Homework 1 – ME 890 Fundamentals of Modern Control Theory

Q1. Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0).$$

The solution is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau,$$

$$\dot{\Phi}(t, \tau) = A(t)\Phi(t, \tau), \ \ \Phi(\tau, \tau) = I.$$

Prove:

$$\frac{d}{d\tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau).$$

Q2. Prove:

- If $AB = BA$, then $e^{A+B} = e^A e^B$.
- The inverse of e^{At} is e^{-At}.

Q3. Given

$$\dot{x}(t) = Ax(t) + Bu(t),$$

the zero-order-hold discretized model is

$$x_{k+1} = Fx_k + Gu_k.$$

Prove:

$$F = e^{Ah}, \qquad G = \left[I + \frac{Ah}{2!} + \frac{A^2h^2}{3!} + \frac{A^3h^3}{4!} + \cdots\right]Bh.$$

Further, prove: if A is invertible,

$$G = A^{-1}(e^{Ah} - I)B.$$

Q4. Given

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Suppose the holder holds $u(t)$ over $[t, t+h)$, where $t = kh$ is the k-th sampling instant, through

$$u(\tau) = u(t) + \frac{u(t+h) - u(t)}{h} \cdot (\tau - t), \quad t \leq \tau < t + h.$$

(Doesn't this holder make more sense? But note this holder must have access to future information due to the use of $u(t+h)$.) Show the obtained discrete-time model.

Q5. Consider discretizing the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

where $A(t)$ and $B(t)$ is piece-wise continuous, and $u(t)$ is subject to the zero-order holder. That is,

$$A(\tau) = A(t), \quad t \leq \tau < t + h,$$

$$B(\tau) = B(t), \quad t \leq \tau < t + h,$$

$$u(\tau) = u(t), \quad t \leq \tau < t + h,$$

where t is the sampling time instant. Suppose $t = kh$, where k is the sampling time index. We rewrite the above as

$$A(\tau) = A_k, \quad t \leq \tau < t + h,$$

$$B(\tau) = B_k, \quad t \leq \tau < t + h,$$

$$u(\tau) = u_k, \quad t \leq \tau < t + h,$$

Determine F_k and G_k of the corresponding discrete-time model

$$x_{k+1} = F_k x_k + G_k u_k.$$