Homework 2 – ME 890 Fundamentals of Modern Control Theory

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1 Problem Statement

Q1: A discrete-time linear system can be expressed by the following transfer function:

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}\, u(t) = \frac{b_1 q^{-1} + \cdots + b_m q^{-m}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}}\, u(t),$$

where q^{-1} is the unit backward shift operator (refer to the z-transform). An often-used model in system identification is the autoregressive exogenous input (ARX) model:

$$A(q^{-1})y(t) = B(q^{-1})u(t) + e(t),$$

where $e(t)$ represents an error or noise.

(1) **Rewriting the ARX Model:** Show that the ARX model can be rewritten in the following linear regression form:

$$y(t) = \varphi^\top(t)\theta + e(t),$$

where

$$\varphi(t) = \begin{bmatrix} -y(t-1) \\ \vdots \\ -y(t-n) \\ u(t-1) \\ \vdots \\ u(t-m) \end{bmatrix}, \quad \theta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

(2) **Parameter Identification:** Identify the parameter vector θ using both the batch least squares (LS) and the recursive least squares (RLS) methods for a dryer system at

`https://homes.esat.kuleuven.be/ smc/daisy/daisydata.html`.

2 Solution

2.1 (1) Rewriting the ARX Model in Linear Regression Form

The ARX model is given by:

$$A(q^{-1})y(t) = B(q^{-1})u(t) + e(t),$$

with

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}, \quad B(q^{-1}) = b_1 q^{-1} + \cdots + b_m q^{-m}.$$

Expanding the left-hand side yields:

$$y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = b_1 u(t-1) + \cdots + b_m u(t-m) + e(t).$$

Rearrange the terms to isolate $y(t)$:

$$y(t) = -a_1 y(t-1) - \cdots - a_n y(t-n) + b_1 u(t-1) + \cdots + b_m u(t-m) + e(t).$$

By defining the regression (feature) vector

$$\varphi(t) = \begin{bmatrix} -y(t-1) \\ \vdots \\ -y(t-n) \\ u(t-1) \\ \vdots \\ u(t-m) \end{bmatrix}$$

and the parameter vector

$$\theta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_m \end{bmatrix},$$

we obtain the compact linear regression form:

$$y(t) = \varphi^\top(t)\theta + e(t).$$

2.2 (2) Parameter Identification via Least Squares

Batch Least Squares (LS)

For a dataset containing N samples, proceed as follows:

- Use 70 percent of the data for identifying the parameters and use the remaining 30 percent for testing the accuracy.
- Define the regression matrix $\Phi \in \mathbb{R}^{N \times (n+m)}$ as:

  $$\Phi = \begin{bmatrix} \varphi^\top(1) \\ \varphi^\top(2) \\ \vdots \\ \varphi^\top(N) \end{bmatrix},$$

  and let the output vector be:

  $$Y = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}.$$

- The batch LS estimate of θ is then given by:

$$\hat{\theta}_{\text{batch}} = (\Phi^\top \Phi)^{-1} \Phi^\top Y.$$

This method minimizes the mean squared error over the entire dataset.

Recursive Least Squares (RLS)

When data arrive sequentially, the RLS algorithm is particularly useful. Initialize the parameter estimate $\hat{\theta}(0)$ and the covariance matrix $P(0)$ (commonly chosen as $P(0) = \delta I$ with a large δ). Then, for each new data point at time t, update as follows:

$$K(t) = \frac{P(t-1)\varphi(t)}{1 + \varphi^\top(t) P(t-1) \varphi(t)},$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t) \left[y(t) - \varphi^\top(t) \hat{\theta}(t-1) \right],$$

$$P(t) = P(t-1) - K(t) \varphi^\top(t) P(t-1),$$

These equations allow the model to adapt online to new data, providing a computationally efficient means of tracking time-varying parameters. Note that RLS and LS achieve exactly the same solutions.

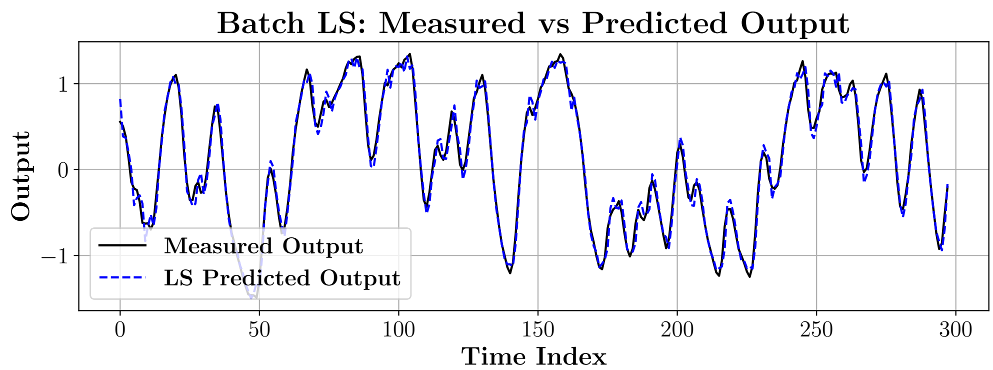
Application to the Dryer Dataset

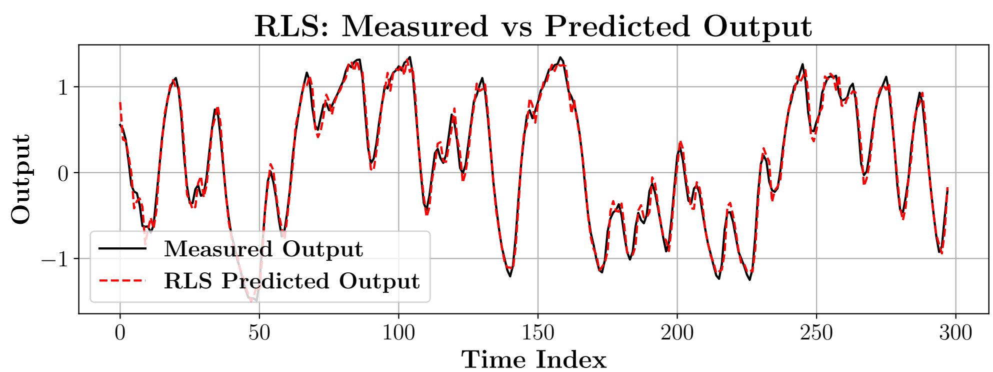
To estimate θ for the dryer system:

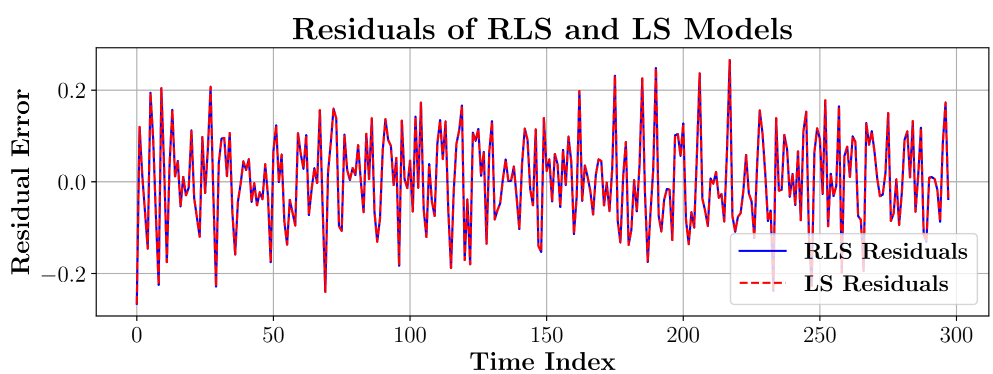
1. Preprocess the data by extracting the input $u(t)$ and output $y(t)$ signals.
2. For each time instant, construct the regressor $\varphi(t)$ from past outputs and inputs.
3. For batch LS:
   - Form the matrix Φ and vector Y as described.
   - Compute the estimate $\hat{\theta}_{\text{batch}} = (\Phi^\top \Phi)^{-1} \Phi^\top Y$.
4. For RLS:
   - Choose an initial guess $\hat{\theta}(0)$ and $P(0)$.
   - Update $\hat{\theta}(t)$ recursively using the equations provided.

Table 1: Comparison of Batch LS and RLS Estimation Results

	Batch LS	RLS
Estimate of θ	$[-1.68586212,\ 0.750728244,\ 0.000997468642,\ 0.0411371357]$	$[-1.68576352,\ 0.750632073,\ 0.000997421932,\ 0.0411412996]$
Validation RMSE	0.0982	0.0982
Fit Percentage	86.98%	86.98%


Batch LS: Measured vs Predicted Output
Output
Time Index
Measured Output
LS Predicted Output


RLS: Measured vs Predicted Output
Output
Time Index
Measured Output
RLS Predicted Output


Residuals of RLS and LS Models
Residual Error
Time Index
RLS Residuals
LS Residuals