Homework 3 – ME 890 Fundamentals of Modern Control Theory

Q1. Prove that the following system is globally asymptotically stable, and show the phase portraits with different initial conditions.

$$\dot{x}_1 = -x_1 + x_2^2,$$

$$\dot{x}_2 = -(x_1 + 1)x_2.$$

Q2. The objective of this problem is to analyze the convergence of the gradient algorithm for finding a local minimum of a function. Let $f: R^n \to R$ and assume that x^* is a local minimum; i.e., $f(x^*) < f(x)$ for all x close enough but not equal to x^*. Assume that f is continuously differentiable. Let $g^\top: R \to R^n$ be the gradient of f:

$$g^\top = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}.$$

It follows from elementary Calculus that $g(x^*) = 0$.

If one has a good estimate of x^*, then it is argued that the solution to the dynamic system:

$$\dot{x} = -g(x)$$

with $x(0)$ close to x^* will give $x(t)$ such that

$$\lim_{t\to\infty} x(t) = x^*.$$

(a) Use Lyapunov stability analysis methods to prove the above argument.

(b) System $\dot{x} = -g(x)$ is usually solved numerically by the discrete-time system

$$x(k+1) = x(k) - \alpha_k \cdot g(x_k),$$

where α_k is the step size. Use Lyapunov stability analysis methods for discrete-time systems to give a possible choice for α_k so that

$$\lim_{t\to\infty} x(k+1) = x^*.$$

(c) Analyze directly the gradient algorithm for the function

$$f(x) = \frac{1}{2} x^\top Q x, \qquad Q > 0.$$

Suppose applying $x(k+1) = x(k) - \alpha \cdot g(x_k)$ using a real constant α. Find out tight bounds on α

Hint: Descent Lemma: For a differentiable function $h(x): R^n \to R$, if $\|\nabla h(x) - \nabla h(y)\| \leq L$ for any x and y and $L > 0$, then

$$h(y) \leq h(x) + \nabla^\top h(x) \cdot (y - x) + \frac{L}{2}\|y - x\|^2.$$

Q3. Recall Homework 2 which deals with system identification for a linear system via recursive least squares. We are interested in convergence analysis for this algorithm using Lyapunov stability analysis. Now consider

$$y_k = \phi_k^\top \theta,$$

where ϕ_k is the regression vector and θ is the parameter vector. The recursive least squares algorithm to estimate θ is

$$\hat{\theta}_{k+1} = \hat{\theta}_k + K_k\left(y_k - \phi_k^\top \hat{\theta}_k\right),$$

$$K_k = P_{k+1}\phi_k,$$

$$P_{k+1} = P_k - \frac{P_k \phi_k \phi_k^\top P_k}{1 + \phi_k^\top P_k \phi_k}.$$

The estimation error $\tilde{\theta}_k = \hat{\theta}_k - \theta$, which is governed by

$$\tilde{\theta}_{k+1} = (I - K_k \phi_k^\top)\, \tilde{\theta}_k.$$

Prove that the dynamics of $\tilde{\theta}_k$ is (marginally) stable.