Dynamic Systems and Control
Homework 4 Solutions

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Q1:

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2^2, \\
\dot{x}_2 &= -(x_1 + 1)x_2.
\end{aligned}
$$

The only equilibrium point is obviously at the origin $(0, 0)$ and this is unique.

Lyapunov function candidate:

$$
\begin{aligned}
V(x_1, x_2) &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0 \\
V(0, 0) &= 0, \quad \text{only for (0,0)}, \\
\dot{V}(x_1, x_2) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1(-x_1 + x_2^2) + x_2(-(x_1 + 1)x_2) \\
&= -x_1^2 + x_1x_2^2 - x_1x_2^2 - x_2^2 \\
&= -x_1^2 - x_2^2 < 0 \quad \text{for all } (x_1, x_2) \neq (0, 0).
\end{aligned}
$$

Because $V(x)$ is positive definite on the entire state space, and has the additional property that

$$
V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty,
$$

and $\dot{V}(x)$ is negative definite on the entire state space, then the equilibrium point at the origin is *globally asymptotically stable.*

Q2:

(a)

Choose

$$V(x) = f(x) - f(x^*).$$

$$V(x) > 0 \quad \forall x \neq x^*, \quad \text{since} \quad x^* \quad \text{is a local minimum.}$$

$$\dot{V}(x) = \nabla f(x)^\top \dot{x} = \nabla f(x)^\top(-g(x)) = -\|\nabla f(x)\|^2 \leq 0.$$

$\dot{V}(x)$ is negative definite (except at x^*, where $\dot{V}(x^*) = 0$). By Lyapunov's theorem, x^* is locally asymptotically stable. if $f(x) \to \infty$ as $\|x\| \to \infty$ the system is globally asymptotically stable and since $\dot{x} = -\nabla f(x)$ remains near x^* and converges to it, so $\lim_{t\to\infty} x(t) = x^*$.

(b)

Choose

$$V(x) = f(x) - f(x^*).$$

If

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|,$$

then:

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2}\|y - x\|^2.$$

By substituting $y = x(k+1) = x(k) - \alpha_k g(x(k))$, we get:

$$f(x(k+1)) \leq f(x(k)) - \alpha_k \|\nabla f(x(k))\|^2 + \frac{L}{2}\alpha_k^2 \|\nabla f(x(k))\|^2.$$

Thus,

$$\Delta V \leq -\alpha_k \left(1 - \frac{L\alpha_k}{2}\right) \|\nabla f(x(k))\|^2.$$

For $\Delta V \leq 0$, we require:

$$1 - \frac{L\alpha_k}{2} \geq 0 \quad \Rightarrow \quad \alpha_k \leq \frac{2}{L}.$$

$$0 < \alpha_k < \frac{2}{L},$$

at each step.

(c)

$$g(x) = \nabla f(x) = Qx, \quad x(k+1) = x(k) - \alpha Q x(k) = (I - \alpha Q)x(k).$$

For convergence, we require:

$$\|I - \alpha Q\| < 1.$$

Let λ_i be the eigenvalues of Q. Stability requires:

$$|1 - \alpha\lambda_i| < 1 \quad \forall i.$$

$$|1 - \alpha\lambda_i| < 1 \Rightarrow -1 < 1 - \alpha\lambda_i < 1 \Rightarrow 0 < \alpha\lambda_i < 2$$

This implies:

$$0 < \alpha < \frac{2}{\lambda_{\max}(Q)},$$

where $\lambda_{\max}(Q)$ is the largest eigenvalue of Q.

Q3

Lyapunov function :

$$V_k = \tilde{\theta}_k^\top P_k^{-1} \tilde{\theta}_k$$

where $\tilde{\theta}_k = \theta - \hat{\theta}_k$ is the parameter estimation error. The error evolution is given by:

$$\tilde{\theta}_{k+1} = (I - K_k \phi_k^\top) \tilde{\theta}_k$$

$$P_{k+1} = P_k - \frac{P_k \phi_k \phi_k^\top P_k}{1 + \phi_k^\top P_k \phi_k}$$

Using the Woodbury matrix identity (we had it in HW1), we derive:

$$P_{k+1}^{-1} = P_k^{-1} + \phi_k \phi_k^\top \tag{1}$$

Compute the temporal difference:

$$\begin{aligned} V_{k+1} - V_k &= \tilde{\theta}_{k+1}^\top P_{k+1}^{-1} \tilde{\theta}_{k+1} - \tilde{\theta}_k^\top P_k^{-1} \tilde{\theta}_k \\ &= \tilde{\theta}_k^\top (I - K_k \phi_k^\top)^\top P_{k+1}^{-1} (I - K_k \phi_k^\top) \tilde{\theta}_k - \tilde{\theta}_k^\top P_k^{-1} \tilde{\theta}_k \end{aligned}$$

Substitute P_{k+1}^{-1} from (1) and $K_k = \frac{P_k \phi_k}{1 + \phi_k^\top P_k \phi_k} = P_{k+1} \phi_k$ (we had it in HW2):

$$V_{k+1} - V_k = \tilde{\theta}_k^\top \left[(I - K_k \phi_k^\top)^\top (P_k^{-1} + \phi_k \phi_k^\top)(I - K_k \phi_k^\top) - P_k^{-1} \right] \tilde{\theta}_k \tag{2}$$

Define

$$A \triangleq I - K_k \phi_k^\top.$$

Since $A^\top = I - \phi_k K_k^\top$, we have

$$V_{k+1} - V_k = \tilde{\theta}_k^\top \Big[A^\top (P_k^{-1} + \phi_k \phi_k^\top) A - P_k^{-1} \Big] \tilde{\theta}_k.$$

$$A^\top (P_k^{-1} + \phi_k \phi_k^\top) A = (I - \phi_k K_k^\top)(P_k^{-1} + \phi_k \phi_k^\top)(I - K_k \phi_k^\top).$$

Expanding this product term by term:

$$\begin{aligned} A^\top (P_k^{-1} + \phi_k \phi_k^\top) A = {} & (P_k^{-1} + \phi_k \phi_k^\top) \\ & - (P_k^{-1} + \phi_k \phi_k^\top) K_k \phi_k^\top \\ & - \phi_k K_k^\top (P_k^{-1} + \phi_k \phi_k^\top) \\ & + \phi_k K_k^\top (P_k^{-1} + \phi_k \phi_k^\top) K_k \phi_k^\top. \end{aligned} \tag{3}$$

substituting (3) in (2) we obtain:

$$A^\top (P_k^{-1} + \phi_k \phi_k^\top) A - P_k^{-1} = \phi_k \phi_k^\top - (P_k^{-1} + \phi_k \phi_k^\top) K_k \phi_k^\top \tag{4}$$

$$- \phi_k K_k^\top (P_k^{-1} + \phi_k \phi_k^\top) \tag{5}$$

$$+ \phi_k K_k^\top (P_k^{-1} + \phi_k \phi_k^\top) K_k \phi_k^\top. \tag{6}$$

We know that

$$K_k = \frac{P_k \phi_k}{1 + \phi_k^\top P_k \phi_k}.$$

We now verify the identity

$$(P_k^{-1} + \phi_k \phi_k^\top) K_k = \phi_k.$$

Verification:

$$\begin{aligned}
(P_k^{-1} + \phi_k \phi_k^\top) K_k &= (P_k^{-1} + \phi_k \phi_k^\top) \frac{P_k \phi_k}{1 + \phi_k^\top P_k \phi_k} \\
&= \frac{1}{1 + \phi_k^\top P_k \phi_k} \Big(P_k^{-1} P_k \phi_k + \phi_k \phi_k^\top P_k \phi_k \Big) \\
&= \frac{1}{1 + \phi_k^\top P_k \phi_k} \Big(\phi_k + \phi_k \big(\phi_k^\top P_k \phi_k \big) \Big) \\
&= \frac{\phi_k \Big(1 + \phi_k^\top P_k \phi_k \Big)}{1 + \phi_k^\top P_k \phi_k} = \phi_k.
\end{aligned}$$

Using this result we can write equations in (4),(5),(6) as

$$(P_k^{-1} + \phi_k \phi_k^\top) K_k = \phi_k.$$

$$\begin{aligned}
(P_k^{-1} + \phi_k \phi_k^\top) K_k \phi_k^\top &= \phi_k \phi_k^\top, \\
\phi_k K_k^\top (P_k^{-1} + \phi_k \phi_k^\top) &= \phi_k \phi_k^\top.
\end{aligned}$$

$$\phi_k K_k^\top (P_k^{-1} + \phi_k \phi_k^\top) K_k \phi_k^\top = \phi_k \big(\phi_k^\top K_k \big) \phi_k^\top, \tag{7}$$

and since

$$\phi_k^\top K_k = \frac{\phi_k^\top P_k \phi_k}{1 + \phi_k^\top P_k \phi_k},$$

Eq. (7) cann can be rewreitten as

$$\phi_k \left(\frac{\phi_k^\top P_k \phi_k}{1 + \phi_k^\top P_k \phi_k} \right) \phi_k^\top.$$

Substituting these into our expansion (2):

$$\begin{aligned}
A^\top (P_k^{-1} + \phi_k \phi_k^\top) A - P_k^{-1} &= \phi_k \phi_k^\top - \phi_k \phi_k^\top - \phi_k \phi_k^\top \\
&\quad + \phi_k \left(\frac{\phi_k^\top P_k \phi_k}{1 + \phi_k^\top P_k \phi_k} \right) \phi_k^\top \\
&= \phi_k \phi_k^\top \left(\frac{\phi_k^\top P_k \phi_k}{1 + \phi_k^\top P_k \phi_k} - 1 \right).
\end{aligned}$$

Observe that

$$\frac{\phi_k^\top P_k \phi_k}{1 + \phi_k^\top P_k \phi_k} - 1 = -\frac{1}{1 + \phi_k^\top P_k \phi_k}.$$

Thus,

$$A^\top (P_k^{-1} + \phi_k \phi_k^\top) A - P_k^{-1} = -\frac{1}{1 + \phi_k^\top P_k \phi_k} \, \phi_k \phi_k^\top.$$

Substitute back into the original expression (2):

$$\begin{aligned} V_{k+1} - V_k &= \tilde{\theta}_k^\top \left[-\frac{1}{1 + \phi_k^\top P_k \phi_k} \, \phi_k \phi_k^\top \right] \tilde{\theta}_k \\ &= -\frac{1}{1 + \phi_k^\top P_k \phi_k} \, \tilde{\theta}_k^\top \phi_k \phi_k^\top \tilde{\theta}_k \\ &= -\frac{\left(\phi_k^\top \tilde{\theta}_k \right)^2}{1 + \phi_k^\top P_k \phi_k} \leq 0. \end{aligned}$$

So the error dynamics is marginally stable.