Homework 3 – ME 890 Fundamentals of Modern Control Theory

Q1. Reproduce the simulation results presented in the following paper (attached):

- F.-C. Chen and H. K. Khalil, “Adaptive control of nonlinear systems using neural networks,” *International Journal of Control*, vol. 55, no. 6, pp. 1299-1317, 1992.

You may also find it helpful to review a related paper by the same authors on reference tracking:

- F.-C. Chen and H. K. Khalil, “Adaptive control of a class of nonlinear discrete-time systems using neural networks,” *IEEE Transactions on Automatic Control*, vol. 40, no. 5, pp. 791-801, 1995.

INT. J. CONTROL, 1992, VOL. 55, NO. 6, 1299–1317

Adaptive control of nonlinear systems using neural networks

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Layered networks are used in a nonlinear adaptive control problem. The plant is an unknown feedback-linearizable discrete-time system, represented by an input–output model. A state space model of the plant is obtained to define the zero dynamics, which are assumed to be stable. A linearizing feedback control is derived in terms of some unknown nonlinear functions. To identify these functions, it is assumed that they can be modelled by layered neural networks. The weights of the networks are updated and used to generate the control. A local convergence result is given. Computer simulations verify the theoretical result.

1. Introduction

Linearization by feedback (Isidori 1989) is a promising approach to the control of nonlinear systems. The essence of the idea is to transform a state space model of the plant into new coordinates where nonlinearities can be cancelled (fully or partially) by feedback. The major challenge in performing such cancellation is the need to know precise models of the nonlinearities. One approach to address this challenge is to use adaptive control where the controller learns the nonlinearities on line. This idea has been investigated for continuous-time systems (Taylor *et al.* 1989, Sastry and Isidori 1989) assuming that the nonlinearities can be parametrized linearly in some unknown parameters. In this paper we investigate a similar scheme for discrete-time systems, but we do not assume that the nonlinearities depend linearly on unknown parameters. Instead, we explore the use of layered neural networks (Rumelhart *et al.* 1986) to model the nonlinearities. The network comprises fixed (sigmoid-type) nonlinearities and adjustable weights which appear nonlinearly.

Layered neural networks have good potential for control applications because they can approximate nonlinear functions. Previous applications are available in the work of Psaltis *et al.* (1988), Zeman *et al.* (1989), Li and Slotine (1989), Narendra and Parthasarathy (1990), Chen (1990). Recently, it has been shown by Funahashi (1989), Cybenko (1988), Hornik *et al.* (1989), and Hecht-Nielson (1989), using different techniques, that layered neural networks can approximate any "well-behaved" nonlinear function to any desired accuracy. The theorem shown by Funahashi is quoted here.

Theorem 1: *Let $\phi(x)$ be a non-constant, bounded and monotonically increasing continuous function. Let K be a compact subset of $\mathbb{R}^n$ and $f(x_1, ..., x_n)$ be a real valued continuous function on K. Then for any $\varepsilon > 0$, there exists an integer N and real*

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constants c_i, $\sigma_i (i = 1, ..., N)$, $w_{ij}(i = 1, ..., N, j = 1, ..., n)$ *such that*

$$\tilde{f}(x_1, ..., x_n) = \sum_{i=1}^{N} c_i \phi \left(\sum_{j=1}^{n} w_{ij} x_j - \sigma_i \right) \tag{1}$$

satisfies $\max_{x \in K} |f(x_1, ..., x_n) - \tilde{f}(x_1, ..., x_n)| < \varepsilon$, *where the* σ_i*'s in* (1) *are bias weights.*

However, these results do not give an estimate of the number of neurons needed to approximate a nonlinear function given a specified error bound, nor do they say how to choose the weights. The difficulty of adjusting the weights in the network $\tilde{f}$ to reduce approximation errors arise from the fact that the weights appear nonlinearly in $\tilde{f}$. Let w denote a vector containing all variable weights in $\tilde{f}$. With the error defined as

$$E = (f - \tilde{f})^2$$

the back-propagation algorithm (Rumelhart *et al.* 1986) provides a neat method for calculating $\partial E / \partial w$, and then the weights can be updated as

$$w(k+1) = w(k) - \mu \left. \frac{\partial E}{\partial w} \right|_{w(k)} \tag{2}$$

where $w(k)$ represents the estimate of w at the kth training iteration. From the mathematical point of view, the back-propagation algorithm is a standard delta rule, but it allows us to modify hidden weights in the network.

We review the minimum phase property of nonlinear discrete-time systems in § 2. The adaptive control system using neural networks and a convergence result are described in § 3. Section 4 shows simulation results. Conclusions are given in § 5. A converse Lyapunov theorem for discrete-time systems, which is needed in the convergence proof in § 3, is provided in the Appendix.

2. Minimum-phase property

The concepts of zero dynamics and the minimum phase property for nonlinear continuous-time systems were introduced by Isidori and coworkers (Isidori 1989). They were adapted to the discrete-time case by Monaco and Normand-Cyrot (1987). Consider a single-input/single-output system of the form:

$$\left. \begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \right\} \tag{3}$$

It was shown by (Monaco and Normand-Cyrot 1987) that if certain conditions are satisfied, then there exist a state feedback control

$$u = \gamma(x, r) \tag{4}$$

and a change of coordinate $z = T(x)$ such that the closed-loop system is described by

$$\left. \begin{aligned} z_1(k+1) &= Az_1(k) + Br(k) \\ z_2(k+1) &= F(z_1(k), z_2(k), r(k)) \\ y(k) &= Cz_1(k) \end{aligned} \right\} \tag{5}$$

where (A, B, C) is a controllable–observable triple.

If system (5) starts from $z_1(0) = 0$ and $r \equiv 0$, then $z_1 \equiv 0$ and the plant output stays at zero. The motion of the system is determined by the dynamics of z_2, which gives rise to the notion of zero dynamics.

Definition: The *zero dynamics* of system (5) are defined to be

$$z_2(k+1) = F(0, z_2(k), 0) \tag{6}$$

The system is said to be *minimum phase* if the zero dynamics have an asymptotically stable equilibrium at the origin. □

In this paper we will strengthen this definition by requiring the origin to be globally exponentially stable.

3. Adaptive control using neural networks

We consider an adaptive regulation problem for a single-input/single-output relative-degree-one system

$$\begin{aligned} y_{k+1} = &f(y_k, \ldots, y_{k-n+1}, u_{k-1}, \ldots, u_{k-m}) \\ &+ g(y_k, \ldots, y_{k-n+1}, u_{k-1}, \ldots, u_{k-m})u_k \end{aligned} \tag{7}$$

Choosing the state variables as

$$\begin{aligned} z_{11}(k) &= y_{k-n+1} \\ &\vdots \\ z_{1n}(k) &= y_k \\ z_{21}(k) &= u_{k-m} \\ &\vdots \\ z_{2m}(k) &= u_{k-1} \end{aligned}$$

one obtains the state space model

$$\begin{aligned} z_{11}(k+1) &= z_{12}(k) \\ &\vdots \\ z_{1n}(k+1) &= f(z(k)) + g(z(k))u_k \\ z_{21}(k+1) &= z_{22}(k) \\ &\vdots \\ z_{2m}(k+1) &= u_k \end{aligned} \tag{8}$$

$$y(k) = z_{1n}(k) \tag{9}$$

It is noted that with the feedback control

$$u_k = \frac{1}{g(z(k))}[-f(z(k)) + r(k)]$$

the system takes the form (5), with

$$\left.\begin{aligned} A &= \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \cdots & 0 & 1 \\ 0 & & \cdots & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ \text{and} \quad C &= [0 \quad 0 \quad \ldots \quad 0 \quad 1] \end{aligned}\right\} \tag{10}$$

Assumption 1: $f(z)$ is smooth (differentiable a sufficient number of times) and vanishes at the origin.

Assumption 2: $g(z)$ is smooth and bounded away from zero over S, a compact subset of $\mathbb{R}^{n+m}$.

Assumption 3: The system is minimum phase. By that we mean the zero dynamics

$$z_{21}(k+1) = z_{22}(k)$$
$$\vdots$$
$$z_{2m}(k+1) = \left.\frac{-f(z(k))}{g(z(k))}\right|_{z_{1i}=0}$$

has a globally exponentially stable equilibrium point at the origin. It can be shown (see Claim 1 in the Appendix) that there exists a Lyapunov function $V_2(z_2(k))$ such that

$$c_1|z_2(k)|^2 \leqslant V_2(z_2(k)) \leqslant c_2|z_2(k)|^2 \tag{11}$$

$$V_2(z_2(k+1)) - V_2(z_2(k)) \leqslant -\alpha|z_2(k)|^2 \tag{12}$$

$$\left|\frac{\partial V_2(z)}{\partial z}\right| \leqslant L|z| \tag{13}$$

in some ball $B_{\rho 2} \subset \mathbb{R}^m$.

Assumption 4: The nonlinear functions f can be exactly represented by a multi-layer neural network $\hat{f}$ on the compact set S. $\hat{f}$ employs the *hyperbolic tangent* function in nonlinear neurons and contains no bias weights such that $\hat{f}$ vanishes at the the origin. In the case of a three-layer neural network,

$$f(z(k)) = \hat{f}(z(k), \boldsymbol{w}) = \sum_{i=1}^{p} w_i H\left(\sum_{j=1}^{m+n} w_{ij} z_j\right) \tag{14}$$

Assumption 5: The function g can be exactly represented by a multi-layer neural network $\hat{g}$ on the compact set S. The nonlinear neurons of $\hat{g}$ contains no bias weights. However, there is a bias weight added to the linear neuron at the output layer such that $\hat{g}$ can be bounded away from zero at the origin. In the case of a three-layer neural network,

$$g(z(k)) = \hat{g}(z(k), \boldsymbol{v}) = v_0 + \sum_{i=1}^{q} v_i H\left(\sum_{j=1}^{m+n} v_{ij} z_j\right) \tag{15}$$

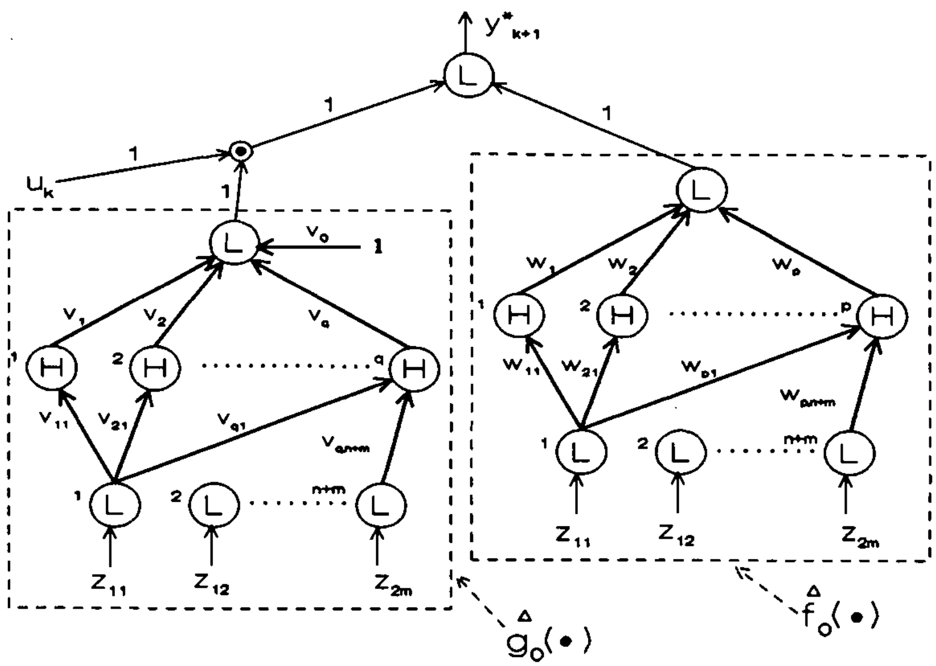

Figure 1. The neural network model.

The w and v in (14) and (15) are vectors containing variable weights in the neural networks. Let $\Theta = [w' \ v']'$. Now the plant (7) can be rewritten as

$$y_{k+1} = \hat{f}(z(k), w) + \hat{g}(z(k), v)u_k \tag{16}$$

The estimate of the plant is

$$y^*_{k+1} = \hat{f}(z(k), w(k)) + \hat{g}(z(k), v(k))u_k \tag{17}$$

where $w(k)$ and $v(k)$ are the estimates of w and v at time k. Figure 1 shows the model (17) constructed by a three-layer neural network. There is no theoretical evidence about how well three-layer neural networks without bias weights can approximate nonlinear functions. We expect that they can deal with certain classes of systems. Some evidence from simulation is provided later.

The purpose of the control is to regulate the plant output to zero asymptotically. At each time step, the control

$$u_k = -\frac{\hat{f}(z(k), w(k))}{\hat{g}(z(k), v(k))} \tag{18}$$

is applied to the plant. Then the weights are updated. Define the cost function to be

$$J_k = (y^*_{k+1} - y_{k+1})^2$$

The effect of adjusting weights on the cost function can be revealed by the following gradient:

$$\nabla_{\Theta(k)} J_k = (2(y^*_{k+1} - y_{k+1}) \begin{bmatrix} \left(\dfrac{\partial \hat{f}(z(k), w(k))}{\partial w(k)}\right)' \\ \left(\dfrac{\partial \hat{g}(z(k), v(k))}{\partial v(k)}\right)' u_k \end{bmatrix} \tag{19}$$

The weights are updated as follows:

$$\Theta(k+1) = \Theta(k) - \frac{\mu}{2r_k} \nabla_{\Theta(k)} J_k \tag{20}$$

where μ is a positive constant and

$$r_k = 1 + \left\| \begin{bmatrix} \left(\dfrac{\partial \hat{f}(z(k), w(k))}{\partial w(k)}\right)' \\ \left(\dfrac{\partial \hat{g}(z(k), v(k))}{\partial v(k)}\right)' u_k \end{bmatrix} \right\|^2$$

Notice that y^*_{k+1} in (19) equals zero, because the control (18), which is calculated from the model, can exactly cancel the model dynamics. Next a local convergence result is provided.

Theorem 2: *Given any constant $\rho > 0$, there exist positive constants $\rho_1 = \rho_1(\rho)$, $\rho_2 = \rho_2(\rho)$, and $\delta^* = \delta^*(\rho)$ such that if Assumption* 1 *is satisfied, Assumptions* 2, 4, 5 *are satisfied on $S \supset B_{\rho_1}$, Assumption* 3 *is satisfied on B_{ρ_2}, and*

$$|z(0)| \leqslant \rho$$

$$|\Theta(0) - \Theta| \leqslant \delta < \delta^*$$

then

$$y_k \to 0 \quad as \quad k \to \infty$$

Proof

Step 1: *The closed-loop control system*

The control u_k is defined in (18) as $-\hat{f}(z(k), w(k))/\hat{g}(z(k), v(k))$. Assume that $\hat{g}(z(k), v(k))$ is of the same sign as $\hat{g}(z(k), v)$ and bounded away from zero. Later we will show that this is true for all $\hat{g}(z(k), v(k))$ if $\Theta(0)$ is sufficiently close to Θ. Substituting u_k into (8) and (9), one gets

$$\begin{aligned} z_{11}(k+1) &= z_{12}(k) \\ &\vdots \\ z_{1n}(k+1) &= \left[\hat{f}(z(k), w) + \hat{g}(z(k), v) \left(-\frac{\hat{f}(z(k), w(k))}{\hat{g}(z(k), v(k))} \right) \right]_1 \\ z_{21}(k+1) &= z_{22}(k) \\ &\vdots \\ z_{2,m-1}(k+1) &= z_{2m}(k) \\ z_{2m}(k+1) &= -\frac{\hat{f}(z_1(k), z_2(k), w(k))}{\hat{g}(z_1(k), z_2(k), v(k))} \\ y(k) &= z_{1n}(k) \end{aligned} \tag{21}$$

The functions $\hat{f}$ and $\hat{g}$ in (21) and their derivatives are continuously differentiable infinitely many times. The term $[\cdot]_1$ vanishes when $\boldsymbol{\Theta}(k) = \boldsymbol{\Theta}$ and the derivative of $\hat{f}(z(k), \mathbf{w})/\hat{g}(z(k), \mathbf{v})$ with respect to $\boldsymbol{\Theta}$ is zero when $z(k) = 0$. The latter property can be seen from (14) and (15). Using these properties, we have

$$\begin{aligned}
[\cdot]_1 &= \left[\hat{f}(z(k), \mathbf{w}) + \hat{g}(z(k), \mathbf{v}) \left(-\frac{\hat{f}(z(k), \mathbf{w}(k))}{\hat{g}(z(k), \mathbf{v}(k))} \right) \right] \\
&= \left[\hat{f}(z(k), \mathbf{w}) + \hat{g}(z(k), \mathbf{v}) \left(-\frac{\hat{f}(z(k), \mathbf{w}(k))}{\hat{g}(z(k), \mathbf{v}(k))} \right) \right] \\
&\quad - \left[\hat{f}(z(k), \mathbf{w}) + \hat{g}(z(k), \mathbf{v}) \left(-\frac{\hat{f}(z(k), \mathbf{w}}{\hat{g}(z(k), \mathbf{v}} \right) \right] \\
&= \hat{g}(z(k), \mathbf{v}) \left[\frac{\hat{f}(z(k), \mathbf{w})}{\hat{g}(z(k), \mathbf{v})} - \frac{\hat{f}(z(k), \mathbf{w}(k))}{\hat{g}(z(k), \mathbf{v}(k))} \right] \\
&= \hat{g}(z(k), \mathbf{v}) \frac{\partial}{\partial \boldsymbol{\Theta}} \left[\frac{\hat{f}(z(k), \mathbf{w})}{\hat{g}(z(k), \mathbf{v})} \right] \Bigg|_{\boldsymbol{\Theta} + (1-\zeta)(\boldsymbol{\Theta}(k) - \boldsymbol{\Theta})} (\boldsymbol{\Theta}(k) - \boldsymbol{\Theta})
\end{aligned}$$

where the last equality follows from the Mean Value Theorem (Rudin 1976). Therefore, on any compact set of $(z, \boldsymbol{\Theta})$,

$$|[\cdot]_1| \leqslant k_1 |\tilde{\boldsymbol{\Theta}}(k)| \cdot |z(k)|, \quad \text{where } \tilde{\boldsymbol{\Theta}}(k) = \boldsymbol{\Theta}(k) - \boldsymbol{\Theta} \tag{22}$$

Step 2. *To choose a Lyapunov function associated with* z_1

$$z_1(k+1) = A z_1(k) + B[\cdot]_1$$

where A and B have been described in (10). A is a stable matrix (since all eigenvalues are at the origin). Therefore, given any symmetric $Q > 0$, $\exists$ a symmetric $P > 0$ such that $A'PA - P = -Q$. Choose the Lyapunov function

$$V_1(z_1(k)) = z_1'(k) P z_1(k)$$

Then, using (22),

$$\begin{aligned}
V_1(z_1(k+1)) - V_1(z_1(k)) &= -z_1'(k) Q z_1(k) + 2 z_1(k)' A' P \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} [\cdot]_1 \\
&\quad + [\cdot]_1^2 [0 \ \ \dots \ \ 1] P \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \\
&\leqslant -z_1'(k) Q z_1(k) + k_3 |\tilde{\boldsymbol{\Theta}}(k)| \, |z(k)|^2 + k_4 |\boldsymbol{\Theta}(k)|^2 \, |z(k)|^2
\end{aligned} \tag{23}$$

Step 3: *To choose a Lyapunov function associated with* z_2

The plant dynamics associated with z_2 is

$$
\begin{aligned}
z_{21}(k+1) &= z_{22}(k)\\
&\ \vdots\\
z_{2,m-1}(k+1) &= z_{2m}(k)\\
z_{2m}(k+1) &= \frac{-\hat{f}(z_1(k), z_2(k), w(k))}{\hat{g}(z_1(k), z_2(k), v(k))}\\
&= \frac{-\hat{f}(0, z_2(k), w(k))}{\hat{g}(0, z_2(k), v(k))}\\
&\quad + \left[\frac{-\hat{f}(z_1(k), z_2(k), w(k))}{\hat{g}(z_1(k), z_2(k), v(k))} - \frac{-\hat{f}(0, z_2(k), w(k))}{\hat{g}(0, z_2(k), v(k))}\right]_a\\
&= \frac{-\hat{f}(0, z_2(k), w)}{\hat{g}(0, z_2(k), v)}\\
&\quad + \left[\frac{-\hat{f}(z_1(k), z_2(k), w(k))}{\hat{g}(z_1(k), z_2(k), v(k))} - \frac{-\hat{f}(0, z_2(k), w(k))}{\hat{g}(0, z_2(k), v(k))}\right]_a\\
&\quad + \left[\frac{\hat{f}(0, z_2(k), w(k))}{\hat{g}(0, z_2(k), v(k))} - \frac{-\hat{f}(0, z_2(k), w)}{\hat{g}(0, z_2(k), v)}\right]_b
\end{aligned}
$$

By using the techniques similar to those we used in showing (22), we can arrive at

$$|[\,\cdot\,]_a \leqslant c_3|z_1(k)| \tag{24}$$

and

$$|[\,\cdot\,]_b| \leqslant c_4|\tilde{\boldsymbol{\Theta}}(k)|\cdot|z_2(k)| \tag{25}$$

on any compact set of $(z, \boldsymbol{\Theta})$. Let

$$s_k = \left[\ z_{22}(k), \ldots, z_{2m}(k), -\frac{\hat{f}(0, z_2(k), w)}{\hat{g}(0, z_2(k), v)}\right]'$$

and

$$q_k = [0, \ldots, 0, [\,\cdot\,]_a + [\,\cdot\,]_b]'$$

Applying (24), (25), (12) and (13), we have

$$
\begin{aligned}
V_2(z_2(k+1)) - V_2(z_2(k)) &= V_2(s_k + q_k) - V_2(z_2(k))\\
&= [V_2(s_k) - V_2(z_2(k))] + V_2(s_k + q_k) - V_2(s_k)\\
&\leqslant -k_5|z_2(k)|^2 + \hat{k}|\zeta s_k + (1-\zeta)q_k|\cdot|q_k|\\
&\leqslant -k_5|z_2(k)|^2 + \hat{k}_1|s_k|\cdot|q_k| + \hat{k}_2|q_k|^2\\
&\leqslant -k_5|z_2(k)|^2 + k_6|z_1(k)|\cdot|z_2(k)| + k_7|z_1(k)|^2\\
&\quad + c'_1|z_2(k)|^2|\tilde{\boldsymbol{\Theta}}(k)| + c'_2|z_1(k)|\cdot|z_2(k)|\,|\tilde{\boldsymbol{\Theta}}(k)|\\
&\quad + c'_3|z_2(k)|^2|\tilde{\boldsymbol{\Theta}}(k)|^2\\
&\leqslant -k_5|z_2(k)|^2 + k_6|z_1(k)|\cdot|z_2(k)| + k_7|z_1(k)|^2\\
&\quad + k_{11}|z(k)|^2|\tilde{\boldsymbol{\Theta}}(k)| + k_{12}|z(k)|^2|\tilde{\boldsymbol{\Theta}}(k)|^2
\end{aligned} \tag{26}
$$

where the first inequality follows from (12) and the Mean Value Theorem (Rudin 1976) with $0 < \zeta < 1$.

Step 4: *To combine Step* 3 *and Step* 4

Let $V(z(k)) = V_1(z_1(k)) + \beta V_2(z_2(k))$. Then, from (23) and (26),

$$\begin{aligned} V(z(k+1)) - V(z(k)) &\leqslant -z_1'(k)Qz_1(k) + k_3|\tilde{\boldsymbol{\Theta}}(k)|\,|z(k)|^2 + k_4|\tilde{\boldsymbol{\Theta}}(k)|^2\,|z(k)|^2 \\ &\quad - \beta k_5|z_2(k)|^2 + \beta k_6|z_1(k)|\,|z_2(k)| + \beta k_7|z_1(k)|^2 \\ &\quad + \beta k_{11}|z(k)|^2|\tilde{\boldsymbol{\Theta}}(k)| + \beta k_{12}|z(k)|^2\,|\tilde{\boldsymbol{\Theta}}(k)|^2 \\ &\leqslant [(-k_{10} + \beta k_7)|z_1|^2 - \beta k_5|z_2(k)|^2 + \beta k_6|z_1(k)|\,|z_2(k)|] \\ &\quad + (k_3 + \beta k_{11})|\tilde{\boldsymbol{\Theta}}(k)|z(k)|^2 + (k_4 + \beta k_{12})|\tilde{\boldsymbol{\Theta}}(k)|^2\,|z(k)|^2 \\ &\leqslant -k_1'|z(k)|^2 + k_3'|\tilde{\boldsymbol{\Theta}}(k)|\,|z(k)|^2 + k_4'|\tilde{\boldsymbol{\Theta}}(k)|^2\,|z(k)|^2 \end{aligned} \tag{27}$$

The last inequality is true if β is small enough.

Step 5: *A Lyapunov-type function related to weight convergence*

Rewrite the updating rule (20) as

$$\boldsymbol{\Theta}(k+1) = \boldsymbol{\Theta}(k) + \frac{\mu}{r_k}\Delta_k$$

where

$$\Delta_k = y_{k+1}\begin{bmatrix} \left(\dfrac{\partial \hat{f}(z(k), w(k))}{\partial w(k)}\right)' \\ \left(\dfrac{\partial \hat{g}(z(k), v(k))}{\partial v(k)}\right)' u_k \end{bmatrix}$$

Then

$$\tilde{\boldsymbol{\Theta}}(k+1) = \tilde{\boldsymbol{\Theta}}(k) + \frac{\mu}{r_k}\Delta_k$$

and the inner product of $\tilde{\boldsymbol{\Theta}}_{k+1}$ by itself is

$$\tilde{\boldsymbol{\Theta}}(k+1)'\tilde{\boldsymbol{\Theta}}(k+1) = \tilde{\boldsymbol{\Theta}}(k)'\tilde{\boldsymbol{\Theta}}(k) + \frac{2\mu}{r_k}\tilde{\boldsymbol{\Theta}}(k)'\Delta_k + \frac{\mu^2}{r_k^2}\Delta_k'\Delta_k \tag{28}$$

Notice that

$$y_{k+1} = \tilde{f}(z(k), w) + \hat{g}(z(k), v)u_k \tag{29}$$

and

$$0 = \hat{f}(z(k), w(k)) + \hat{g}(z(k), v(k))u_k \tag{30}$$

After subtracting (30) from (29), y_{k+1} can be rewritten as

$$y_{k+1} = -\tilde{\boldsymbol{\Theta}}(k)'\left.\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\right|_{\boldsymbol{\Theta} + (1-\zeta)(\boldsymbol{\Theta}(k) - \boldsymbol{\Theta}) \overset{\text{def}}{=} \Gamma(k)} \tag{31}$$

Now let us investigate the term $\tilde{\boldsymbol{\Theta}}(k)'\Delta_k$ in (28). It can be quickly verified that

$$\tilde{\boldsymbol{\Theta}}(k)'\Delta_k = \tilde{\boldsymbol{\Theta}}(k)'\left.\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\right|_{\boldsymbol{\Theta}_k} y_{k+1}$$

Then, making use of (31), we have

$$
\begin{aligned}
\tilde{\boldsymbol{\Theta}}(k)'\,\Delta_k &= \tilde{\boldsymbol{\Theta}}(k)'\,\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\bigg|_{\boldsymbol{\Theta}(k)} y_{k+1} \\
&= \tilde{\boldsymbol{\Theta}}(k)'\,\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\bigg|_{\Gamma(k)} y_{k+1} + \left[\tilde{\boldsymbol{\Theta}}(k)'\,\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\bigg|_{\boldsymbol{\Theta}(k)} y_{k+1}\right. \\
&\quad \left. - \tilde{\boldsymbol{\Theta}}(k)'\,\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\bigg|_{\Gamma(k)} y_{k+1}\right] \\
&= -y_{k+1}^2 + \left[\tilde{\boldsymbol{\Theta}}(k)'\,\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\bigg|_{\boldsymbol{\Theta}(k)} - \tilde{\boldsymbol{\Theta}}(k)'\,\frac{\partial y_{k+1}}{\partial \boldsymbol{\Theta}}\bigg|_{\Gamma(k)}\right] y_{k+1} \\
&= -y_{k+1}^2 + \left[\sum_{i=1}^{p} \tilde{w}_i \left(\frac{\partial \hat{f}}{\partial w_i}\bigg|_{\boldsymbol{\Theta}(k)} - \frac{\partial \hat{f}}{\partial w_i}\bigg|_{\Gamma(k)}\right)\right. \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{m+n} \tilde{w}_{ij} \left(\frac{\partial \hat{f}}{\partial w_i}\bigg|_{\boldsymbol{\Theta}(k)} - \frac{\partial \hat{f}}{\partial w_i}\bigg|_{\Gamma(k)}\right) \\
&\quad + \left\{\sum_{i=1}^{q} \tilde{v}_i \left(\frac{\partial \hat{g}}{\partial v_i}\bigg|_{\boldsymbol{\Theta}(k)} - \frac{\partial \hat{g}}{\partial v_i}\bigg|_{\Gamma(k)}\right)\right. \\
&\quad \left.\left. \times \sum_{i=1}^{q}\sum_{j=1}^{m+n} \tilde{v}_{ij} \left(\frac{\partial \hat{g}}{\partial v_{ij}}\bigg|_{\boldsymbol{\Theta}(k)} - \frac{\partial \hat{g}}{\partial v_{ij}}\bigg|_{\Gamma(k)}\right)\right\} u_k \right] y_{k+1}
\end{aligned}
\tag{32}
$$

The functions and parameters used in the last equality of (32) are defined in (14) and (15). By virtue of the special properties of the *hyperbolic tangent* function, each

$$
\tilde{w}\left(\frac{\partial \hat{f}}{\partial w}\bigg|_{\boldsymbol{\Theta}(k)} - \frac{\partial \hat{f}}{\partial w}\bigg|_{\Gamma(w)}\right) \quad \text{or} \quad \tilde{v}\left(\frac{\partial \hat{g}}{\partial v}\bigg|_{\boldsymbol{\Theta}(k)} - \frac{\partial \hat{g}}{\partial v}\bigg|_{\Gamma(k)}\right)
$$

term is of the order $|\tilde{\boldsymbol{\Theta}}(k)|^2 \cdot |z(k)|$. Moreover, the control u_k, as defined in (18), is of the order $|z(k)|$ and y_{k+1} is of the order $|\tilde{\boldsymbol{\Theta}}(k)| \cdot |z(k)|$ (see (22)).
So,

$$
\tilde{\boldsymbol{\Theta}}(k)'\,\Delta_k \leqslant -y_{k+1}^2 + k'|\tilde{\boldsymbol{\Theta}}(k)|^3(|z(k)|^2 + |z(k)|^3) \tag{33}
$$

Next, check the term $(\mu^2/r_k^2)\,\Delta_k'\,\Delta_k$ in (28).

$$
\frac{\mu^2}{r_k^2}\,\Delta_k'\,\Delta_k = \frac{\mu^2}{r_k^2}\left\|\begin{bmatrix} \left(\dfrac{\partial \hat{f}(T(z(k)), \boldsymbol{w}(k))}{\partial \boldsymbol{w}(k)}\right)' \\ \left(\dfrac{\partial \hat{g}(T(z(k)), \boldsymbol{v}(k))}{\partial \boldsymbol{v}(k)}\right)' u_k \end{bmatrix}\right\| y_{k+1}^2
$$

Since

$$
r_k = 1 + \left\|\begin{bmatrix} \left(\dfrac{\partial \hat{f}(T(z(k)), \boldsymbol{w}(k))}{\partial \boldsymbol{w}(k)}\right)' \\ \left(\dfrac{\partial \hat{g}(T(z(k)), \boldsymbol{v}(k))}{\partial \boldsymbol{v}(k)}\right)' u_k \end{bmatrix}\right\|^2
$$

we have

$$
\frac{\mu^2}{r_k^2}\,\Delta_k'\,\Delta_k < \frac{\mu^2}{r_k}\,y_{k+1}^2
$$

It can be shown that r_k is bounded on any compact set of $(z, \boldsymbol{\Theta})$. Setting $\mu = 1$, the equation (28) becomes

$$\tilde{\boldsymbol{\Theta}}(k+1)'\tilde{\boldsymbol{\Theta}}(k+1) - \tilde{\boldsymbol{\Theta}}(k)'\tilde{\boldsymbol{\Theta}}(k) \leqslant -k_8 y_{k+1}^2 + k_9|\tilde{\boldsymbol{\Theta}}(k)|^3(|z(k)|^2 + |z(k)|^3) \quad (34)$$

Final step: A Lyapunov function for the overall system

Choose the Lyapunov function

$$\tilde{V}(k) = \tilde{\boldsymbol{\Theta}}(k)'\tilde{\boldsymbol{\Theta}}(k) + \gamma^2 V(z(k))$$

By (27) and (34),

$$\begin{aligned}\tilde{V}(k+1) - \tilde{V}(k) \leqslant{} & -k_8 y_{k+1}^2 + k_9|\tilde{\boldsymbol{\Theta}}(k)|^3(|z(k)|^2 + |z(k)|^3) \\ & + \gamma^2(-k_1'|z(k)|^2 + k_3'|\tilde{\boldsymbol{\Theta}}(k)||z(k)|^2 + k_4'|\tilde{\boldsymbol{\Theta}}(k)|^2|z(k)|^2)\end{aligned} \quad (35)$$

Suppose $|z(0)| \leqslant K$. Then, by (11) and the definition of $V(z(k))$, there exists a constant d_1 such that

$$V(z(0)) \leqslant d_1 K^2$$

Consider the set

$$S = \left\{ \begin{pmatrix} z \\ \tilde{\boldsymbol{\Theta}} \end{pmatrix} \middle| \tilde{\boldsymbol{\Theta}}'\tilde{\boldsymbol{\Theta}} + \gamma^2 V \leqslant c^2 \right\}$$

It can be verified that if γ is chosen to be

$$\gamma = \frac{c}{K\sqrt{2d_1}} \quad (36)$$

then

$$|\tilde{\boldsymbol{\Theta}}(0)| \leqslant \frac{c}{\sqrt{2}} \quad \text{and} \quad |z(0)| \leqslant K \Rightarrow \begin{pmatrix} z(0) \\ \tilde{\boldsymbol{\Theta}}(0) \end{pmatrix} \in S$$

Next, we show that if c is chosen small enough, then the set S is an invariant set. For any $\begin{pmatrix} z(k) \\ \tilde{\boldsymbol{\Theta}}(k) \end{pmatrix} \in S$,

$$|\tilde{\boldsymbol{\Theta}}(k)| \leqslant c \quad (37)$$

and

$$|\sqrt{V(z(k))}\,| \leqslant K\sqrt{2d_1}$$

Again, by (11) and the definition of $V(z(k))$, there exists a constant d_2 such that

$$|z(k)| \leqslant d_2|\sqrt{V(z(k))}\,| \leqslant d_2 K\sqrt{2d_1} \quad (38)$$

Since S is a compact set, inequality (33) holds over S. Substituting (36), (37) and (38) into (35), one gets

$$\begin{aligned}\tilde{V}(k+1) - \tilde{V}(k) \leqslant{} & -k_8 y_{k+1}^2 - |z(k)|^2 \frac{k_1' c^2}{K^2(2d_1)} \\ & \times \left[1 - \frac{c}{k_1'}(k_3' + k_4' c + 2k_9 K^2 d_1 + 2d_1 k_9 K^3 d_2\sqrt{2d_1}) \right]\end{aligned} \quad (39)$$

It is obvious that there exists a c_0 such that if $c \leqslant c_0$, then (39) can be rewritten as

$$\tilde{V}(k+1) - \tilde{V}(k) \leqslant -k_8 y_{k+1}^2 - \tilde{k}|z(k)|^2 \quad (40)$$

and the set S is an invariant set. Since

$$\begin{pmatrix} z(0) \\ \Theta(0) \end{pmatrix} \in S, \quad \begin{pmatrix} z(k) \\ \tilde{\Theta}(k) \end{pmatrix} \in S \quad \forall k \geqslant 0$$

Furthermore, suppose $|\hat{g}(z(k), v)| \geqslant l_0 > 0$. Then, from the continuous property of the g function, given any $0 < l < l_0$, there exists a constant c_{00} such that

$$|\Theta(k) - \Theta| < \frac{c_{00}}{\sqrt{2}} \rightarrow |\hat{g}(z(k), v(k)) - \hat{g}(z(k), v)| < l_0 - l$$

Choose $c = \min(c_0, c_{00})$. For all initial conditions in S, $|\Theta(k) - \Theta|$ is less than $c_{00}/\sqrt{2}$; hence, $\hat{g}(z(k), v(k))$ is of the same sign as $\hat{g}(z(k), v)$ and bounded away from zero for all k. Finally, (40) implies that

$$\tilde{V}(k) \rightarrow \bar{V} \quad \text{as } k \rightarrow \infty \tag{41}$$

and therefore

$$y_k \rightarrow 0 \quad \text{as } k \rightarrow \infty \tag{42}$$

□

4. Simulation

Simulation programs are written in Microsoft C and run on an IBM PC compatible machine. We assume that the unknown plant is of the form

$$y_{k+1} = f(\cdot) + gu_k$$

and the sign of the constant g is known. The neural network in Fig. 2 (with two hidden layers, each containing seven nonlinear neurons) is used in the control

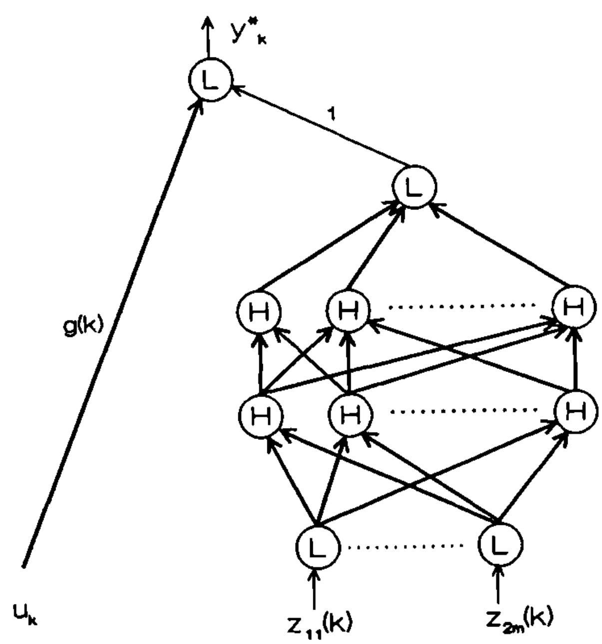

Figure 2. The neural network model used in simulations.

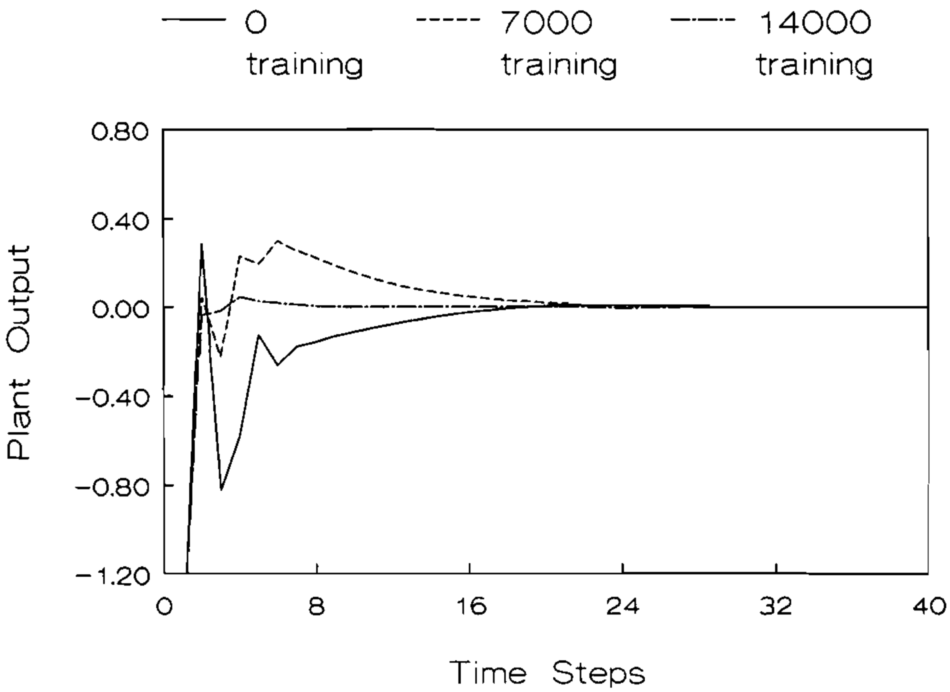

Figure 3.

system to regulate the output of the unknown plant to zero. For all simulations, the initial weights of neural networks are chosen randomly between $(-0{\cdot}1, 0{\cdot}1)$, with $\hat{g}(0)$ selected to be of the same sign as g.

Part 1: The neural network is used to model the plant

$$y_{k+1} = \frac{1{\cdot}5 y_k y_{k-1}}{1 + y_k^2 + y_{k-1}^2} + 0{\cdot}7 \sin[0{\cdot}5(y_k + y_{k-1})] \cos[0{\cdot}5(y_k + y_{k-1})] + 1{\cdot}2 u_k \tag{43}$$

Before the neural network is used for control purposes, it may go through a training process using a standard back-propagation algorithm. During the training, the control u_k is selected randomly from the interval $[-1{\cdot}7, 1{\cdot}7]$. The weights of the neural network at the end of the training are taken as the initial weights of the regulation process. When the neural network is used in feedback regulation, the updating rule is switched to (20), where μ is set to be one. Simulation results in Fig. 3 show how the different amount of training on the neural network would affect the performance of closed-loop regulation. An important message from Fig. 3 is that the plane output is regulated to zero even when the neural network is not pretrained.

Part 2: In this paper it is assumed that the nonlinearities of the plant are unknown but can be modelled by neural networks. There may be cases when the nonlinear functions of the plant are known, but the coefficients attached to these nonlinear functions are unknown. This situation has been considered recently in the continuous-time set-up by Sastry and Isidori (1989), and will be referred to as the analytic approach here. The purpose of this part of the simulation is to compare the robustness properties of the neural network approach and the analytic approach.

Suppose the plant is

$$y_{k+1} = \frac{1 \cdot 5 y_k y_{k-1}}{1 + y_k^2 + y_{k-1}^2} + 1 \cdot 2 u_k \tag{44}$$

The neural network model has been described in Part 1. In the analytic method, (44) is modelled by

$$y_{k+1} = a(k) \frac{1 \cdot 5 y_k y_{k-1}}{1 + y_k^2 + y_{k-1}^2} + b(k) u_k \tag{45}$$

where $a(k)$ and $b(k)$ are unknown coefficients, and $b(0)$ is selected to be positive. Both the neural net model and the analytic model are trained to approximate (44) before they are applied to regulation problems.

Suppose the actual plant is

$$y_{k+1} = \frac{1 \cdot 5 y_k y_{k-1}}{1 + y_k^2 + y_{k-1}^2} + w \cdot \sin\left[y_k + y_{k-1}\right] + 1 \cdot 2 u_k \tag{46}$$

where $w \cdot w\sin[\cdot]$ is an error term that represents the deviation of the actual plant from the design model (44). Figures 4 to 6 compare the plant outputs resulting from the neural network controller and the analytic controller when $w = 0{\cdot}0$, $0{\cdot}1$ and $0{\cdot}4$, respectively. The neural network control can always bring the plant output

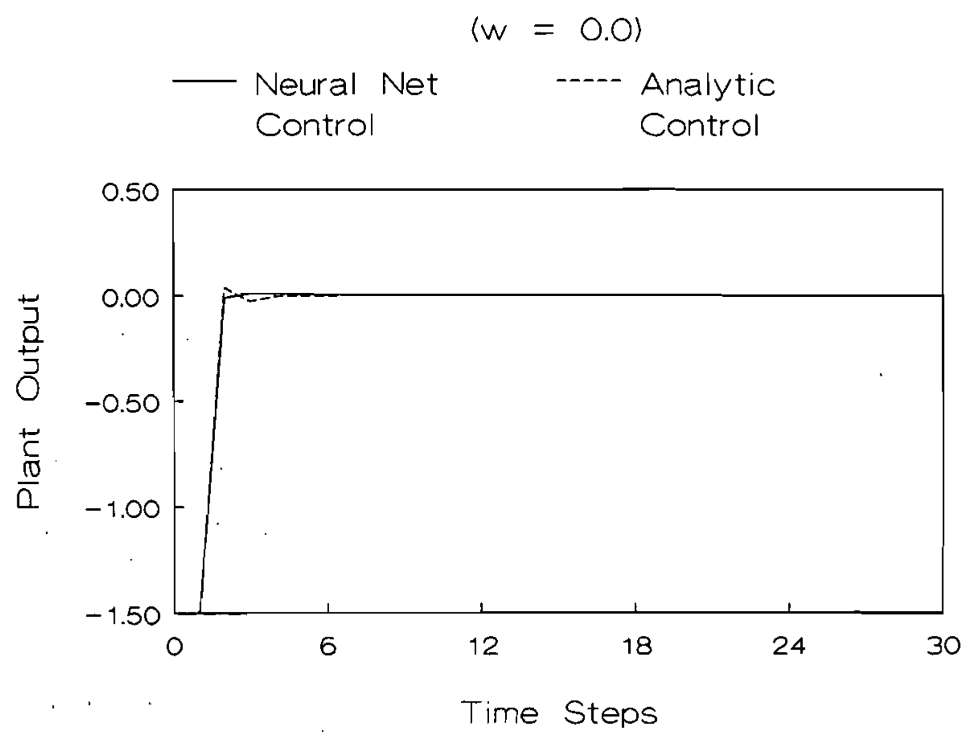

Figure 4.

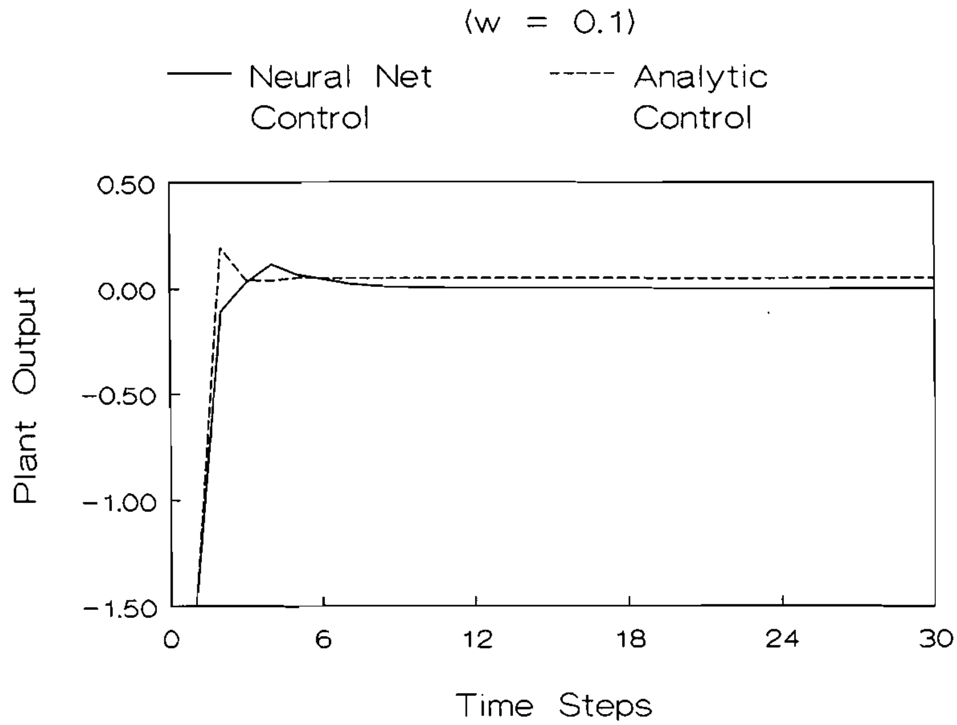

Figure 5.

to zero in all of these cases, while errors, which are roughly proportional to the magnitude of w, are observed when the analytic controller is used.

5. Conclusion

This paper is our first attempt to address issues related to 'adaptive control using neural networks' in a theoretical framework. The main challenge in this problem is the fact that the output of the neural network depends nonlinearily on the weights. The updating rule (20) is typical of rules used in adaptive control of linear systems (see for example Kumar and Variaya 1986, p. 294). As a consequence of using a linear learning rule in a nonlinear learning problem, our convergence result is local with respect to the weights; that is, we require the initial weights to be sufficiently close to the exact weights. We emphasize, however, that the convergence result is non-local with respect to the initial states of the plant, which could belong to any compact set. It is clear from the proof that the larger the initial states of the plant, the more restrictive the requirement on the initial weights.

We have made some unrealistic assumptions in this paper. Among them are that the function f_0 vanishes at the origin, the unknown system can be exactly modelled by the neural network, and that the neural network contains no bias weights. Under these assumptions we were able to show that the regulation error converges to zero asymptotically if the initial weight errors are small enough. These restrictive assumptions can be relaxed, but the learning rule needs to be modified. We have

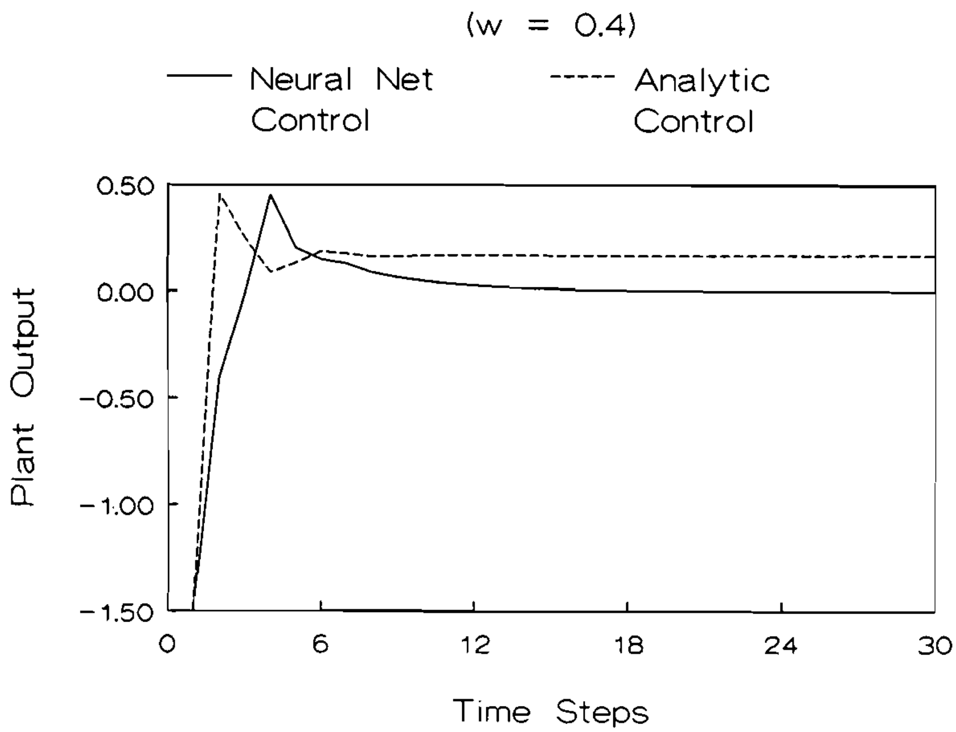

Figure 6.

previously described a dead-zone modification (Chen and Khalil 1991). The trade-off is that in that work we could no longer show that the tracking error tends to zero. Instead, we showed that the error converges to a small ball whose size is determined by the size of the dead zone.

Finally, we point out that the restriction of the class of systems to be of relative degree one is made for convenience. We previously treated systems of higher relative degree (Chen and Khalil 1991).

Appendix

Claim one: If the discrete-time system

$$x(k+1) = f(x(k)), \quad f(0) = 0$$

is exponentially stable at the origin and $f(x(k))$ is differentiable, then there exists a Lyapunov function $V(x(k))$ such that

$$c_1'|x(k)|^2 \leqslant V(x(k)) \leqslant c_2'|x(k)|^2 \tag{A 1}$$

$$V(x(k+1)) - V(x(k)) \leqslant -\alpha|x(k)|^2 \tag{A 2}$$

$$\left|\frac{\partial V(x)}{\partial x}\right| \leqslant L|x| \tag{A 3}$$

on any compact set, where c_1', c_2' and α are positive constants.

Proof: If $\boldsymbol{x}(0) = \boldsymbol{x}$, then

$$\begin{aligned} \boldsymbol{x}(1) &= f(\boldsymbol{x}) \\ \boldsymbol{x}(2) &= f(\boldsymbol{x}(1)) = f^2(\boldsymbol{x}) \\ &\vdots \\ \boldsymbol{x}(k) &= f^k(\boldsymbol{x}) \end{aligned}$$

The system is exponentially stable at the origin; therefore there exist $c > 0$ and $0 < \gamma < 1$ such that

$$|f^k(\boldsymbol{x})| \leqslant c\gamma^k |\boldsymbol{x}|$$

Select the Lyapunov function as

$$\begin{aligned} V(\boldsymbol{x}) &= \sum_{k=0}^{N-1} [f^k(\boldsymbol{x})]'[f^k(\boldsymbol{x})] \\ &= \boldsymbol{x}'\boldsymbol{x} + [f(\boldsymbol{x})]'[f(\boldsymbol{x})] + [f^2(\boldsymbol{x})]'[f^2(\boldsymbol{x})] + \cdots \end{aligned} \tag{A 4}$$

$V(\boldsymbol{x})$ is positive definite, since

$$V(\boldsymbol{x}) \geqslant 0$$

and

$$V(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = 0$$

To show that (A 1) is true, consider

$$\begin{aligned} V(\boldsymbol{x}) &= \sum_{k=0}^{N-1} [f^k(\boldsymbol{x})]'[f^k(\boldsymbol{x})] \\ &= \sum_{k=0}^{N-1} |f^k(\boldsymbol{x})|^2 \\ &\leqslant \sum_{k=0}^{N-1} c^2\gamma^{2k}|\boldsymbol{x}|^2 \\ &= c^2|\boldsymbol{x}|^2 \sum_{k=0}^{N-1} (\gamma^2)^k \\ &\leq c^2\left(\frac{1-\gamma^{2N}}{1-\gamma^2}\right)|\boldsymbol{x}|^2 \end{aligned}$$

Adding the fact that $V(\boldsymbol{x}) \geqslant \boldsymbol{x}'\boldsymbol{x}$, we have

$$|\boldsymbol{x}|^2 \leqslant V(\boldsymbol{x}) \leqslant c^2\left(\frac{1-\gamma^{2N}}{1-\gamma^2}\right)|\boldsymbol{x}|^2 \tag{A 5}$$

Next we show that (A 2) is true

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= \sum_{k=0}^{N-1} [f^{k+1}(x)]'[f^{k+1}(x)] - \sum_{k=0}^{N-1} [f^k(x)]'[f^k(x)] \\ &= \sum_{j=1}^{N} [f^j(x)]'[f^j(x)] - \sum_{k=0}^{N-1} [f^k(x)]'[f^k(x)] \\ &= [f^N(x)]'[f^N(x)] - x'x \\ &\leqslant c^2\gamma^{2N}|x|^2 - |x|^2 \\ &= -[1 - c^2\gamma^{2N}]|x|^2 \end{aligned}$$

where $[1 - c^2\gamma^{2N}]$ can be made positive if N is chosen large enough.

Finally we show that (A 3) is true. Since $f(x)$ is smooth, $(\partial/\partial x)$ $[f^k(x)]$ is bounded on any compact set of x. Hence,

$$\begin{aligned} \left| \frac{\partial}{\partial x} [f^k(x)]'[f^k(x)] \right| &= \left| 2[f^k(x)]' \frac{\partial}{\partial x} [f^k(x)] \right| \\ &\leqslant c_1 \gamma^k |x|, \quad c_1 > 0 \end{aligned}$$

Therefore

$$\begin{aligned} \left| \frac{\partial V}{\partial x} \right| &\leqslant \sum_{k=0}^{N-1} \left| \frac{\partial}{\partial x} [f^k(x)]'[f^k(x)] \right| \\ &\leqslant \sum_{k=0}^{N-1} c_1 \gamma^k |x| \\ &\leqslant c_1 \frac{1-\gamma^N}{1-\gamma} |x| \end{aligned} \qquad \text{(A 6)}$$

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