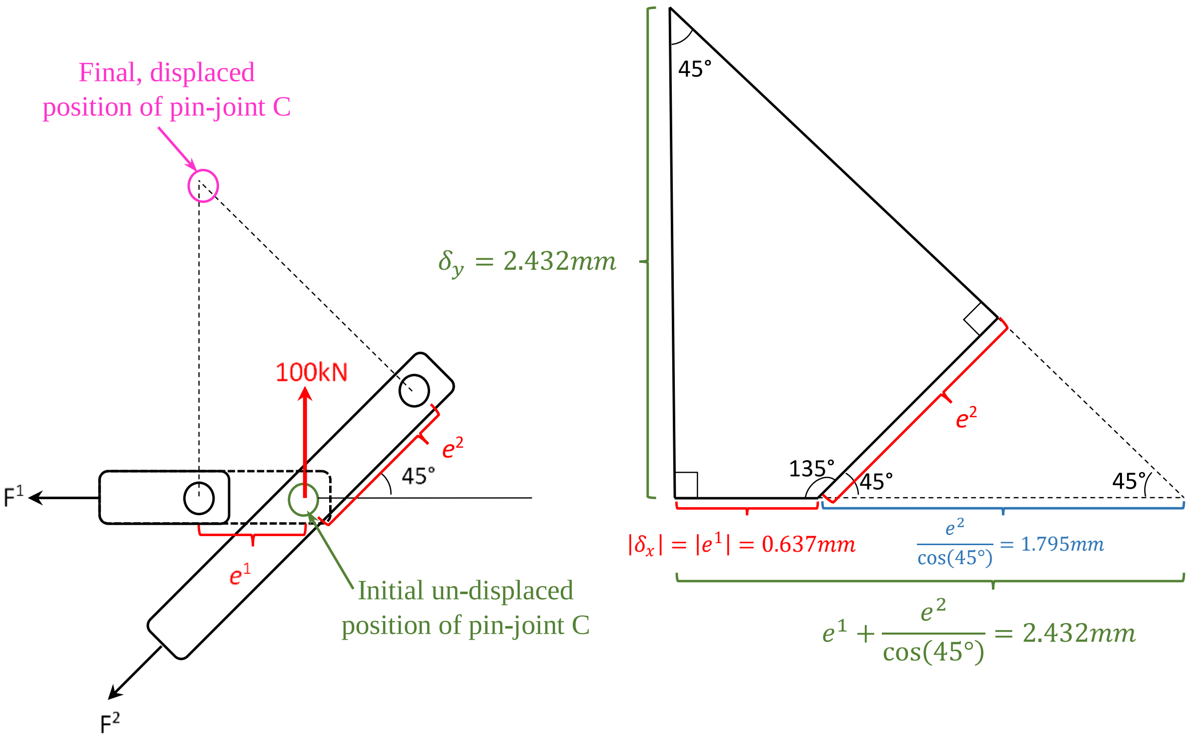
Calculate elongation of each element based upon the internal loads.

Member 1: $e^1 = \frac{F^1L^1}{A^1E^1} = \frac{(-100{,}000)(10)}{\left(\frac{\pi}{4}0.1^2\right)(200\times10^9)} = -6.36619\times10^{-3}m = -0.637mm$

Member 2: $e^2 = \frac{F^2L^2}{A^2E^2} = \frac{(141{,}420)(14.1)}{\left(\frac{\pi}{4}0.1^2\right)(200\times10^9)} = 1.2694\times10^{-3}m = 1.27mm$

Now, we need to manually interpret these individual element deflections to determine the displacements between the initial and final locations of pin-joint C.

After some trigonometry and messing about with an awkward quadrilateral shape, we can determine that pin-joint C displaces horizontally to the left by 0.637mm and vertically upwards by 2.432mm.

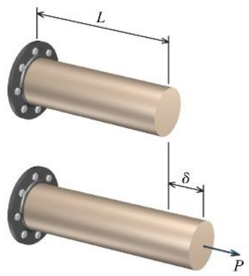


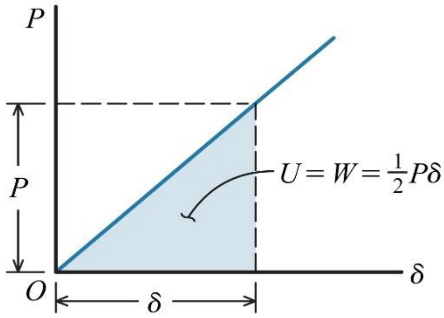
A simple, statically-determinate structure like this can be solved by hand, but it gets complicated quickly and we would like a more general approach. The first such approach that we will consider is the Work-Energy Method for Single Loads.

Work-Energy Method for Single Loads

When a load is applied and a body deforms, work is done on the body. The **external forces** are said to do **external work**. The **internal reactions** to these loads does **internal work**, often referred to as **strain energy**. Think about strain energy like the potential energy stored in a stretched/strained spring.

Work of a force: Work is defined as the product of a force times the distance that it moves, in the direction of that force.





$$W = \frac{1}{2} P\delta$$

The principle of conservation of energy states that energy is neither created nor destroyed, rather transformed from one state to another. Therefore, the work done by the external force is stored within the material as strain energy. Provided no energy is lost as heat, then we can say that the strain energy, U, is equal in magnitude to the external work, W:

$$U = W = \int_0^{\delta_1} P d\delta$$

For axial deformation, we have the equation for internal strain energy as:

$$U = \frac{1}{2} P\delta$$

For prismatic bars, with constant cross-section, δ = PL/AE, giving an expression for strain energy in terms of applied force, P:

$$U = \frac{P^2 L}{2AE}$$

The Work-Energy Method for Single Loads can be used to determine the deformations of a structural member for **very selected conditions**:

- The member or structure must be loaded by a single external concentrated force
- Corresponding displacements can only be determined at the location of the single load and in the direction that the load acts

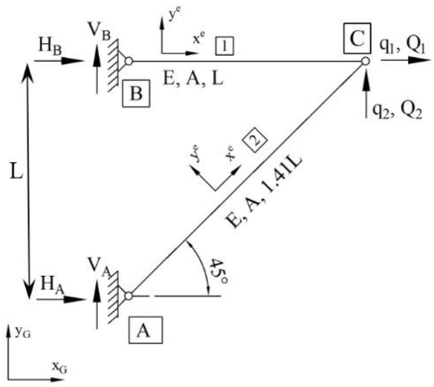
Why are we restricted to a single external load? The equation above (*W* = *U*) is the only equation we have available to us. The strain energy, *U*, will be a single number. The work, *W*, performed by the external load is also only a single number.

Consequently, if more than one external force is applied, then the term for *W* will have more than one unknown deflection or rotation angle.

➔ One equation, two (or more) unknowns = not solvable, without extending the methodology.

Simple Axial Bar Problem – Solved using the Work-Energy Method for Single Loads

External forces $Q_1 = 0$, $Q_2 = 100kN$ act on the structure below. Find the structural deflections at pin-joint C. Assume $L_1 = 10m$ and $L_2 = 14.1m$, and both members are made of 100mm diameter solid steel bars with elastic modulus, $E = 200$ GPa.



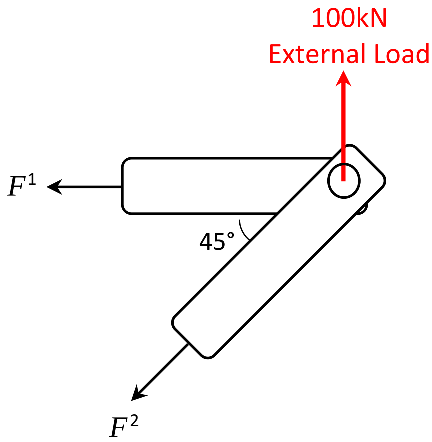
The first steps are the same as before. We consider force equilibrium at the top right pin joint:

Vertical Force Equilibrium: $\sum F_y = 0$: $\quad 100kN - F^2\cos(45°) = 0$

$$F^2 = \frac{100kN}{\cos(45°)} = 141.42kN$$

Horizontal Force Equilibrium: $\sum F_x = 0$: $-F^1 - F^2\cos(45°) = 0$

$$F^1 = -F^2\cos(45°) = -100kN$$



Based upon these solved internal forces, we can calculate the internal strain energy within the structure.

Member 1: $U^1 = \frac{(F^1)^2 L^1}{2A^1E^1} = \frac{(-100{,}000)^2(10)}{2\left(\frac{\pi}{4}0.1^2\right)(200\times10^9)} = 31.83\,J$

Member 2: $U^2 = \frac{(F^2)^2 L^2}{2A^2E^2} = \frac{(141{,}420)^2(14.1)}{2\left(\frac{\pi}{4}0.1^2\right)(200\times10^9)} = 89.76\,J$

The total internal strain energy is: $U^{Total} = U^1 + U^2 = 31.83\,J + 89.76\,J = 121.59\,J$

Equating external work to internal strain energy: $U^{Total} = W^{External} = \frac{1}{2}P\delta_y = 121.59\,J$

And we can then solve for the vertical deflection at the point that the load is applied:

$$\delta_y = \frac{2U^{Total}}{P} = \frac{2(121.59)}{100{,}000} = 2.432\times10^{-3}m = 2.432mm$$

This solved vertical deflection is the same as result that we obtained before, but without all the manual intervention, without the awkward quadrilateral, and without the pesky triangles and trigonometry.

<u>However</u>, this method (as it stands) cannot give us the horizontal deflection at pin-joint C. This method (as it stands) also breaks down as soon as we have more than one load acting on the structure.

Method of Virtual Work

The Method of Virtual Work is an important extension of the Work-Energy Method for Single Loads. The Principle of Virtual Work allows us to solve for deflections for multiply-loaded structures, and at locations other than those where external loads are applied. The extension to the method is actually relatively simple to implement, but can be difficult to understand conceptually.

The Principle of Virtual Work states:

> *If a deformable body is in equilibrium under a virtual force system and remains in equilibrium while it is subjected to a set of small, compatible deformations, then the external virtual work done by the virtual external forces acting through the real external displacements (or rotations) is equal to the virtual internal work done by the virtual internal forces acting through the real internal displacements (or rotations).*

But what does that actually mean? That statement made my brain hurt…

Procedurally, if we have a structure subjected to a single-load and we want the deflection at a different location to that of the load (or at that point but not in the direction of that load), or for multiply-loaded structures, then we can apply a virtual external load.

We use the principle of superposition to independently consider the real loads and the virtual loads and how each load is developed within the structure.

We typically define the virtual external load to have a value of 1.0 (N or kN) to make life easier. We have to choose something and choosing this value makes the calculations a little bit simpler.

When used to analyse a truss structure, this process has several steps:

- First, take the real external loads applied only (ignore the virtual load for now) and work out the internal forces developed within each member.
- Then, ignore the real external loads and only consider the chosen virtual external load applied to the structure. This virtual load is applied at the point that you wish to know the deflection and acts in the direction in which you wish to know the deflection. Work out the internal forces that would be induced by the application of this virtual external load (we call these the virtual internal forces).
- Apply the Principle of Virtual Work and equate the Virtual Internal Work (the work done by the internal virtual forces moving through the real displacements) to the Virtual External Work (the work done by the virtual external load moving through the real external displacement).

The principle can be expressed in an equation as:

$$\begin{matrix}\text{Virtual}\\\text{External}\\\text{Loads}\end{matrix} \times \begin{matrix}\text{Real External}\\\text{Displacements}\end{matrix} = \sum\left(\begin{matrix}\text{Virtual}\\\text{Internal}\\\text{Forces}\end{matrix} \times \begin{matrix}\text{Real Internal}\\\text{Displacements}\end{matrix} \right)$$

For a compound truss with homogenous (constant elastic modulus) and prismatic (constant cross-section) members, this principle can be expressed as:

$$1 \cdot \Delta = \sum_i f^i \left(\frac{F^i L^i}{A^i E^i} \right)$$

where:

1 = virtual external unit load acting in the direction desired to obtain deflection, Δ.

Δ = real joint displacement caused by the real loads that act on the truss.

f^i = virtual internal force created within truss member i when the truss is loaded with only the single virtual external virtual load (ignoring any real external loads that act on the truss)

F^i = real internal force created within truss member i when the truss is loaded with all the real loads (ignoring the virtual load)

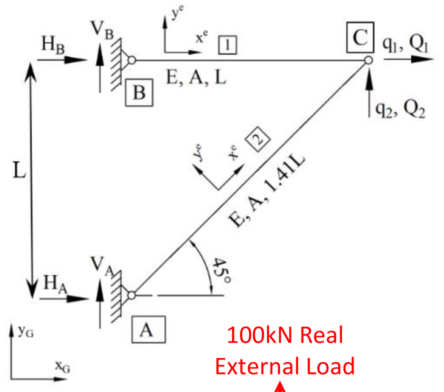
L^i = the length of truss member i

A^i = the cross-sectional area of truss member i

E^i = the elastic modulus of truss member i

Simple Axial Bar Problem – Solved using the Method of Virtual Work

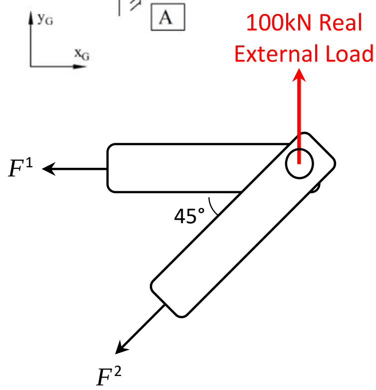
External forces $Q_1 = 0$, $Q_2 = 100kN$ act on the structure below. Find the structural deflections at pin-joint C. Assume $L_1 = 10m$ and $L_2 = 14.1m$, and both members are made of 100mm diameter solid steel bars with elastic modulus, $E = 200$ GPa.



The first steps are the same as before. We consider force equilibrium at the top right pin joint:

Vertical Force Equilibrium: $\sum F_y = 0: \quad 100kN - F^2\cos(45°) = 0$

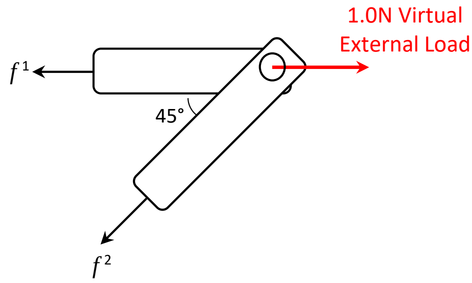
$$F^2 = \frac{100kN}{\cos(45°)} = 141.42kN$$



Horizontal Force Equilibrium: $\sum F_x = 0: -F^1 - F^2\cos(45°) = 0$

$$F^1 = -F^2\cos(45°) = -100kN$$

We wish to determine the horizontal deflection at pin-joint C, so we will now ignore the real externally applied load and then apply a unit virtual load in the horizontal direction at C. The corresponding internal loads within each member are the virtual internal forces.



Vertical Force Equilibrium: $\sum f_y = 0: \quad -f^2\cos(45°) = 0$

$$f^2 = \frac{-0}{\cos(45°)} = 0N$$

Horizontal Force Equilibrium:

$$\sum F_x = 0: -f^1 - f^2\cos(45°) + 1.0N = 0N$$

$$f^1 = 1.0 - f^2\cos(45°) = 1.0N$$

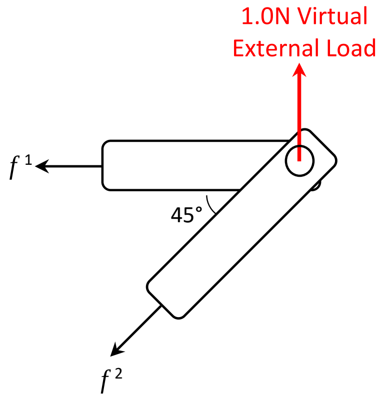
Member	*Length, L (m)*	*Area, A (m²)*	*Elastic Modulus, E (Pa)*	*Real Internal Force, F (N)*	*Virtual Internal Force, f (N)*	$f^i\left(\frac{F^iL^i}{A^iE^i}\right)$ *(kN²m)*
1	*10*	*7.854 x 10⁻³*	*200 x 10⁹*	*-100,000*	*1.0*	*-6.366x10⁻⁴*
2	*14.1*	*7.854 x 10⁻³*	*200 x 10⁹*	*141,421*	*0*	*0*
					$\sum_i f^i\left(\frac{F^iL^i}{A^iE^i}\right)$	*-6.366x10⁻⁴*

$$1 \cdot \Delta = \sum_i f^i\left(\frac{F^iL^i}{A^iE^i}\right) \rightarrow \Delta = \sum_i f^i\left(\frac{F^iL^i}{A^iE^i}\right) = -6.366 \times 10^{-4}m = -0.6366mm$$

What if instead, we had wanted the vertical displacement (like we did earlier), what would that look like?

The first part where we calculate the real internal force does not change (the real loads have not changed in this instance), so we will not repeat that part.

We now wish to determine the **vertical** deflection at pin-joint C, so we again ignore the real externally applied load and apply a unit virtual load, this time in the **vertical** direction at C. The corresponding internal loads within each member are the virtual internal forces.



Vertical Force Equilibrium: $\sum f_y = 0: \quad -f^2\cos(45°) + 1.0 = 0$

$$f^2 = \frac{1.0}{\cos(45°)} = 1.4142N$$

Horizontal Force Equilibrium:

$$\sum F_x = 0: \; -f^1 - f^2\cos(45°) = 0N$$

$$f^1 = -f^2\cos(45°) = -1.0N$$

Member	*Length, L (m)*	*Area, A (m^2)*	*Elastic Modulus, E (Pa)*	*Real Internal Force, F (N)*	*Virtual Internal Force, f (N)*	$f^i\left(\frac{F^iL^i}{A^iE^i}\right)$ *(kN^2m)*
1	*10*	*7.854×10^{-3}*	*200×10^9*	*-100,000*	*-1.0*	*$6.366\text{x}10^{-4}$*
2	*14.1*	*7.854×10^{-3}*	*200×10^9*	*141,421*	*1.4142*	*$1.795\text{x}10^{-3}$*
					$\sum_i f^i\left(\frac{F^iL^i}{A^iE^i}\right)$	*$2.432\text{x}10^{-3}$*

$$1 \cdot \Delta = \sum_i f^i\left(\frac{F^iL^i}{A^iE^i}\right) \rightarrow \Delta = \sum_i f^i\left(\frac{F^iL^i}{A^iE^i}\right) = 2.432 \times 10^{-3}m = 2.432mm$$

This simple example will hopefully give you some confidence with the theory behind the principle of virtual displacements and how they are applied. This structure is simple and statically-determinate, so we can easily solve for real and virtual internal forces within each structural member. However, if we want to apply this to more complex statically-indeterminate structures, then we will need to develop a better/more extensive computational framework to solve such problems.

The principle of virtual displacements is used in the derivation of our stiffness matrices for bar elements, and then again later when we do more complex elements that include bending.

Derive the element stiffness matrix in a different (and simpler) way

(16) Bar element formulation can be obtained in a simpler way using the Principle of Virtual Displacements (PVD). Given Equation (6):

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

Using force boundary conditions:

$$-EA\frac{du}{dx}\bigg|_{x=0} = f_1$$

$$-EA\frac{du}{dx}\bigg|_{x=L} = f_2$$

(17) Multiply both sides of Equation (16) by virtual (small) displacement, δu:

$$\delta u\left(-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right) = \delta u(0) = 0 \quad \forall \quad x = 0 \to L$$

(18) Integrate over element domain $\Omega = (0 \to L)$ which is a less strict requirement than Equation (17) above:

$$\int_0^L \delta u\left(-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right)dx = \int_0^L \delta u\left(EA\frac{d^2u}{dx^2}\right)dx = 0$$

And integrate by parts so that u and δu have the same order of differentiation (currently δu – zero order and u is second order):

$$\left[\delta u\, EA\frac{du}{dx}\right]\bigg|_0^L - \int_0^L \left(\frac{d}{dx}\delta u\right)\left(EA\,\frac{du}{dx}\right)dx = 0$$

$$\delta u(L)EA\frac{du}{dx} - \delta u(0)EA\frac{du}{dx} - \int_0^L \left(\frac{d}{dx}\delta u\right)\left(EA\,\frac{du}{dx}\right)dx = 0$$

And after applying boundary conditions:

$$\delta u(L)f_2 - \delta u(0)(-f_1) - \int_0^L (\delta u')\left(EA\,\frac{du}{dx}\right)dx = 0$$

$$\boxed{\int_0^L (\delta u')\left(EA\,\frac{du}{dx}\right)dx = \delta u(0)(f_1) + \delta u(L)f_2}$$

It's called the "weak form" because it must be true over integrated terms only.

(19) Now we use given or known shape functions, $\Psi(x) = [\psi_1(x) \quad \psi_2(x)]$, to substitute for:

$$u(x) = \Psi(x)\,\mathrm{d} = \psi_1 d_1 + \psi_2 d_2$$

$$\delta u(x) = \Psi(x)\,\delta\mathrm{d} = \psi_1 \delta d_1 + \psi_2 \delta d_2$$

Or,

$$u(x)^T = d^T \Psi(x)^{\mathrm{T}} = \psi_1 d_1 + \psi_2 d_2$$

$$\delta u(x)^T = \delta \mathrm{d}^{\mathrm{T}} \Psi(x)^{\mathrm{T}} = \psi_1 \delta d_1 + \psi_2 \delta d_2$$

where $\psi_1 = \left(1 - \frac{x}{L}\right)$; $\psi_2 = \frac{x}{L}$ are pre-defined interpolation/shape functions.

(20) Equation (18) becomes:

$$\int (\delta d^T \Psi'(x)^T) EA\, (\Psi'(x) d)\, dx = \delta d^T f = \delta d^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where $\Psi'(x) = \frac{d}{dx}(\Psi(x)) = \begin{pmatrix} \frac{d}{dx}\psi_1 & \frac{d}{dx}\psi_2 \end{pmatrix}$

$$\cancel{\delta d^T} \int (\Psi'(x)^T EA\, \Psi'(x)\, dx) d = \cancel{\delta d^T} f$$

$$\int (\Psi'(x)^T EA\, \Psi'(x) dx) d = f$$

$$\boxed{\begin{gathered} K^e d^e = f^e \\ K^e = \int_0^L (\Psi'(x)^T EA\, \Psi'(x) dx) \\ \text{where } \Psi' = \begin{pmatrix} -\frac{1}{L} & \frac{1}{L} \end{pmatrix} \end{gathered}}$$

(21) Deriving the final form of bar element stiffness matrix

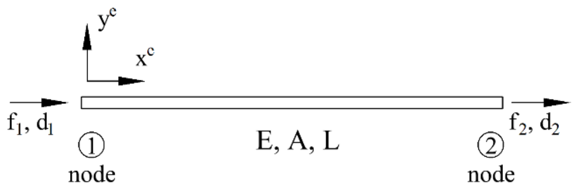
$$K^e = \int_0^L \begin{pmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{pmatrix} EA \begin{pmatrix} -\frac{1}{L} & \frac{1}{L} \end{pmatrix} dx = \frac{EA}{L^2} \int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

$$= \frac{EA}{L^2} \left(\begin{bmatrix} L & -L \\ -L & L \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\boxed{K^e = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Just as before…but <u>this</u> derivation makes K^e easy to calculate given shape functions $\psi(x)$ to relate or interpolate motion along the element from nodal (end) displacement values.

Summary



Axial bar element stiffness matrix derived:

$$f^e_{\{2\times1\}} = K^e_{\{2\times2\}}\, d^e_{\{2\times1\}}$$

$$f^e = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad d^e = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$K^e = \int_0^L (\Psi'(x)^T EA\, \Psi'(x) dx) = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{K^e_{\{2\times2\}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Element stiffness matrix, K^e, is a function of *E, A, L*, and the given shape functions.

Axial forces along bar assumed to be constant, ie: $p(x) = 0$.

Captures only axial deformation and force displacement behaviour. It does not carry moments and has no rotations.

Stiffness matrix is derived based on shape functions:

$$\Psi = [\psi_1(x) \quad \psi_2(x)]$$

$$\psi_1(x) = 1 - \frac{x}{L}; \qquad \psi_2(x) = \frac{x}{L}$$

Bar element axial deformation at any point x can be estimated:

$$u(x) = \psi_1(x)d_1 + \psi_2(x)d_2 = \Psi d^e$$

Bar element coordinate axes are defined by (x^e, y^e) where x^e is aligned along element and y^e-perpendicular to the longitudinal axis of the element.

Final notes:

- Bars (and beams) are 1D elements, so shape functions are along only that 1 dimension- i.e. $\psi =$ function of x, $\psi(x)$, and not $\psi(x, y)$ if we had a plate instead.
- This process is very general. Given a governing ODE and boundary conditions as well as known exact solution or estimated shape functions, $u = \Psi d$, then this process holds in general for a very wide range of finite element types.
- Hence, knowing this approach will allow you to understand (almost) any finite element you might encounter now in a software package (or in future).
- Recall though that our governing ODE in this axial bar case only was for "small" deflections and thus, our K^e matrix and $f^e = K^e d^e$ is also only valid for small deflections.

Questions

1) How do I make a model of something real?

2) What if bar element axes (x^e, y^e) do not match the global model reference coordinates?

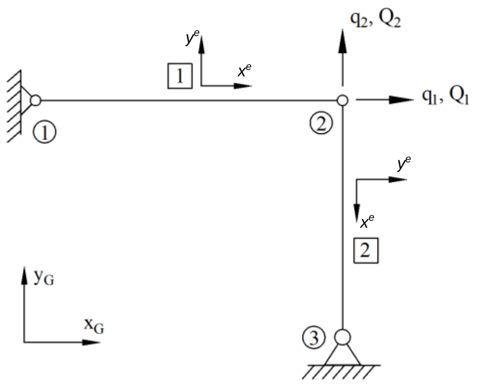
3) What if elements do not all line up in coordinate system?

Answer

Model an **assembly** of a series of element stiffness matrices into a full model.

Example 1. Analysis of truss frames in global coordinate system

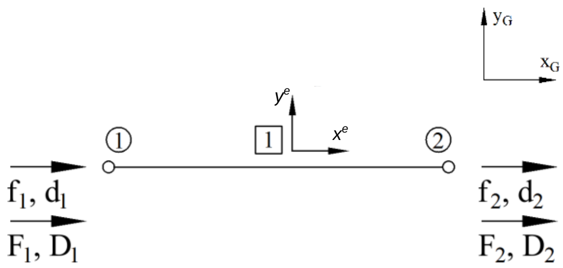
(1) A Truss structure pinned at supports is modelled using two bar elements:



- The overall structure's motion/deformation can be described by **2 DOFs** (q_1 and q_2)
- Q_1 and Q_2 are the external loads applied at corresponding global DOFs (q_1 and q_2)
- Element #2 local coordinate axis (x^e, y^e) does **not** align with global coordinate axes (x_G, y_G).

(2) Truss structures can be broken down into separate bar elements with corresponding local DOFs and nodal forces:

Element #1

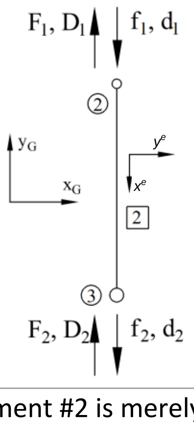


d_1 and d_2 – element nodal deformations in local coordinate axis system.
D_1 and D_2 – element nodal deformations in global coordinate axis system.
f_1 and f_2 – element nodal forces in local co-ordinates (axial forces for bar elements).
F_1 and F_2 – element nodal forces in global coordinate system.

Element #1 local (x^e, y^e) and global (x_G, y_G) coordinate axes **match**, so:

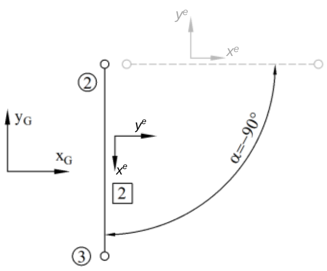
$$d_1 = D_1 \ \& \ f_1 = F_1$$
$$d_2 = D_2 \ \& \ f_2 = F_2$$

Element #2

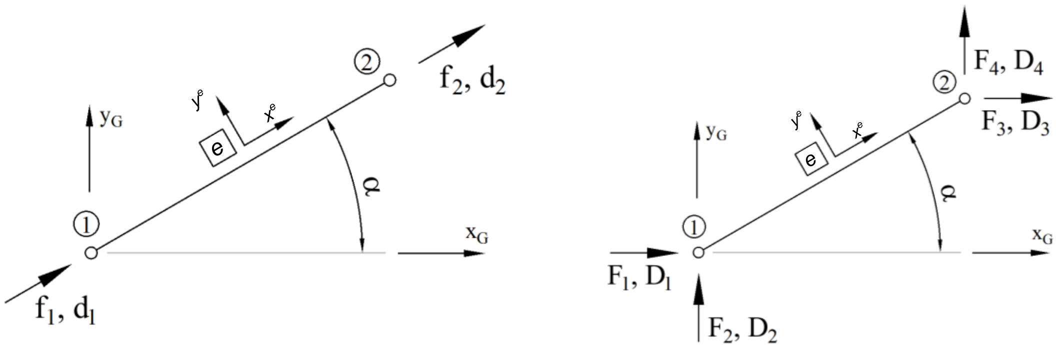


- Element #2 local (x^e, y^e) and global (x_G, y_G) coordinate axes **do not match**.
- What do we do??
- Nothing matches and this is devolving into chaos!

(3) Note that Element #2 is merely rotated through -90deg.



(4) For a general case we can write:



(5) We need to find a transformation matrix, which rotates the bar element by angle, α. Let:

$$\boxed{\boldsymbol{c = \cos\alpha \quad \& \quad s = \sin\alpha}}$$

Hence we can find the relation between local element displacements d^e and nodal element displacement, D^e:

$$d_1 = D_1 \cos(\alpha) + D_2 \sin(\alpha) = c\, D_1 + s\, D_2$$
$$d_2 = D_3 \cos(\alpha) + D_4 \sin(\alpha) = c\, D_3 + s\, D_4$$

Or in matrix form:

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \end{bmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} = \Lambda^e\, D^e$$

Where the matrix that 'transforms' the co-ordinate system is (unsurprisingly) called the transformation matrix:

$$\Lambda^e = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \end{bmatrix}$$

(6) Element nodal displacements, D^e, and element nodal forces, F^e, (in global co-ordinates) can be transformed into corresponding displacements, d^e, and internal forces, f^e in local co-ordinates:

$$d^e{}_{\{2x1\}} = \Lambda^e{}_{\{2x4\}}\ D^e{}_{\{4x1\}}$$

$$F^e{}_{\{4x1\}} = \Lambda^{e^T}{}_{\{4x2\}}\ f^e{}_{\{2x1\}}$$

$$\Lambda^e{}_{\{2x4\}} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \end{bmatrix}$$

Bar element transformation matrix, Λ^e, is general and can be applied to any bar element.

(7) What about the stiffness matrix K^e?

- If an element is rotated α degrees the original 2×2 stiffness matrix, K^e, is no longer valid as it must now be able to process pieces of the 4 (not 2) displacements, D^e, and forces, F^e.
- Element nodal forces in the global coordinate system can be found using the relation:

$$F^e = \widehat{K}^e \, D^e$$

  Where D^e is a 4×1 vector representing element nodal displacements, and $\widehat{K}^e$ is 4×4 element global stiffness matrix.

- How can we transform element local stiffness matrix, $K^e_{\{2\times2\}}$, into element global stiffness matrix, $\widehat{K}^e_{\{4\times4\}}$?

(8) Recall,

$$K^e_{\{2\times2\}} = \int_0^L \Psi'(x)^T EA \, \Psi'(x) dx$$

where Ψ represents element shape functions in local coordinates (x^e, y^e) for $u(x) = \Psi(x)\, d$.

(9) Hence, in global coordinates (x_G, y_G) element displacement at any point x can be calculated:

$$\hat{u} = \Psi(x)\, d^e = \Psi(x)\, (\Lambda\, D^e) = (\Psi(x)\, \Lambda)\, D^e$$

(10) Therefore, in formulating element stiffness matrix in global coordinates we can write:

$$\widehat{K}^e_{\{4\times4\}} = \int_0^L \Lambda^T \Psi'(x)^T EA \, \Psi'(x) \Lambda \, dx = \Lambda^T \left(\int_0^L \Psi'(x)^T EA \, \Psi'(x) \, dx \right) \Lambda$$

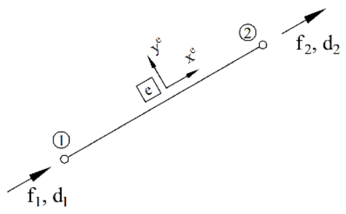
$$\boxed{\begin{aligned} \widehat{K}^e_{\{4\times4\}} &= \Lambda^T_{\{4\times2\}} K^e_{\{2\times2\}} \Lambda_{\{2\times4\}} \\ K^e_{\{2\times2\}} &= \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \Lambda_{\{2x4\}} &= \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \end{aligned}}$$

Or

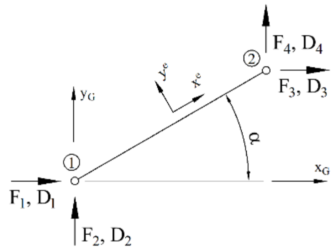
$$\boxed{\widehat{K}^e_{\{4\times4\}} = \Lambda^T_{\{4\times2\}} K^e_{\{2\times2\}} \Lambda_{\{2\times4\}} = \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}}$$

Converting between element deflections in local and global coordinates

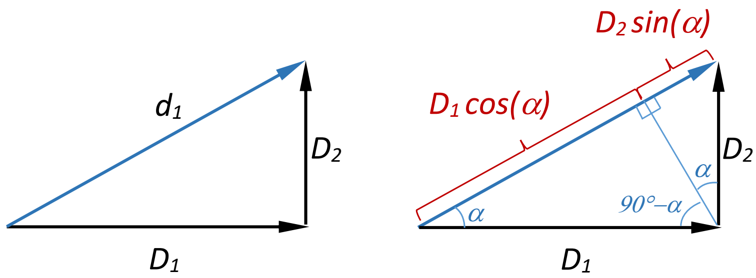
Element DOF in local (element) co-ordinates



Element DOF in global co-ordinates



Let's look at the vector sum of displacements, D_1^e and D_2^e, which must be equal to d_1^e



$$d_1 = D_1\cos(\alpha) + D_2\sin(\alpha)$$

And similarly:

$$d_2 = D_3\cos(\alpha) + D_4\sin(\alpha)$$

And in matrix form, this is expressed as $d^e = \Lambda^e\, D^e$, which can be written as:

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \end{bmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix}$$

However, if you instead want to transform from local co-ordinates d^e to global co-ordinates, D^e, the equation is:

$D^e = {\Lambda^e}^T d^e$ - But why is it this equation? Why do we use ${\Lambda^e}^T$, why not the inverse, ${\Lambda^e}^{-1}$?

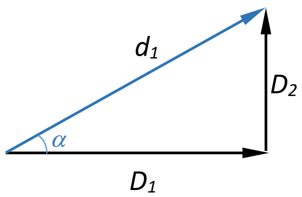
Well, first up, Λ^e is rectangular, not square, so the inverse doesn't exist. That's a good reason to not use the inverse, but why the transpose? Does that actually work? Well, let's take a look…

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} = \begin{bmatrix} \cos(\alpha) & 0 \\ \sin(\alpha) & 0 \\ 0 & \cos(\alpha) \\ 0 & \sin(\alpha) \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

If we write out these equations, we get:

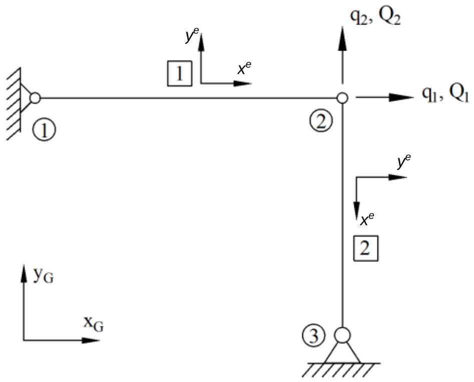
$$D_1 = d_1\cos(\alpha) + 0d_2$$
$$D_2 = d_1\sin(\alpha) + 0d_2$$
$$D_3 = 0d_1 + d_2\cos(\alpha)$$
$$D_4 = 0d_1 + d_2\sin(\alpha)$$

Recall our displacement vector triangle (which can also be applied to node 2).



Example 1 Revisited. Assembly + Analysis of Plane (2D) Trusses

- Global structure
- Identification of DOFS, q
- Identifying external loads, Q

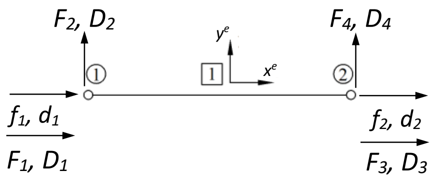


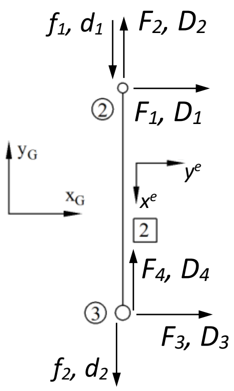
Element Freebody Diagrams: **Element #1** **Element #2**

Break structure into elements

Identify element DOFs, d

- Local axes, (x^e, y^e)
- Angle of rotation, α





Transformation matrix

$$\Lambda = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

$$\alpha = 0°$$
$$c = \cos\alpha = \cos(0°) = 1$$
$$s = \sin\alpha = \sin(0°) = 0$$

$$\Lambda^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\alpha = -90° \text{ or } +270°$$
$$c = \cos(-90°) = 0$$
$$s = \sin(-90°) = -1$$

$$\Lambda^2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Element local deformation

$$d^1 = \begin{pmatrix} d_1^1 \\ d_2^1 \end{pmatrix} = \Lambda^1 D^1 = \begin{pmatrix} D_1^1 \\ D_3^1 \end{pmatrix} \qquad d^2 = \begin{pmatrix} d_1^2 \\ d_2^2 \end{pmatrix} = \Lambda^2 D^2 = \begin{pmatrix} -D_2^2 \\ -D_4^2 \end{pmatrix}$$

Element nodal forces

$$F^1 = \begin{pmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{pmatrix} = (\Lambda^1)^T f^1 = \begin{pmatrix} f_1^1 \\ 0 \\ f_2^1 \\ 0 \end{pmatrix} \qquad F^2 = \begin{pmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{pmatrix} = (\Lambda^2)^T f^2 = \begin{pmatrix} 0 \\ -f_1^2 \\ 0 \\ -f_2^2 \end{pmatrix}$$

We now have information that explains how the degrees of freedom for each element, in local co-ordinates (d_1,d_2) correspond to the degrees of freedom for each element in global co-ordinates (D_1, D_2, D_3, and D_4)

We can use the information obtained on the previous page to formulate the element stiffness equation in global co-ordinates for each element, using the follow equations:

$K^e_{\{2\times2\}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\widehat{K}^e_{\{4\times4\}} = \Lambda^T_{\{4\times2\}} K^e_{\{2\times2\}} \Lambda_{\{2\times4\}}$	$F^e = \widehat{K}^e\, D^e$

However, that will only get us so far, as the equation for each element is still independent from other elements, does not include information about how these elements connect to each other, and does not include information about support points/boundary conditions.

We need to solve the overall structure as a system, but how do we do that?

How do we get from element-level equations using F^e, D^e & $\widehat{K}^e$ to structure-level equations that use Q, q, and K_G?

Solving global structure for motions	$Q_{\{n\times1\}} = K_{G\{n\times n\}} q_{\{n\times1\}}$ K_G- $n \times n$ global structure stiffness matrix, where n is the number of global DOFs in structure

The answer is using **assembly matrices**, A^e, for each element.

$$D^e_{\{4\times1\}} = (A^e)^T_{\{4\times n\}}\, q_{\{n\times1\}}$$

$$Q^e_{\{n\times1\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times4\}}\, F^e_{\{4\times1\}}$$

$$K_{G\{n\times n\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times4\}}\ \widehat{K}^e_{\{4\times4\}}\ (A^e)^T{}_{\{4\times n\}}$$

$$n_e - \text{total number of elements}$$

$$n - \text{total number of global DOF (the q's)}$$

That looks easy (maybe)! But, what is this assembly A^e and how does it map local element displacements and forces in global coordinates (D^e, F^e) into final model DOFs (the q's)??

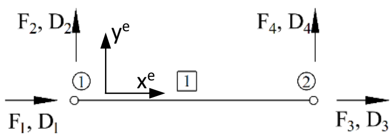
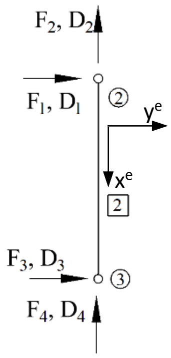
- Since it "maps" force and displacement it is made of 0s and 1s only.

In this example, in the global level the structure has 2 DOFs, where each element has 4 DOFs. Hence, the assembly matrices in this example will be 2×4 matrix containing 0s and 1s.

columns = # element DOFs in global co-ordinates

rows = # global DOFs in the overall structure (the # of q's)

Element Assembly Matrix

	Element #1	Element #2
Element DOF, D^e, and nodal forces, F^e		
Formulation of assembly matrices	$A^1 = \begin{array}{cc} & \begin{array}{cccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$ which means, $D_3^1 = q_1$ and $D_4^1 = q_2$	$A^2 = \begin{array}{cc} & \begin{array}{cccc} D_1^2 & D_2^2 & D_3^2 & D_4^2 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$ which means, $D_1^2 = q_1$ and $D_2^2 = q_2$

(6) Applied external force is equal to the sum of element forces:

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \sum_{e=1}^{n_e} A^e F^e = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{pmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{pmatrix}$$

$$Q = \begin{pmatrix} F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \end{pmatrix}$$

(7) The overall structure's stiffness matrix (referred to as the Global Stiffness Matrix K_G) can be found by assembling element global stiffness matrices:

$$K_G = \sum_{e=1}^{n_e} A^e \widehat{K}^e (A^e)^T = A^1 \widehat{K}^1 (A^1)^T + A^2 \widehat{K}^2 (A^2)^T$$

(8) And recall that element global stiffness matrix is defined:

$$\widehat{K}^e_{\{4\times4\}} = {\Lambda^e}^T_{\{4\times2\}} K^e_{\{2\times2\}} \Lambda^e_{\{2\times4\}}$$

$$\widehat{K}^e_{\{4\times4\}} = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$c = \cos\alpha \text{ and } s = \sin\alpha$$

Element #1	Element #2
$\alpha = 0^o$	$\alpha = -90^o$
$\widehat{K}^1 = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\widehat{K}^2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

(9) Element global stiffness matrix contribution to the total stiffness matrix of the whole structure is:

Element #1	Element #2
$K_G^1 = A^1\widehat{K}^1 {A^1}^T$	$K_G^2 = A^2\widehat{K}^2 {A^2}^T$
$K_G^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$K_G^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$K_G^1 = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$K_G^2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(10) Total, overall, global stiffness matrix (KG) that represents the stiffness of the overall structure, including all elements and the boundary conditions (support points). This is simply obtained by summing up the contribution from each element.

$$K_G = K_G^1 + K_G^2 = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(11) Now structure can be solved for displacements:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = K_G \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = K_G^{-1}\, Q = \frac{L}{EA}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{L}{EA}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

Which can be separated out to:

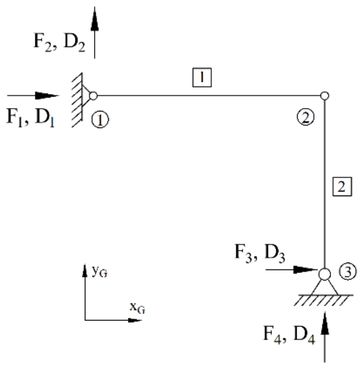
$$q_1 = \frac{L}{EA}Q_1 \qquad\qquad q_2 = \frac{L}{EA}Q_2$$

This answer for q_1 relies only on Element 1 boundary conditions. Element 2 is perpendicular to Element 1 and provides no resistance to the q_1 DOF as it is a bar element having no flexural stiffness. Similarly, the displacement in the q_2 DOF is a function of Q_2 and stiffness of Element #2 only.

Essentially, the matrix equation reduces down to two independent equations, each of which is a simple axial force-deformation equation.

The equations are only independent and separable due to the 90° rotation between them and the fact that we are dealing with pin-jointed bar elements.

(12) What about reaction loads?



What are the reaction forces F_1, F_2 at joint #1?

What are the reaction forces F_3, F_4 at joint #3?

(13) We know:

$$F^e = \widehat{K}^e D^e$$

where $\widehat{K}^e$ is element stiffness matrix in global co-ordinates and D_e is the vector of element nodal displacements, again in global co-ordinates.

To use the equation above, we need to know what deflections have occurred within each element. Thankfully, the assembly matrix that we have already defined can be used to 'extract' or 'select' the relevant pieces of the overall (now solved) deflection vector, q.

$$D^e = (A^e)^T q$$

hence, the element nodal forces can be found:

$$F^e = \widehat{K}^e (A^e)^T q$$

Nodal forces for element #1:

$$F^1 = \begin{pmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{pmatrix} = \widehat{K}^1 (A^1)^T q = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Q_1 L}{EA} \\ \frac{Q_2 L}{EA} \end{bmatrix} = \begin{pmatrix} -Q_1 \\ 0 \\ Q_1 \\ 0 \end{pmatrix}$$

Nodal forces for element #2:

$$F^2 = \begin{pmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{pmatrix} = \hat{K}^2 (A^2)^T q = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{Q_1 L}{EA} \\ \frac{Q_2 L}{EA} \end{bmatrix} = \begin{pmatrix} 0 \\ Q_2 \\ 0 \\ -Q_2 \end{pmatrix}$$

Some Final General Comments

Assembly matrix, A^e, maps model global DOF (q_i) to element DOF in global coordinates (D_i). As this step also defines the total number of structural DOFs, the assembly matrix applies the boundary conditions (support points) as well.

In the previous example we get 2 global DOF (q_1, q_2) and global element displacements in global coordinates, $D^e = (D_1^e \quad D_2^e \quad D_3^e \quad D_4^e)^T$

$A^e \quad =$

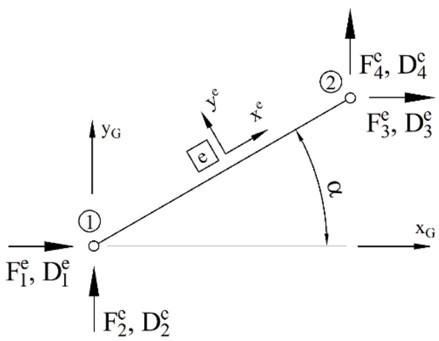
	D_1^e	D_2^e	D_3^e	D_4^e
q_1				
q_2				

Where each row i of the assembly matrix A^e is constructed:

$$\left.\begin{array}{l} 0 = q_i \neq D_i \\ 1 = q_i = D_i \end{array}\right\}$$

Therefore, only 0 or 1 in each field. Each row will be all zeros or have one "1" as each possible movement/DOF, q_i, cannot equal more than one D_i.

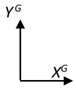
When labelling D_i, F_i recall we must match global coordinates. Hence, for generic case shown below, $X_G \rightarrow D_1^e, D_3^e$ and $Y_G \rightarrow D_2^e, D_4^e$. **This is a fixed nomenclature and approach**.

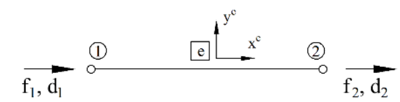
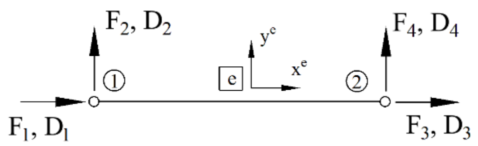
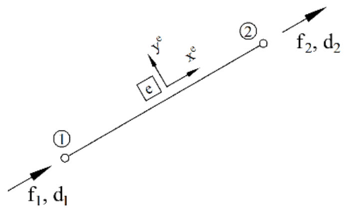
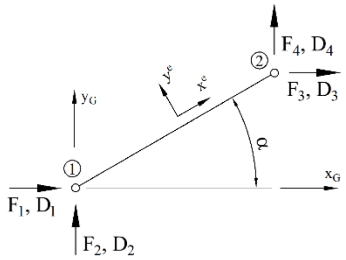
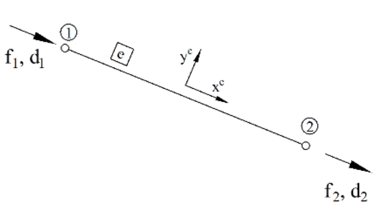
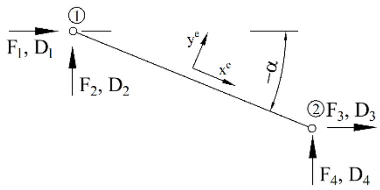
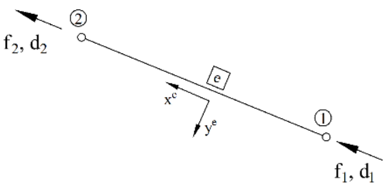
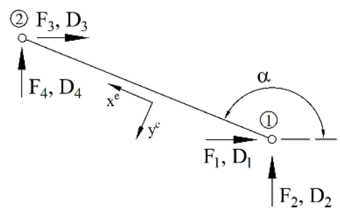


Local vs Global Element Degrees of Freedom

<table>
<tr><td><u>Element DOF in local (element) co-ordinates</u>

Local x^e at node 1 → d_1
Local x^e at node 2 → d_2</td><td><u>Element DOF in global co-ordinates</u>


Global X^G at node 1 → D_1
Global Y^G at node 1 → D_2

Global X^G at node 2 → D_3
Global Y^G at node 2 → D_4</td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
</table>

Note: you can choose any element end as a start node #1, which will define f_1, d_1. The angle of rotation, α, will depend on the choice of start node #1 and end node #2.

Problem 1 summary: All major steps and equations.

(1) Define possible motions/degrees-of-freedom of the overall structure: $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

and external forces applied to the overall structure: $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$

(2) Define element displacement-deformation relationship by creating local and global stiffness matrices for each element:

Local coordinate system	**Global coordinate system**
$f^e{}_{\{2\times1\}} = K^e{}_{\{2\times2\}}\ d^e{}_{\{2\times1\}}$	$F^e{}_{\{4\times1\}} = \widehat{K}^e{}_{\{4\times4\}}\ D^e{}_{\{4\times1\}}$
$K^e{}_{\{2\times2\}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\widehat{K}^e{}_{\{4\times4\}} = \Lambda^{e^T}{}_{\{4\times2\}}\ K^e_{\{2\times2\}}\ \Lambda^e{}_{\{2\times4\}}$
	$\Lambda^e{}_{\{2\times4\}} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$

(3) Define element assembly matrix, A^e, (by hand) for each element:

Number of element DOF in global co-ordinates

A^e =

	D_1^e	D_2^e	D_3^e	D_4^e
q_1				
q_2				
…				
q_n				

Number of structure DOF

(4) Assemble the total global stiffness matrix of the whole system:

$$K_{G\,\{n\times n\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times4\}}\ \widehat{K}^e_{\{4\times4\}}\ (A^e)^T{}_{\{4\times n\}}$$

n_e − total number of elements

n − total number of DOF of the overall structure

(5) Solve the system for motions/displacements:

$$Q = K_G\, q$$
$$\downarrow$$
$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (K_G)^{-1}\, Q$$

(6) Find the reaction forces:

$$F^e = \widehat{K}^e\ D^e = \widehat{K}^e (A^e)^T q$$

where

$$D^e = (A^e)^T q$$

(7) Find element motions:

$$d^e = \Lambda^e\ D^e$$

and element nodal forces:

$$F^e = (\Lambda^e)^T\ f^e$$

Summary of Different Stiffness Matrices

There are several levels of stiffness matrix, which can potentially be confusing. This table is included to help you understand the significance of the different forms of stiffness matrix and how additional information about the structure is introduced as we go through the process to build up the overall global stiffness matrix, K_G, that defines the overall structure, including information about stiffness contributions from every elements and the boundary conditions.

Different forms of Stiffness Matrices	Information on Element Material and Geometric Properties	Information on Element Orientation	Information on how the element is connected to other elements and/or to support points	Information on more than one element
K^e - Element stiffness matrix in local (element) co-ordinates	YES ✓	NO ×	NO ×	NO ×
$\widehat{K}^e = \Lambda^{e^T} K^e \Lambda^e$ Element stiffness matrix in global co-ordinates	YES ✓	YES ✓ Introduced through the transformation matrix	NO ×	NO ×
$K_G^e = A^e \widehat{K}^e (A^e)^T$ This element's contribution to the global stiffness matrix	YES ✓	YES ✓	YES ✓ Introduced through the assembly matrix	NO ×
$K_G = \sum_{e=1}^{n_e} K_G^e$ The overall global stiffness matrix. Captures all the stiffness terms of the total structure	YES ✓	YES ✓	YES ✓	YES ✓ Information on every element and every support point.

Let’s consider a larger pin-jointed truss made up of ten bar elements that is supported by a two pin-joints at the left edge of the structure. We will assume that every element is made of the same homogenous material and that all elements have the same cross-sectional area, expressed mathematically as:

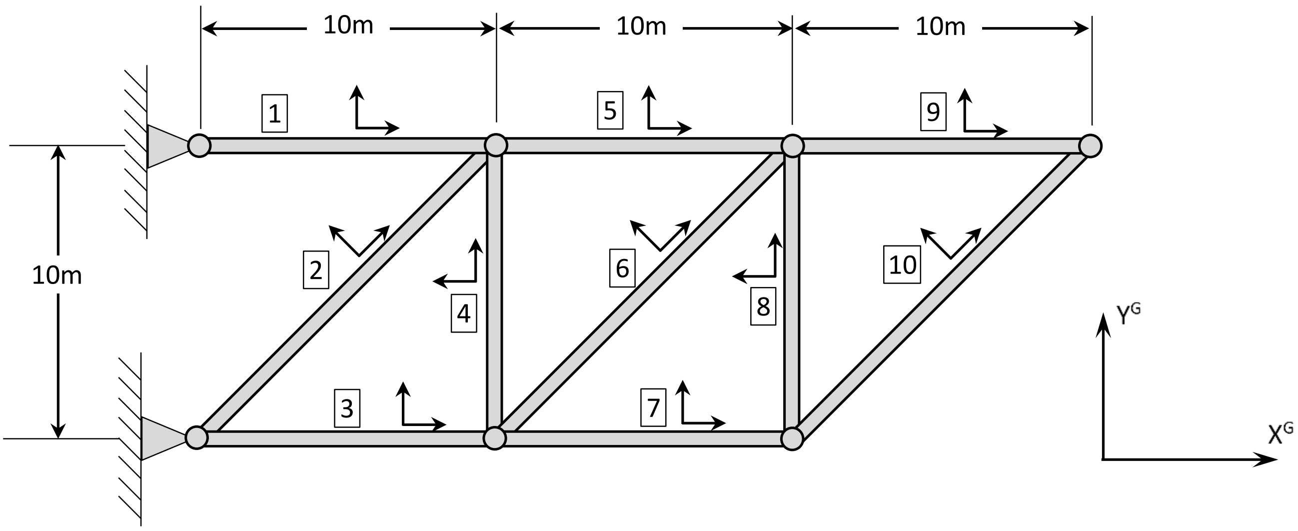
$$E^1 = E^2 = E^3 = E^4 = E^5 = E^6 = E^7 = E^8 = E^9 = E^{10} = E$$

$$A^1 = A^2 = A^3 = A^4 = A^5 = A^6 = A^7 = A^8 = A^9 = A^{10} = A$$

We will assume that the truss includes a square grid, so that the lengths of many elements are the same:

$$L^1 = L^3 = L^4 = L^5 = L^7 = L^8 = L^9$$

and $$L^2 = L^6 = L^{10}$$



As the local stiffness matrix in local coordinates (K^e) depends only on only geometric (A and L) and material (E) properties (and not element orientation or connectivity within the structure), then:

$$K^1 = K^3 = K^4 = K^5 = K^7 = K^8 = K^9$$

and $$K^2 = K^6 = K^{10}$$

However, we will now introduce element orientation information (via the transformation matrix) when we calculate the element stiffness matrix in global coordinate ($\widehat{K}^e = \Lambda^{e^T} \ K^e \ \Lambda^e$). Elements with the same geometric and material properties **and** the same orientation will have the same $\widehat{K}$

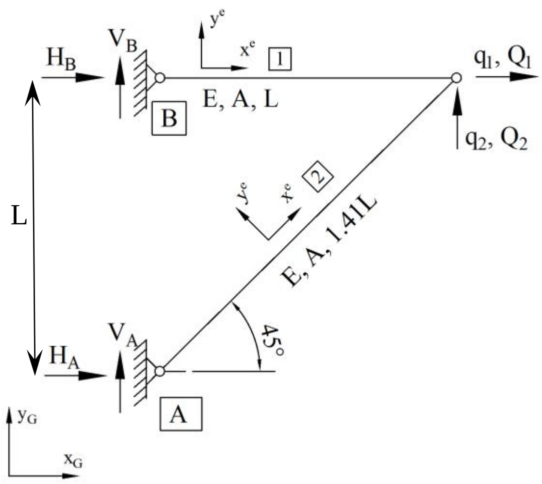
$\widehat{K}^1 = \widehat{K}^3 = \widehat{K}^5 = \widehat{K}^7 = \widehat{K}^9$	$\widehat{K}^4 = \widehat{K}^8$	$\widehat{K}^2 = \widehat{K}^6 = \widehat{K}^{10}$

Once we introduce the element connectivity information through the assembly matrices (and noting that the assembly matrix for every element will be unique as every element connects into the structure differently), then the contribution of each element to the global stiffness matrix ($K_G^e = A^e \ \ \widehat{K}^e \ \ (A^e)^T$) will be unique, such that:

$$K_G^1 \neq K_G^2 \neq K_G^3 \neq K_G^4 \neq K_G^5 \neq K_G^6 \neq K_G^7 \neq K_G^8 \neq K_G^9 \neq K_G^{10}$$

Problem 2

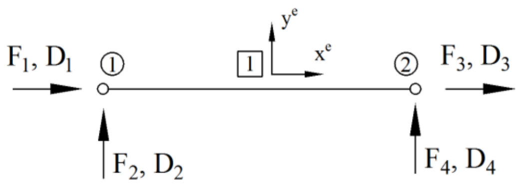
External forces $Q_1 = 0$, $Q_2 = 100kN$ act on the structure below. Find the structural deflections, the reaction forces at supports A & B, and element internal forces. Assume $L_1 = 10m$ and $L_2 = 14.1m$, and both members are made of 100mm diameter solid steel bars with elastic modulus, $E = 200$ GPa.



Solution

Finding local element stiffness matrix for each element:

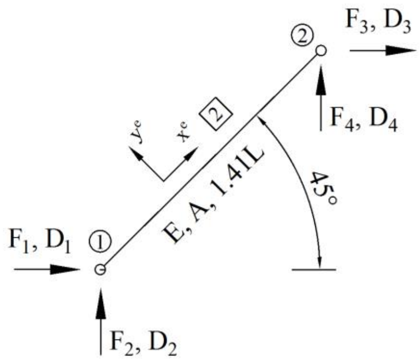
<u>Element #1 Free-body Diagram:</u>



$$K^1 = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{K}^1 = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element #2



$$K^2 = \frac{EA}{1.41L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\widehat{K}^2 = \frac{EA}{1.41L}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

(3) Creating assembly matrix for each element:

$A^1 =$

	D_1^1	D_2^1	D_3^1	D_4^1
q_1	0	0	1	0
q_2	0	0	0	1

$A^2 =$

	D_1^2	D_2^2	D_3^2	D_4^2
q_1	0	0	1	0
q_2	0	0	0	1

(4) Assembling global element matrices to form a total stiffness matrix:

$$K_G = A^1\,\widehat{K}^1\,(A^1)^T + A^2\;\widehat{K}^2\;(A^2)^T$$

$$K_G = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{EA}{1.41L}\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1.35 & 0.35 \\ 0.35 & 0.35 \end{bmatrix}$$

(5) Solving the structure for deflections:

$$Q = K_G\, q$$

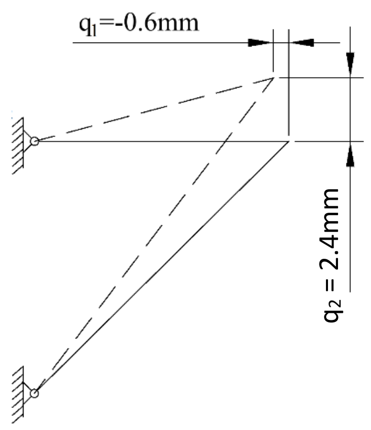
$$\begin{pmatrix} 0 \\ 1e^5 \end{pmatrix} = \frac{EA}{L}\begin{bmatrix} 1.35 & 0.35 \\ 0.35 & 0.35 \end{bmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$K_G^{-1} = \frac{L}{EA}\begin{bmatrix} 1 & -1 \\ -1 & 3.86 \end{bmatrix}$$

Hence, the deflection is equated as:

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = K_G^{-1}\, Q = \frac{L}{EA}\begin{bmatrix} 1 & -1 \\ -1 & 3.86 \end{bmatrix}\begin{pmatrix} 0 \\ 1e^5 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -0.0006 \\ 0.0024 \end{pmatrix}$$



Element #1 is **compressed** as $q_1 = -0.6mm < 0$
Element #2 is **stretched** as $q_2 = 2.4mm > 0$

(6) Finding element nodal forces: $F^e = \widehat{K}^e\, D^e = \widehat{K}^e (A^e)^T q$

<u>Element #1</u>

$$F^1 = \begin{pmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{pmatrix} = \widehat{K}^1 (A^1)^T q = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -0.006 \\ 0.0024 \end{pmatrix}$$

$$\begin{pmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{pmatrix} = \begin{pmatrix} 1e^5 \\ 0 \\ -1e^5 \\ 0 \end{pmatrix}$$

Where reaction forces at support B, H_B & V_B, are represented by element #1 nodal forces F_1^1 & F_2^1, respectively:

$$H_B = F_1^1 = 100kN$$
$$V_B = F_2^1 = 0$$

<u>Element #2:</u>

$$F^2 = \begin{pmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{pmatrix} = \widehat{K}^2 (A^2)^T q = \frac{EA}{1.41L} \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -0.0006 \\ 0.0025 \end{pmatrix}$$

$$\begin{pmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{pmatrix} = \begin{pmatrix} -1e^5 \\ -1e^5 \\ 1e^5 \\ 1e^5 \end{pmatrix}$$

Where reaction forces at support A, H_A & V_A, are represented by element #2 nodal forces F_1^2 & F_2^2:

$$H_A = F_1^2 = -100kN$$
$$V_A = F_2^2 = -100kN$$

(7) Finding local element deformation, d:

$$d^e = \Lambda^e \cdot D^e = \Lambda^e \cdot (A^e)^T q$$

<u>Element #1 deformation:</u>

$$d^1 = \Lambda^1 \, (A^1)^T q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -0.0006 \\ 0.0025 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.0006 \end{pmatrix}$$

Element #2 deformation:

$$d^2 = \Lambda^2 \, (A^2)^T q = \begin{bmatrix} 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0.707 & 0.707 \end{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -0.0006 \\ 0.0025 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0013 \end{pmatrix}$$

Hence, element #1 **compressed** by:

$$\Delta L^1 = d_2^1 - d_1^1 = -0.0006 - 0 = -0.0006m \rightarrow \boldsymbol{-0.6mm}$$

And element #2 **stretched** by:

$$\varDelta L^2 = d_2^2 - d_1^2 = 0.0013 - 0 = 0.0013m \rightarrow \mathbf{1.3}\boldsymbol{mm}$$

(8) Finding element strains: $\varepsilon^e = \frac{\varDelta L^e}{L^e}$

Element #1 strain:

$$\varepsilon^1 = \frac{\varDelta L^1}{L^1} = \frac{-0.0006}{10} = -6e^{-5}$$

Element #2 strain:

$$\varepsilon^2 = \frac{\varDelta L^2}{L^2} = \frac{0.0013}{14.1} = 9e^{-5}$$

(9) Finding element forces in local (element) co-ordinates:

$$f^e = K^e\, d^e$$

Element #1 local forces:

$$f^1 = K^1\, d^1 = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ -0.0006 \end{pmatrix} = \begin{pmatrix} 1e^5 \\ -1e^5 \end{pmatrix} N = \begin{pmatrix} 100\ kN \\ -100\ kN \end{pmatrix}$$

Element #2 local forces:

$$f^2 = K^2\, d^2 = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0.0013 \end{pmatrix} = \begin{pmatrix} -1.41e^5 \\ 1.41e^5 \end{pmatrix} N = \begin{pmatrix} -141\ kN \\ 141\ kN \end{pmatrix}$$

How to code it?

(1) Create functions for bar elements to calculate the bar element stiffness matrix in local co-ordinates, K^e, and the bar element stiffness matrix in global co-ordinates, $\widehat{K}^e$.

(2) Create assembly matrix, A^e, for each element by hand.

(3) Assemble total stiffness matrix of the whole system by:

$$K_{G\{n\times n\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times 4\}}\, \widehat{K}^e_{\{4\times 4\}}\ (A^e)^T{}_{\{4\times n\}}$$

$$n_e - \text{total number of elements}$$

$$n - \text{total number of overall (structural) DOF}$$

(4) Solve equation $Q = K_G \cdot q$ to get displacements, q. Use the matrix inverse or backslash operator

(5) Calculate element nodal forces in global co-ordinates

$$F^e = \widehat{K}^e\ D^e = \widehat{K}^e (A^e)^T q$$

for any given element so you can pick out and solve reaction loads.

(6) Calculate element nodal displacements in local co-ordinates:

$$d^e = \Lambda^e\, D^e = \Lambda^e\, (A^e)^T\, q$$

(7) Calculate element internal forces in local co-ordinates:

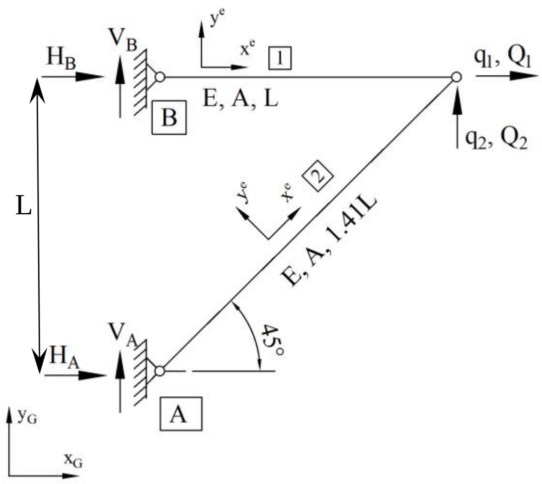
$$f^e = K^e\, d^e$$

and strains:

$$\varepsilon^e = \frac{d_2^e - d_1^e}{L^e}$$

Problem 2 solved using Python

Given the bars are 100mm diameter solid steel with $E = 200$ GPa, and $L = 10m$, and external forces $Q_1 = 0$, $Q_2 = 100kN$, find the reaction forces at supports A & B, and element internal forces.



Element 1	Element 2
$K1 = local\_bar(E, A, L) =$ $K^1 = 1 \times 10^8 \times \begin{bmatrix} 1.5708 & -1.5708 \\ -1.5708 & 1.5708 \end{bmatrix}$	$K2 = local\_bar(E, A, L) =$ $K^2 = 1 \times 10^8 \times \begin{bmatrix} 1.1140 & -1.1140 \\ -1.1140 & 1.1140 \end{bmatrix}$
$K1hat, Lambda1 = global\_bar(K1{,}0)$ $\widehat{K}^1 = 1 \times 10^8$ $\begin{vmatrix} 1.5708 & 0 & -1.5708 & 0 \\ 0 & 0 & 0 & 0 \\ -1.5708 & 0 & 1.5708 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ $\Lambda^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$K2hat, Lambda2 = global\_bar(K2{,}45)$ $\widehat{K}^2 = 1 \times 10^7$ $\begin{vmatrix} 5.5702 & 5.5702 & -5.5702 & -5.5702 \\ 5.5702 & 5.5702 & -5.5702 & -5.5702 \\ -5.5702 & -5.5702 & 5.5702 & 5.5702 \\ -5.5702 & -5.5702 & 5.5702 & 5.5702 \end{vmatrix}$ $\Lambda^2 = \begin{bmatrix} 0.7071 & 0.7071 & 0 & 0 \\ 0 & 0 & 0.7071 & 0.7071 \end{bmatrix}$
Assembly Matrix (generated manually): $A^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	Assembly Matrix (generated manually): $A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$K_G^1 = A^1 \widehat{K}^1 (A^1)^T =$ $K_G^1 = 1 \times 10^8$ $\begin{vmatrix} 1.5708 & 0 \\ 0 & 0 \end{vmatrix}$	$K_G^2 = A^2 \widehat{K}^2 (A^2)^T =$ $K_G^2 = 1 \times 10^7$ $\begin{vmatrix} 5.5702 & 5.5702 \\ 5.5702 & 5.5702 \end{vmatrix}$

Finding total stiffness matrix:

$$K_G = K_G^1 + K_G^2$$

$$K_G = 1 \times 10^8 \times \begin{vmatrix} 2.1278 & 0.5570 \\ 0.5570 & 0.5570 \end{vmatrix}$$

Creating external force vector:

$$Q = [0; \quad 100{,}000]$$

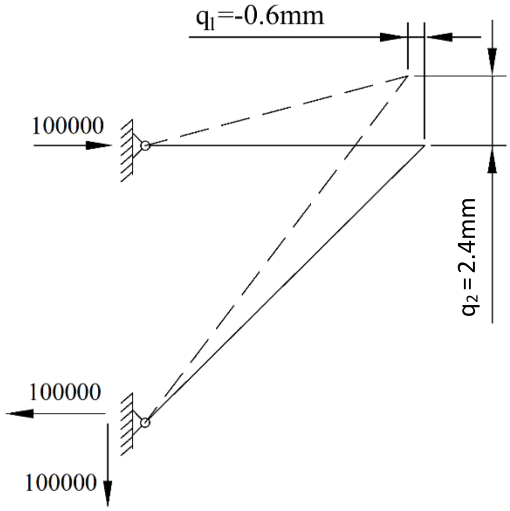
Solving equation for motions:

$$Q = K_G\, q \rightarrow q = inv(K_G)\, Q$$

$$q = \begin{bmatrix} -0.0006 \\ 0.0024 \end{bmatrix}$$

Element #1 nodal force vector (in global co-ordinates): $F^1 = \widehat{K}^1 D^1 = \widehat{K}^1\, ((A^1)^T\, q) = \begin{pmatrix} 100{,}000 \\ 0 \\ -100{,}000 \\ 0 \end{pmatrix}$	Element #2 nodal force vector (in global co-ordinates): $F^2 = \widehat{K}^2 D^2 = \widehat{K}^2\, ((A^2)^T\, q) = \begin{pmatrix} -100{,}000 \\ -100{,}000 \\ 100{,}000 \\ 100{,}000 \end{pmatrix}$
Element #1 nodal displacement vector (in local co-ordinates): $d^1 = \Lambda^1\, (A^1)^T\, q = \begin{pmatrix} 0 \\ -0.6366 \end{pmatrix} \times 10^{-3}$ where $\Lambda^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is by definition given, using α = 0°, and used in the second step when generating K1hat	Element #2 nodal displacement vector (in local co-ordinates): $d^2 = \Lambda^2\, (A^2)^T\, q = \begin{pmatrix} 0 \\ 0.0013 \end{pmatrix}$ $\Lambda^2 = \begin{bmatrix} 0.7071 & 0.7071 & 0 & 0 \\ 0 & 0 & 0.7071 & 0.7071 \end{bmatrix}$ is by definition given, using α = 45°, and used in the second step when generating K2hat
Element #1 strain: $\varepsilon^1 = \frac{d_2^1 - d_1^1}{L^1} = \frac{(-0.6366 - 0) \times 10^{-3}}{10} = -6.36 \times 10^{-5}$	Element #2 strain: $\varepsilon^2 = \frac{d_2^2 - d_1^2}{L^2} = \frac{(0.0013 - 0)}{14.1} = 9 \times 10^{-5}$
Element #1 nodal force vector (in local co-ordinates): $f^1 = K^1 d^1 = \begin{pmatrix} 100{,}000 \\ -100{,}000 \end{pmatrix}$	Element #2 nodal force vector (in local co-ordinates): $f^2 = K^2 d^2 = \begin{pmatrix} -1.4142 \\ 1.4142 \end{pmatrix} \times 10^5$

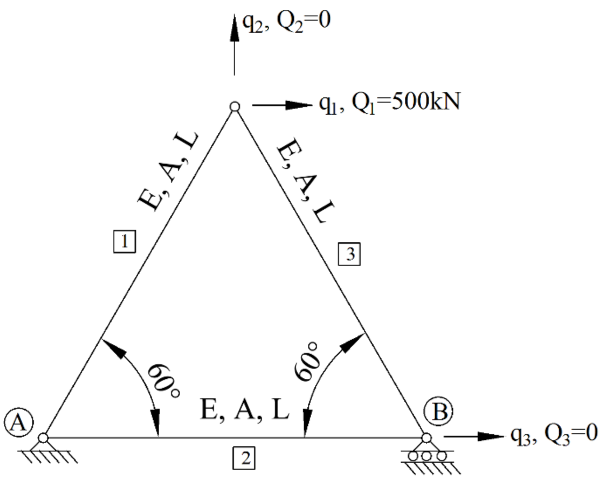
Plotting a deformed shape and reaction forces:



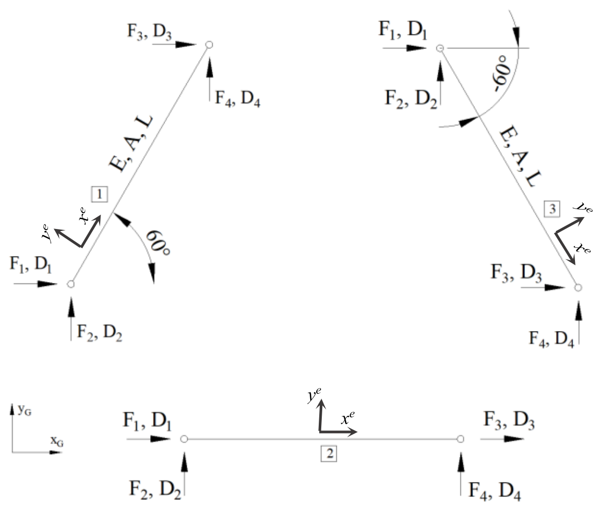
Problem 3

A truss made up of three bar elements is supported by a pin at point A and a roller at point B. A horizontal load of 500kN is applied at the top of the structure. Assume all members are $L = 5m$ long, have a circular hollow section with 100mm outside diameter and 10mm wall thickness, and are made of aluminium with $E = 70GPa$. Solve for structural deflections, element forces and deflections, and determine the support reaction forces.

(1) Define possible motions/DOF, q, and identify external forces, Q:



(2) Defining global element DOF, D, and nodal forces, F on element free-body diagrams.



(3) Defining global element stiffness matrix, $\widehat{K}^e$, for each element:

Element #1 ($\alpha^1 = 60°$):

$$\widehat{K}^1 = \frac{EA}{L}\begin{bmatrix} 0.25 & 0.43 & -0.25 & -0.43 \\ 0.43 & 0.75 & -0.43 & -0.75 \\ -0.25 & -0.43 & 0.25 & 0.43 \\ -0.43 & -0.75 & 0.43 & 0.75 \end{bmatrix}$$

Element #2 ($\alpha^2 = 0°$):

$$\widehat{K}^2 = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element #3 ($\alpha^3 = -60°$):

$$\widehat{K}^3 = \frac{EA}{L}\begin{bmatrix} 0.25 & -0.43 & -0.25 & 0.43 \\ -0.43 & 0.75 & 0.43 & -0.75 \\ -0.25 & 0.43 & 0.25 & -0.43 \\ 0.43 & -0.75 & -0.43 & 0.75 \end{bmatrix}$$

(4) Define element assembly matrices:

Element #1

$$A^1 = \begin{array}{c|cccc} & D_1^1 & D_2^1 & D_3^1 & D_4^1 \\ \hline q_1 & 0 & 0 & 1 & 0 \\ q_2 & 0 & 0 & 0 & 1 \\ q_3 & 0 & 0 & 0 & 0 \end{array}$$

Element #2

$$A^2 = \begin{array}{c|cccc} & D_1^2 & D_2^2 & D_3^2 & D_4^2 \\ \hline q_1 & 0 & 0 & 0 & 0 \\ q_2 & 0 & 0 & 0 & 0 \\ q_3 & 0 & 0 & 1 & 0 \end{array}$$

Element #3

$$A^3 = \begin{array}{c|cccc} & D_1^3 & D_2^3 & D_3^3 & D_4^3 \\ \hline q_1 & 1 & 0 & 0 & 0 \\ q_2 & 0 & 1 & 0 & 0 \\ q_3 & 0 & 0 & 1 & 0 \end{array}$$

(5) Obtaining total stiffness matrix, K_G, by assembling global stiffness matrix of each element, $\widehat{K}^e$:

$$K_G = \sum_{e=1}^{3} A^e \ \widehat{K}^e \, (A^e)^T$$

$$K_G = A^1\widehat{K}^1(A^1)^T \ + \ A^2\widehat{K}^2(A^2)^T \ + \ A^3\widehat{K}^3(A^3)^T$$

Element #1 contribution to the total stiffness matrix:

$$K_G^1 = A^1\widehat{K}^1(A^1)^T = \frac{EA}{L}\begin{bmatrix} 0.25 & 0.43 & 0 \\ 0.43 & 0.75 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element #2 contribution:

$$K_G^2 = A^2\widehat{K}^2(A^2)^T = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Element #3 contribution:

$$K_G^3 = A^3\widehat{K}^3(A^3)^T = \frac{EA}{L}\begin{bmatrix} 0.25 & -0.43 & -0.25 \\ -0.43 & 0.75 & 0.43 \\ -0.25 & 0.43 & 0.25 \end{bmatrix}$$

Hence, the total stiffness matrix is equal to a sum of element contributions:

$$K_G = K_G^1 + K_G^2 + K_G^3 = \frac{EA}{L}\begin{bmatrix} 0.5 & 0 & -0.25 \\ 0 & 1.5 & 0.43 \\ -0.25 & 0.43 & 1.25 \end{bmatrix}$$

(6) Solving the systems for deformations:

$$Q = K_G q \quad \rightarrow \quad q = K_G^{-1} Q$$

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{L}{EA}\begin{pmatrix} 1125 \\ -72 \\ 250 \end{pmatrix} \times 10^3 \ m$$

(7) Finding element nodal forces:

$$F^e = \widehat{K}^e \ D^e = \widehat{K}^e (A^e)^T q$$

Element #1 nodal forces (in global co-ordinates):

$$F^1 = \widehat{K}^1(A^1)^T q = \begin{pmatrix} -250{,}000 \\ -433{,}000 \\ 250{,}000 \\ 433{,}000 \end{pmatrix} N$$

Element #2 nodal forces (in global co-ordinates):

$$F^2 = \widehat{K}^2(A^2)^T q = \begin{pmatrix} -250{,}000 \\ 0 \\ 250{,}000 \\ 0 \end{pmatrix} N$$

Element #3 nodal forces (in global co-ordinates):

$$F^3 = \widehat{K}^3(A^3)^T q = \begin{pmatrix} 250{,}000 \\ -433{,}000 \\ -250{,}000 \\ 433{,}000 \end{pmatrix} N$$

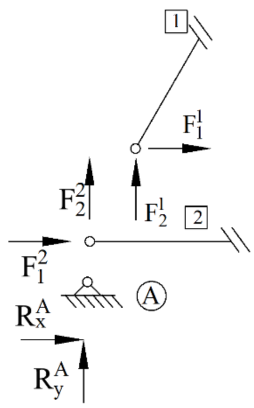
(8) Reaction forces at supports:

$F^1 = \begin{pmatrix} -250{,}000 \\ -433{,}000 \\ 250{,}000 \\ 433{,}000 \end{pmatrix} N$	$F^2 = \begin{pmatrix} -250{,}000 \\ 0 \\ 250{,}000 \\ 0 \end{pmatrix} N$	$F^3 = \begin{pmatrix} 250{,}000 \\ -433{,}000 \\ -250{,}000 \\ 433{,}000 \end{pmatrix} N$

Reaction forces can be found by superimposing corresponding global element forces, F^e.

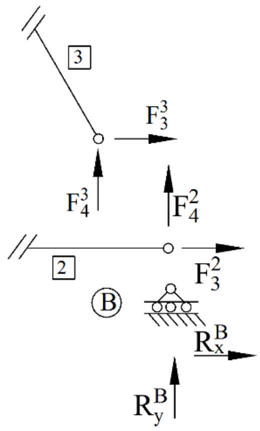
Reaction forces at support A:

$$R_x^A = F_1^1 + F_1^2 = -250000 + (-250000) = -50000N = -500kN$$
$$R_y^A = F_2^1 + F_2^2 = -433000 + 0 = -433000N = -433kN$$



Reaction forces at support B:

$$R_x^B = F_3^2 + F_3^3 = 250000 + (-250000) = 0N$$
$$R_y^B = F_4^2 + F_4^3 = 0 + 433000 = 433000N = 433kN$$



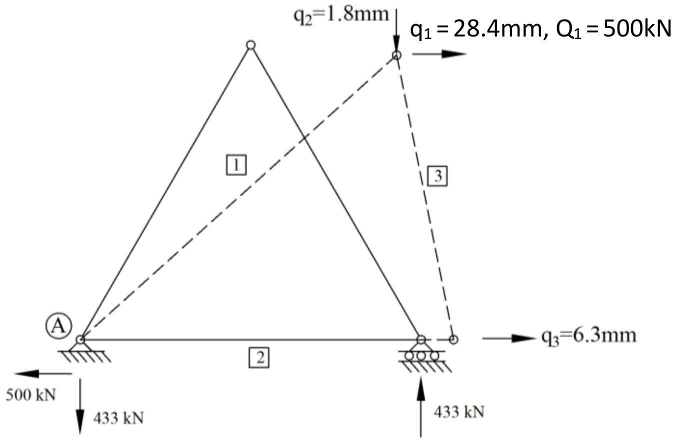
(9) Plotting deformed shape and reaction loads:

System deformations are calculated in Step (6). Recall that all members are $L = 5m$ long, circular hollow core 100mm outside diameter and 10mm wall thickness, made of aluminium, $E = 70GPa$, system deformation equals:

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{L}{EA}\begin{pmatrix} 1125 \\ -72 \\ 250 \end{pmatrix} m = \frac{5}{(70 \times 10^9)(2.83 \times 10^{-3})}\begin{pmatrix} 1125 \\ -72 \\ 250 \end{pmatrix} = \begin{pmatrix} 0.0284 \\ -0.0018 \\ 0.0063 \end{pmatrix} m$$

where the cross sectional area of a hollow core tube is:

$$A = \frac{\pi}{4}\left(D_e^2 - D_i^2\right) = \frac{\pi}{4}\left(0.1^2 - (0.1 - 2 \times 0.01)^2\right) = 2.83 \times 10^{-3} m^2$$



(10) Calculating local element deformations: $d^e = \Lambda^e D^e = \Lambda^e (A^e)^T q$

Element #1 to #3 local deformations:

$$d^1 = \begin{pmatrix} 0 \\ 500{,}000 \end{pmatrix}\frac{L}{EA} = \begin{pmatrix} 0 \\ 0.0126 \end{pmatrix} m$$

$$d^2 = \begin{pmatrix} 0 \\ 250{,}000 \end{pmatrix}\frac{L}{EA} = \begin{pmatrix} 0 \\ 0.0063 \end{pmatrix} m$$

$$d^3 = \begin{pmatrix} 625{,}000 \\ 125{,}000 \end{pmatrix}\frac{L}{EA} = \begin{pmatrix} 0.0158 \\ 0.0032 \end{pmatrix} m$$

(11) Finding bar element strains and stresses:

$$\varepsilon^e = \frac{d_2^e - d_1^e}{L^e} \qquad\qquad \sigma^e = E^e \varepsilon^e$$

Element #1 to #3 strains and stresses: $\varepsilon^1 = \frac{d_2^1 - d_1^1}{L^1} = \frac{0.0126-0}{5} = 0.0025$

$$\sigma^1 = E^1\,\varepsilon^1 = (70 \times 10^9)(0.0025) = 176.8\ MPa$$

$$\varepsilon^2 = 0.0013\ \rightarrow \sigma^2 = 88.4\ MPa$$

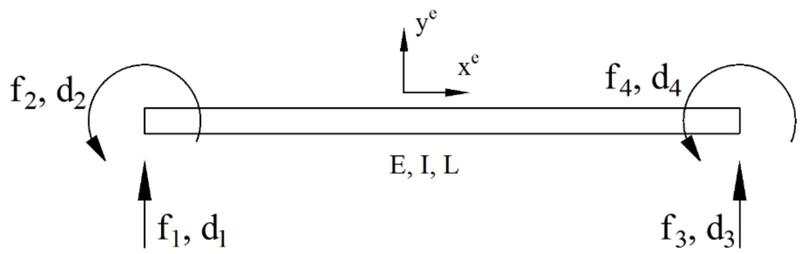
$$\varepsilon^3 = -0.0025\ \rightarrow\ \sigma^3 = -176.8\ MPa$$

BEAM ELEMENTS

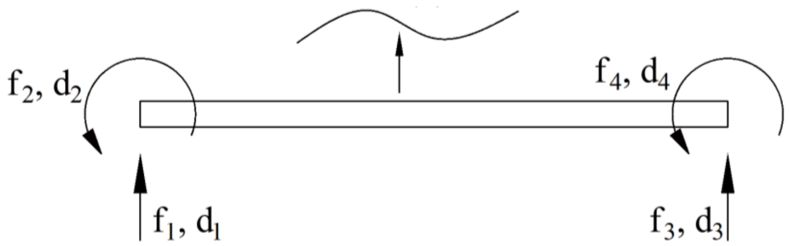
So far, we have only considered pin-jointed bar elements that only carry axial loads. We will now consider beam elements, which carry shear loads and moments.

(1) Beam elements:

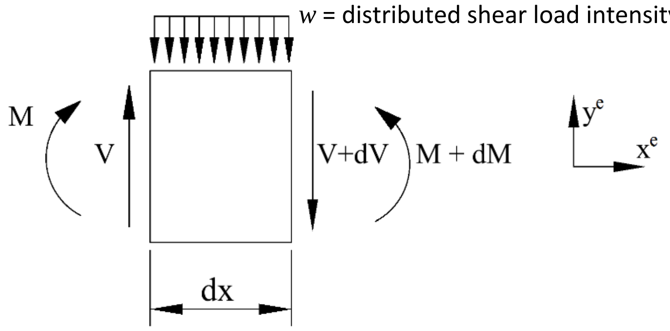
- Are defined by length, L, elastic modulus, E, and the second moment of area of the cross-section for bending, I.
- Carry moment and shear.
- Have **no axial component,** as they are considered **rigid axially**.
- Have 4 local DOF per element (2 transverse (shear) deflections and 2 rotations).



(2) Beam elements are formulated assuming the beam is “slender”, $L \gg (a, b)$, where a and b are the cross-sectional dimensions.



(4) Consider a small finite element in equilibrium:



(a) Force equilibrium in y^e axis:

$$\sum F_y = V - (V + dV) - w\,dx = 0$$
$$w = \frac{dV}{dx}$$

(b) Bending moment equilibrium:

$$\sum M_z = -M - (w\,dx)\,\frac{dx}{2} - (V + dV)\,dx + (M + dM) = 0$$
$$-\frac{1}{2}\,w\,dx^2 - V\,dx - dV\,dx + dM = 0$$

Assuming the length of the selected segment, dx, is infinitesimally small, the following assumptions can be made:

$$dx^2 \to 0$$

$$dV\,dx \to 0$$

Which leads to:

$$\frac{dM}{dx} = V$$

(c) From ENME 202 we know $M = EI\frac{d^2v}{dx^2}$ (the moment-curvature relationship).

(d) Hence, shear force can be expressed:

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)$$

(e) Similarly, the uniformly distributed load intensity can be approximated:

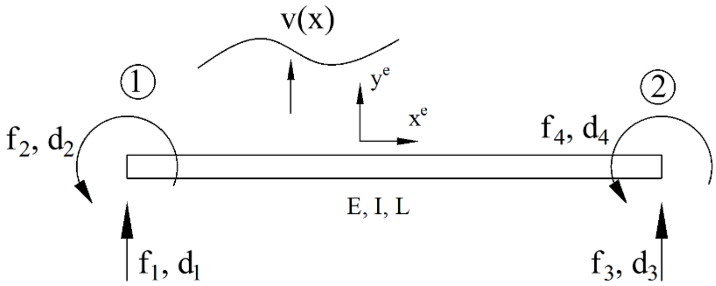
$$w = \frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)$$

(5) As we did before for axial bar elements, we will assume external uniformly distributed load is equal to 0 (for now):

$$w(x) = 0$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = 0$$

(6) Hence, given E, I, f_1, f_2, f_3, f_4 over an element, find the deformation function, $v(x)$, such that:



$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = 0$$

$$\left.\begin{aligned} EI\left.\frac{d^3v}{dx^3}\right|_{x=0} &= f_1 \\ -EI\left.\frac{d^2v}{dx^2}\right|_{x=0} &= f_2 \end{aligned}\right\} \text{BC at end 1}$$

$$\left.\begin{aligned} EI\left.\frac{d^3v}{dx^3}\right|_{x=L} &= f_3 \\ -EI\left.\frac{d^2v}{dx^2}\right|_{x=L} &= f_4 \end{aligned}\right\} \text{BC at end 2}$$

(7) Using the principle of virtual displacement (PVD):

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = 0 \quad \rightarrow \quad \delta v\left(\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\right) = 0$$

$$\int_0^L \delta v\left(\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\right)dx = 0$$

(8) Integration by parts twice leads to:

$$\delta v\left(\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\right)\Bigg|_0^L - \int_0^L \frac{d}{dx}\delta v\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)dx = 0$$

$$-\delta v(0)f_1 - \delta v(L)f_3 - \left[\frac{d}{dx}\delta v\, EI\frac{d^2v}{dx^2}\right]_0^L + \int_0^L \frac{d^2}{dx^2}\delta v\left(EI\frac{d^2v}{dx^2}\right)dx = 0$$

$$-\delta v(0)f_1 - \delta v(L)f_3 - \delta v'(L)\, f_4 - \delta v'(0)f_2 + \int_0^L \delta v'' EI v'' dx = 0$$

The following “weak form” of internal virtual work of beam is obtained:

$$\boxed{\int_0^L \delta v'' EI v'' dx = \delta v(0)f_1 + \delta v'(0)f_2 + \delta v(L)f_3 + \delta v'(L)\, f_4}$$

(9) Now, assume that deflection at any point along the beam can be calculated using a set of shape functions, N:

$$v(x) = N\,d \qquad\qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

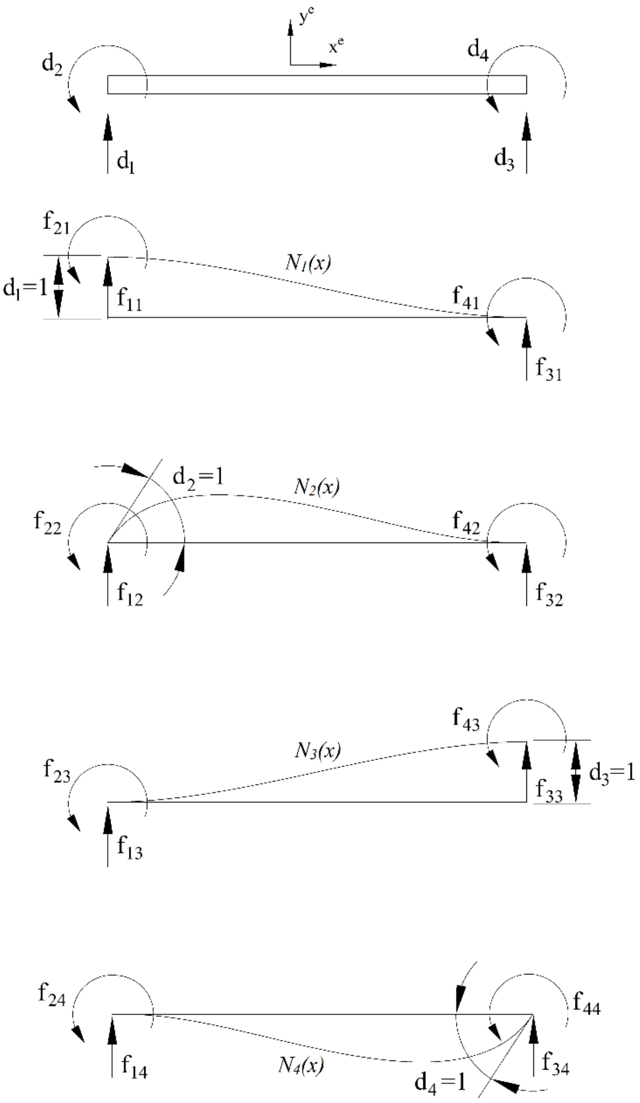
(10) Assume that:

$$N = [N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)]$$

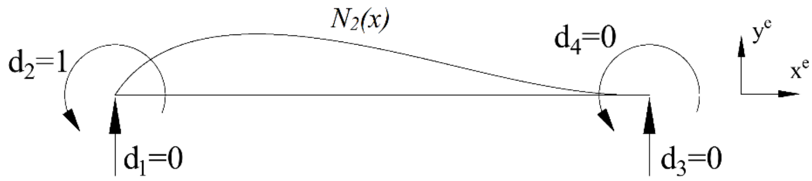
Where N can be expressed as a set of Hermitian polynomials:

$$N_i(x) = Ax^3 + Bx^2 + Cx + D$$

Where A, B, C, D are found from boundary conditions for each displaced shape below.



(11) Shape function representing $d_2 = 1$ displacement shape can be derived as follows:



Begin with the general 3rd order polynomial: $N_i(x) = Ax^3 + Bx^2 + Cx + D$

The apply the following boundary conditions:

$$N_2(x=0) = d_1 = 0$$

$$\left.\frac{dN_2}{dx}\right|_{x=0} = d_2 = 1$$

$$N_2(x=L) = d_3 = 0$$

$$\left.\frac{dN_2}{dx}\right|_{x=L} = d_4 = 0$$

Using the boundary conditions:

BC1: $$N_2(0) = A(0)^3 + B(0)^2 + C(0) + D = 0 \\ \downarrow \\ D = 0$$

BC2: $$N_2'(0) = 3A(0)^2 + 2B(0) + C = 1 \\ \downarrow \\ C = 1$$

BC3: $$N_2(L) = A(L)^3 + B(L)^2 + C(L) + D = 0 \\ \downarrow \\ AL^2 + BL + 1 = 0$$

BC4: $$N_2'(L) = 3A(L)^2 + 2B(L) + C = 0 \\ \downarrow \\ 3AL^2 + 2BL + 1 = 0$$

Constants A and B can be found using equations for boundary conditions BC3 and BC4, which yields:

$$A = \frac{1}{L^2} \quad B = -\frac{2}{L}$$

The following shape function is obtained:

$$\boxed{N_2(x) = \frac{x^3}{L^2} - \frac{2x^2}{L} + x}$$

(12) Applying a similar approach for the other deflected shapes yields $N_1(x), N_3(x), N_4(x)$:

$$\boxed{\begin{aligned} N_1(x) &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ N_2(x) &= \frac{x^3}{L^2} - \frac{2x^2}{L} + x \\ N_3(x) &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ N_4(x) &= \frac{x^3}{L^2} - \frac{x^2}{L} \end{aligned}}$$

(13) Hence, the transverse deflection along the length of the beam from multilple deflection components can be given by:

$$v(x) = N\, d^e = \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

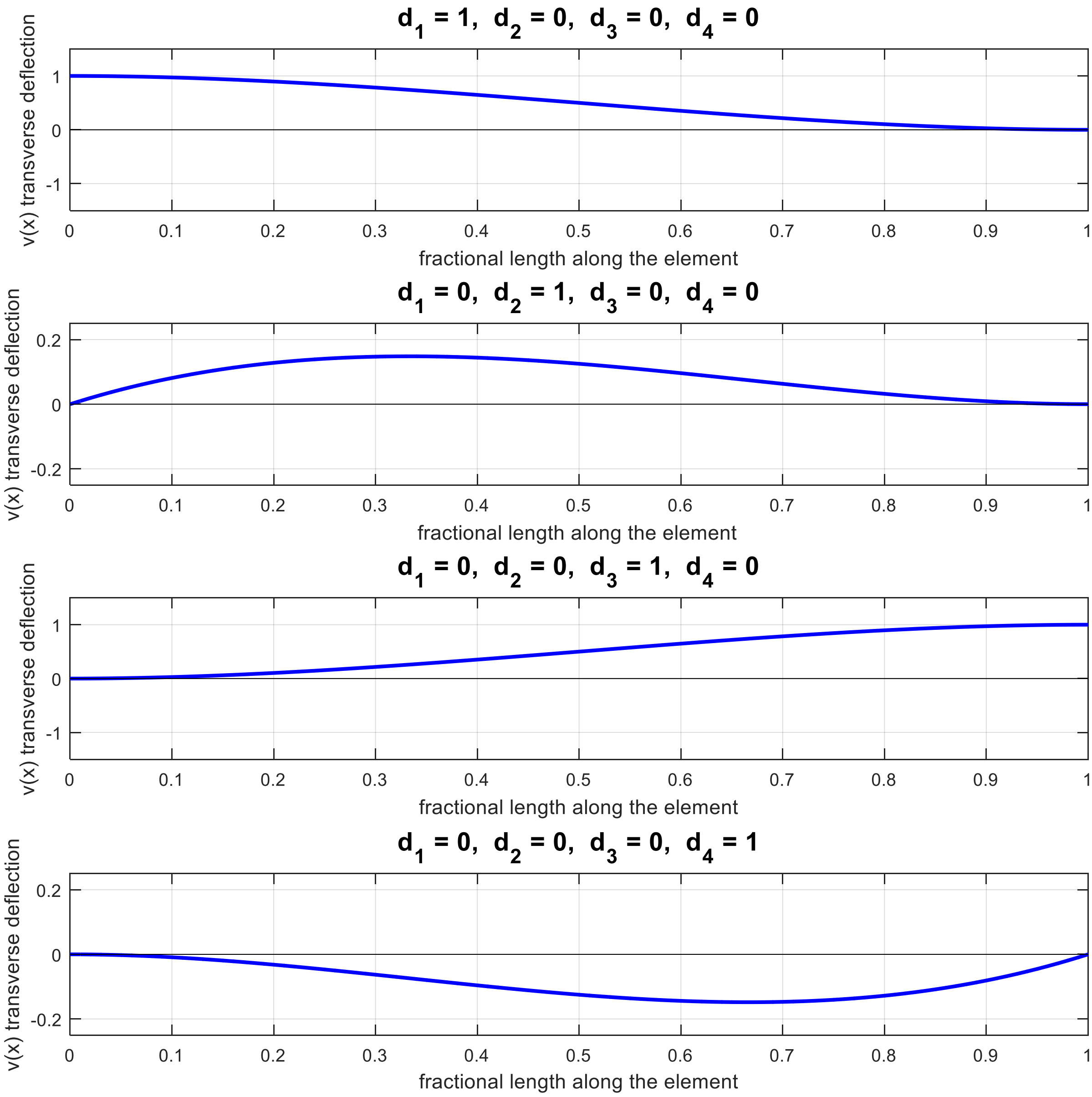
which yields:

$$v(x) = N_1(x)d_1 + N_2(x)d_2 + N_3(x)d_3 + N_4(x)d_4$$

(14) Shape functions are derived based on the following assumptions:

- “Small’ motions
- Bernoulli-Euler beam theory, so aspect ratio is >5-10, meaning not applicable to squat, short beam elements.
- The transverse deflection profile, $v(x)$ can only be weighted combinations of d_1, d_2, d_3, d_4 using $N(x)$ shape functions.

Let's plot the four shape functions (set each of the four displacements to a value of 1.0, with all others zero, just like we used in the derivation of the shape functions)



$$N_1(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$$

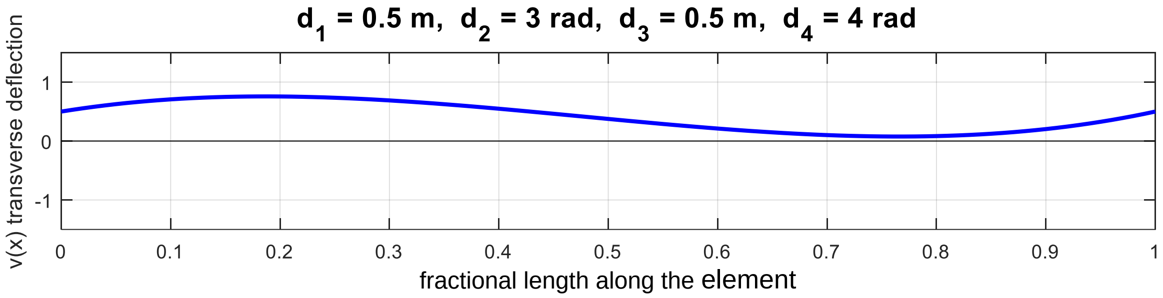
$$N_2(x) = \frac{x^3}{L^2} - \frac{2x^2}{L} + x$$

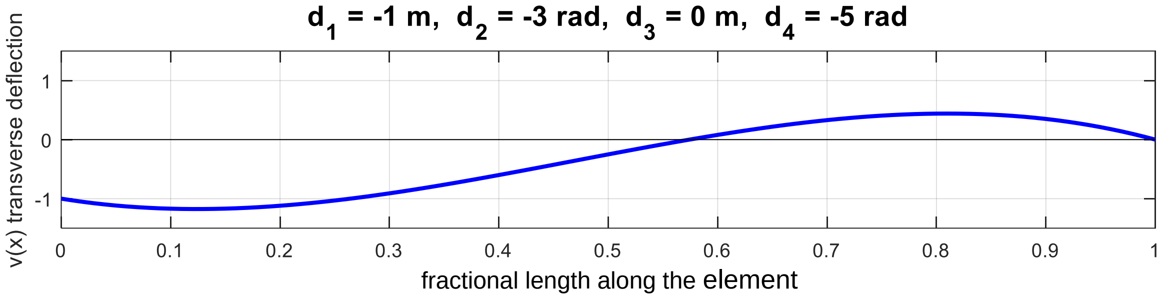
$$N_3(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

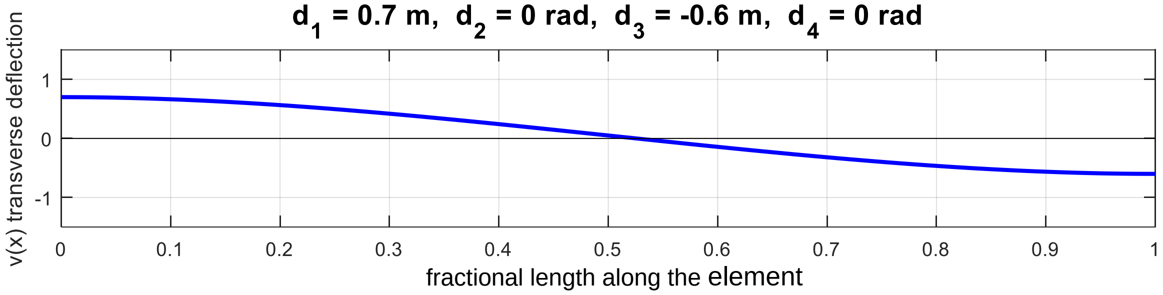
$$N_4(x) = \frac{x^3}{L^2} - \frac{x^2}{L}$$

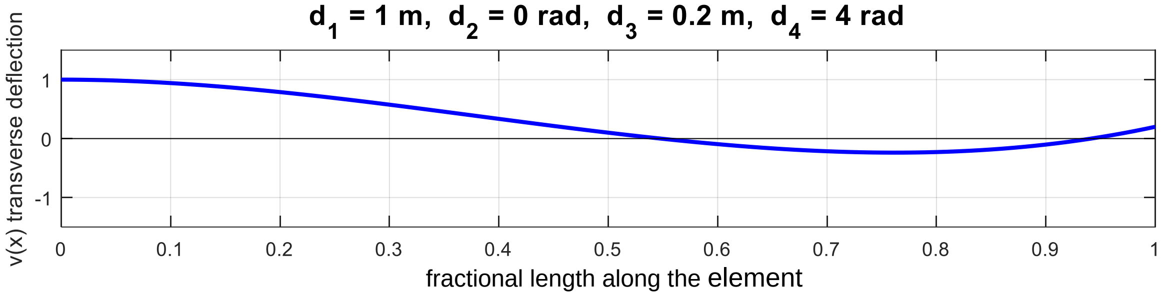
Below is some example nodal deflection values and the corresponding deflected shapes. The deflections are not small (so this violates our assumptions) but the deflections are exaggerated to display some of the allowable deflected shapes.

$$v(x) = N\,d^e = \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$



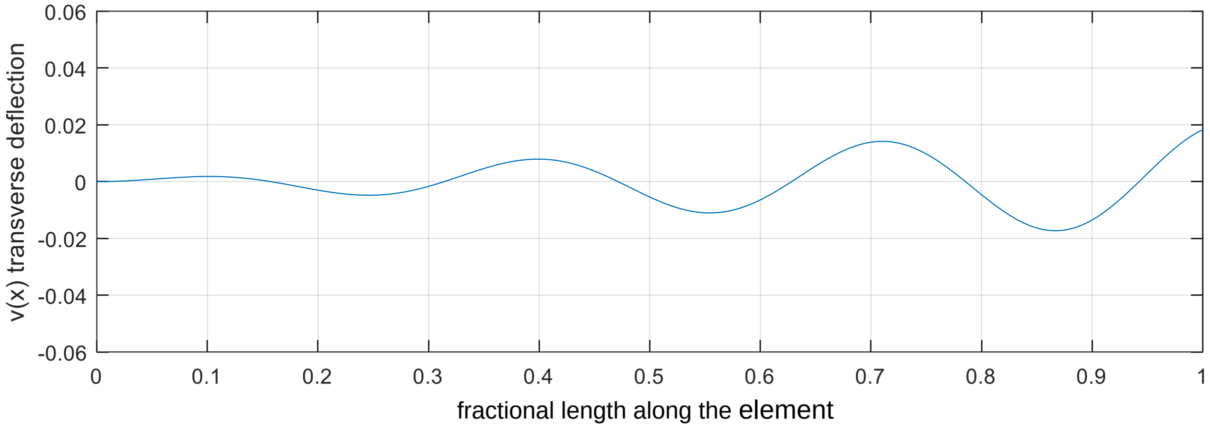


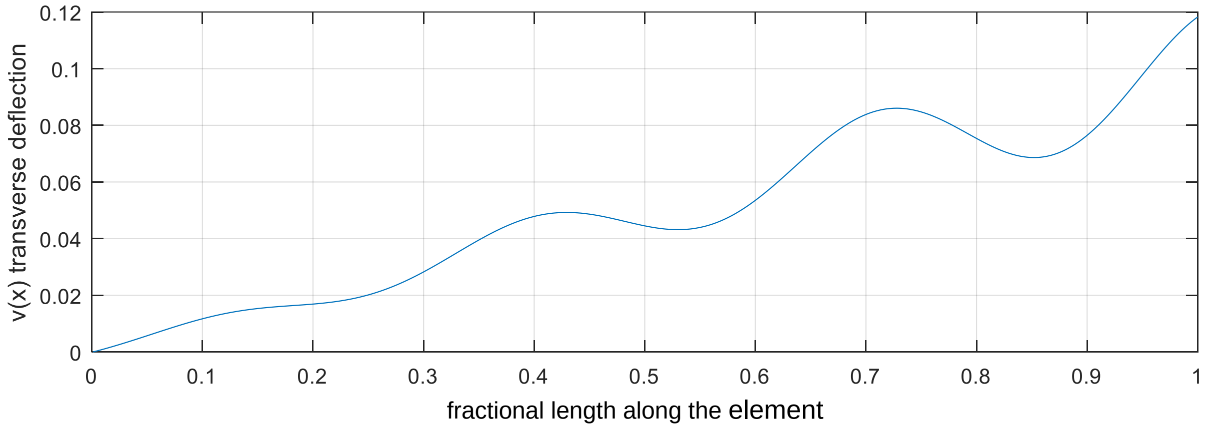




Do you think it is possible for the beam element that we have derived to capture the displaced profile below?

If you are modelling a structure where this sort of deflected profile is expected, what should you do? What must you be careful of?





These shape functions represent the internal reaction mechanisms that occur within the beam and enable us to relate loads at the ends of the element (the nodes) to the corresponding deflections.

We can now use these shape functions to define a stiffness matrix for this beam element, using a closely similar method to what we did for the bar element.

(15) Repeat and rewrite Equation (8)

$$\int_0^L \delta v'' EI v'' dx = \delta d^{e^T} f^e$$

where:

$$\delta v = N\, \delta d^e \qquad and \qquad \delta v'' = N''(x)\, \delta d^e$$

substituting:

$$\int_0^L \left(\delta d^{e^T} \left(N''(x)\right)^T \; EI \; \left(N''(x)\right) d^e\right) dx = \delta d^{e^T} f^e$$

$$\downarrow$$

$$\delta d^T \left(\int_0^L \left(N''(x)\right)^T \; EI \; \left(N''(x)\right) dx\right) d^e = \delta d^T f^e$$

$$\downarrow$$

$$\boxed{K^e = \int_0^L \left(\left(N''(x)\right)^T \; EI \; N''(x)\right) dx}$$

where:

$$N_1(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$$
$$N_2(x) = \frac{x^3}{L^2} - \frac{2x^2}{L} + x$$
$$N_3(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$
$$N_4(x) = \frac{x^3}{L^2} - \frac{x^2}{L}$$

so that:

$$N_1''(x) = -\frac{6}{L^2} + \frac{12x}{L^3}$$
$$N_2''(x) = \frac{6x}{L^2} - \frac{4}{L}$$
$$N_3''(x) = \frac{6}{L^2} - \frac{12x}{L^3}$$
$$N_4''(x) = \frac{6x}{L^2} - \frac{2}{L}$$

We can apply the shape function associated with each deflected shape to generate a corresponding stiffness term, and in doing so we will generate the stiffness matrix for this element.

(16) For example, ij^{th} component of K^e matrix can be obtained:

$$K_{ij}^e = \int_0^L N_i''(x)^T \ EI \ N_j''(x)\, dx$$

For example,

$$\begin{aligned}
K_{22}^e &= EI\int_0^L \left(\frac{6x}{L^2}-\frac{4}{L}\right)\left(\frac{6x}{L^2}-\frac{4}{L}\right)\, dx \\
&= EI\int_0^L \left(\frac{36x^2}{L^4}-\frac{48x}{L^3}+\frac{16}{L^2}\right)\, dx \\
&= EI\left[\frac{36x^3}{3L^4}-\frac{48x^2}{2L^3}+\frac{16x}{L^2}\right]_0^L \\
&= EI\left[\frac{36L^3}{3L^4}-\frac{48L^2}{2L^3}+\frac{16L}{L^2}\right] \\
&= EI\left[\frac{12}{L}-\frac{24}{L}+\frac{16}{L}\right] \\
&= \frac{4EI}{L}
\end{aligned}$$

(17) If we repeat the procedure above for all $i = 1 \dots 4$ and $j = 1 \dots 4$ possible motions, we will obtain beam element stiffness matrix in local (element co-ordinates), K^e:

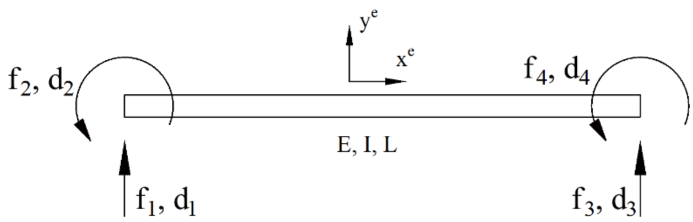
$$\boxed{K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}}$$

Note: $\boldsymbol{K_{ji} = K_{ij}}$ meaning stiffness matrix is always symmetric.

$$f^e = K^e d^e$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

(18) Notes on beam element stiffness matrix:



$$f^e = K^e d^e \ \rightarrow \text{in element coordinate system}$$

$$K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

- The order of terms within this matrix is based upon the degree of freedom numbering in the diagram above. If we had numbered these degrees of freedom differently, the stiffness terms within this matrix would be in a different order. Therefore, to be able to use this stiffness matrix, we **<u>must</u>** follow the same numbering sequence that was used during the derivation.

- Term EI/L^3 represents the flexural rigidity of the beam, which in some texts is noted as:

$$\bar{\alpha} = \frac{EI}{L^3}$$

- Beam element stiffness matrix, K^e, is symmetric. This means force at one end has same effect at the other end, no matter which end is loaded.

- Stiffness matrix is derived assuming E, I, L are constant over beam length. If E or I varies the integral in Equation (15) must be for $E(x)$ and/or $I(x)$ resulting in a different stiffness matrix, K^e.

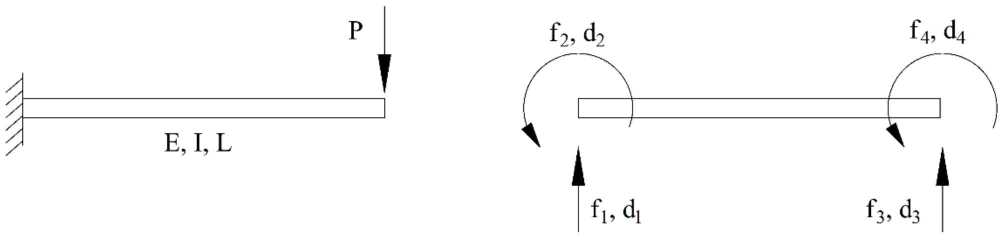
- Is valid for long, slender beams whose response/deformation is predominantly flexural with aspect ratio > 5-10.

- Accounts for small motions as stiffness matrix is derived ignoring dx^2 term in (4).

- Assumes beam is axially rigid, which implies $EA/L \gg EI/L^3$ which is typically true when the beam is long and slender so that L is relatively large.

- K^e is not full rank as boundary conditions have not yet been applied and the element is therefore not constrained (just like with bar elements).

Beam Element Example 1. Simple cantilever beam to prove the concept



(1) Global forces can be calculated:

$$f^e = K^e\, d^e$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

However, cantilever beam shown above is rigidly fixed on the left side, meaning $d_1 = d_2 = 0$ and f_1, f_2 will be reaction loads. As a result, we can ignore first 2 rows and columns of the K^e matrix (ie: ignoring the stiffness terms which relate to the zero displacement d_1 and the zero rotation d_2 at the fixed support), leading to:

$$\begin{pmatrix} f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P \\ 0 \end{pmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{pmatrix} d_3 \\ d_4 \end{pmatrix}$$

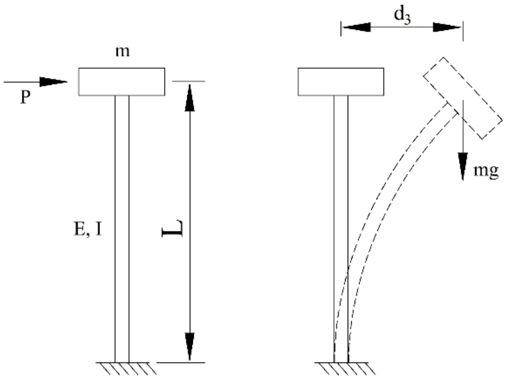
Displacements, d, can be found by re-arranging this equation:

$$\begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \frac{L^3}{EI}\,\frac{1}{48L^2 - 36L^2}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{pmatrix} -P \\ 0 \end{pmatrix} = \frac{L^3}{EI}\,\frac{1}{12L^2}\begin{pmatrix} -4L^2P \\ -6LP \end{pmatrix} = \frac{L}{12EI}\begin{pmatrix} -4L^2P \\ -6LP \end{pmatrix}$$

$$\begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} \dfrac{-PL^3}{3EI} \\ \dfrac{-PL^2}{2EI} \end{pmatrix}$$

Which matches analytical solutions in books for tip deflection d_3 and rotation d_4.

Beam Element Example 2. Vertical cantilever beam with lumped mass on top



If we consider this (very simplified) 2 DOF building (or bridge pier) model. What have we neglected?

Answer: Gravity!

Ignoring axial effects at rest/in the undeformed shape, the gravity force mg has no impact. But deflected, it adds a further base moment:

$$M = mg \times d_3$$

which will increase deflection d_3 and thus, we would have to iterate to get a solution.

If instead, we simply said:

$$\begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \frac{L}{12EI}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{pmatrix} P \\ mg\,\dfrac{PL^3}{3EI} \end{pmatrix}$$

$$d_3 = \frac{PL^3}{3EI}\left(1 + \frac{mgL^2}{2EI}\right)$$

The added term $\frac{mgL^2}{2EI}$ can amount to 20-30% more displacement!

Consider if:

$$E = 200\ GPa \qquad L = 6\ m \qquad mg = 40kN$$
$$I = 2 \times 10^{-5}\ m^4 \qquad P = 10kN$$

Horizontal displacement due to horizontal force alone:

$$d_3 = \frac{PL^3}{3EI} = \frac{(10 \times 10^3)\ (6^3)}{3(200 \times 10^9)(2 \times 10^{-5})} = 0.18m$$

Horizontal displacement due to horizontal force AND gravity force:

$$d_3 = \left(1 + \frac{mgL^2}{2EI}\right)\frac{PL^3}{3EI} = \left(1 + \frac{(40 \times 10^3)(6^2)}{2(200 \times 10^9)(2 \times 10^{-5})}\right)\frac{(10 \times 10^3)\ (6^3)}{3(200 \times 10^9)(2 \times 10^{-5})} = 0.212m$$

- Hence, addition of gravity forces results in 18% increase in horizontal displacement.
- In earthquake terms a 10 kN lateral load in a 40kN building is ~0.25g peak ground acceleration (PGA), which is a relatively large earthquake.
- A 0.5g earthquake would then have increased this effect.
- Again, it is a non-linear problem because $d_3 = f(d_3)$ and an iterative solution is required, but the effect can be much larger.

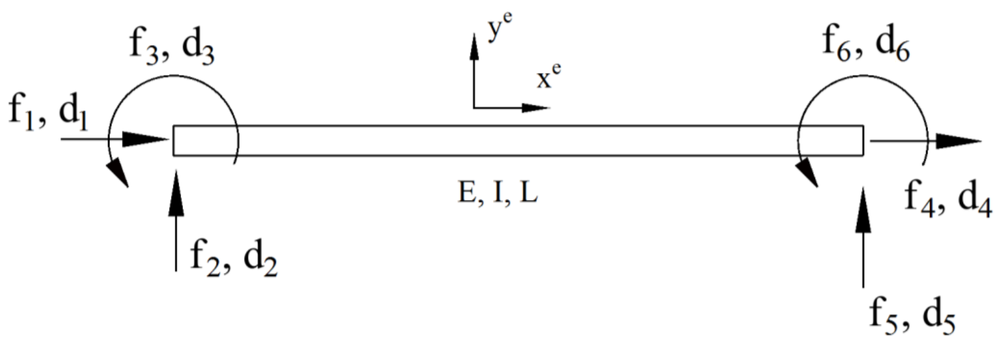
FRAME ELEMENTS

We now know how to model bar elements that carry only axial loads, and how to model beam elements that only carry shear loads and moments. For a general element, we want to be able to consider all possible loads simultaneously. We will now define FRAME elements, which are able to support axial and shear forces, as well as bending moments.

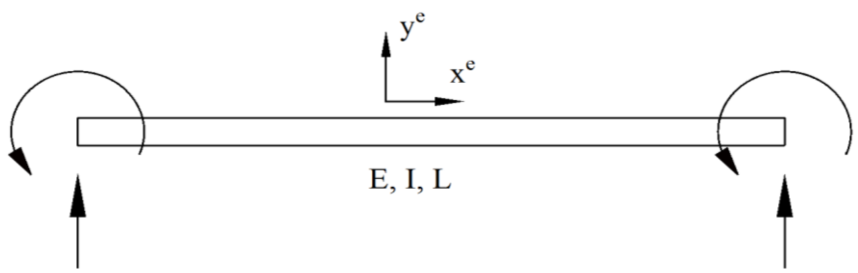
Does that mean that we need to do all those integrals and shape functions again?? **Thankfully, NO!**

The stiffness matrix for a FRAME element can be obtained from the **Principle of Linear Superposition.**

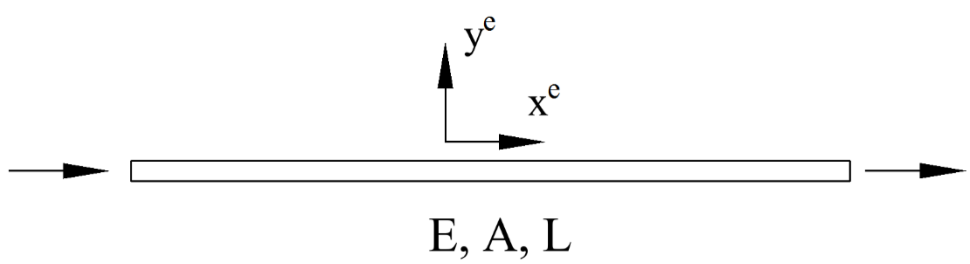
A 6 DOF FRAME Element:



is simply the addition of a 4 DOF BEAM Element: =

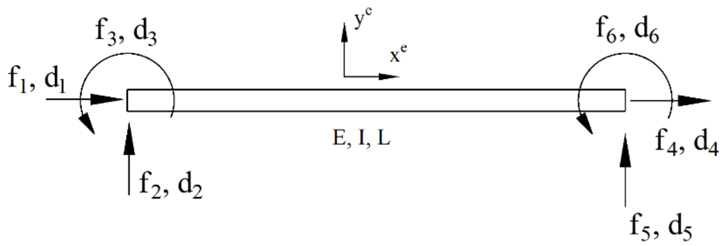


and a 2 DOF BAR Element +



<u>Assumption</u>: No coupling between axial terms (d_1, d_4, and f_1, f_4) and flexural (bending) terms (d_2, d_3, d_5, d_6, and f_2, f_3, f_5, f_6). Each set of terms can be calculated independently.

What does a FRAME element stiffness matrix look like?



$$f^e{}_{\{6\times1\}} = K^e{}_{\{6\times6\}} d^e{}_{\{6\times1\}} \rightarrow \text{in element coordinate system}$$

Frame Element Stiffness Matrix in Local Co-ordinates:

$$K^e{}_{\{6\times6\}} = \frac{EI}{L^3}\begin{bmatrix} \beta & 0 & 0 & -\beta & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\beta & 0 & 0 & \beta & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \quad \text{where: } \beta = \frac{AL^2}{I}$$

Again, the stiffness matrix above is in local (element) co-ordinates. However, it can be transformed into global co-ordinates in a very similar way to a bar element. The stiffness matrix for a FRAME element, in global co-ordinates is calculated using the following (hopefully familiar looking) equation:

$$\hat{K}^e{}_{\{6\times6\}} = \Lambda^{eT}{}_{\{6\times6\}}\, K^e_{\{6\times6\}}\, \Lambda^e{}_{\{6\times6\}}$$

But now the transformation matrix looks a little different:

Frame Element Transformation Matrix

$$\Lambda^e{}_{\{6\times6\}} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_{\{3\times3\}} & 0_{\{3\times3\}} \\ 0_{\{3\times3\}} & \lambda_{\{3\times3\}} \end{bmatrix} \; where: \; \lambda_{\{3\times3\}} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All the other relationships remain the same as before (but now the vectors/matrices are larger):

$$f^e{}_{\{6\times1\}} = K^e{}_{\{6\times6\}} d^e{}_{\{6\times1\}}$$

$$F^e{}_{\{6\times1\}} = \widehat{K}^e{}_{\{6\times6\}} D^e{}_{\{6\times1\}}$$

$$d^e{}_{\{6\times1\}} = \Lambda^e{}_{\{6\times6\}}\; D^e{}_{\{6\times1\}}$$

$$F^e{}_{\{6\times1\}} = \Lambda^{e^T}{}_{\{6\times6\}}\; f^e{}_{\{6\times1\}}$$

All the matrix equations are exactly the same as what we used for bar elements. The size of the matrices/vectors has now changed, but the way that they multiply together remains unchanged.

Furthermore, assembly of a structure using FRAME elements is also just like before:

$$D^e{}_{\{6\times1\}} = (A^e)^T{}_{\{6\times n\}}\; q_{\{n\times1\}}$$

$$Q_{\{n\times1\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times6\}} F^e{}_{\{6\times1\}}$$

$$K_{G\{n\times n\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times6\}}\; \widehat{K}^e_{\{6\times6\}}\; (A^e)^T{}_{\{6\times n\}}$$

$$n_e - \text{total number of elements}$$

$$n - \text{total number of overall (structural) DOF}$$

For a FRAME element

columns = # element DOFs in global co-ordinates
(6 DOF for a FRAME element)

rows = # global DOFs
in the overall structure
(the # of q's)

Element Assembly Matrix

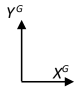
And just like before:

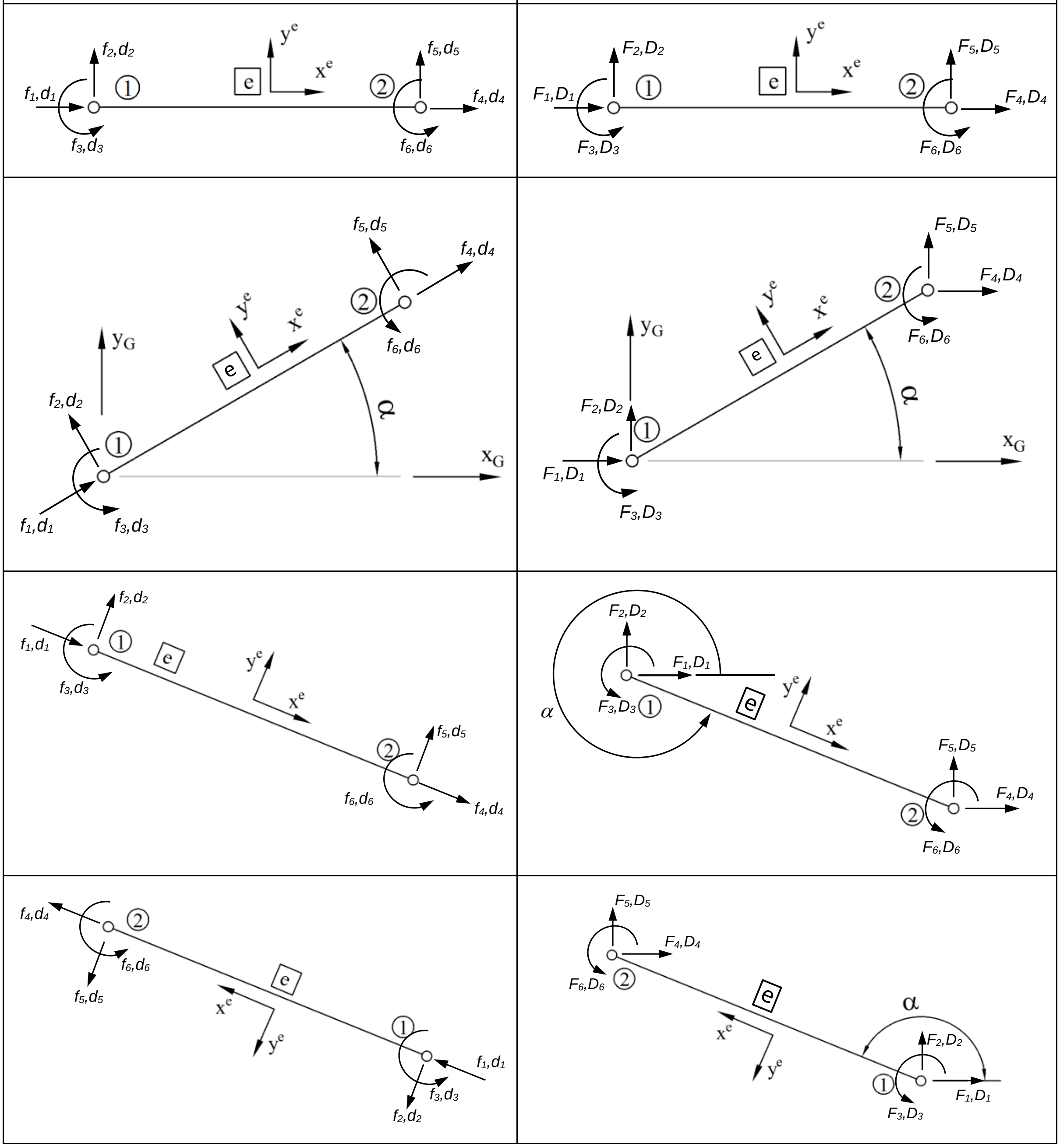
$$Q_{\{n\times1\}} = K_{G\{n\times n\}} q_{\{n\times1\}}$$

Note: you can make a FRAME element behave like a BEAM element in an assembly, simply by constraining the axial degrees of freedom ($d_1$and d_4), enforcing the element to be axially rigid and preventing any axial deformation. Therefore, the FRAME element essentially supersedes the BEAM element and we will deal with FRAME elements from here.

Local vs Global Element Degrees of Freedom for a FRAME element

Element DOF in local (element) co-ordinates	Element DOF in global co-ordinates
Local x^e at node 1 → d_1 Local y^e at node 1 → d_2 Local z^e at node 1 → d_3 Local x^e at node 2 → d_4 Local y^e at node 2 → d_5 Local z^e at node 2 → d_6	Global X^G at node 1 → D_1 Global Y^G at node 1 → D_2 Global Z^G at node 1 → D_3 Global X^G at node 2 → D_4 Global Y^G at node 2 → D_5 Global Z^G at node 2 → D_6





Again, you can choose any element end as a start node #1, which will define $f_1, f_2, f_3,\ d_1, d_2, d_3$
The angle of rotation, α, will depend on the choice of start node #1 and end node #2.

Summary of the Steps in the Finite Element Solution Method

(1) Determine the allowable global degrees of freedom (the qs) and the corresponding global forcing terms (the Qs) based upon the number of nodes in the overall structure and the type of support points. Label and number these on a diagram of the overall structure using a systematic approach.

(2) Draw free-body diagrams of each element, labelling the element displacements and forcing terms in global co-ordinates (the D^e and F^e components) as well as the angle of orientation, α^e.

(3) Create the element stiffness matrix in local co-ordinates, K^e, from E, A, I, and L.

(4) Using the element orientation angle α^e, create the element transformation matrix, Λ^e, and use that to generate the bar element stiffness matrix in global co-ordinates, $\widehat{K}^e = \Lambda^{eT}\, K^e\, \Lambda^e$

(5) Write out/create the assembly matrix, A^e, for each element by hand (by inspection).

(6) Create the individual element contributions to the global stiffness matrix:

$$K^e_{G\,\{n\times n\}} = A^e{}_{\{n\times 6\}}\, \widehat{K}^e_{\{6\times 6\}}\, (A^e)^T{}_{\{6\times n\}}$$

(7) Sum up these individual element contributions to get the overall global stiffness matrix:

$$K_{G\{n\times n\}} = \sum_{e=1}^{n_e} A^e{}_{\{n\times 6\}}\ \widehat{K}^e_{\{6\times 6\}}\ (A^e)^T{}_{\{6\times n\}}$$

n_e − total number of elements

n − total number of overall (structural) DOF

(8) Solve the overall structural equation $q = (K_G)^{-1}Q$ to get the solved displacement vector, q.

(9) Calculate element nodal forces in global co-ordinates for any given element so you can pick out and solve reaction loads.

$$F^e = \widehat{K}^e\ D^e = \widehat{K}^e (A^e)^T q$$

(10) Calculate element nodal displacements and forces in local co-ordinates to enable stress and strain calculations:

$$d^e = \Lambda^e\, D^e = \Lambda^e\, (A^e)^T\, q$$

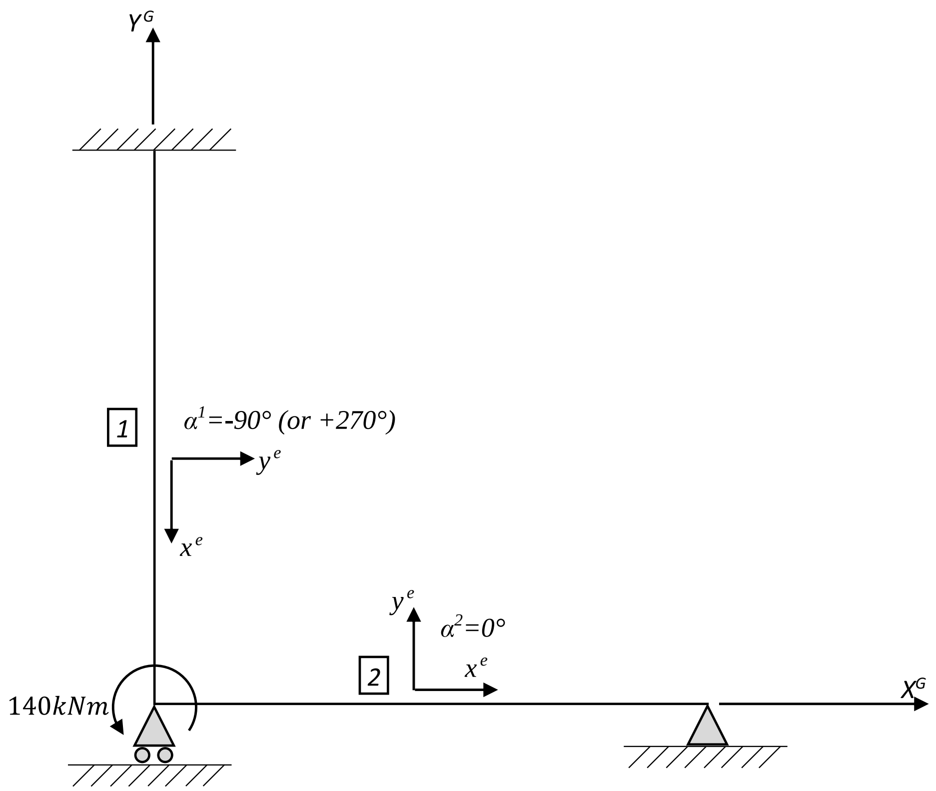
$$f^e = K^e\, d^e$$

Problem 4 – A simple FRAME element structure

A simple structure consists of two FRAME elements. The top support is a fixed/clamped support, the bottom left support is a knife-edge support on a roller (able to provide a vertical reaction force, but not any horizontal force or moment) and the support at the bottom right is a pin-support that prevents translation but not rotation.

The only external load applied to the structure is a $140kNm$ concentrated moment acting at the bottom left corner. Find the structural deflections, the reaction forces at the three supports and the element internal forces.

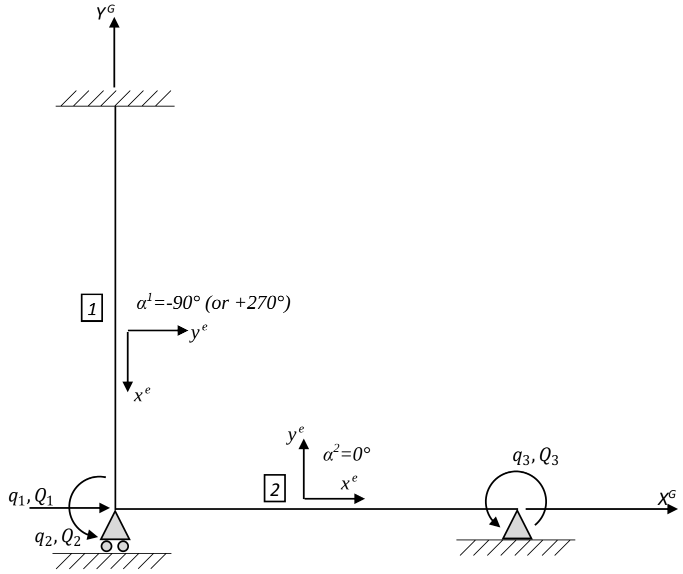
Assume $L_1 = L_2 = 10m$, and both members have $I_1 = I_2 = 5 \times 10^{-4}\ m^4$, $A_1 = A_2 = 1 \times 10^{-5}\ m^2$ and the same elastic modulus of $E = 200$ GPa.



Solution

Determine (and label) the allowable overall structural degrees of freedom:

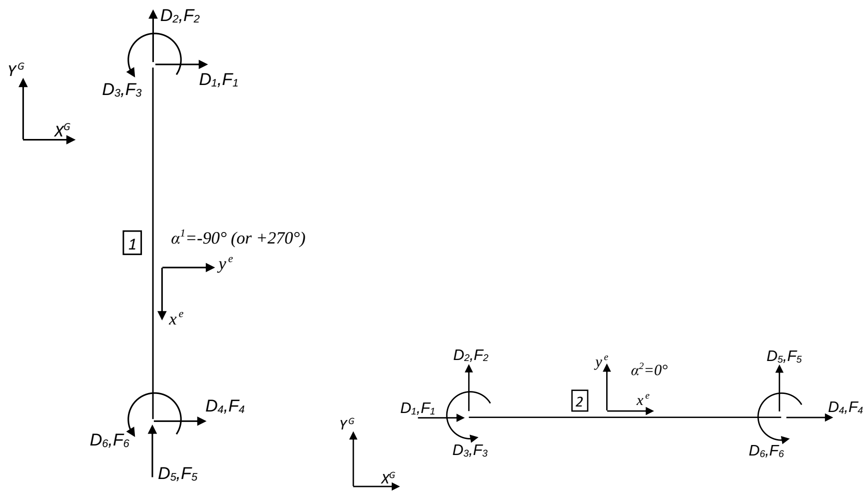
- Global structure
- Identification of DOFs, q
- Identifying external loads, Q



Element Freebody Diagrams: **Element #1** **Element #2**

Break the structure into elements

Identify element DOFs, D and angle of rotation, α



Transformation matrix

$$\Lambda^e_{\{6\times6\}} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = -90° \text{ or } +270°$$
$$c = \cos(-90°) = 0$$
$$s = \sin(-90°) = -1$$

$$\Lambda^1_{\{6\times6\}} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = 0°$$
$$c = \cos\alpha = \cos(0°) = 1$$
$$s = \sin\alpha = \sin(0°) = 0$$

$$\Lambda^2_{\{6\times6\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Element stiffness matrix in local (element) co-ordinates	$K^1_{\{6\times6\}} = K^2_{\{6\times6\}} = 1\times10^5 \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 12 & 60 & 0 & -12 & 60 \\ 0 & 60 & 400 & 0 & -60 & 200 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -12 & -60 & 0 & 12 & -60 \\ 0 & 60 & 200 & 0 & -60 & 400 \end{bmatrix}$
Element stiffness matrix in global co-ordinates	$\widehat{K}^1 = 1\times10^5 \begin{bmatrix} 12 & 0 & 60 & -12 & 0 & 60 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 60 & 0 & 400 & -60 & 0 & 200 \\ -12 & 0 & -60 & 12 & 0 & -60 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 60 & 0 & 200 & -60 & 0 & 400 \end{bmatrix}$ $\widehat{K}^2 = 1\times10^5 \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 12 & 60 & 0 & -12 & 60 \\ 0 & 60 & 400 & 0 & -60 & 200 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -12 & -60 & 0 & 12 & -60 \\ 0 & 60 & 200 & 0 & -60 & 400 \end{bmatrix}$
Element Assembly Matrices	$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$ which means, $D_4^1 = q_1$ and $D_6^1 = q_2$ $A^2 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^2 & D_2^2 & D_3^2 & D_4^2 & D_5^2 & D_6^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$ which means, $D_1^2 = q_1, D_3^2 = q_2$ and $D_6^2 = q_3$

Create the Global Stiffness Matrix (K_G) by assembling element global stiffness matrices:

$$K_G = \sum_{e=1}^{n_e} A^e \widehat{K}^e A^{e^T} = A^1 \widehat{K}^1 A^{1^T} + A^2 \widehat{K}^2 A^{2^T}$$

$$K_G = 1 \times 10^5 \begin{bmatrix} 14 & -60 & 0 \\ -60 & 800 & 200 \\ 0 & 200 & 400 \end{bmatrix}$$

Define the global forcing vector, Q, from the information given in the question:

$$Q = \begin{Bmatrix} 0 \\ 140{,}000 \\ 0 \end{Bmatrix}$$

Now the structure can be solved for displacements:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = K_G^{-1}\, Q = \begin{Bmatrix} 0.0135\, m \\ 0.0032\, rad \\ -0.0016\, rad \end{Bmatrix} = \begin{Bmatrix} 13.5484\, mm \\ 3.1613\, mrad \\ -1.5806\, mrad \end{Bmatrix}$$

Now that the system is solved, we can extract the relevant 'pieces' to determine the element deflections:

$$D^e = \left(A^e\right)^T q$$

$$D^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_1 \\ 0 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13.5484mm \\ 0 \\ 3.1613mrad \end{Bmatrix} \qquad D^2 = \begin{Bmatrix} q_1 \\ 0 \\ q_2 \\ 0 \\ 0 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 13.5484mm \\ 0 \\ 3.1613mrad \\ 0 \\ 0 \\ -1.5806mrad \end{Bmatrix}$$

These element deflection vectors (in global co-ordinates) can now be transformed into element co-ordinates by the transformation matrix:

$$d^e = \Lambda^e D^e$$

$$d^1 = \Lambda^1 D^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 13.5484mm \\ 3.1613mrad \end{Bmatrix} \qquad d^2 = \Lambda^2 D^2 = \begin{Bmatrix} q_1 \\ 0 \\ q_2 \\ 0 \\ 0 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 13.5484mm \\ 0 \\ 3.1613mrad \\ 0 \\ 0 \\ -1.5806mrad \end{Bmatrix}$$

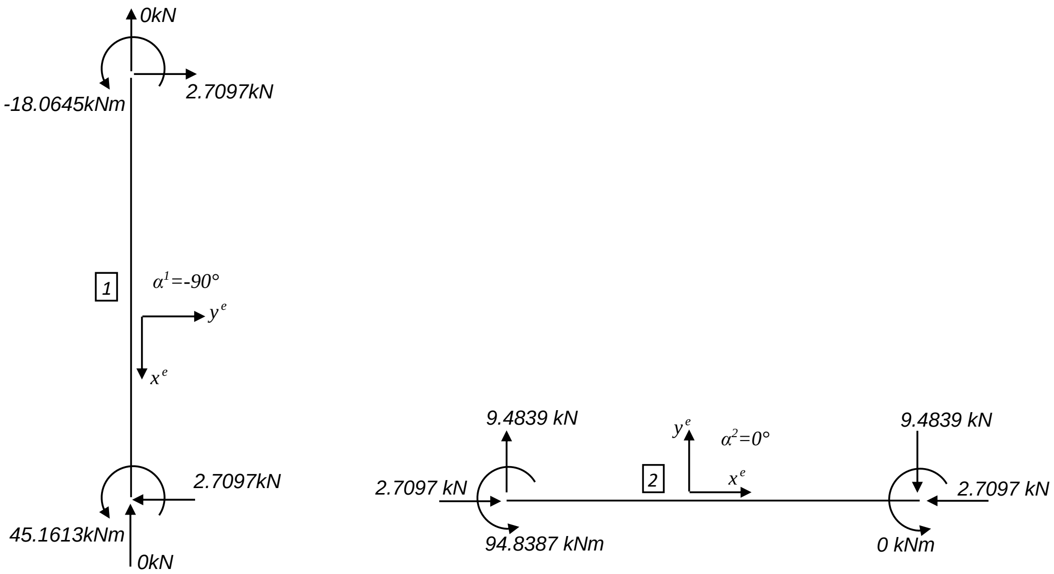
The element forcing vectors can now be calculated by $f^e = K^e\, d^e$

$$f^1 = K^1 d^1 = \begin{Bmatrix} 0\,kN \\ 2.7097\,kN \\ -18.0645\,kNm \\ 0\,kN \\ -2.7097\,kN \\ 45.1613\,kNm \end{Bmatrix} \qquad f^2 = K^2 d^2 = \begin{Bmatrix} 2.7097\,kN \\ 9.4839\,kN \\ 94.8387\,kNm \\ -2.7097\,kN \\ -9.4839\,kN \\ 0\,kNm \end{Bmatrix}$$

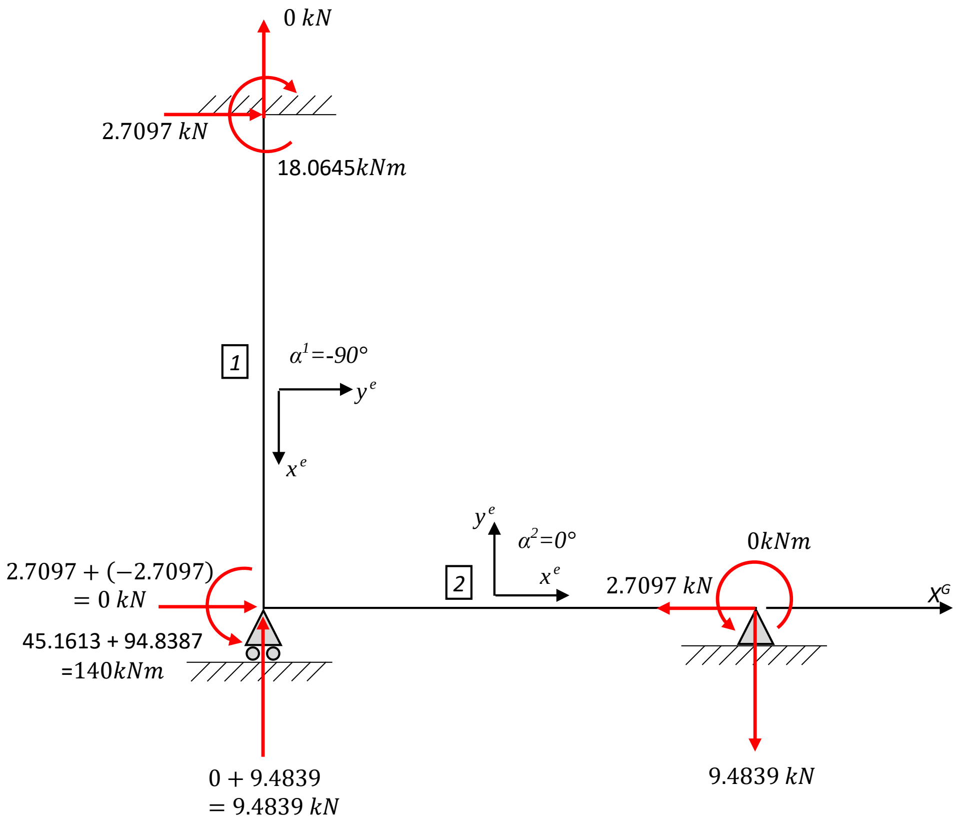
Reaction loads: When adding element forcing terms to calculate reaction loads, it is best to use element force vectors global co-ordinates (F^e), which can be calculated two ways: $F^e = \widehat{K}^e D^e$ or $F^e = (\Lambda^e)^T f^e$:

$$F^1 = \left(\Lambda^1\right)^T f^1 = \left(\Lambda^1\right)^T \left(K^1 d^1\right) = \begin{Bmatrix} 2.7097\,kN \\ 0\,kN \\ -18.0645\,kNm \\ -2.7097\,kN \\ 0\,kN \\ 45.1613\,kNm \end{Bmatrix} \qquad F^2 = \left(\Lambda^2\right)^T f^2 = \left(\Lambda^2\right)^T \left(K^2 d^2\right) = \begin{Bmatrix} 2.7097\,kN \\ 9.4839\,kN \\ 94.8387\,kNm \\ -2.7097\,kN \\ -9.4839\,kN \\ 0\,kNm \end{Bmatrix}$$

Using the original element free-body diagrams, we can plot these forces onto an element diagram:

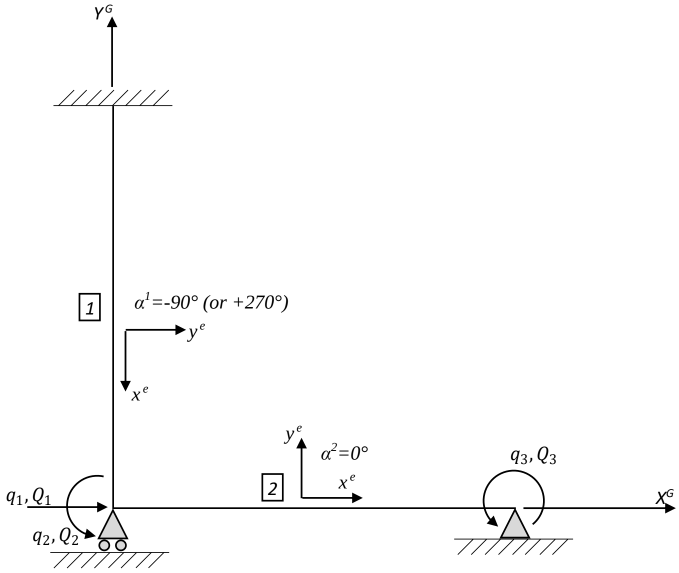


Plot the combined element reaction forces onto a schematic diagram of the overall structure:



Problem 4 – In Python:

Given $L_1 = L_2 = 10m$, and both members have $I_1 = I_2 = 5 \times 10^{-4}\, m^4$, $A_1 = A_2 = 1 \times 10^{-5}\, m^2$ and the same elastic modulus of $E = 200$ GPa.



Element 1	Element 2
$K1 = local\_frame(E, I, A, L) =$ $K^1_{\{6\times6\}} = 1\times10^5 \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 12 & 60 & 0 & -12 & 60 \\ 0 & 60 & 400 & 0 & -60 & 200 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -12 & -60 & 0 & 12 & -60 \\ 0 & 60 & 200 & 0 & -60 & 400 \end{bmatrix}$	$K2 = local\_frame(E, I, A, L) =$ $K^2_{\{6\times6\}} = 1\times10^5 \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 12 & 60 & 0 & -12 & 60 \\ 0 & 60 & 400 & 0 & -60 & 200 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -12 & -60 & 0 & 12 & -60 \\ 0 & 60 & 200 & 0 & -60 & 400 \end{bmatrix}$
$K1hat, Lambda1 = global\_frame(K1, -90)$ $\widehat{K}^1 = 1\times10^5 \begin{bmatrix} 12 & 0 & 60 & -12 & 0 & 60 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 60 & 0 & 400 & -60 & 0 & 200 \\ -12 & 0 & -60 & 12 & 0 & -60 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 60 & 0 & 200 & -60 & 0 & 400 \end{bmatrix}$ $\Lambda^1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$K2hat, Lambda2 = global\_frame(K2,0)$ $\widehat{K}^2 = 1\times10^5 \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 12 & 60 & 0 & -12 & 60 \\ 0 & 60 & 400 & 0 & -60 & 200 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -12 & -60 & 0 & 12 & -60 \\ 0 & 60 & 200 & 0 & -60 & 400 \end{bmatrix}$ $\Lambda^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
Assembly Matrix (generated manually): $A^1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Assembly Matrix (generated manually): $A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$K_G^1 = A^1 \hat{K}^1 (A^1)^T =$ $$K_G^1 = 1 \times 10^5 \begin{bmatrix} 12 & -60 & 0 \\ -60 & 400 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$	$K_G^2 = A^2 \hat{K}^2 (A^2)^T =$ $$K_G^2 = 1 \times 10^5 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 400 & 200 \\ 0 & 200 & 400 \end{bmatrix}$$

$$K_G = K_G^1 + K_G^2$$

$$K_G = 1 \times 10^5 \begin{bmatrix} 14 & -60 & 0 \\ -60 & 800 & 200 \\ 0 & 200 & 400 \end{bmatrix}$$

Define the global forcing vector, Q, from the information given in the question:

$$Q = \begin{Bmatrix} 0 \\ 140{,}000 \\ 0 \end{Bmatrix}$$

Solve for the overall structural deflections:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = K_G^{-1}\, Q = \begin{Bmatrix} 0.0135\, m \\ 0.0032\, rad \\ -0.0016\, rad \end{Bmatrix} = \begin{Bmatrix} 13.5484\, mm \\ 3.1613\, mrad \\ -1.5806\, mrad \end{Bmatrix}$$

Extract the components of the structural deflections for each element, using $D^e = \left(A^e\right)^T q$ to get the element nodal deflection vectors (in global co-ordinates):

$$D^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_1 \\ 0 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13.5484mm \\ 0 \\ 3.1613mrad \end{Bmatrix} \qquad D^2 = \begin{Bmatrix} q_1 \\ 0 \\ q_2 \\ 0 \\ 0 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 13.5484mm \\ 0 \\ 3.1613mrad \\ 0 \\ 0 \\ -1.5806mrad \end{Bmatrix}$$

The element deflection vectors (in global co-ordinates) above can now be transformed into the element deflection vectors (in element co-ordinates) by the transformation matrix by: $d^e = \Lambda^e\, D^e$

$$d^1 = \Lambda^1 D^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 13.5484mm \\ 3.1613mrad \end{Bmatrix} \qquad d^2 = \Lambda^2 D^2 = \begin{Bmatrix} q_1 \\ 0 \\ q_2 \\ 0 \\ 0 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 13.5484mm \\ 0 \\ 3.1613mrad \\ 0 \\ 0 \\ -1.5806mrad \end{Bmatrix}$$

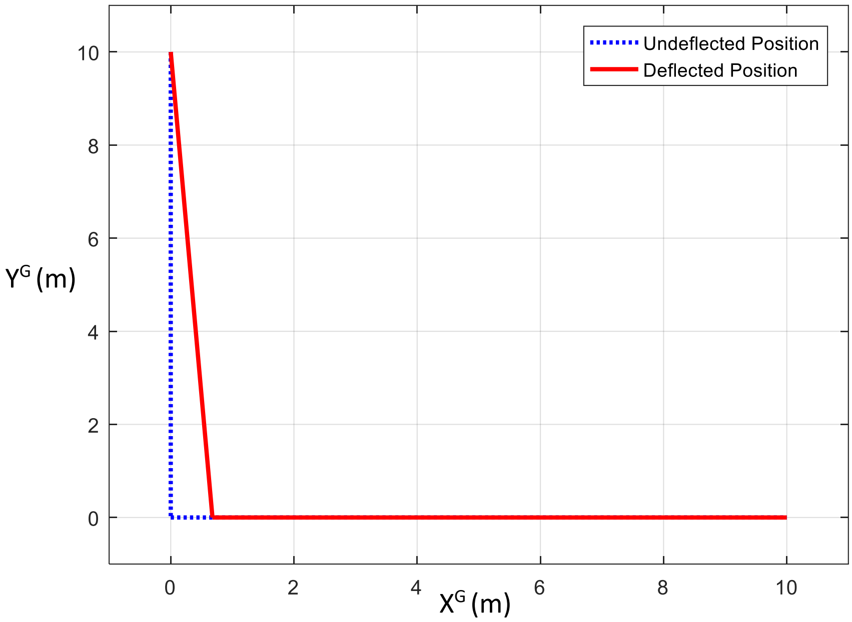
The element nodal forcing vectors (in element co-ordinates) can now be calculated by $f^e = K^e d^e$

$$f^1 = K^1 d^1 = \begin{Bmatrix} 0\,kN \\ 2.7097\,kN \\ -18.0645\,kN.m \\ 0\,kN \\ -2.7097\,kN \\ 45.1613\,kNm \end{Bmatrix} \qquad f^2 = K^2 d^2 = \begin{Bmatrix} 2.7097\,kN \\ 9.4839\,kN \\ 94.8387\,kNm \\ -2.7097\,kN \\ -9.4839\,kN \\ 0\,kNm \end{Bmatrix}$$

The element nodal forcing vectors (in global co-ordinates) can be calculated using either of these two methods: $F^e = \widehat{K}^e D^e$ or $F^e = (\Lambda^e)^T f^e$:

$$F^1 = \begin{Bmatrix} 2.7097\,kN \\ 0\,kN \\ -18.0645\,kNm \\ -2.7097\,kN \\ 0\,kN \\ 45.1613\,kNm \end{Bmatrix} \qquad F^2 = \begin{Bmatrix} 2.7097\,kN \\ 9.4839\,kN \\ 94.8387\,kNm \\ -2.7097\,kN \\ -9.4839\,kN \\ 0\,kNm \end{Bmatrix}$$

Deflected shape, using a displacement amplification of 50 (deflections 50x larger than reality):



An important note about element connections

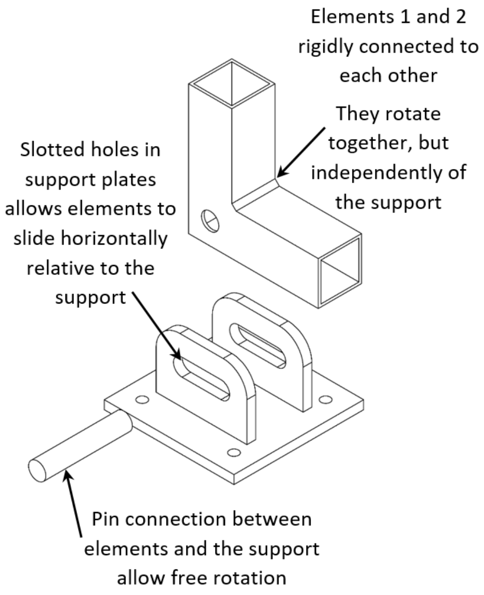
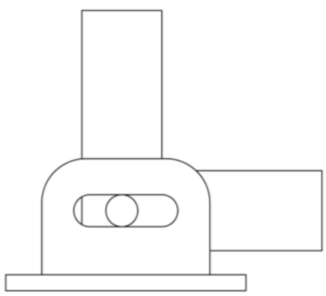
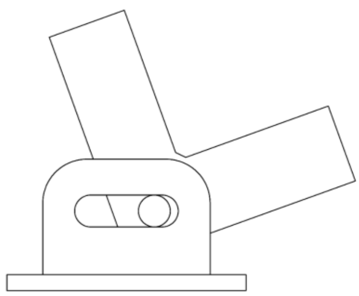
Frame elements are always assumed to be rigidly connected to one another. Any time two frame elements meet, the frame element formulation we are using assumes that these two elements are rigidly connection, such that the end of the two elements undergo the same horizontal and vertical translations, as well as the same rotation. In the example we have just done, the bottom left hand corner is shown as a roller support that does not prevent translation or rotation. We have assigned an allowable global degree of freedom at this node that corresponds to horizontal translation (we called this q_1) along with an allowable global rotation at this node (we called this q_2).

However, it is important to note that in our assembly matrices, we linked the horizontal (X^G) degree of freedom at the bottom of element 1 (D_4^1) and the horizontal (X^G) degree of freedom at the left edge of element 2 (D_1^2) to the same global degree of freedom (q_1). When we solve our system, there will be only one value within the final global deflection vector (q) that represents horizontal translation at this node, and the two elements are mathematically constrained to undergo the exact same horizontal deflection.

The same is true for rotations. In our assembly matrices, we linked the rotation (Z^G) degree of freedom at the bottom of element 1 (D_6^1) and the rotation (Z^G) degree of freedom at the left edge of element 2 (D_3^2) to the same global degree of freedom (q_2). Again, there will be only one value within the final global deflection vector (q) that represents rotation at this node, and the two elements are mathematically constrained to undergo the exact same rotation.

This means that the two elements translate and rotate together. There is no allowable relative translation or allowable relative rotation between the elements, even though there is an allowable translation and rotation between the elements and the supports.

Physically, this might look something like this (among several other potential options):

<table>
<tr><th>Exploded view of the connection</th><th>Un-displaced side elevation view</th><th>Displaced side elevation view</th></tr>
<tr><td>
</td><td></td><td></td></tr>
</table>

How to plot the full deflected shape of a frame element?

We've said all along that the purpose of shape functions is to be able to model a continuous element by only solving for the deflections at the nodal points. Therefore, we can use these solved nodal deflections, and the shape functions that we used within our derivations, to plot the full deflected shape of a frame element. We will now go through each step required to plot this deflected shape.

Step 1: We will generate a function that plots the deflected shape of a generic frame element. We will call this function the same number of times as we have frame elements (for Problem 4, this is two).

Suggested function format, assuming that the code has been initially set up with:

```
import numpy as np
import matplotlib.pyplot as plt
```

```
plot_deflected_shape(node1XG,node1YG,node2XG,node2YG,d_e,Disp_mag,N_points):
```

Suggested Inputs:

```
node1XG - the X-global (XG) location of node 1 for the current element
    (relative to whatever origin location that you have chosen)
node1YG - the Y-global (YG) location of node 1 for the current element
    (relative to whatever origin location that you have chosen)
node2XG - the X-global (XG) location of node 2 for the current element
    (relative to whatever origin location that you have chosen)
node2YG - the Y-global (YG) location of node 2 for the current element
    (relative to whatever origin location that you have chosen)
d_e - the solved 6x1 deflection vector for this element, in element co-ordinates
Disp_mag - a chosen displacement magnification factor to exaggerate the
      deflected shape
N_points - the number of points within the element at which we will evaluate the
      deflected shape. This number needs to be enough (minimum 10-20) if we want
      to obtain a smooth curve of the deflected shape.
```

You could also add inputs for element length, L, and element orientation angle, α, or these values can be calculated within the code from the X_G and Y_G co-ordinates of the two ends.

Step 2: We need to generate a vector of points within the element co-ordinate system (x^e). These are the points at which we will evaluate the transverse and axial deflections and eventually plot.

Suggested code: `x_e = np.linspace(0,L,N_points)`

Step 3: Using the newly defined x_e variable, evaluate the axial and shear deflection shape functions at each of these x_e points. Note that all shape functions will now be vectors, where each element of these vectors represents the shape function evaluated at "N_points" points within the elements.

Axial Shape Functions:	Transverse Shape Functions:
$\psi_1(x) = \left(1-\frac{x}{L}\right);\ \ \psi_2(x) = \frac{x}{L}$	$N_1(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$ $N_2(x) = \frac{x^3}{L^2} - \frac{2x^2}{L} + x$ $N_3(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$ $N_4(x) = \frac{x^3}{L^2} - \frac{x^2}{L}$

Step 4: Calculate the axial, $u(x)$, and transverse, $v(x)$, displacements (at all points along the element) using the shape functions, the vector of x values, and the element deflection vector, d^e (frame element numbering).

Element Axial Displacement, $u(x) = \psi_1(x)d_1 + \psi_2(x)d_4$

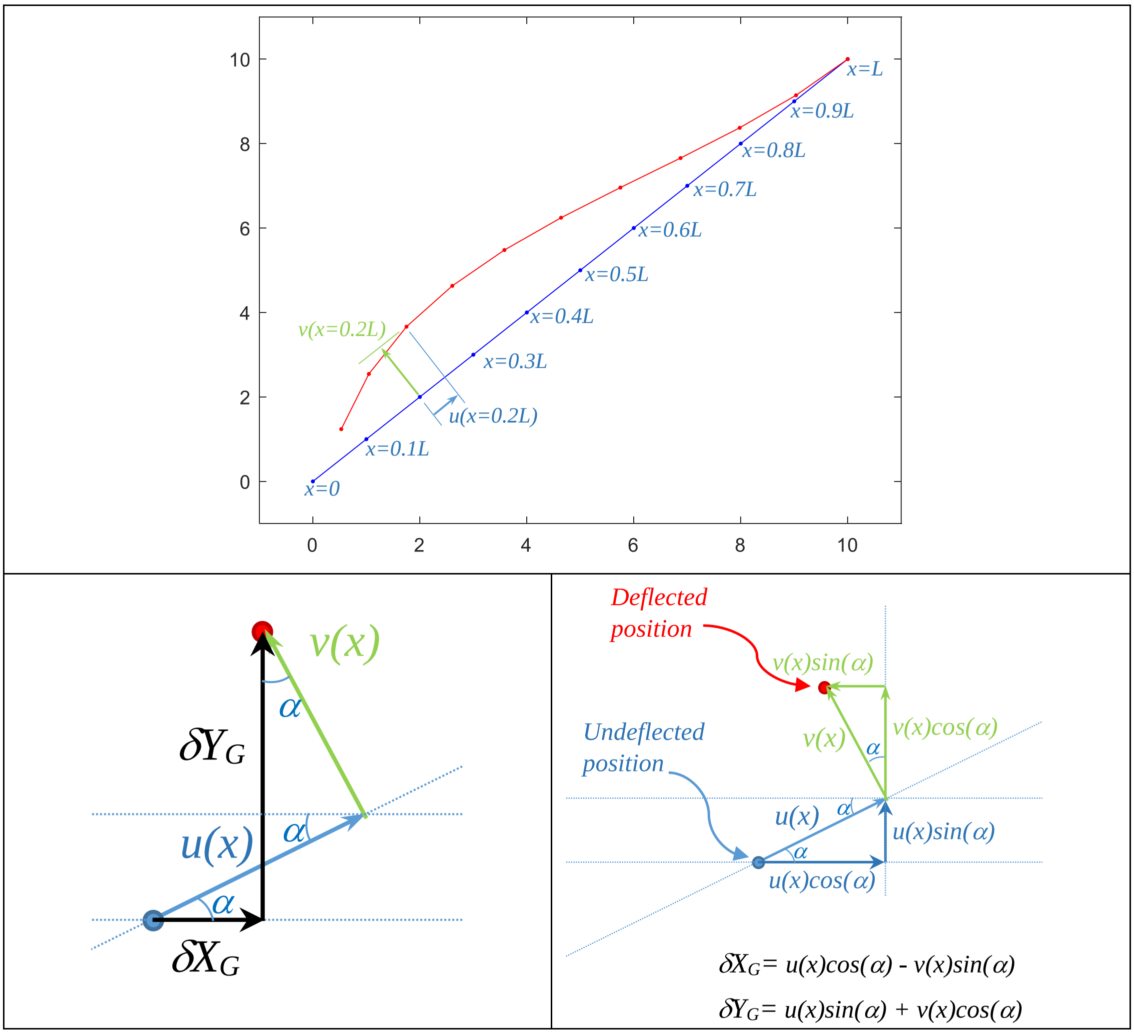
Element Transverse Displacement, $v(x) = N\, d^e = \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix}$

$$v(x) = N_1(x)d_2 + N_2(x)d_3 + N_3(x)d_5 + N_4(x)d_6$$

Step 5: Transform these displacements from element co-ordinates into global co-ordinates, using this subset of the transpose of the frame element transformation matrix.

Deflections_XG = $u(x)\cos(\alpha) - v(x)\sin(\alpha)$
Deflections_YG = $u(x)\sin(\alpha) + v(x)\cos(\alpha)$

These variables represent how much each of the points within the element has deflected in the X_G and Y_G directions. If you're wondering where these equations came from, consider an element inclined at angle α (in this case $\alpha \approx 45°$) and the internal deflection evaluated at 11 points (N_points = 11), which will break the element up into 10 segments. The *u(x)* and *v(x)* values evaluated in Step 4 represent the axial and transverse deflections of each internal point within the element, in element co-ordinates.



Step 6: Generate a baseline vector of points in X_G and Y_G that represent the un-deflected baseline of the element.

Suggested code:

```
Undeflected_baseline_XG = np.linspace(node1XG,node2XG,N_points)
Undeflected_baseline_YG = np.linspace(node1YG,node2YG,N_points)
```

Step 7: Calculate a vector of points in X_G and Y_G that represent the deflected position of all points within the element.

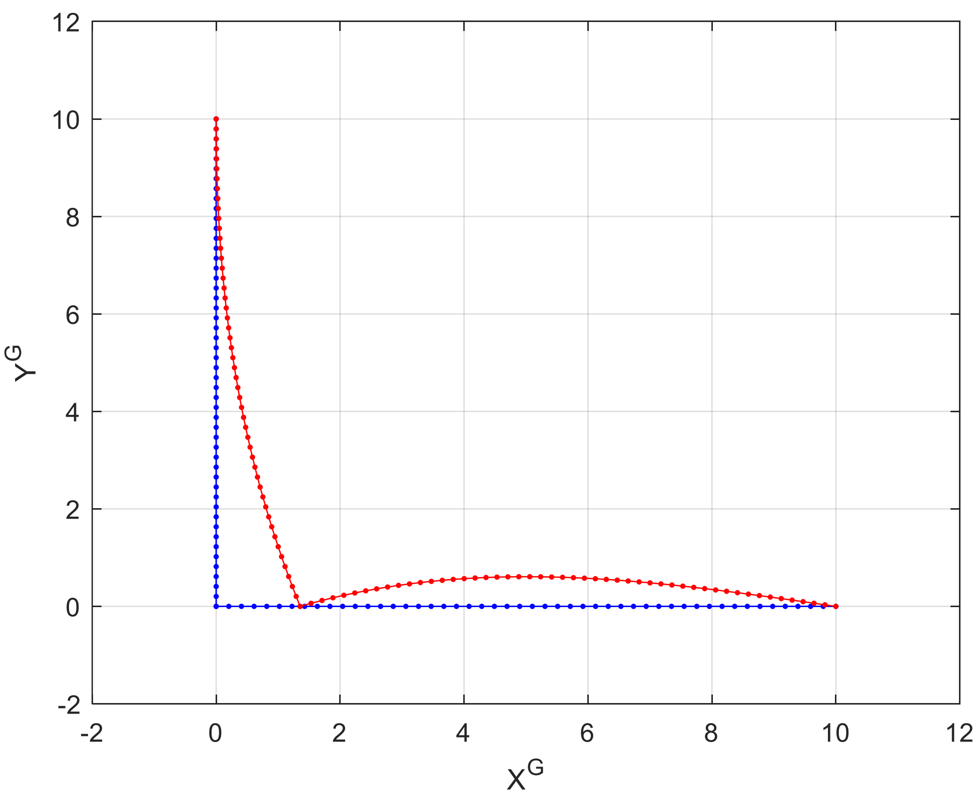
Suggested code:

```
Deflected_XG = Undeflected_baseline_XG + Disp_multiplier*Deflections_XG
Deflected_YG = Undeflected_baseline_YG + Disp_multiplier*Deflections_YG
```

Step 8: Plot the result (using your define prefix, such as plt.plot(….).

```
plt.plot(Undeflected_baseline_XG,Undeflected_baseline_YG,'b.-')
plt.plot(Deflected_XG,Deflected_YG,'r.-')
```

Using this approach the plot of the deflected position of the structure in Problem 4 now looks like (using a displacement multiplier of 100):

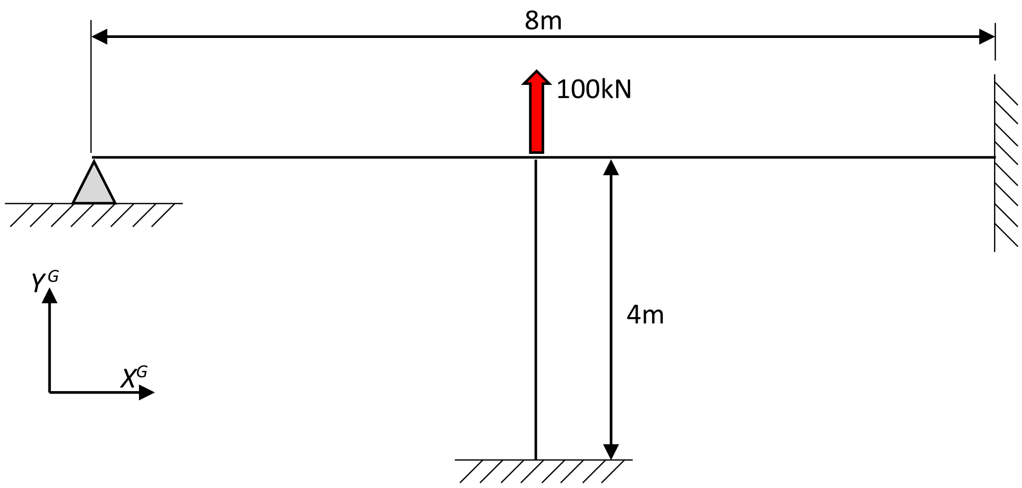


Problem 5 – Another (relatively) simple FRAME element structure

A simple structure consists of three FRAME elements. The right and bottom supports are fixed/clamped, and the left support is a knife-edge support that is able to prevent translation, but cannot restrain rotation or provide any moment reaction.

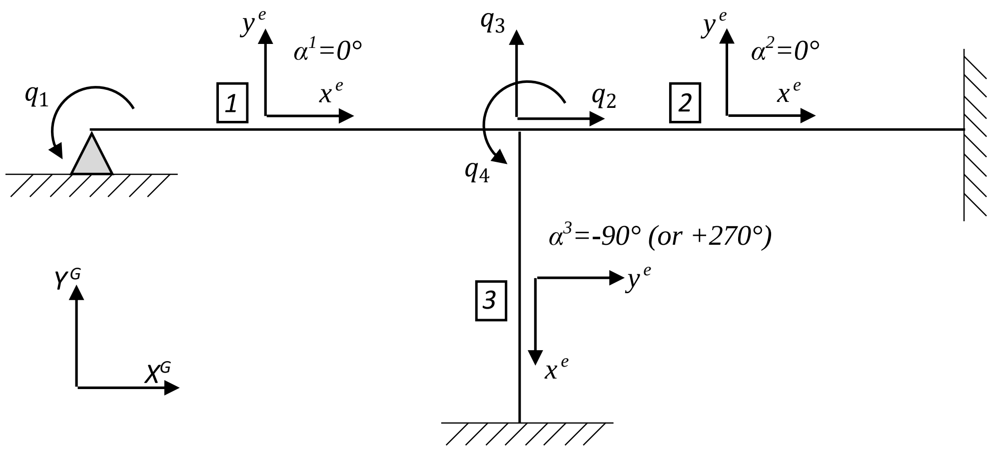
The only external load applied to the structure is a $100kN$ concentrated load acting upwards at the middle of the top of the structure. Find the structural deflections, the reaction forces at the three supports and the element internal forces.

Assume that all members have $A = 1 \times 10^{-5}\ m^2$, $I = 5 \times 10^{-4}\ m^4$ and an elastic modulus of $E =$ 200 GPa.

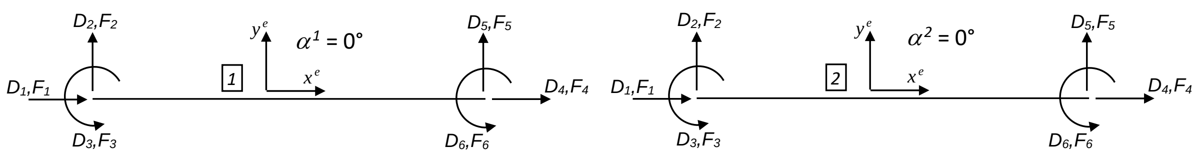


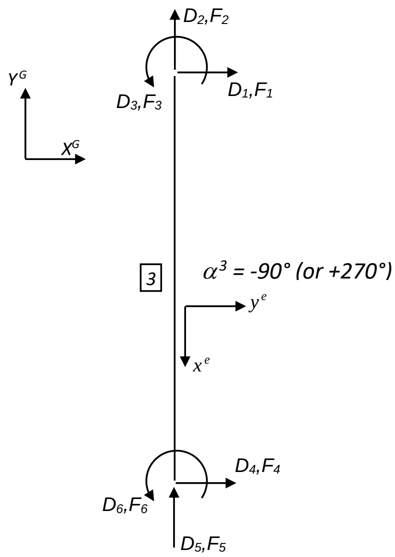
Solution

Determine (and label) the allowable overall structural degrees of freedom:



Element Freebody Diagrams





	$\alpha = 0°$	$\alpha = -90°$ or $+270°$
Transformation matrix	$\Lambda^1 = \Lambda^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$\Lambda^3 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Element stiffness matrix in local (element) co-ordinates	$$K^1 = K^2 = K^3 = 1\times10^3 \begin{bmatrix} 500 & 0 & 0 & -500 & 0 & 0 \\ 0 & 18750 & 37500 & 0 & -18750 & 37500 \\ 0 & 37500 & 100000 & 0 & -37500 & 50000 \\ -500 & 0 & 0 & 500 & 0 & 0 \\ 0 & -18750 & -37500 & 0 & 18750 & -37500 \\ 0 & 37500 & 50000 & 0 & -37500 & 100000 \end{bmatrix}$$
Element stiffness matrix in global co-ordinates	$$\hat{K}^1 = \hat{K}^2 = 1\times10^3 \begin{bmatrix} 500 & 0 & 0 & -500 & 0 & 0 \\ 0 & 18750 & 37500 & 0 & -18750 & 37500 \\ 0 & 37500 & 100000 & 0 & -37500 & 50000 \\ -500 & 0 & 0 & 500 & 0 & 0 \\ 0 & -18750 & -37500 & 0 & 18750 & -37500 \\ 0 & 37500 & 50000 & 0 & -37500 & 100000 \end{bmatrix}$$ $$\hat{K}^3 = 1\times10^3 \begin{bmatrix} 18750 & 0 & 37500 & -18750 & 0 & 37500 \\ 0 & 500 & 0 & 0 & -500 & 0 \\ 37500 & 0 & 100000 & -37500 & 0 & 50000 \\ -18750 & 0 & -37500 & 18750 & 0 & -37500 \\ 0 & -500 & 0 & 0 & 500 & 0 \\ 37500 & 0 & 50000 & -37500 & 0 & 100000 \end{bmatrix}$$
Element Assembly Matrices	$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad A^2 = A^3 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Create the Global Stiffness Matrix (K_G) by assembling element global stiffness matrices:

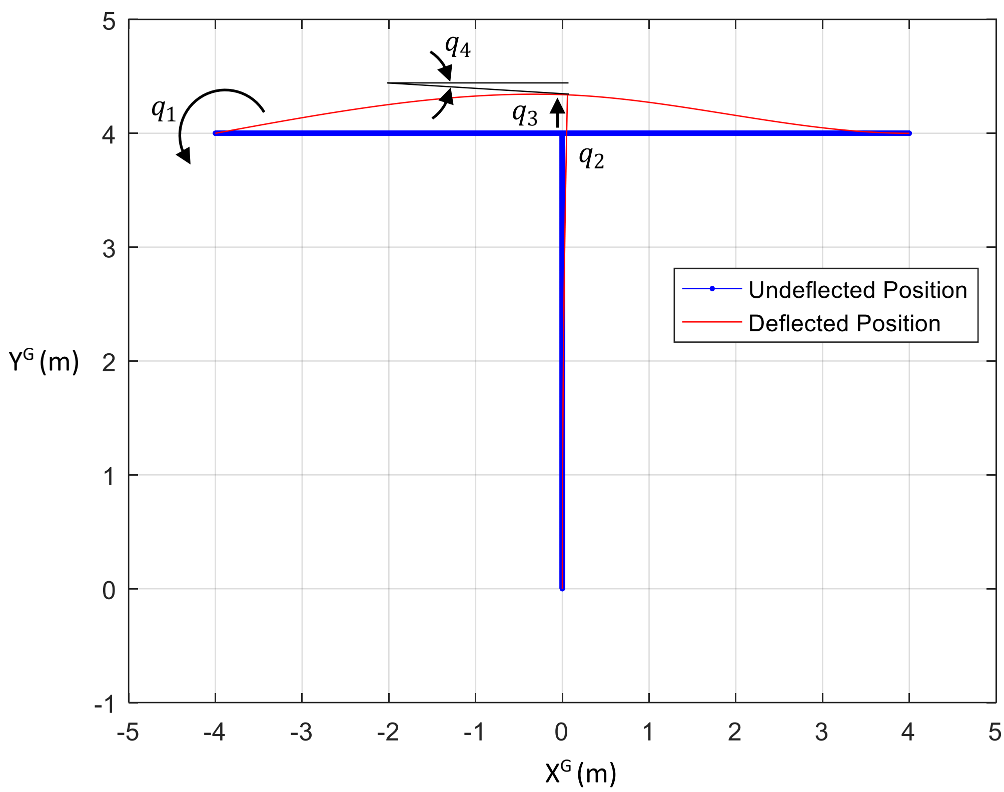
$$K_G = 1\times10^7 \begin{bmatrix} 10 & 0 & -3.75 & 5 \\ 0 & 1.975 & 0 & 3.75 \\ -3.75 & 0 & 3.8 & 0 \\ 5 & 3.75 & 0 & 30 \end{bmatrix}$$

Define the global forcing vector, Q, from the information given in the question:

$$Q = \begin{Bmatrix} 0 \\ 0 \\ 100,000N \\ 0 \end{Bmatrix}$$

Now the structure can be solved for displacements:

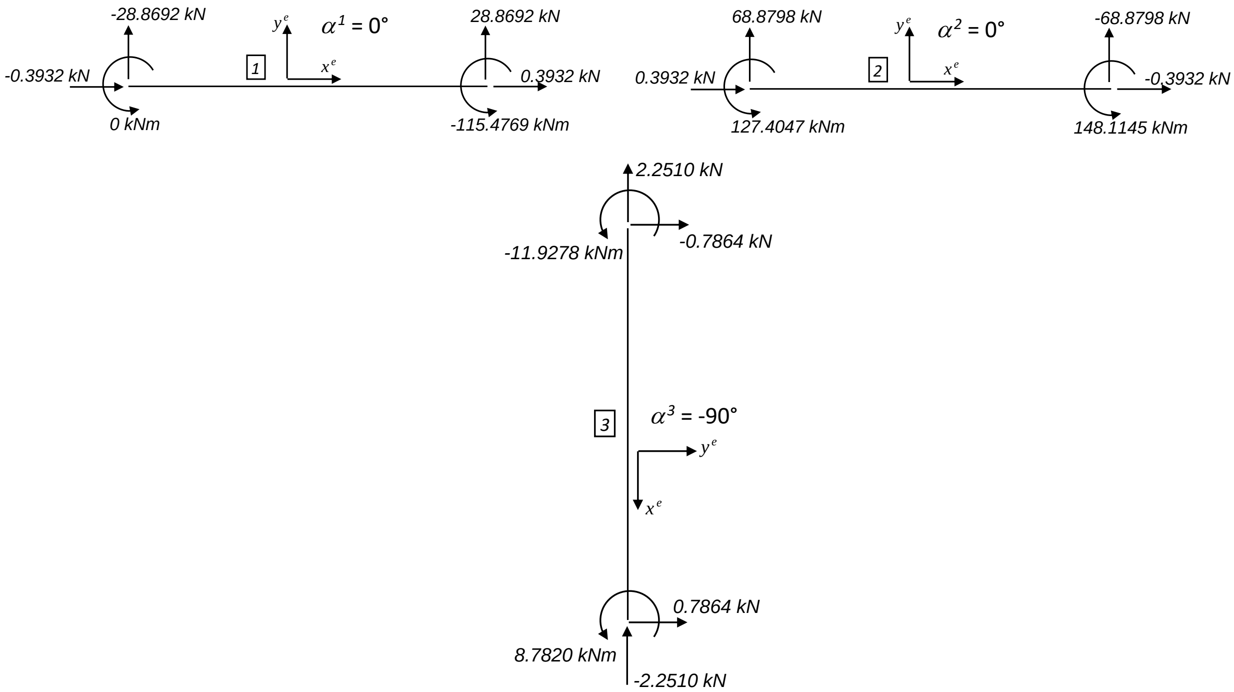
$$q = \begin{Bmatrix} 0.0019 \\ 0.0008 \\ 0.0045 \\ -0.0004 \end{Bmatrix} = \begin{Bmatrix} 1.8953\,mrad \\ 0.7864\,mm \\ 4.5020\,mm \\ -0.4142\,mrad \end{Bmatrix}$$



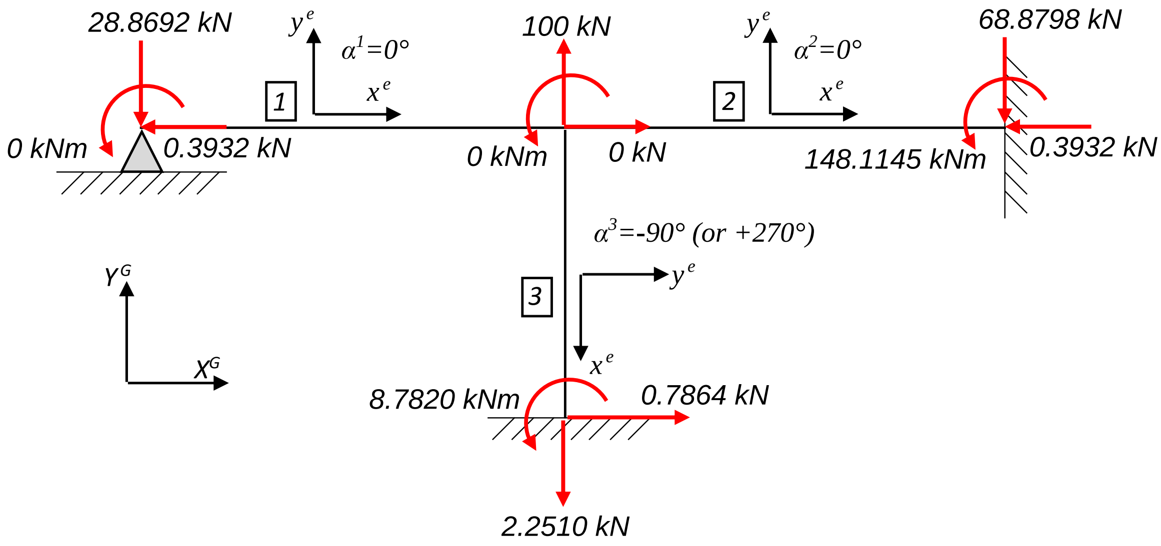
Reaction loads: When adding element forcing terms to calculate reaction loads, it is best to use element force vectors global co-ordinates (F^e), which can be calculated two ways: $F^e = \widehat{K}^e D^e$ or $F^e = (\Lambda^e)^T f^e$:

$$F^1 = \begin{Bmatrix} -0.3932\,kN \\ -28.8692\,kN \\ 0\,kNm \\ 0.3932\,kN \\ 28.8692\,kN \\ -115.4769\,kNm \end{Bmatrix} \qquad F^2 = \begin{Bmatrix} 0.3932\,kN \\ 68.8798\,kN \\ 127.4047\,kNm \\ -0.3932\,kN \\ -68.8798\,kN \\ 148.1145\,kNm \end{Bmatrix} \qquad F^3 = \begin{Bmatrix} -0.7864\,kN \\ 2.2510\,kN \\ -11.9278\,kNm \\ 0.7864\,kN \\ -2.2510\,kN \\ 8.7820\,kNm \end{Bmatrix}$$

Using the original element free-body diagrams, we can plot these forces onto an element diagram:



Plot the combined element reaction forces onto a schematic diagram of the overall structure:



Summary of Different Stiffness Matrices

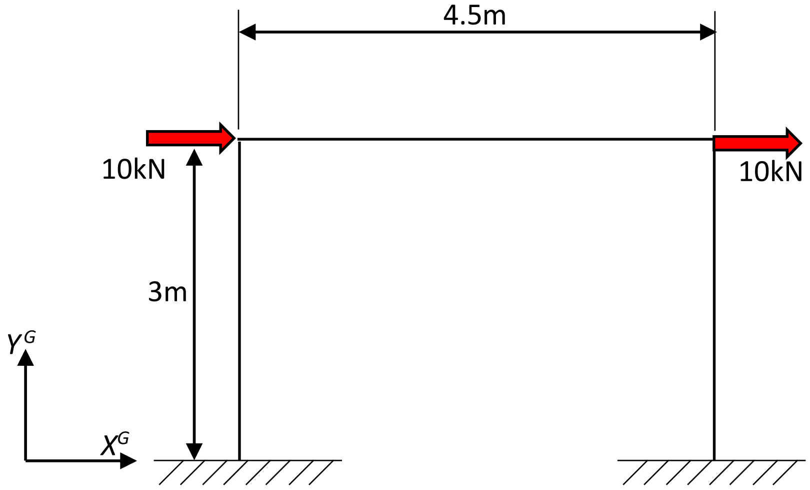
There are several levels of stiffness matrix, which can potentially be confusing. This table is included to help you understand the significance of the different forms of stiffness matrix and how additional information about the structure is introduced as we go through the process to build up the overall global stiffness matrix, K_G, that defines the overall structure.

Different forms of Stiffness Matrices	Information on Element Material and Geometric Properties	Information on Element Orientation	Information on how the element is connected to other elements and/or to support points	Information on more than one element
K^e - Element stiffness matrix in local (element) co-ordinates	YES ✓	NO ×	NO ×	NO ×
$\widehat{K}^e = {\Lambda^e}^T K^e \Lambda^e$ Element stiffness matrix in global co-ordinates	YES ✓	YES ✓ introduced through the transformation matrix	NO ×	NO ×
$K_G^e = A^e \widehat{K}^e (A^e)^T$ This element's contribution to the global stiffness matrix	YES ✓	YES ✓	YES ✓ Introduced through the assembly matrix	NO ×
$K_G = \sum_{e=1}^{n_e} K_G^e$ The overall global stiffness matrix. Captures all the stiffness terms of the total structure	YES ✓	YES ✓	YES ✓	YES ✓ Information on every element and every support point.

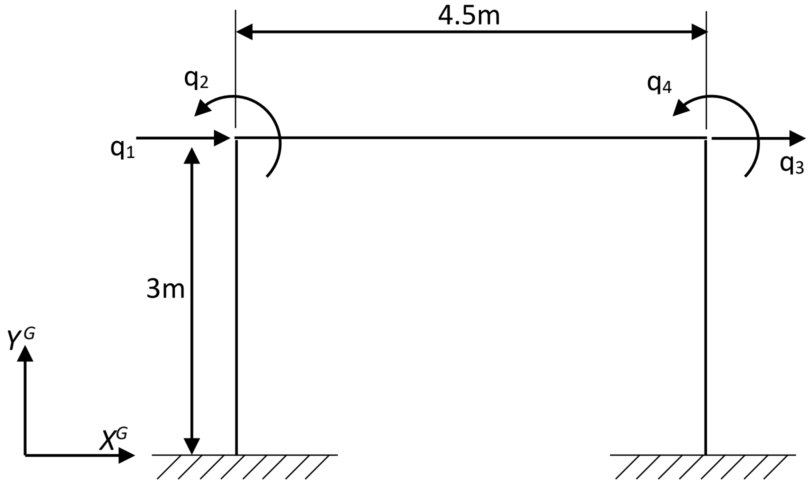
Problem 6 – A 'portal frame' structure representing a simple building.

A simple structure consists of three FRAME elements. The two supports are fixed/clamped. The only external loads applied to the structure are two $10kN$ concentrated loads acting horizontally to the right at the top of the structure (representing an earthquake or wind load). Find the structural deflections and the reaction forces.

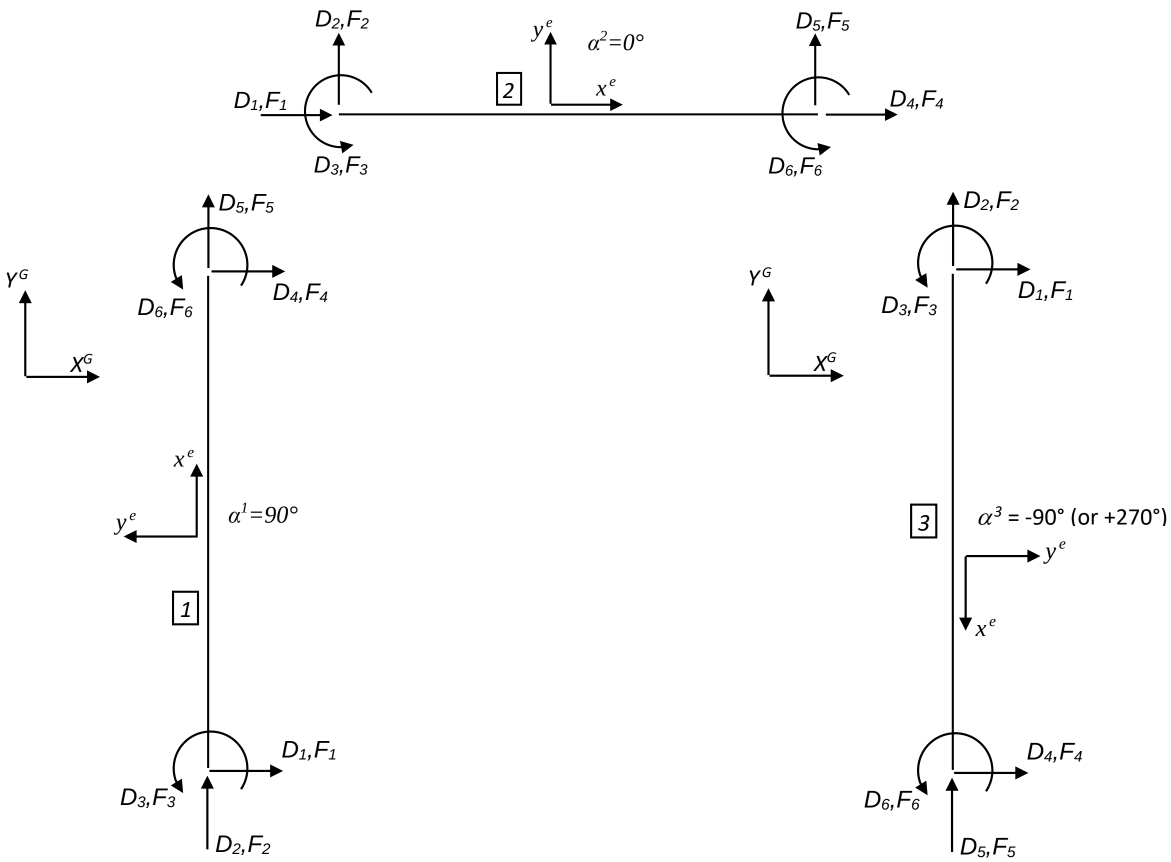
Assume that all members have $I = 1 \times 10^{-5}\ m^4$, $A = 5 \times 10^{-4}\ m^2$ and an elastic modulus of $E =$ 200 GPa.



Since there is no vertical loading applied, let's assume that the columns are rigid axially and we can ignore the vertical degrees of freedom, giving 4 overall structural DOFs. That shouldn't matter, should it? Let's take a look.



Element Free-body Diagrams:



q_1 aligns with D_4^1 and D_1^2

q_2 aligns with D_6^1 and D_3^2

q_3 aligns with D_4^2 and D_1^3

q_4 aligns with D_6^2 and D_3^3

$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad A^2 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^2 & D_2^2 & D_3^2 & D_4^2 & D_5^2 & D_6^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad A^3 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^3 & D_2^3 & D_3^3 & D_4^3 & D_5^3 & D_6^3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Create the overall global stiffness matrix.

$$K_G^{\,1} = 1\times10^6 \begin{bmatrix} 0.8889 & 1.3333 & 0 & 0 \\ 1.3333 & 2.6667 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_G^{\,2} = 1\times10^7 \begin{bmatrix} 2.2222 & 0 & -2.2222 & 0 \\ 0 & 0.1778 & 0 & 0.0889 \\ -2.2222 & 0 & 2.2222 & 0 \\ 0 & 0.0889 & 0 & 0.1778 \end{bmatrix}$$

$$K_G^{\,3} = 1\times10^6 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8889 & 1.3333 \\ 0 & 0 & 1.3333 & 2.6667 \end{bmatrix}$$

$$K_G = 1\times10^7 \begin{bmatrix} 2.3111 & 0.1333 & -2.2222 & 0 \\ 0.1333 & 0.4444 & 0 & 0.0889 \\ -2.2222 & 0 & 2.3111 & 0.1333 \\ 0 & 0.0889 & 0.1333 & 0.4444 \end{bmatrix}$$

Applied loads:

$$Q = \begin{Bmatrix} 10,000N \\ 0Nm \\ 10,000N \\ 0Nm \end{Bmatrix}$$

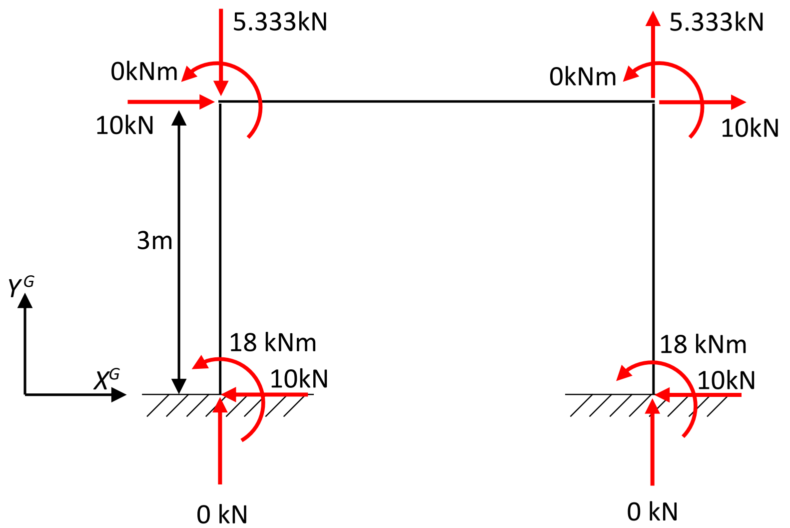
Solve for structural displacements:

$$q = \begin{Bmatrix} 0.0180 \\ -0.0045 \\ 0.0180 \\ -0.0045 \end{Bmatrix} = \begin{Bmatrix} 18.0\,mm \\ -4.5\,mrad \\ 18.0\,mm \\ -4.5\,mrad \end{Bmatrix}$$

Element forcing vectors:

$$F^1 = \begin{Bmatrix} -10\,kN \\ 0\,kN \\ 18\,kN.m \\ 10\,kN \\ 0\,kN \\ 12\,kN.m \end{Bmatrix} \qquad F^2 = \begin{Bmatrix} 0\,kN \\ -5.3333\,kN \\ -12\,kN.m \\ 0\,kN \\ 5.3333\,kN \\ -12\,kN.m \end{Bmatrix} \qquad F^3 = \begin{Bmatrix} 10\,kN \\ 0\,kN \\ 12\,kN.m \\ -10\,kN \\ 0\,kN \\ 18\,kN.m \end{Bmatrix}$$

Reaction loads:



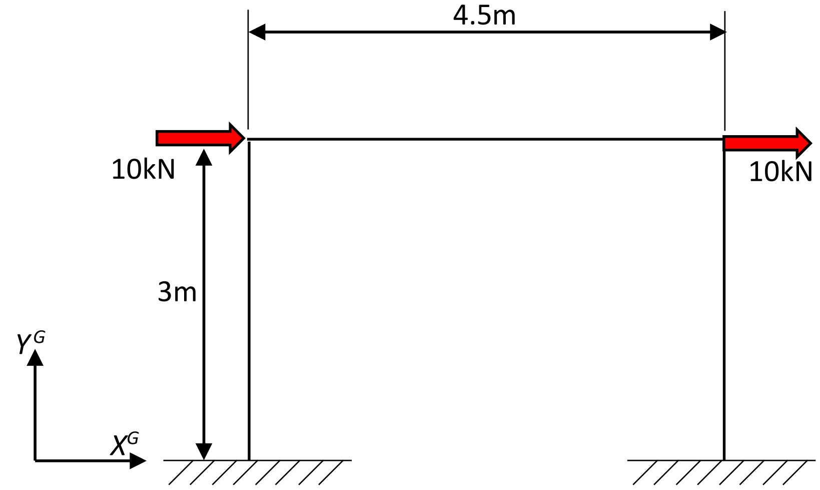
The reaction loads are symmetrical, as expected. The vertical reaction forces at the support points are zero, as we constrained the columns to not deflect vertically (assumed that they were axially rigid). No vertical motion means the columns did not stretch or compress vertically, which means that they don't carry any vertical loads, which means no vertical component of the reaction forces.

However, is this realistic? Or did we miss something……

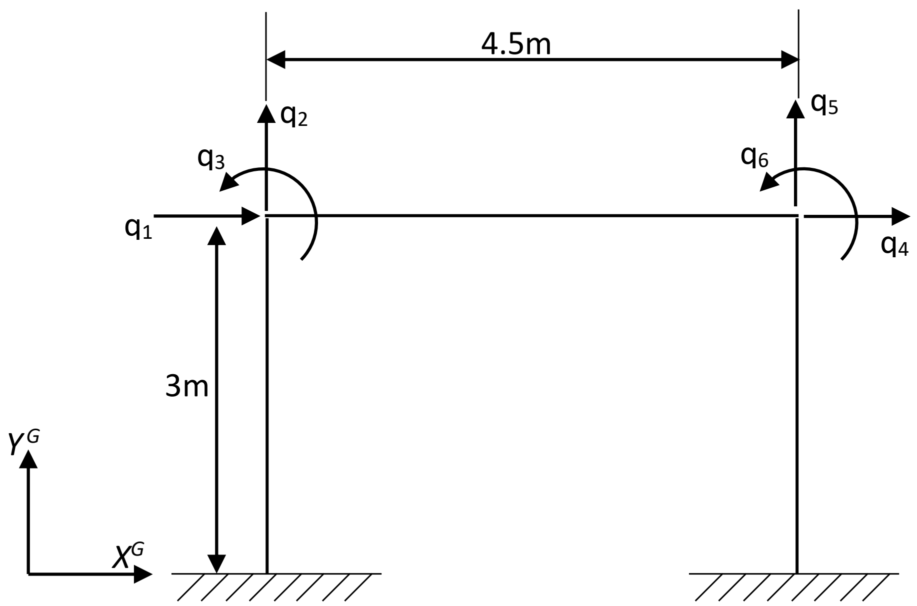
Let's re-work the problem without assuming axial rigidity.

Problem 6 – Reworked.

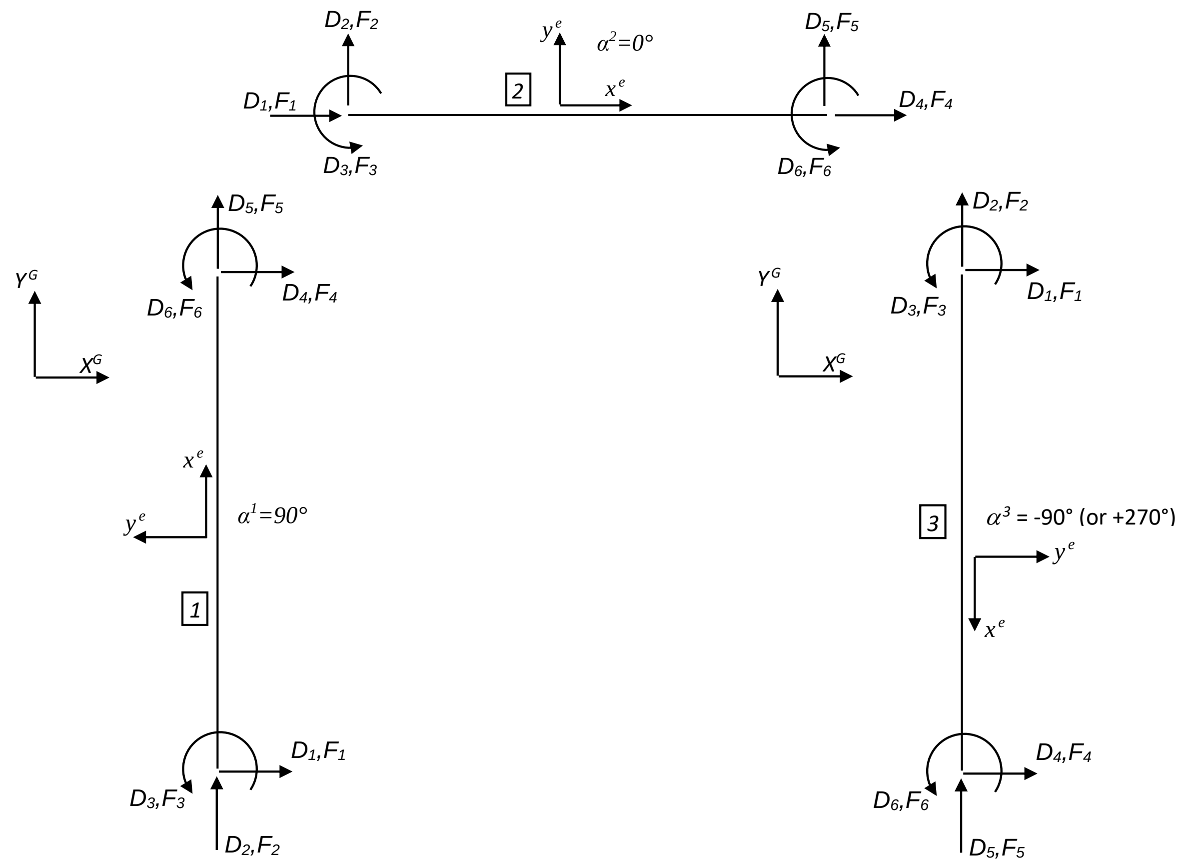
Re-do the problem, but include the vertical degrees of freedom (ie: do not assume axial rigidity of the columns). All the element properties and dimensions will remain the same as before.



Since we are no longer assuming axial rigidity, we will need to include structural degrees of freedom (q s) in the vertical direction. This will result in six global degrees of freedom for the revised structure.



Element Free-body Diagrams:



q_1 aligns with D_4^1 and D_1^2

q_2 aligns with D_5^1 and D_2^2

q_3 aligns with D_6^1 and D_3^2

q_4 aligns with D_4^2 and D_1^3

q_5 aligns with D_5^2 and D_2^3

q_6 aligns with D_6^2 and D_3^3

$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad A^2 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^2 & D_2^2 & D_3^2 & D_4^2 & D_5^2 & D_6^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad A^3 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^3 & D_2^3 & D_3^3 & D_4^3 & D_5^3 & D_6^3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$K_G{}^1 = 1\times10^7\begin{bmatrix} 0.0889 & 0 & 0.1333 & 0 & 0 & 0 \\ 0 & 3.3333 & 0 & 0 & 0 & 0 \\ 0.1333 & 0 & 0.2667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_G{}^2 = 1\times10^7\begin{bmatrix} 2.2222 & 0 & 0 & -2.2222 & 0 & 0 \\ 0 & 0.0263 & 0.0593 & 0 & -0.0263 & 0.0593 \\ 0 & 0.0593 & 0.1778 & 0 & -0.0593 & 0.0889 \\ -2.2222 & 0 & 0 & 2.2222 & 0 & 0 \\ 0 & -0.0263 & -0.0593 & 0 & 0.0263 & -0.0593 \\ 0 & 0.0593 & 0.0889 & 0 & -0.0593 & 0.1778 \end{bmatrix}$$

$$K_G{}^3 = 1\times10^7\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0889 & 0 & 0.1333 \\ 0 & 0 & 0 & 0 & 3.3333 & 0 \\ 0 & 0 & 0 & 0.1333 & 0 & 0.2667 \end{bmatrix}$$

$$K_G = 1\times10^7\begin{bmatrix} 2.3111 & 0 & 0.1333 & -2.2222 & 0 & 0 \\ 0 & 3.3597 & 0.0593 & 0 & -0.0263 & 0.0593 \\ 0.1333 & 0.0593 & 0.4444 & 0 & -0.0593 & 0.0889 \\ -2.2222 & 0 & 0 & 2.3111 & 0 & 0.1333 \\ 0 & -0.0263 & -0.0593 & 0 & 3.3597 & -0.0593 \\ 0 & 0.0593 & 0.0889 & 0.1333 & -0.0593 & 0.4444 \end{bmatrix}$$

Applied loads:

$$Q = \begin{Bmatrix} 10,000N \\ 0 \\ 0 \\ 10,000N \\ 0 \\ 0 \end{Bmatrix}$$

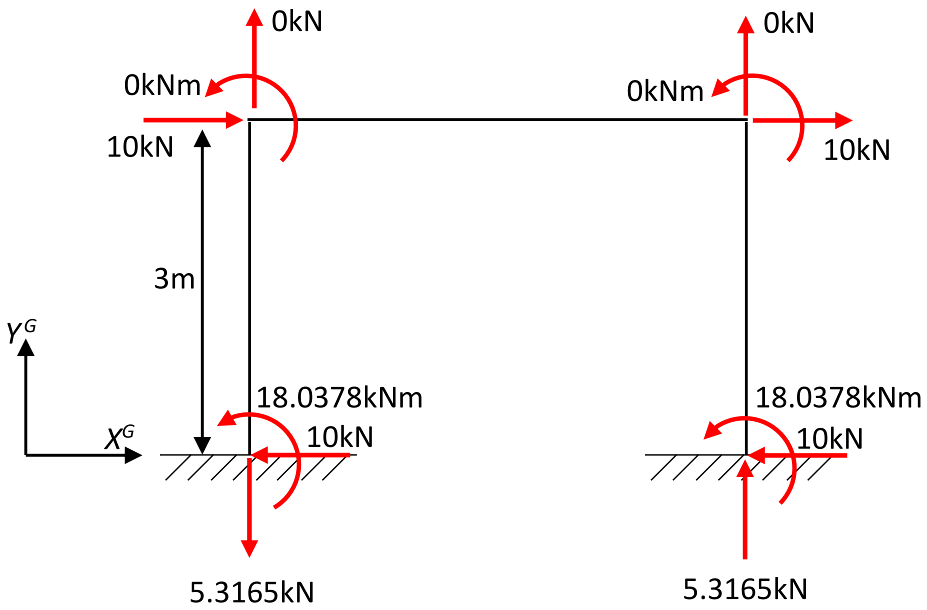
Solve for structural displacements:

$$q = \begin{Bmatrix} 0.0181 \\ 0.0002 \\ -0.0046 \\ 0.0181 \\ -0.0002 \\ -0.0046 \end{Bmatrix} = \begin{Bmatrix} 18.0851mm \\ 0.1595mm \\ -4.5567mrad \\ 18.0851mm \\ -0.1595mm \\ -4.5567mrad \end{Bmatrix}$$

Element forcing vectors (in global co-ordinates):

$$F^1 = \begin{Bmatrix} -10\,kN \\ -5.3165\,kN \\ 18.0378\,kN.m \\ 10\,kN \\ 5.3165\,kN \\ 11.9622\,kN.m \end{Bmatrix} \qquad F^2 = \begin{Bmatrix} 0\,kN \\ -5.3165\,kN \\ -11.9622\,kN.m \\ 0\,kN \\ 5.3165\,kN \\ -11.9622\,kN.m \end{Bmatrix} \qquad F^3 = \begin{Bmatrix} 10\,kN \\ -5.3165\,kN \\ 11.9622\,kN.m \\ -10\,kN \\ 5.3165\,kN \\ 18.0378\,kN.m \end{Bmatrix}$$

Revised reaction loads, based upon the new model:



The reaction loads are still symmetrical, as expected. However, the vertical reaction forces at the support points are no longer zero. The overall structural system generates a force-couple through opposing axial forces in each column. This outcome induced a vertical load at the supports, which was ‘lost’ when we assumed axial rigidity. The base moments are now slightly different as well, as the force couple from the column axial forces affects the overall equilibrium of the structure.

What is the moral of the story?

Be careful when you make assumptions such as axial rigidity. You may simplify the system (reduce the number of global degrees of freedom) but you can introduce unintended consequences.

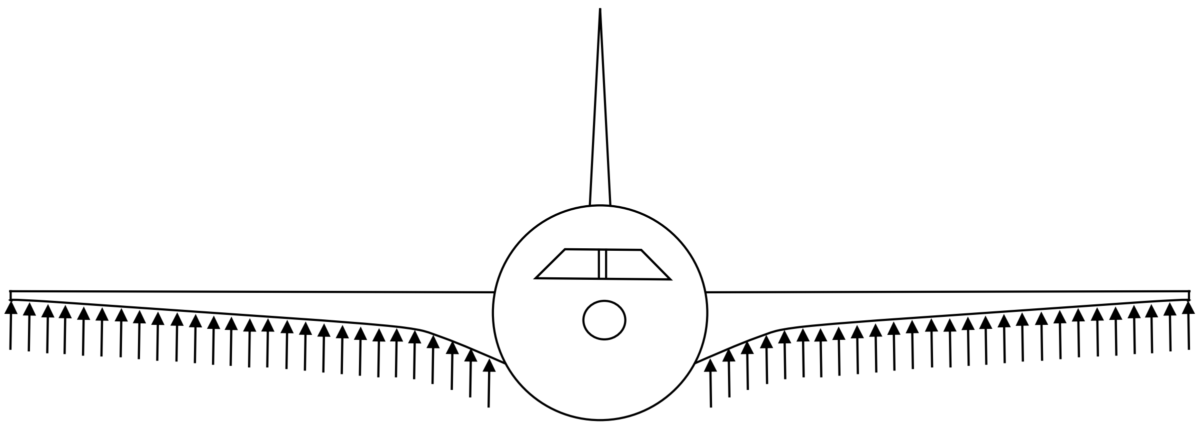
Distributed Loading

In our original derivation of the stiffness matrices for both bar and beam elements, we assumed no distributed load acted along the element. This meant that we were assuming a constant shear force acts along a beam element and a constant axial force acts along a bar element. We will now look at how we can allow for a situation where these distributed loads might act on a given element.

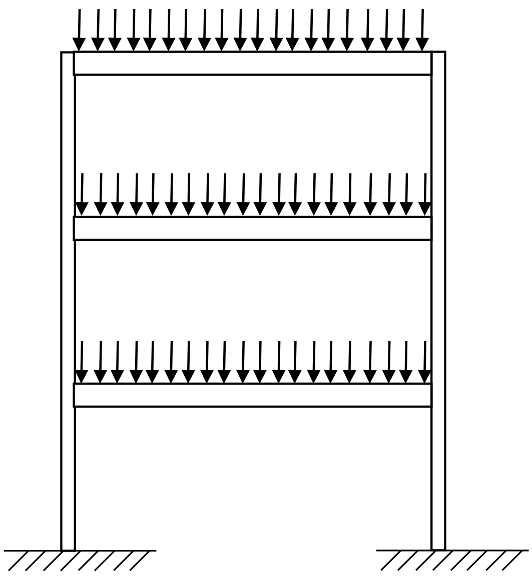
Our method of solving for structural deflections is based upon loads only being applied at nodal points, where elements connect to each other, or where they connect to support points.

Therefore, we will investigate how we can **turn distributed loads into equivalent nodal loads** that induce the same deflections within the elements and all the same support reactions as the distributed loads, but can be applied at nodal points instead.

Distributed shear loads can include lift in aircraft wings:



Or gravity loads from building floors which apply a distributed shear load along a beam:



How do we deal with this type of distributed load and turn them into equivalent nodal loads that can be included in the overall applied load vector, Q?

Recall, that during our derivation for the beam element stiffness matrix on page 62, we used:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = 0$$

Where the right hand side was zero because we assumed that the distributed load intensity $w(x)$ was zero.

We also used:

$$v(x) = N\ d^e = \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

Where the shape functions were defined:

$$\boxed{\begin{aligned} N_1(x) &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ N_2(x) &= \frac{x^3}{L^2} - \frac{2x^2}{L} + x \\ N_3(x) &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ N_4(x) &= \frac{x^3}{L^2} - \frac{x^2}{L} \end{aligned}}$$

Which provide us with the following definition for the element stiffness matrix:

$$\boxed{K^e = \int_0^L \left(\left(N''(x)\right)^T\ EI\ N''(x) \right)\, dx}$$

Which gave us:

$$\boxed{K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}}$$

The key steps to note about this process are:

$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = 0$	We assumed that all distributed loads were equal to zero
$K^e = \int_0^L \left(\left(N''(x)\right)^T\ EI\ N''(x) \right)\, dx$	The integral term is the internal virtual work, integrated over the length of the element

The derivation we originally used (back on page 63) applied the principle of virtual displacement. However, the right hand side is no longer zero, as we are no longer assuming the distributed load intensity to be zero. We will assume that the distributed shear load intensity along the element is defined by some function $w(x)$. Therefore, we now have:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x) \quad \rightarrow \quad \delta v\left(\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\right) = \delta v\ (w(x))$$

Integrating along the element gives:

$$\int_0^L \delta v\left(\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\right)dx = \int_0^L \delta v\ (w(x))\,dx$$

Noting that the right hand side can be modified using the relationship $\delta v = N(x)\delta d = \delta d^T(N(x))^T$ to give:

$$\int_0^L \delta v\left(\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\right)dx = \int_0^L (\delta d^T(N(x))^T)\ (w(x))\,dx$$

Which can be integrated (the left hand side by parts) to give:

$$\delta d^T \int_0^L \left((N''(x))^T\ \ EI\ \ N''(x)\right)\ dx$$
$$= \int_0^L (\delta d^T(N(x))^T)\ (w(x))\,dx + \delta v(0)f_1 + \delta v'(0)f_2 + \delta v(L)f_3 + \delta v'(L)\,f_4$$

Which can be re-arranged to:

$$\delta d^T\left(\underbrace{\int_0^L \left(N''^T(x)\ EI\ N''(x)\right) dx\ d^e}_{K^e d^e} \underbrace{-f^e}_{\substack{f^e \\ \text{applied} \\ \text{nodal} \\ \text{forces}}} - \underbrace{\int_0^L N^T(x)w(x)dx}_{\substack{f^e_{eq} \\ \text{equivalent} \\ \text{nodal forces} \\ \text{representing} \\ \text{distributed} \\ \text{load effects}}}\right) = 0$$

The element-level stiffness equation can be expressed as a function of both directly applied nodal load and the equivalent nodal loads that represent distributed loads:

$$K^e d^e = f^e + f^e_{eq}$$

Where the equivalent nodal loads are defined:

$$\boxed{f^e_{eq} = \int_0^L N^T(x)\ w(x)dx}$$

Note that this vector is expressed in local (element) co-ordinates and must be transformed before being applied to the structure.

To transform the equivalent nodal loads into global co-ordinates:

$$F_{eq}^{e} = \Lambda^{e^T} f_{eq}^{e}$$

However, while this equivalent load vector is now in global co-ordinates, it is still at the element-level. We need to use the assembly matrix to link it to the structure-level applied forces.

$$Q_{eq}^{e} = A^{e} F_{eq}^{e} \ = \ A^{e}\left(\Lambda^{e^T} f_{eq}^{e}\right)$$

Where Q_{eq}^{e} is the contribution to the overall applied force vector that comes from this element. The overall structural applied force vector that represents all equivalent nodal terms, Q_{eq}, is defined:

$$Q_{eq} = \sum_{i=1}^{e} A^{e} F_{eq}^{e}$$

And the structure-level stiffness equations is now defined:

$$\left(Q + Q_{eq}\right) = K_G \, q$$

The same Q vector we've always used, representing concentrated loads (forces and moments) applied directly at nodal points

The new contribution to the structure-level forcing terms that contains equivalent nodal forcing terms, representing the distributed loads

We now have a way of incorporating distributed loads into our system of equations. **Everything else is just as it was before (thankfully!)**

Note: We only used the basic 4DOF BEAM element formulation (with the 4x4 K^{e}matrix from pages 63-64) in this derivation and NOT the full 6DOF FRAME element matrix. This is due to the fact that a distributed shear load only affects flexural loads and NOT axial loads, **as we assume no coupling between flexural and axial deformations**.

So far, we have not used any specific definition of the distributed loading profile. We have simply said that it follows some profile along the beam, $w(x)$.

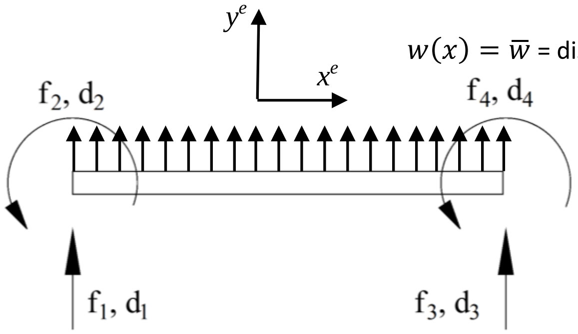
We will now consider some specific versions of this general definition.

A Uniformly Distributed Load (UDL)

For a Uniformly Distributed Load, the function for the distributed load intensity is constant, with an intensity which we will define as $\bar{w}$, such that $w(x) = \bar{w}$. We will now find the equivalent nodal force vector $f^e_{eq_{\{4x1\}}}$ that corresponds to this distributed loading case.

We will define the UDL to act upwards in the positive y^e direction. If the load is downward, in the negative y^e direction, then the value of UDL intensity, $\bar{w}$, will be negative.

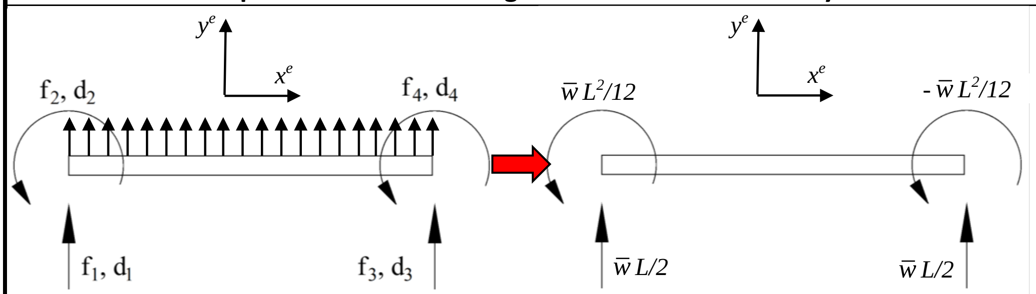
$w(x) = \bar{w}$ = distributed shear load intensity



$$f^e_{eq,UDL} = \int_0^L \left(N^T(x)\, w(x)\right) dx = \int_0^L \left(N^T(x)\, \bar{w}\right) dx = \int_0^L \begin{Bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ \frac{x^3}{L^2} - \frac{2x^2}{L} + x \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ \frac{x^3}{L^2} - \frac{x^2}{L} \end{Bmatrix} (\bar{w})\, dx$$

$$f^e_{eq,UDL} = (\bar{w}) \begin{Bmatrix} x - \frac{x^3}{L^2} + \frac{x^4}{2L^3} \\ \frac{x^4}{4L^2} - \frac{2x^3}{3L} + \frac{x^2}{2} \\ \frac{x^3}{L^2} - \frac{x^4}{2L^3} \\ \frac{x^4}{4L^2} - \frac{x^3}{3L} \end{Bmatrix} \Bigg|_{x=0}^{x=L} = (\bar{w}) \begin{Bmatrix} \left(L - \frac{L^3}{L^2} + \frac{L^4}{2L^3}\right) - \left(0 - \frac{0^3}{L^2} + \frac{0^4}{2L^3}\right) \\ \left(\frac{L^4}{4L^2} - \frac{2L^3}{3L} + \frac{L^2}{2}\right) - \left(\frac{0^4}{4L^2} - \frac{2(0)^3}{3L} + \frac{0^2}{2}\right) \\ \left(\frac{L^3}{L^2} - \frac{L^4}{2L^3}\right) - \left(\frac{0^3}{L^2} + \frac{0^4}{2L^3}\right) \\ \left(\frac{L^4}{4L^2} - \frac{L^3}{3L}\right) - \left(\frac{0^4}{4L^2} - \frac{0^3}{3L}\right) \end{Bmatrix} = (\bar{w}) \begin{Bmatrix} \frac{L}{2} \\ \frac{L^2}{12} \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix}$$

Equivalent nodal loading definition for a Uniformly Distributed Load (UDL)



$$f^e_{eq,UDL} = \begin{Bmatrix} \bar{w}L/2 \\ \bar{w}L^2/12 \\ \bar{w}L/2 \\ -\bar{w}L^2/12 \end{Bmatrix}$$

Note: It is assumed that the UDL acts upwards in the positive y^e direction (a positive value of $\bar{w}$).

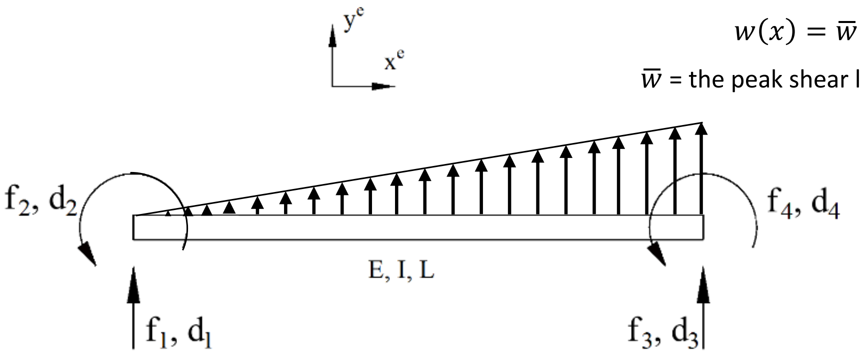
Linearly Varying Distributed Load (LVL)

For a Linearly Varying Distributed Load, the function for the linearly varying load intensity is a simple linear function of x, with a peak intensity at the right hand node (node 2) which we will define as $\overline{w}$, such that $w(x) = \overline{w}\left(\frac{x}{L}\right)$. We will now find the equivalent nodal force vector $f^{e}_{eq,LVL_{\{4x1\}}}$ that corresponds to this distributed loading case.

Again, we will define the LVL to act upwards in the positive y^e direction. If the load is downward, in the negative y^e direction, then the value of peak LVL intensity, $\overline{w}$, will be negative.

$$w(x) = \overline{w}\left(\frac{x}{L}\right)$$

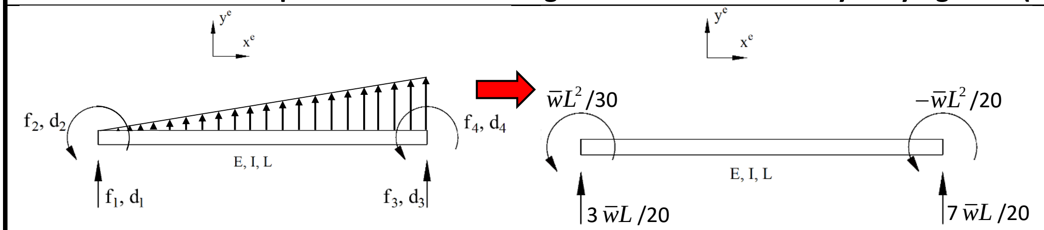
$\overline{w}$ = the peak shear load intensity



$$f^{e}_{eq,LVL} = \int_0^L \left(N^T(x)\,w(x)\right)dx = \int_0^L \left(N^T(x)\left(\overline{w}\right)\left(\frac{x}{L}\right)\right)dx = \left(\frac{\overline{w}}{L}\right)\int_0^L \begin{Bmatrix} x - \frac{3x^3}{L^2} + \frac{2x^4}{L^3} \\ \frac{x^4}{L^2} - \frac{2x^3}{L} + x^2 \\ \frac{3x^3}{L^2} - \frac{2x^4}{L^3} \\ \frac{x^4}{L^2} - \frac{x^3}{L} \end{Bmatrix} dx$$

$$f^{e}_{eq,LVL} = \left(\frac{\overline{w}}{L}\right) \left.\begin{Bmatrix} \frac{x^2}{2} - \frac{3x^4}{4L^2} + \frac{2x^5}{5L^3} \\ \frac{x^5}{5L^2} - \frac{2x^4}{4L} + \frac{x^3}{3} \\ \frac{3x^4}{4L^2} - \frac{2x^5}{5L^3} \\ \frac{x^5}{5L^2} - \frac{x^4}{4L} \end{Bmatrix}\right|_{x=0}^{x=L} = \left(\frac{\overline{w}}{L}\right) \begin{Bmatrix} \left(\frac{L^2}{2} - \frac{3L^4}{4L^2} + \frac{2L^5}{5L^3}\right) - \left(\frac{0^2}{2} - \frac{3(0)^4}{4L^2} + \frac{2(0)^5}{5L^3}\right) \\ \left(\frac{L^5}{5L^2} - \frac{2L^4}{4L} + \frac{L^3}{3}\right) - \left(\frac{0^5}{5L^2} - \frac{2(0)^4}{4L} + \frac{(0)^3}{3}\right) \\ \left(\frac{3L^4}{4L^2} - \frac{2L^5}{5L^3}\right) - \left(\frac{3(0)^4}{4L^2} + \frac{2(0)^5}{5L^3}\right) \\ \left(\frac{L^5}{5L^2} - \frac{L^4}{4L}\right) - \left(\frac{0^5}{5L^2} - \frac{0^4}{4L}\right) \end{Bmatrix} = \begin{Bmatrix} \frac{3\overline{w}L}{20} \\ \frac{\overline{w}L^2}{30} \\ \frac{7\overline{w}L}{20} \\ -\frac{\overline{w}L^2}{20} \end{Bmatrix}$$

Equivalent nodal loading definition for a Linearly Varying Load (LVL)



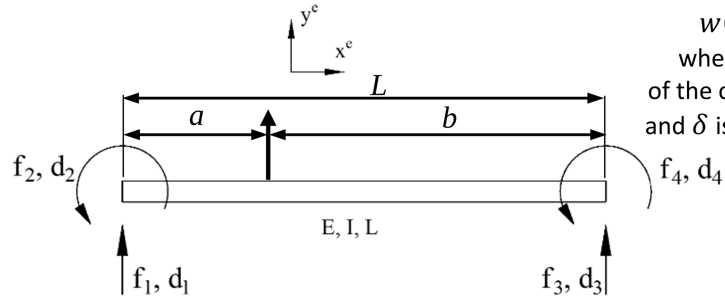
$$f^{e}_{eq,LVL} = \begin{Bmatrix} 3\overline{w}L/20 \\ \overline{w}L^2/30 \\ 7\overline{w}L/20 \\ -\overline{w}L^2/20 \end{Bmatrix}$$

Note: It is assumed that the LVL acts upwards in the positive y^e direction (a positive value of $\overline{w}$).

Point loads applied within an element

Until now, we have assumed that concentrated loads are only applied at nodal points, where elements connect to one another, or to support points. How can we include concentrated loads applied at some point along the element?

For the general case of a concentrated shear load applied at a distance a from node 1, the distributed load intensity is defined as $w(x) = (\bar{w})\delta(x-a)$ where $\bar{w}$ is the magnitude of the concentrated load and δ is the Dirac delta function, such that $w(x) = \bar{w}$ at $x = a$ and $w(x) = 0$ everywhere else. Again, we will define the concentrated load to act upwards in the positive y^e direction. If the load is downward, in the negative y^e direction, then the value of load intensity, $\bar{w}$, will be negative.



$$w(x) = \bar{w}\,\delta(x-a)$$

where $\bar{w}$ = the magnitude of the concentrated shear force and δ is the Dirac delta function

$$f^e_{eq,PL} = \int_0^L \left(N^T(x)\,w(x)\right)dx = \int_0^L \left(N^T(x)\left(\bar{w}\right)\delta\left(x-a\right)\right)dx = \bar{w}\,N^T(x)\Big|_{x=a}$$

$$f^e_{eq,PL} = \bar{w}\left.\begin{Bmatrix} 1-\dfrac{3x^2}{L^2}+\dfrac{2x^3}{L^3} \\ \dfrac{x^3}{L^2}-\dfrac{2x^2}{L}+x \\ \dfrac{3x^2}{L^2}-\dfrac{2x^3}{L^3} \\ \dfrac{x^3}{L^2}-\dfrac{x^2}{L} \end{Bmatrix}\right|_{x=a} = \bar{w}\begin{Bmatrix} 1-\dfrac{3a^2}{L^2}+\dfrac{2a^3}{L^3} \\ \dfrac{a^3}{L^2}-\dfrac{2a^2}{L}+a \\ \dfrac{3a^2}{L^2}-\dfrac{2a^3}{L^3} \\ \dfrac{a^3}{L^2}-\dfrac{a^2}{L} \end{Bmatrix}$$

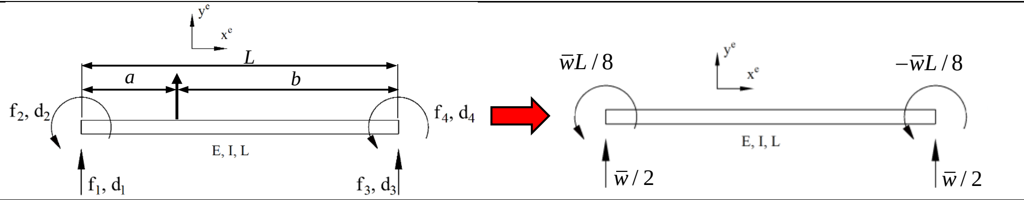
Just plug in the appropriate value of a to get the corresponding equivalent nodal force vector to represent the concentrated shear load

Let's also consider **<u>a special case of a mid-span point load</u>**, where $a = L/2$.

$$f^e_{eq,PL} = \bar{w}\begin{Bmatrix} 1-\dfrac{3(L/2)^2}{L^2}+\dfrac{2(L/2)^3}{L^3} \\ \dfrac{(L/2)^3}{L^2}-\dfrac{2(L/2)^2}{L}+(L/2) \\ \dfrac{3(L/2)^2}{L^2}-\dfrac{2(L/2)^3}{L^3} \\ \dfrac{(L/2)^3}{L^2}-\dfrac{(L/2)^2}{L} \end{Bmatrix} = \bar{w}\begin{Bmatrix} 1-\dfrac{3}{4}+\dfrac{1}{4} \\ \dfrac{L}{8}-\dfrac{L}{2}+\dfrac{L}{2} \\ \dfrac{3}{4}-\dfrac{1}{4} \\ \dfrac{L}{8}-\dfrac{L}{4} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\bar{w}}{2} \\ \dfrac{\bar{w}L}{8} \\ \dfrac{\bar{w}}{2} \\ -\dfrac{\bar{w}L}{8} \end{Bmatrix}$$

A special case where $a = L/2$

Equivalent nodal loading definition for a mid-span point load, where $a = L/2$



$$f^e_{eq,PL} = \begin{Bmatrix} \bar{w}/2 \\ \bar{w}L/8 \\ \bar{w}/2 \\ -\bar{w}L/8 \end{Bmatrix}$$

Note: We assume that the point load acts upwards in the positive y^e direction (a positive value of $\bar{w}$).

Extension to Frame elements

In these derivations for equivalent nodal loadings, we've been using the stiffness matrix for a BEAM element. As a result, the equivalent nodal forcing vectors are 4x1 in size and do not include axial terms.

The obvious question from here is: How do we modify these equivalent nodal force vectors so that we can use them with FRAME elements?

That's easy, as we assume no coupling between axial and shear/bending terms! We simply add in zeros that correspond with the axial term for a FRAME element

Equivalent nodal loading definition for a Uniformly Distributed Load (UDL)		
$f^e_{eq,UDL\,\{4\times1\}} = \begin{Bmatrix} \bar{w}L/2 \\ \bar{w}L^2/12 \\ \bar{w}L/2 \\ -\bar{w}L^2/12 \end{Bmatrix}$	$\Rightarrow$	$f^e_{eq,UDL\,\{6\times1\}} = \begin{Bmatrix} 0 \\ \bar{w}L/2 \\ \bar{w}L^2/12 \\ 0 \\ \bar{w}L/2 \\ -\bar{w}L^2/12 \end{Bmatrix} \begin{array}{l} \rightarrow \text{No axial contribution} \\ \\ \\ \rightarrow \text{No axial contribution} \\ \\ \\ \end{array}$

Equivalent nodal loading definition for a Linearly Varying Load (LVL)		
$f^e_{eq,LVL\,\{4\times1\}} = \begin{Bmatrix} 3\bar{w}L/20 \\ \bar{w}L^2/30 \\ 7\bar{w}L/20 \\ -\bar{w}L^2/20 \end{Bmatrix}$	$\Rightarrow$	$f^e_{eq,LVL\,\{6\times1\}} = \begin{Bmatrix} 0 \\ 3\bar{w}L/20 \\ \bar{w}L^2/30 \\ 0 \\ 7\bar{w}L/20 \\ -\bar{w}L^2/20 \end{Bmatrix}$

Equivalent nodal loading definition for a mid-span point load		
$f^e_{eq,PL\{4\times1\}} = \begin{Bmatrix} \bar{w}/2 \\ \bar{w}L/8 \\ \bar{w}/2 \\ -\bar{w}L/8 \end{Bmatrix}$	$\Rightarrow$	$f^e_{eq,PL\{6\times1\}} = \begin{Bmatrix} 0 \\ \bar{w}/2 \\ \bar{w}L/8 \\ 0 \\ \bar{w}/2 \\ -\bar{w}L/8 \end{Bmatrix}$

Which leads us to the next question: What about distributed axial loads? Well, these can be dealt with in a very similar way, but using the axial shape functions we defined earlier, rather than the flexural shape functions.

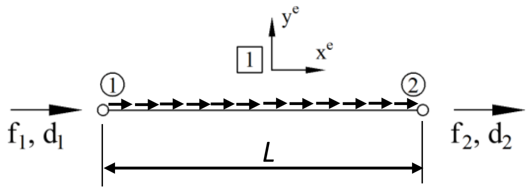
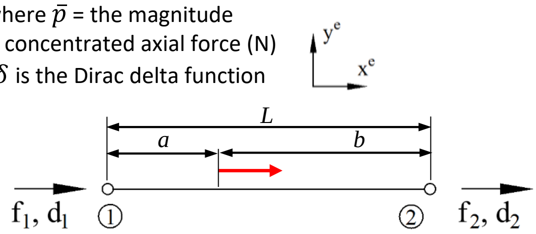
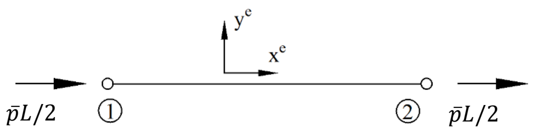
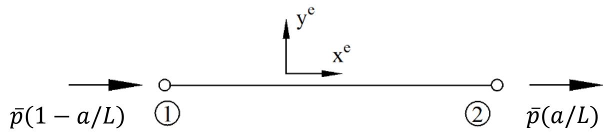
$$f^e_{eq,axial} = \int_0^L \left(\Psi^T(x)\, p(x) \right) dx$$

where $p(x)$=the distributed axial load intensity as a function of x.

and $\Psi(x) = \left[\psi_1(x) \quad \psi_2(x) \right]$ where $\psi_1(x) = \left(1 - \frac{x}{L} \right)$ and $\psi_2(x) = \left(\frac{x}{L} \right)$

Distributed Axial Loads

There are two main cases to consider: a distributed axial load and a concentrated point load applied at some point along the element.

Distributed Axial Load	Concentrated Axial Load
$p(x) = \bar{p}$ where $\bar{p}$ = the magnitude of the distributed axial force (N/m)  	$p(x) = \bar{p}\,\delta(x-a)$ where $\bar{p}$ = the magnitude of the concentrated axial force (N) and δ is the Dirac delta function 
$f^e_{eq,axial} = \int_0^L \left(\begin{pmatrix} 1-(x/L) \\ x/L \end{pmatrix} \bar{p} \right) dx$ $f^e_{eq,axial} = \bar{p} \begin{pmatrix} x-(x^2/2L) \\ x^2/2L \end{pmatrix} \Bigg\|_{x=0}^{x=L}$ $f^e_{eq,axial} = \bar{p} \begin{pmatrix} L/2 \\ L/2 \end{pmatrix}$	$f^e_{eq,axial\ PL} = \int_0^L \left(\begin{pmatrix} 1-(x/L) \\ x/L \end{pmatrix} \bar{p}\,\delta(x-a) \right) dx$ $f^e_{eq,axial\ PL} = \bar{p} \begin{pmatrix} 1-(x/L) \\ x/L \end{pmatrix} \Bigg\|_{x=a}$ $f^e_{eq,axial\ PL} = \bar{p} \begin{pmatrix} 1-(a/L) \\ a/L \end{pmatrix}$
 	
Note: these are 2x1 vectors that only represent the axial portion of a FRAME element. To get the full 6x1, we can again apply the principle of linear superposition, where we assume no linking of axial and flexural terms.	
$f^e_{eq,axial} = \bar{p} \begin{pmatrix} L/2 \\ 0 \\ 0 \\ L/2 \\ 0 \\ 0 \end{pmatrix}$	$f^e_{eq,axial\ PL} = \bar{p} \begin{pmatrix} 1-(a/L) \\ 0 \\ 0 \\ a/L \\ 0 \\ 0 \end{pmatrix}$

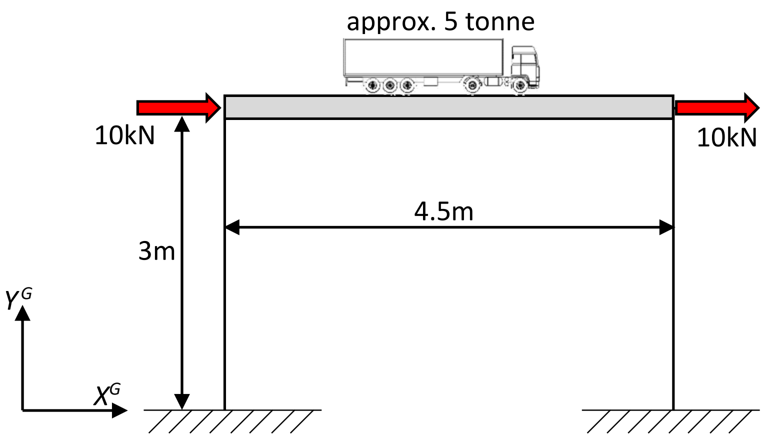
Note: This definition assumes that the point load acts in the positive x^e direction (corresponding to a positive value of $\bar{p}$). If the load acts in the opposite direction, then the value of $\bar{p}$ will be negative.

Therefore, you must keep track of the element co-ordinates when you define the function $p(x)$ for a problem

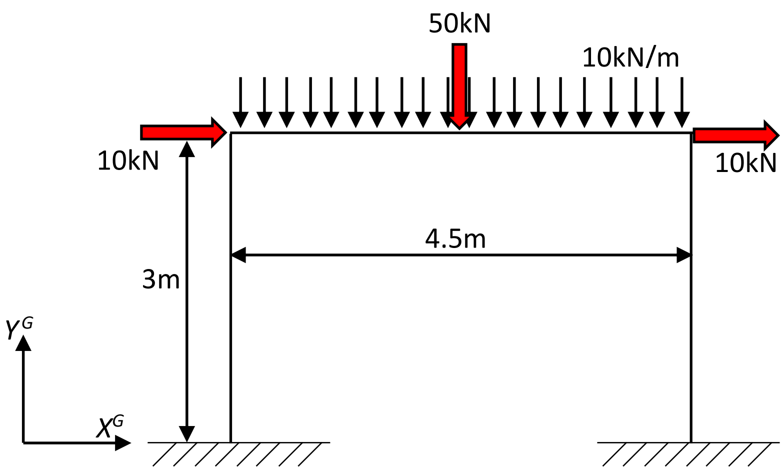
Problem 6 – Revisited.

Let's re-do this problem, but this time we will include some distributed loading terms to see how they get included in the solution. We will use the version of this problem that includes the vertical degrees of freedom (ie: do not assume axial rigidity of the columns).

Again, we will assume that all members have $I = 1 \times 10^{-5}\, m^4$, $A = 5 \times 10^{-4}\, m^2$ and an elastic modulus of $E = 200$ GPa.

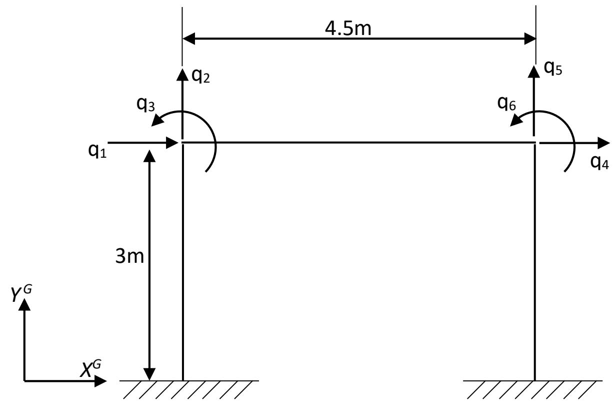


Let's assume that this structure represents the middle section of a bridge. The vertical elements represent bridge piers and the horizontal element the bridge deck. We will include self-weight of the horizontal element (bridge deck) and a concentrated shear load at the mid-span to represent a truck driving across the bridge deck.

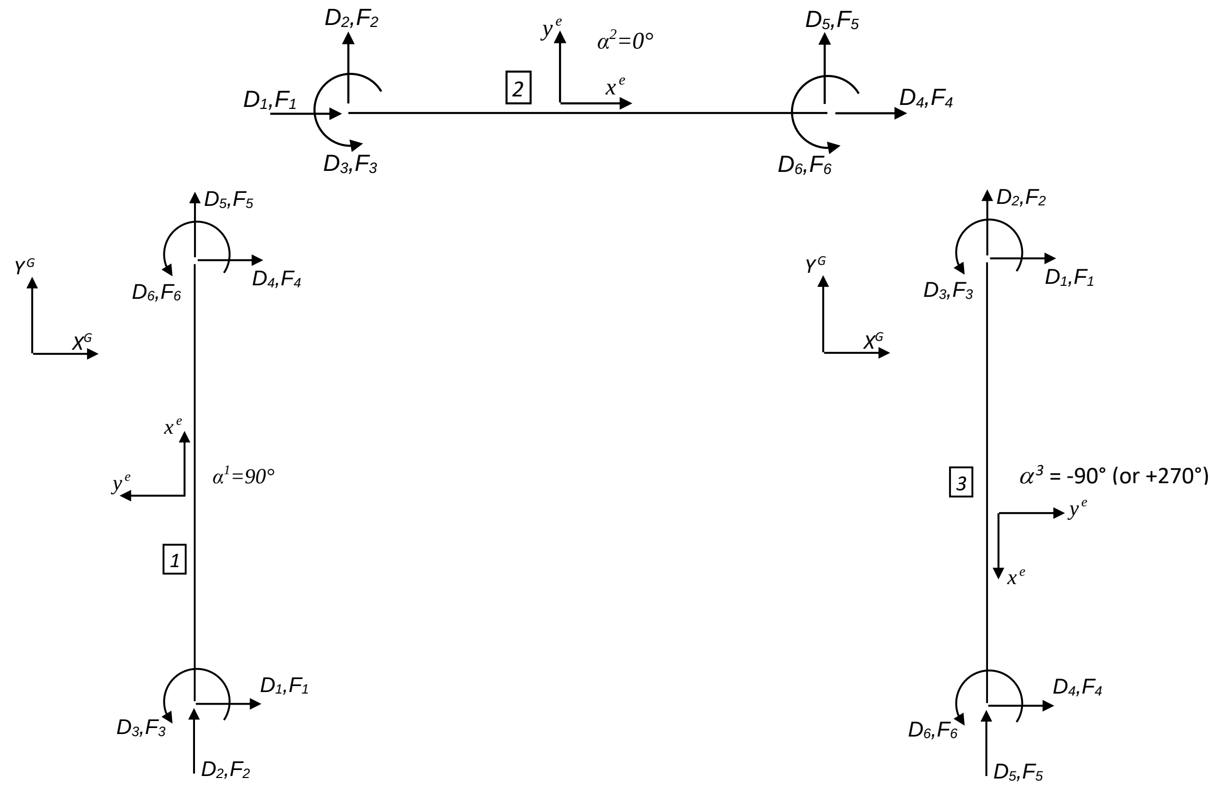


Setting up the structural model is unchanged from what we did last time. The loading conditions haven't changed, only the applied loads, which are represented by the Q vector.

***Note**: this is a repeat of the structure definition presented earlier.*



Element Free-body Diagrams:



q_1 aligns with D_4^1 and D_1^2

q_2 aligns with D_5^1 and D_2^2

q_3 aligns with D_6^1 and D_3^2

q_4 aligns with D_4^2 and D_1^3

q_5 aligns with D_5^2 and D_2^3

q_6 aligns with D_6^2 and D_3^3

$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad A^2 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^2 & D_2^2 & D_3^2 & D_4^2 & D_5^2 & D_6^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad A^3 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^3 & D_2^3 & D_3^3 & D_4^3 & D_5^3 & D_6^3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$K_G^{\ 1} = 1\times10^7 \begin{bmatrix} 0.0889 & 0 & 0.1333 & 0 & 0 & 0 \\ 0 & 3.3333 & 0 & 0 & 0 & 0 \\ 0.1333 & 0 & 0.2667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_G^{\ 2} = 1\times10^7 \begin{bmatrix} 2.2222 & 0 & 0 & -2.2222 & 0 & 0 \\ 0 & 0.0263 & 0.0593 & 0 & -0.0263 & 0.0593 \\ 0 & 0.0593 & 0.1778 & 0 & -0.0593 & 0.0889 \\ -2.2222 & 0 & 0 & 2.2222 & 0 & 0 \\ 0 & -0.0263 & -0.0593 & 0 & 0.0263 & -0.0593 \\ 0 & 0.0593 & 0.0889 & 0 & -0.0593 & 0.1778 \end{bmatrix}$$

$$K_G^{\ 3} = 1\times10^7 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0889 & 0 & 0.1333 \\ 0 & 0 & 0 & 0 & 3.3333 & 0 \\ 0 & 0 & 0 & 0.1333 & 0 & 0.2667 \end{bmatrix}$$

$$K_G = 1\times10^7 \begin{bmatrix} 2.3111 & 0 & 0.1333 & -2.2222 & 0 & 0 \\ 0 & 3.3597 & 0.0593 & 0 & -0.0263 & 0.0593 \\ 0.1333 & 0.0593 & 0.4444 & 0 & -0.0593 & 0.0889 \\ -2.2222 & 0 & 0 & 2.3111 & 0 & 0.1333 \\ 0 & -0.0263 & -0.0593 & 0 & 3.3597 & -0.0593 \\ 0 & 0.0593 & 0.0889 & 0.1333 & -0.0593 & 0.4444 \end{bmatrix}$$

Nothing to this point has changed, as it is still the same structure that we had before. However, from this point forward this will vary from the previous example, as the applied loads have now changed.

The applied loads at the nodal points are the same as before:

$$Q_{nodal} = \begin{Bmatrix} 10,000N \\ 0 \\ 0 \\ 10,000N \\ 0 \\ 0 \end{Bmatrix}$$

However, we must now also consider the distributed loads and how these contribute to the overall applied force vector, Q.

Applying the UDL acting on Element 2:

We are given the input UDL intensity of 10kN/m and the length of element 2 is 4.5m. Note that in this example, the load acts in the negative y^e direction, so the input value of $\overline{w}$ is negative, such that $\overline{w} = -10kN/m$:

$$f_{eq,UDL}^{2} = \begin{Bmatrix} 0 \\ \overline{w}L/2 \\ \overline{w}L^2/12 \\ 0 \\ \overline{w}L/2 \\ -\overline{w}L^2/12 \end{Bmatrix} = \begin{Bmatrix} 0 \\ (-10,000)(4.5)/2 \\ (-10,000)(4.5)^2/12 \\ 0 \\ (-10,000)(4.5)/2 \\ -(-10,000)(4.5)^2/12 \end{Bmatrix} = \begin{Bmatrix} 0N \\ -22,500N \\ -16,875Nm \\ 0N \\ -22,500N \\ 16,875Nm \end{Bmatrix}$$

This vector is still in local (element) co-ordinates, so we need to transform it into global co-ordinates:

$$F_{eq,UDL}^{2} = \left(\Lambda^2\right)^T f_{eq,UDL}^{2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0N \\ -22,500N \\ -16,875Nm \\ 0N \\ -22,500N \\ 16,875Nm \end{Bmatrix} = \begin{Bmatrix} 0N \\ -22,500N \\ -16,875Nm \\ 0N \\ -22,500N \\ 16,875Nm \end{Bmatrix}$$

In this particular case (due to the 0° element rotation) $F_{eq,UDL}^{2} = f_{eq,UDL}^{2}$ as the transformation matrix for element 2 is just a 6x6 identity matrix.

We still need to use the assembly matrix to determine where the forcing terms given in the vector above should be placed within the overall structure forcing term, Q.

$$Q_{eq,UDL} = A^2 F_{eq,UDL}^{2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0N \\ -22,500N \\ -16,875Nm \\ 0N \\ -22,500N \\ 16,875Nm \end{Bmatrix} = \begin{Bmatrix} 0N \\ -22,500N \\ -16,875Nm \\ 0N \\ -22,500N \\ 16,875Nm \end{Bmatrix}$$

Applying the concentrated point load acting on Element 2:

We will now consider the concentrated point load that is applied at the mid-span of element 2.

We are given the input concentrated load intensity of 50kN and the length of element 2 is 4.5m. Note that in this example, this concentrated load acts in the negative y^e direction, so the input value of $\overline{w}$ is negative, such that $\overline{w} = -50kN$:

In local (element) co-ordinates	In global co-ordinates
$f^2_{eq,PL\{6\times1\}} = \begin{Bmatrix} 0 \\ \overline{w}/2 \\ \overline{w}L/8 \\ 0 \\ \overline{w}/2 \\ -\overline{w}L/8 \end{Bmatrix} \begin{Bmatrix} 0 \\ (-50,000)/2 \\ (-50,000)(4.5)/8 \\ 0 \\ (-50,000)/2 \\ -(-50,000)(4.5)/8 \end{Bmatrix} = \begin{Bmatrix} 0N \\ -25,000N \\ -28,125Nm \\ 0N \\ -25,000N \\ 28,125Nm \end{Bmatrix}$	$F^2_{eq,PL} = \left(\Lambda^2\right)^T f^2_{eq,PL} = \begin{Bmatrix} 0N \\ -25,000N \\ -28,125Nm \\ 0N \\ -25,000N \\ 28,125Nm \end{Bmatrix}$

Again, in this particular case (due to the 0° element rotation) $F^2_{eq,PL} = f^2_{eq,PL}$ as the transformation matrix for element 2 is just a 6x6 identity matrix.

We still need to use the assembly matrix to determine where the forcing terms given in the vector above should be placed within the overall structure forcing term, Q.

$$Q_{eq,PL} = A^2 F^2_{eq,PL} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0N \\ -25,000N \\ -28,125Nm \\ 0N \\ -25,000N \\ 28,125Nm \end{Bmatrix} = \begin{Bmatrix} 0N \\ -25,000N \\ -28,125Nm \\ 0N \\ -25,000N \\ 28,125Nm \end{Bmatrix}$$

We can now obtain the overall structure forcing term, Q_{total}, by simply summing the contributions from loads applied directly at nodal points (Q_{nodal}), and the equivalent nodal forcing terms representing the uniformly distributed load ($Q_{eq,UDL}$) and the concentrated, mid-span point load, ($Q_{eq,PL}$), to get:

$$Q_{total} = Q_{nodal} + Q_{eq,UDL} + Q_{eq,PL} = \begin{Bmatrix} 10,000 \\ 0 \\ 0 \\ 10,000 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -22,500 \\ -16,875 \\ 0 \\ -22,500 \\ 16,875 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -25,000 \\ -28,125 \\ 0 \\ -25,000 \\ 28,125 \end{Bmatrix} = \begin{Bmatrix} 10,000N \\ -47,500N \\ -45,000Nm \\ 10,000N \\ -47,500N \\ 45,000Nm \end{Bmatrix}$$

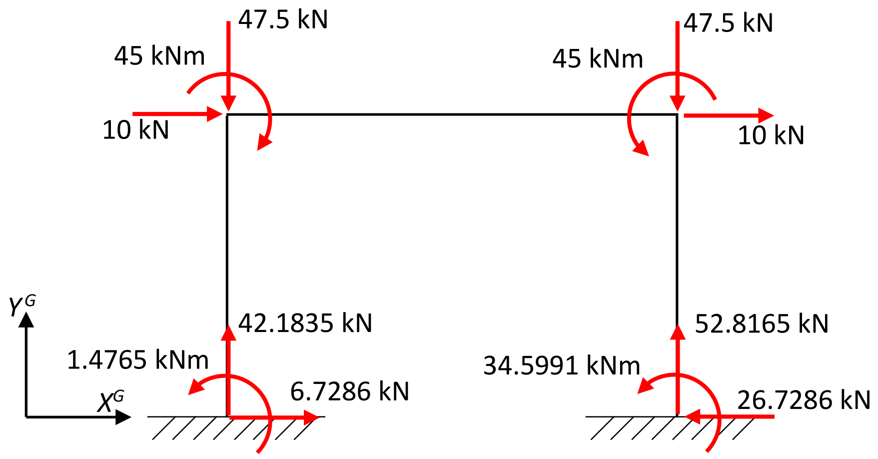
We can now solve for structural displacements, just like before:

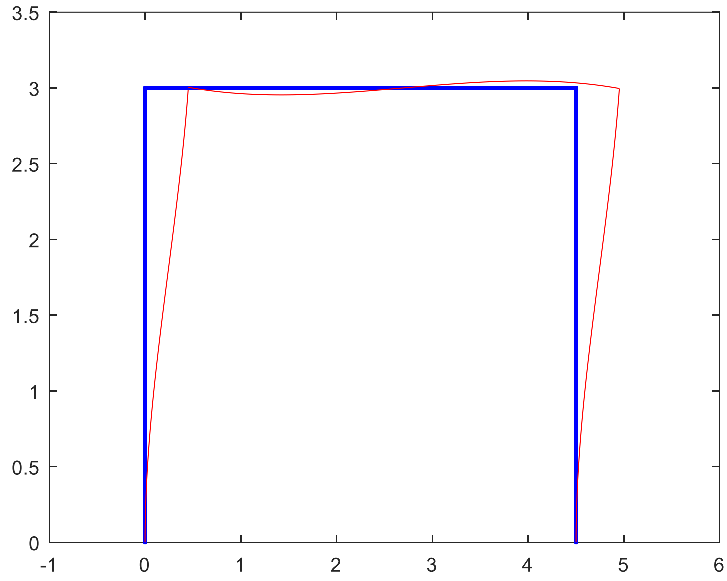
$$q = K_G^{-1} Q_{total} = \begin{Bmatrix} 0.0185 \\ -0.0013 \\ -0.0174 \\ 0.0177 \\ -0.0016 \\ 0.0082 \end{Bmatrix} = \begin{Bmatrix} 18.4615mm \\ -1.2655mm \\ -17.3541mrad \\ 17.7087mm \\ -1.5845mm \\ 8.2407mrad \end{Bmatrix}$$

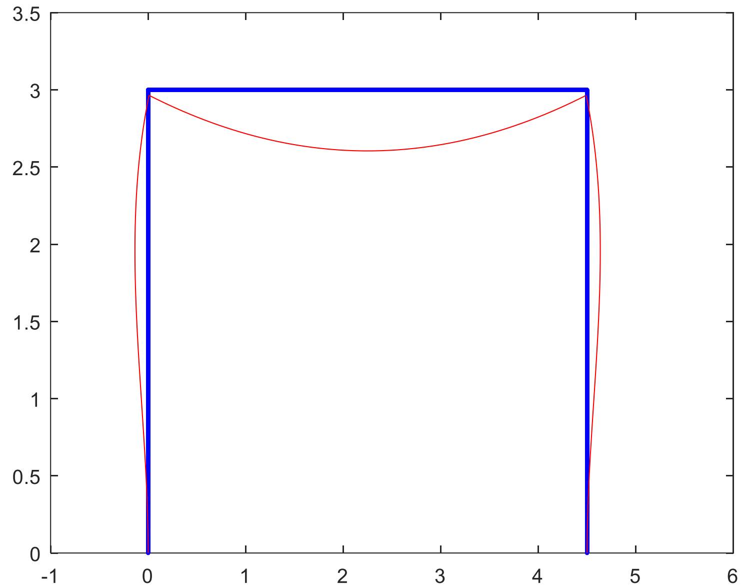
Element forcing vectors (in global co-ordinates):

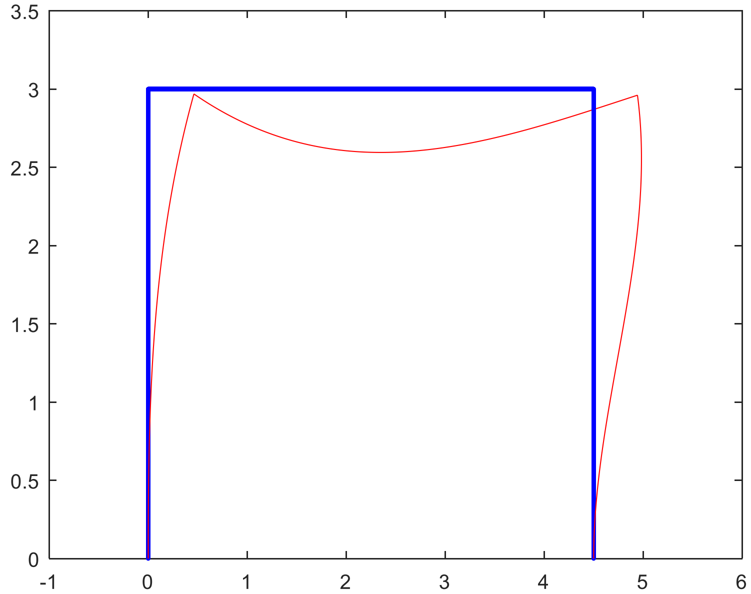
$$F^1 = \begin{Bmatrix} 6.7286\,kN \\ 42.1835\,kN \\ 1.4765\,kN.m \\ -6.7286\,kN \\ -42.1835kN \\ -21.6623\,kN.m \end{Bmatrix} \qquad F^2 = \begin{Bmatrix} 16.7286\,kN \\ -5.3165\,kN \\ -23.3377\,kN.m \\ -16.7286kN \\ 5.3165\,kN \\ -0.5867\,kN.m \end{Bmatrix} \qquad F^3 = \begin{Bmatrix} 26.7286\,kN \\ -52.8165\,kN \\ 45.5867\,kN.m \\ -26.7286\,kN \\ 52.8165kN \\ 34.5991kN.m \end{Bmatrix}$$

Revised reaction loads, based upon the new model:




3.5
3
2.5
2
1.5
1
0.5
0
-1
0
1
2
3
4
5
6

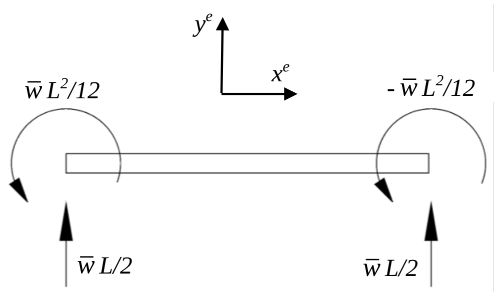

3.5
3
2.5
2
1.5
1
0.5
0
-1
0
1
2
3
4
5
6


3.5
3
2.5
2
1.5
1
0.5
0
-1
0
1
2
3
4
5
6

Distributed Loads and Equivalent Nodal Loading – A Sanity Check

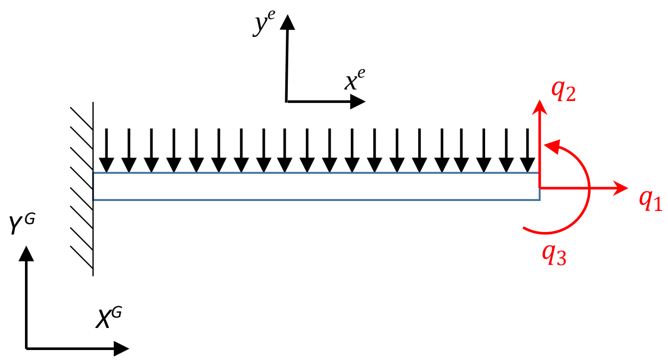
We used our trusty transverse shape functions and performed an integral to determine that a Uniformly Distributed Load (UDL) can be represented as:

$$f^e_{eq,UDL\,\{6\times1\}} = \begin{Bmatrix} 0 \\ \bar{w}L/2 \\ \bar{w}L^2/12 \\ 0 \\ \bar{w}L/2 \\ -\bar{w}L^2/12 \end{Bmatrix}$$



However, does this actually work? Does it match the answer that we would have obtained from a bending deflection calculation, such as what we would have done in ENME202? Or is it just garbage?

Let's consider a simple cantilever subjected to a UDL:



$\bar{w} = -10kN/m$

$E = 200GPa$

$A = 4 \times 10^{-4}\ m^2$

$I = 1 \times 10^{-5}\ m^4$

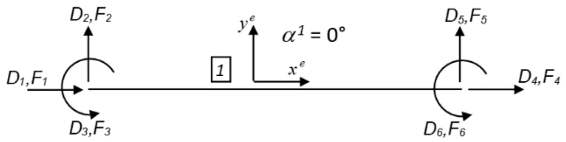
$L = 3m$

We can formulate the bending moment equation, integrate and evaluate the boundary conditions, and we would obtain:

Tip deflection is defined as: $v_{tip} = \dfrac{\bar{w}L^4}{8EI}$

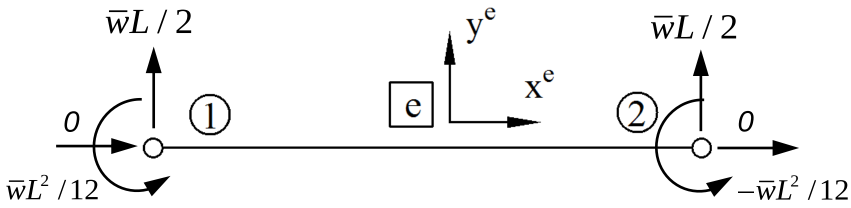
Tip rotation is defined as: $\left.\dfrac{dv}{dx}\right|_{x=L} = \theta_{tip} = \dfrac{\bar{w}L^3}{6EI}$

Now we can construct this specific problem using a finite element formulation.



$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Recall that the equivalent nodal loading to represent the UDL was:



The equivalent nodal loading at the nodal points that represent the uniformly distributed load:

$$f^e_{eq,UDL\,\{6\times1\}} = \begin{Bmatrix} 0 \\ \overline{w}L/2 \\ \overline{w}L^2/12 \\ 0 \\ \overline{w}L/2 \\ -\overline{w}L^2/12 \end{Bmatrix} = \begin{Bmatrix} 0N \\ -15,000N \\ -7500Nm \\ 0N \\ -15,000N \\ +7500Nm \end{Bmatrix}$$

$$Q^1_{eq} = A^1 F^1_{eq} = A^1 \left(\Lambda^{1^T} f^1_{eq}\right) = \begin{Bmatrix} 0N \\ -15,000N \\ +7,500Nm \end{Bmatrix}$$

Solving for deflections yields:

$$q = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = K_G^{-1} Q = \begin{Bmatrix} 0 \\ -0.0506m \\ -0.0225rad \end{Bmatrix}$$

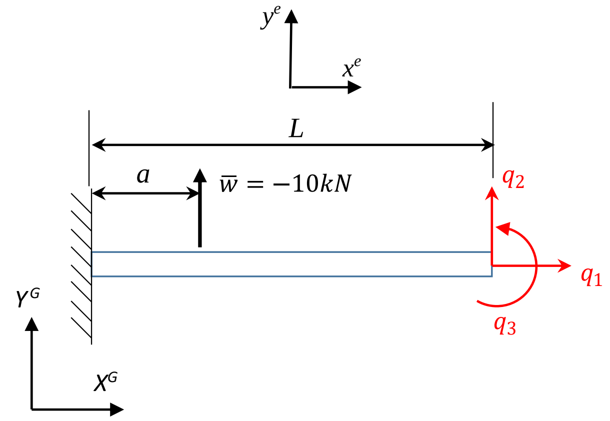
Kehat = Global_FRAME(E, I, A, L, theta) Kehat = 1.0e+07 * 2.6667 0 0 -2.6667 0 0 0 0.0889 0.1333 0 -0.0889 0.1333 0 0.1333 0.2667 0 -0.1333 0.1333 -2.6667 0 0 2.6667 0 0 0 -0.0889 -0.1333 0 0.0889 -0.1333 0 0.1333 0.1333 0 -0.1333 0.2667
A = [0 0 0 1 0 0; 0 0 0 0 1 0 ; 0 0 0 0 0 1]
KG = A*Kehat*A' KG = 1.0e+07 * 2.6667 0 0 0 0.0889 -0.1333 0 -0.1333 0.2667
f_{eq}^1 = [0, -15000, -7500, 0, -15000, 7500]' f_{eq}^1 = 0 -15000 -7500 0 -15000 7500
>> F_{eq}^1= Lambda1'*fp1 **(note Lambda1 = 6x6 identity matrix as α = 0°)** F_{eq}^1 = 0 -15000 -7500 0 -15000 7500
>> Q_{eq}^1 = A* F_{eq}^1 Q_{eq}^1= 0 -15000 7500
>> q = inv(KG)* Q_{eq}^1 q = 0 m – horizontal (axial) deflection at the tip -0.0506 m – vertical (transverse) deflection at the tip -0.0225 rad – rotation at the tip

And from a ENME202 hand analysis, we got:

Transverse tip deflection is defined as: $v_{tip} = \dfrac{\overline{w}L^4}{8EI} = \dfrac{(-10,000)(3^4)}{8(200\times10^9)(1\times10^{-5})} = -0.050625m$

Tip rotation is defined as: $\left.\dfrac{dv}{dx}\right|_{x=L} = \theta_{tip} = \dfrac{\overline{w}L^3}{6EI} = \dfrac{(-10,000)(3^3)}{6(200\times10^9)(1\times10^{-5})} = -0.0225\,rad$

Now, we will look at a **cantilever subjected to a point load part way along the element**:



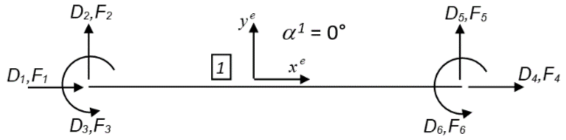
$E = 200GPa$

$A = 4 \times 10^{-4}\ m^2$

$I = 1 \times 10^{-5}\ m^4$

$L = 3m$

$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



If we assume that the load acts at 1.25m from the fixed support (a = 1.25m) and we formulate the bending moment equation, integrate and evaluate the boundary conditions, and we would obtain:

Tip deflection is defined as: $$v_{tip} = \frac{\overline{w}a^2}{6EI}(3L - a) = \frac{(-10,000)(1.25^2)}{6(200\times10^9)(1\times10^{-5})}(3(3) - 1.25) = -0.01009m$$

Using Finite Elements to solve the same problem, the point load can be represented by:

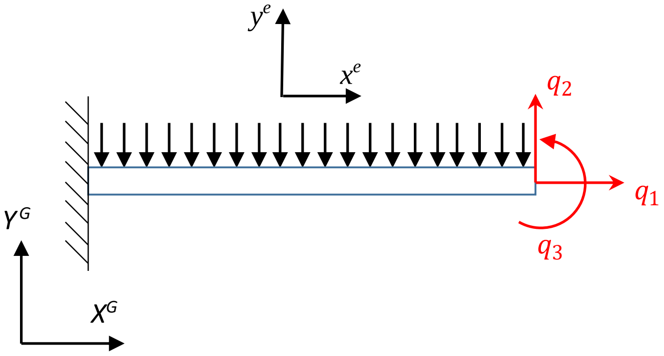
$$f^e_{eq,PL} = \overline{w}\begin{Bmatrix} 0 \\ 1 - \frac{3a^2}{L^2} + \frac{2a^3}{L^3} \\ \frac{a^3}{L^2} - \frac{2a^2}{L} + a \\ 0 \\ \frac{3a^2}{L^2} - \frac{2a^3}{L^3} \\ \frac{a^3}{L^2} - \frac{a^2}{L} \end{Bmatrix} = (-10,000)\begin{Bmatrix} 0 \\ 1 - \frac{3(1.25)^2}{(3)^2} + \frac{2(1.25)^3}{(3)^3} \\ \frac{(1.25)^3}{(3)^2} - \frac{2(1.25)^2}{3} + 1.25 \\ 0 \\ \frac{3(1.25)^2}{(3)^2} - \frac{2(1.25)^3}{(3)^3} \\ \frac{(1.25)^3}{(3)^2} - \frac{(1.25)^2}{3} \end{Bmatrix} = \begin{Bmatrix} 0N \\ -6238.4N \\ 4,253.5Nm \\ 0N \\ -3,761.6N \\ 3,038.2Nm \end{Bmatrix}$$

$$Q^1_{eq,PL} = A^1 F^1_{eq,PL} = A^1\left({\Lambda^1}^T f^1_{eq,PL}\right) = \begin{Bmatrix} 0N \\ -3,761.6N \\ 3,038.2Nm \end{Bmatrix} \Rightarrow \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = K_G^{-1} Q^1_{eq,PL} = \begin{Bmatrix} 0 \\ -0.01009m \\ -0.00391rad \end{Bmatrix}$$

"Lost Loads" and calculation of support reactions using equivalent nodal loading

When we consider support reactions for systems subjected to equivalent nodal loads, there is another aspect that we must consider. This additional consideration only occurs when a distributed load is represented as equivalent nodal loading **AND** one of the nodal points for that element is also a support point.

Let's bring back the simple cantilever beam, subjected to a Uniformly Distributed Load (UDL).



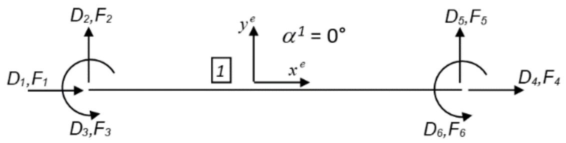
$\bar{w} = -10kN/m$

$E = 200GPa$

$A = 4 \times 10^{-4}\ m^2$

$I = 1 \times 10^{-5}\ m^4$

$L = 3m$



$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Recall that the equivalent nodal loading to represent the UDL was:

$$f^e_{eq,UDL\ \{6\times1\}} = \begin{Bmatrix} 0 \\ \bar{w}L/2 \\ \bar{w}L^2/12 \\ 0 \\ \bar{w}L/2 \\ -\bar{w}L^2/12 \end{Bmatrix} = \begin{Bmatrix} 0N \\ -15,000N \\ -7500Nm \\ 0N \\ -15,000N \\ +7500Nm \end{Bmatrix}$$

$$Q^1_{eq} = A^1 F^1_{eq} = A^1\left(\Lambda^{1^T} f^1_{eq}\right) = \begin{Bmatrix} 0N \\ -15,000N \\ +7,500Nm \end{Bmatrix}$$

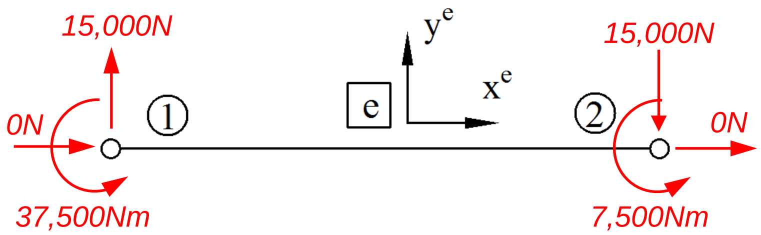
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = K_G^{-1}Q = \begin{Bmatrix} 0 \\ -0.0506m \\ -0.0225rad \end{Bmatrix}$$

Now, let's consider reaction loads....

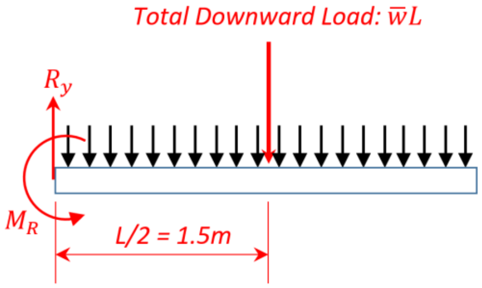
The assembly matrix extracts the deflection results and places those values in the appropriate locations within the displacement vector for element 1 (in global coordinates). We then multiply that result by the element stiffness matrix (in global coordinates) to get element forcing terms (in global coordinates):

$$D^1 = \left(A^1\right)^T q = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0\,m \\ 0\,m \\ 0\,rad \\ 0\,m \\ -0.0506\,m \\ -0.0225\,rad \end{Bmatrix} \qquad F^1 = \hat{K}^1 D^1 = \begin{Bmatrix} 0N \\ 15,000N \\ 37,500Nm \\ 0N \\ -15000N \\ 7,500Nm \end{Bmatrix}$$

We can draw these forces onto an element freebody diagram:



But let's consider overall equilibrium…



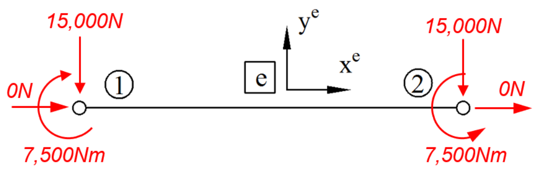
Moment Equilibrium: $\sum M = 0$: $\quad M_R - (\overline{w}L)\left(\frac{L}{2}\right) = 0$

$$M_R = \frac{\overline{w}L^2}{2} = \frac{(10{,}000)(3)^2}{2} = 45{,}000\ Nm$$

Vertical Force Equilibrium: $\sum F_y = 0$: $\quad R_y - (\overline{w}L) = 0$

$$R_y = (\overline{w}L) = (10{,}000)(3) = 30{,}000N$$

But hang on, wait just a minute, that doesn't match what we have above! Why is there a difference? Well, we also need to consider the applied (equivalent) external loads, which were:

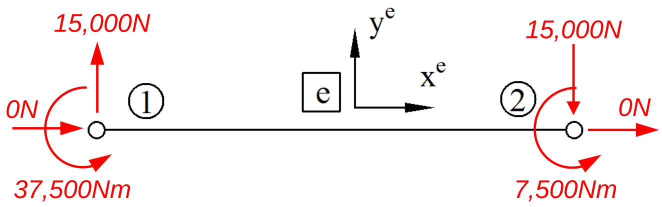


However, these equivalent loads are applied to the element, so that the supports will produce an equal and opposite force/moment to those shown above.

Considering these factors, we can develop this summary for how to deal with "lost loads" created when loads are applied directly to support points.

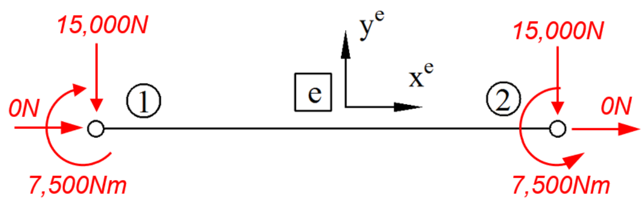
Internal Reactions, F^1:

$$F^1 = \hat{K}^1 D^1 = \hat{K}^1 \left(A^1\right)^T q = \begin{Bmatrix} 0N \\ 15{,}000N \\ 37{,}500Nm \\ 0N \\ -15000N \\ 7{,}500Nm \end{Bmatrix}$$



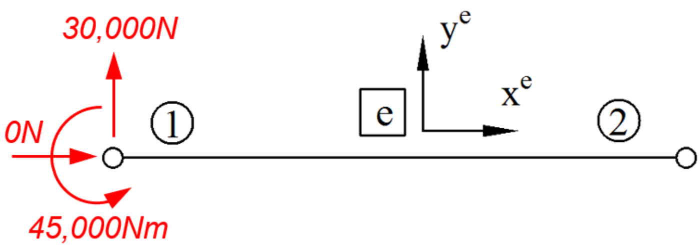
Applied equivalent nodal loading, F^1_{eq}:

$$F^e_{eq,UDL} = \begin{Bmatrix} 0N \\ -15{,}000N \\ -7500Nm \\ 0N \\ -15{,}000N \\ +7500Nm \end{Bmatrix}$$



Total reaction forces, $F^1 - F^1_{eq}$, or can be written as $F^1 + \left(-F^1_{eq}\right)$:

$$F^e_{reactions} = F^e + \left(-F^e_{eq,UDL}\right) = \begin{Bmatrix} 0N \\ 15{,}000N \\ 37{,}500Nm \\ 0N \\ -15{,}000N \\ +7500Nm \end{Bmatrix} + \begin{Bmatrix} 0N \\ 15{,}000N \\ 7500Nm \\ 0N \\ 15{,}000N \\ -7500Nm \end{Bmatrix} = \begin{Bmatrix} 0N \\ 30{,}000N \\ 45{,}000Nm \\ 0N \\ 0N \\ 0Nm \end{Bmatrix}$$



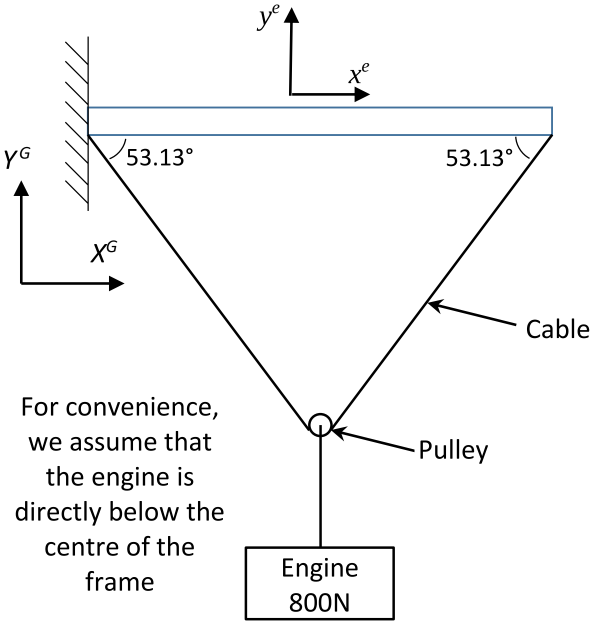
Which matches what we got from the equilibrium analysis, and all is well in the world again!

In most mechanics of materials problems, we don't consider loads being applied directly at supports, so this may seem counter-intuitive.

Overhead Frame and Cable Winch/Pulley

Does this make sense? Let's look at a new example that might make this concept a little easier to understand.

We'll consider a mechanic's workshop that has an overhead frame installed and a cable winch to lift engines out of cars:

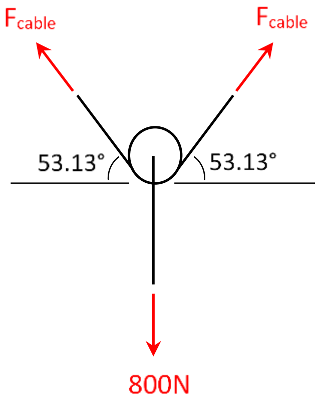


$E = 200GPa$

$A = 4 \times 10^{-4}\ m^2$

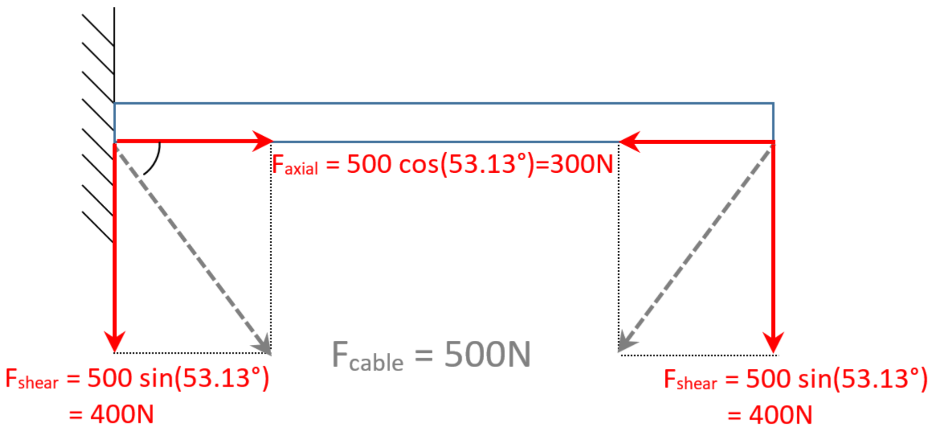
$I = 1 \times 10^{-5}\ m^4$

$L = 3m$

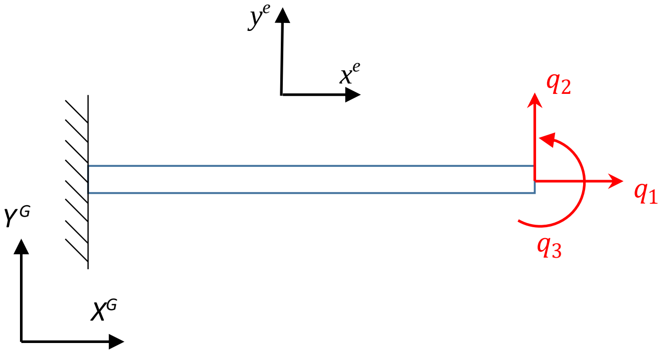


Vertical force equilibrium at the pulley gives: $2F_{cable}\sin(53.13°) = 800$, or $F_{cable} = 500N$.

Conveniently, due to the 53.13° angle (a 3-4-5 triangle), the cable force resolves into:



Let’s bring back the simple cantilever beam finite element model.



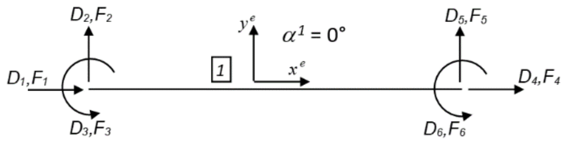
$\bar{w} = -10kN/m$

$E = 200GPa$

$A = 4 \times 10^{-4}\ m^2$

$I = 1 \times 10^{-5}\ m^4$

$L = 3m$



$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The external forcing vector for the element that represents the axial and shear loads from the cable is:

$$f^e = \begin{Bmatrix} 300N \\ -400N \\ 0Nm \\ -300N \\ -400N \\ 0Nm \end{Bmatrix}$$

The system-level forcing vector is:

$$Q_{eq}^1 = A^1 F_{eq}^1 = A^1\left({\Lambda^1}^T f_{eq}^1\right) = \begin{Bmatrix} -300N \\ -400N \\ 0Nm \end{Bmatrix}$$

Solving for deflections yields:

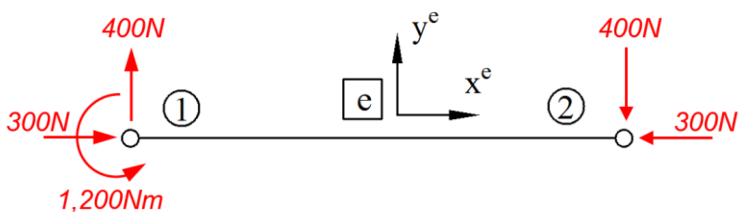
$$q = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = K_G^{-1} Q = \begin{Bmatrix} 0.0000m \\ -0.0018m \\ -0.0009rad \end{Bmatrix} = \begin{Bmatrix} -0.0112mm \\ -1.8000mm \\ -0.900mrad \end{Bmatrix}$$

Use the assembly matrix to get element deflections from global deflections and then forces:

$D^1 = \left(A^1\right)^T q = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0\,mm \\ 0\,mm \\ 0\,mrad \\ -0.0112mm \\ -1.8000mm \\ -0.900mrad \end{Bmatrix}$	$F^1 = \hat{K}^1 q = \begin{Bmatrix} 300N \\ 400N \\ 1200Nm \\ -300N \\ -400N \\ 0Nm \end{Bmatrix}$

We can show the element forcing vector (in global coordinates), F^1 , on an element freebody diagram:

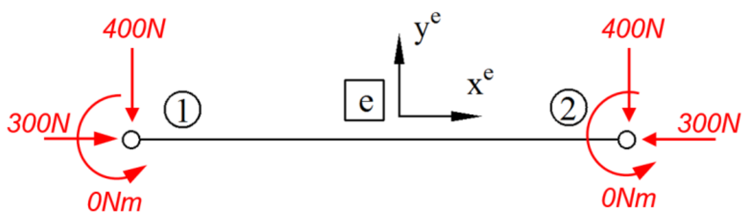
Forces in Element 1, F^1



From internal reactions within Element 1 alone, we get a vertical wall reaction of 400N. This does not match the 800N weight of the engine.

We must also consider the other loads that are applied directly to the support (where the cable anchors to the wall). These loads do not pass through element 1 and are therefore not "seen" by that element (and therefore not included in the element forcing vector).

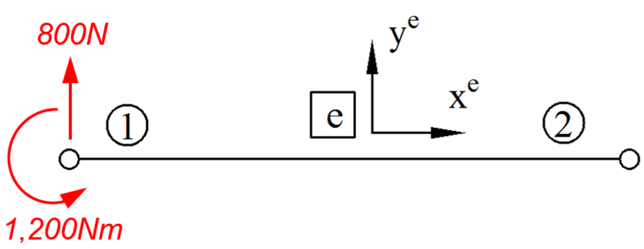
Applied External Loads, F^1_{eq}



Total reaction forces, $F^1 - F^1_{eq}$, or can be written as $F^1 + \left(-F^1_{eq}\right)$:

$$F^e_{reactions} = F^e + \left(-F^e_{eq}\right) = \begin{Bmatrix} 300N \\ 400N \\ 1,200Nm \\ -300N \\ -400N \\ 0Nm \end{Bmatrix} + \begin{Bmatrix} -300N \\ 400N \\ 0Nm \\ 300N \\ 400N \\ 0Nm \end{Bmatrix} = \begin{Bmatrix} 0N \\ 800N \\ 1,200Nm \\ 0N \\ 0N \\ 0Nm \end{Bmatrix}$$

Which we can show on an element freebody diagram:

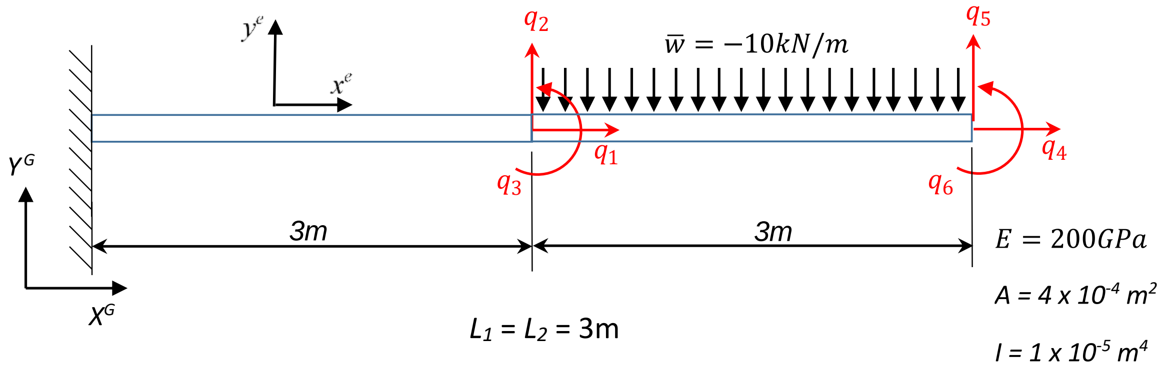


Which matches what we would expect from a simple equilibrium analysis.

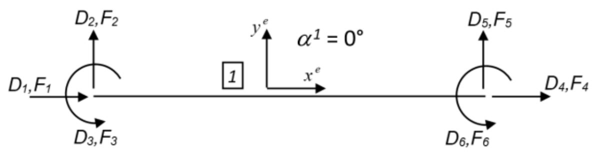
This complication only occurs when external loads are applied directly to support locations.

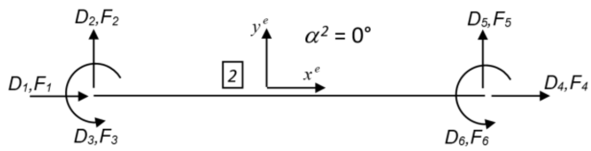
Cantilever Partially Loaded with a UDL

Let's look at a cantilever beam with only part of it subjected to a UDL.



Consider the overall cantilever as two separate elements with the UDL acting on only one element.





$$A^1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} D_1^1 & D_2^1 & D_3^1 & D_4^1 & D_5^1 & D_6^1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad A^2 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} D_1^2 & D_2^2 & D_3^2 & D_4^2 & D_5^2 & D_6^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The UDL acting on element 2 can be expressed as a equivalent nodal loads:

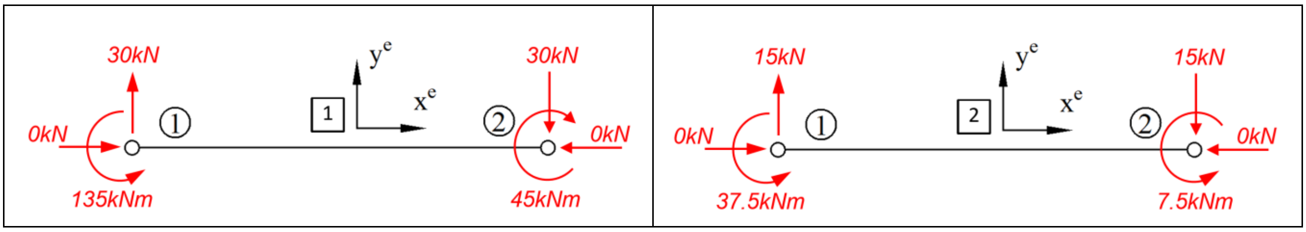
$$f^2_{eq,UDL\,\{6\times1\}} = \begin{Bmatrix} 0 \\ \overline{w}L/2 \\ \overline{w}L^2/12 \\ 0 \\ \overline{w}L/2 \\ -\overline{w}L^2/12 \end{Bmatrix} = \begin{Bmatrix} 0N \\ -15{,}000N \\ -7500Nm \\ 0N \\ -15{,}000N \\ +7500Nm \end{Bmatrix} \qquad Q^2_{eq} = A^2 F^2_{eq} = A^2\left(\Lambda^{2^T} f^2_{eq}\right) = \begin{Bmatrix} 0N \\ -15{,}000N \\ -7500Nm \\ 0N \\ -15{,}000N \\ +7500Nm \end{Bmatrix}$$

$$q = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = K_G^{-1}Q = \begin{Bmatrix} 0m \\ -0.2362m \\ -0.1350rad \\ 0m \\ -0.6919m \\ -0.1575rad \end{Bmatrix}$$

Use the assembly matrices to get element deflections from global deflections and then forces:

Element 1	Element 2
$D^1 = A^{1^T} q = \begin{Bmatrix} 0\,m \\ 0\,m \\ 0\,rad \\ 0\,m \\ -0.2362\,m \\ -0.1350\,rad \end{Bmatrix}$	$D^2 = A^{2^T} q = \begin{Bmatrix} 0m \\ -0.2362m \\ -0.1350rad \\ 0m \\ -0.6919m \\ -0.1575rad \end{Bmatrix}$
$F^1 = \hat{K}^1 q = \begin{Bmatrix} 0\,kN \\ 30\,kN \\ 135\,kNm \\ 0\,kN \\ -30\,kN \\ -45\,kNm \end{Bmatrix}$	$F^2 = \hat{K}^2 q = \begin{Bmatrix} 0\,kN \\ 15\,kN \\ 37.5\,kNm \\ 0\,kN \\ -15\,kN \\ 7.5\,kNm \end{Bmatrix}$

Which we can show on element freebody diagrams:



Overall Summary of the Finite Element Method

We have covered the basic element types that are very commonly used:

a) Bar Elements – carrying only axial loads (no shear force or moment)
b) Beam Elements – carrying only shear and moment loads (no axial terms)
c) Frame Elements – the combination of Bar and Beam elements, carrying axial, moment and shear terms

We used the principle of virtual displacements derivation to create local element stiffness matrices (K^e)

- This is a general approach.
- It is used to derive many different types of elements.
- Overall, the use of shape functions to relate internal element actions to behaviour at nodal points is common and generalisable. You will see this in a lot of various texts.
- Therefore, you can use what you've learned here as the foundation to learn more.

We have only dealt with two-dimensional problems in this section. However, the methods generalise very easily to three-dimensions (it just takes more book-keeping).

Where can we go to from here?

What about dynamics? How do we solve dynamic problems?

- **Mass matrices** can be generated in a very similar way to how distributed loads were converted into equivalent nodal loads – ie: turn distributed masses into equivalent nodal masses and assemble them into the system in a similar way.
- **Damping matrices** are typically defined as a function of the mass and stiffness matrices (take vibrations next year).

Using these matrices, you can set up and solve a dynamic problem as an equation of motion, such as:

$$M\ddot{q} + C\dot{q} + Kq = Q(t)$$

Solve using numerical integration, frequency domain or modal domain analysis, which are covered in the 3rd pro vibrations class.

Everything that we have covered is generalisable to much broader problems and models.