Homework 3 – ME 890 Fundamentals of Modern Control Theory

Ali Vaziri

March 4, 2025

Problem 1

Consider the linear time-varying system

$$\dot{x}(t) \;=\; A(t)\,x(t) \;+\; B(t)\,u(t),$$

with the state–transition matrix $\Phi(t, \tau)$ defined by

$$\frac{\partial}{\partial t}\,\Phi(t,\tau) \;=\; A(t)\,\Phi(t,\tau), \quad \Phi(\tau,\tau) \;=\; I.$$

We also have the identity

$$\Phi(t,\tau)\,\Phi(\tau,t) \;=\; I,$$

where $\Phi(\tau, t) = \Phi(t, \tau)^{-1}$. Differentiate both sides of $\Phi(t, \tau)\,\Phi(\tau, t) = I$ with respect to τ:

$$\frac{\partial}{\partial \tau}\Big[\Phi(t,\tau)\,\Phi(\tau,t)\Big] \;=\; \frac{\partial}{\partial \tau}[I] \;=\; 0.$$

Using the product rule:

$$\frac{\partial}{\partial \tau}\,\Phi(t,\tau)\,\Phi(\tau,t) \;+\; \Phi(t,\tau)\,\frac{\partial}{\partial \tau}\,\Phi(\tau,t) \;=\; 0.$$

Hence,

$$\frac{\partial}{\partial \tau}\,\Phi(t,\tau)\,\Phi(\tau,t) \;=\; -\,\Phi(t,\tau)\,\frac{\partial}{\partial \tau}\,\Phi(\tau,t).$$

By definition, for each fixed t,

$$\frac{d}{d\sigma}\,\Phi(\sigma,t) \;=\; A(\sigma)\,\Phi(\sigma,t), \quad \Phi(t,t) \;=\; I.$$

Setting $\sigma = \tau$ gives

$$\frac{\partial}{\partial \tau}\,\Phi(\tau,t) \;=\; A(\tau)\,\Phi(\tau,t).$$

Substitute this back:

$$\frac{\partial}{\partial \tau}\,\Phi(t,\tau)\,\Phi(\tau,t) \;=\; -\,\Phi(t,\tau)\,\Big[A(\tau)\,\Phi(\tau,t)\Big] \;=\; -\,\Phi(t,\tau)\,A(\tau)\,\Phi(\tau,t).$$

Multiply both sides on the right by $\Phi(\tau, t)^{-1}$, noting that $\Phi(\tau, t)\,\Phi(t, \tau) = I$:

$$\frac{\partial}{\partial \tau}\,\Phi(t,\tau) \;=\; -\,\Phi(t,\tau)\,A(\tau).$$

This completes the proof.

$$\boxed{\frac{d}{d\tau}\,\Phi(t,\tau) \;=\; -\,\Phi(t,\tau)\,A(\tau).}$$

Problem 2

1. If $\mathbf{AB} = \mathbf{BA}$, then $e^{A+B} = e^A e^B$.

Theorem 1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be square matrices of the same dimension. Suppose they commute, i.e.* $\mathbf{AB} = \mathbf{BA}$*. Then*

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}\, e^{\mathbf{B}}.$$

Proof. Recall the definition of the matrix exponential via its power series:

$$e^{\mathbf{X}} = I + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^3}{3!} + \dots$$

Hence,

$$e^{\mathbf{A}+\mathbf{B}} = I + (\mathbf{A}+\mathbf{B}) + \frac{(\mathbf{A}+\mathbf{B})^2}{2!} + \frac{(\mathbf{A}+\mathbf{B})^3}{3!} + \cdots.$$

Because $\mathbf{AB} = \mathbf{BA}$, we can apply the binomial expansion to each power $(\mathbf{A}+\mathbf{B})^n$ exactly as in the scalar commutative case:

$$(\mathbf{A}+\mathbf{B})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k\, \mathbf{B}^{n-k}.$$

Then

$$e^{\mathbf{A}+\mathbf{B}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k \mathbf{B}^{n-k} \right).$$

We can split and regroup terms (interchanging sums is allowed under standard convergence theorems for power series of matrices), to obtain:

$$e^{\mathbf{A}+\mathbf{B}} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} \mathbf{A}^k \mathbf{B}^{n-k}.$$

Meanwhile, the product $e^{\mathbf{A}} e^{\mathbf{B}}$ expands as:

$$e^{\mathbf{A}} e^{\mathbf{B}} = \left(\sum_{m=0}^{\infty} \frac{\mathbf{A}^m}{m!} \right) \left(\sum_{\ell=0}^{\infty} \frac{\mathbf{B}^\ell}{\ell!} \right) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\mathbf{A}^m}{m!} \frac{\mathbf{B}^\ell}{\ell!}.$$

If we set $n = m + \ell$, then $\binom{n}{m}$ appears in regrouping the sums. One shows that each term in the double sum above corresponds exactly to the same binomial coefficients in the expansion of $e^{\mathbf{A}+\mathbf{B}}$. The details rest on the commutativity of $\mathbf{A}$ and $\mathbf{B}$. Thus,

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}\, e^{\mathbf{B}}.$$

□

2. The inverse of e^{At} is e^{-At}.

Theorem 2. *For any square matrix* $\mathbf{A}$ *and scalar* $t \in \mathbb{R}$*,*

$$\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t}.$$

Proof. We first observe that $\mathbf{A}t$ and $-\mathbf{A}t$ trivially commute because they are scalar multiples of the same matrix ($\mathbf{A}$). Hence, by Theorem 1:

$$e^{\mathbf{A}t}\, e^{-\mathbf{A}t} = e^{\mathbf{A}t+(-\mathbf{A}t)} = e^{\mathbf{0}} = I.$$

Since the product of $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$ is the identity matrix I, it follows that each is the inverse of the other. In particular:

$$\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t}.$$

□

Problem 3

Given the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

and assuming a zero-order hold on the input (i.e., $u(t) = u_k$ for $t \in [kh, (k+1)h)$), the solution over one sampling interval is

$$x((k+1)h) = e^{Ah}x(kh) + \left(\int_0^h e^{A\tau}\, d\tau\right) Bu_k.$$

Thus, the discrete-time model is

$$x_{k+1} = F\, x_k + G\, u_k,$$

with

$$F = e^{Ah}, \quad G = \left(\int_0^h e^{A\tau}\, d\tau\right) B.$$

Derivation of the Series Representation for G: The matrix exponential can be expanded as

$$e^{A\tau} = I + A\tau + \frac{A^2\tau^2}{2!} + \frac{A^3\tau^3}{3!} + \cdots.$$

Integrating term-by-term from 0 to h yields

$$\int_0^h e^{A\tau}\, d\tau = \sum_{k=0}^{\infty} \frac{A^k}{k!} \int_0^h \tau^k\, d\tau = \sum_{k=0}^{\infty} \frac{A^k h^{k+1}}{(k+1)!} = h\left[I + \frac{Ah}{2!} + \frac{A^2h^2}{3!} + \frac{A^3h^3}{4!} + \cdots\right].$$

Thus,

$$G = h\left[I + \frac{Ah}{2!} + \frac{A^2h^2}{3!} + \frac{A^3h^3}{4!} + \cdots\right] B.$$

Alternate Expression for G When A Is Invertible: If A is invertible, note that

$$\frac{d}{dh}\left(A^{-1}e^{Ah}\right) = A^{-1}Ae^{Ah} = e^{Ah}.$$

Since $A^{-1}e^{Ah}$ is an antiderivative of e^{Ah} and $e^{A\cdot 0} = I$, it follows that

$$\int_0^h e^{A\tau}\, d\tau = A^{-1}\left(e^{Ah} - I\right).$$

Therefore,

$$G = A^{-1}\left(e^{Ah} - I\right) B.$$

Final Answers:

$$\boxed{F = e^{Ah}, \quad G = h\left[I + \frac{Ah}{2!} + \frac{A^2h^2}{3!} + \frac{A^3h^3}{4!} + \cdots\right] B,}$$

and if A is invertible,

$$\boxed{G = A^{-1}\left(e^{Ah} - I\right) B.}$$

Problem 4

Consider the time-invariant linear system

$$\dot{x}(t) \;=\; A\,x(t) \;+\; B\,u(t),$$

where A is an $(n \times n)$-matrix, B is $(n \times m)$-matrix, and $u(t)$ is an m-dimensional input. For any $t \geq 0$, the exact solution at time $t+h$ can be written via the well-known variation of constants formula:

$$x(t+h) \;=\; e^{A\,h}\,x(t) \;+\; \int_t^{t+h} e^{A\,[\,t+h-\tau\,]}\,B\,u(\tau)\,\mathrm{d}\tau.$$

In many textbooks, one calls $e^{A\,(t+h-\tau)}$ the *state-transition matrix* from τ to $t+h$.

First-Order Hold (FOH) on $u(t)$ Over $[t, t+h]$

Now, assume that $u(\tau)$ is not merely constant (zero-order hold), but *linearly interpolated* between $u(t)$ and $u(t+h)$. Concretely:

$$u(\tau) \;=\; u(t) \;+\; \frac{u(t+h)-u(t)}{h}\,\big[\tau - t\big], \quad \tau \in [\,t,\, t+h\,].$$

Substitute this into the integral:

$$\int_t^{t+h} e^{A\,[\,t+h-\tau\,]}\,B\,\Big[u(t) \;+\; \tfrac{1}{h}\,[\,u(t+h)-u(t)\,](\tau-t)\Big]\,\mathrm{d}\tau.$$

We can split it into two parts:

$$\underbrace{u(t)\int_t^{t+h} e^{A\,[\,t+h-\tau\,]}B\,\mathrm{d}\tau}_{I_1} \;+\; \underbrace{\frac{1}{h}\,[\,u(t+h)-u(t)\,]\int_t^{t+h} e^{A\,[\,t+h-\tau\,]}B\,(\tau-t)\,\mathrm{d}\tau}_{I_2}.$$

Hence, the exact solution becomes

$$x(t+h) \;=\; e^{A\,h}\,x(t) \;+\; \underbrace{\int_t^{t+h} e^{A\,[\,t+h-\tau\,]}B\,\mathrm{d}\tau}_{\Gamma_0}\;u(t) \;+\; \frac{1}{h}\underbrace{\int_t^{t+h} e^{A\,[\,t+h-\tau\,]}B\,(\tau-t)\,\mathrm{d}\tau}_{\Gamma_1}\Big[u(t+h)-u(t)\Big].$$

We define the two FOH integrals:

$$\Gamma_0 \;=\; \int_t^{t+h} e^{A\,[\,t+h-\tau\,]}B\,\mathrm{d}\tau, \quad \Gamma_1 \;=\; \int_t^{t+h} e^{A\,[\,t+h-\tau\,]}B\,(\tau-t)\,\mathrm{d}\tau.$$

Since A is time-invariant, one can verify that Γ_0, Γ_1 are actually independent of t (they depend only on h). Concretely, with the variable shift $\alpha = \tau - t$, it becomes

$$\Gamma_0 = \int_0^h e^{A(h-\alpha)}\,B\,\mathrm{d}\alpha, \quad \Gamma_1 = \int_0^h e^{A(h-\alpha)}\,\alpha\,B\,\mathrm{d}\alpha.$$

Either form is valid. Setting $t = k\,h$, we find

$$x_{k+1} \;=\; x(t+h) \;=\; e^{A\,h}\,x_k \;+\; \Gamma_0\,u_k \;+\; \frac{1}{h}\,\Gamma_1\,\Big[u_{k+1}-u_k\Big].$$

$$x_{k+1} \;=\; e^{A\,h}\,x_k \;+\; \big(\Gamma_0 - \tfrac{1}{h}\Gamma_1\big)u_k \;+\; \big(\tfrac{1}{h}\Gamma_1\big)u_{k+1}.$$

Computing Γ_0 and Γ_1

In LTI systems, one often writes:

$$\Gamma_0 \;=\; \int_0^h e^{A\,\alpha}\,B\,\mathrm{d}\alpha, \quad \Gamma_1 \;=\; \int_0^h \alpha\,e^{A\,\alpha}\,B\,\mathrm{d}\alpha,$$

since Γ_0, Γ_1 do not depend on the particular t.

(a) Series Expansion Approach (Always Valid)

Recall $e^{A\,\alpha} = \sum_{n=0}^{\infty} \frac{(A\,\alpha)^n}{n!}$.

Computing Γ_0.

$$\Gamma_0 \;=\; \int_0^h e^{A\,\alpha}\,B\,\mathrm{d}\alpha \;=\; \int_0^h \Big[\sum_{n=0}^{\infty} \frac{(A\,\alpha)^n}{n!}\Big]\,B\,\mathrm{d}\alpha \;=\; \sum_{n=0}^{\infty} \frac{A^n}{n!} \int_0^h \alpha^n\,\mathrm{d}\alpha \;\; B.$$

Since $\int_0^h \alpha^n\,\mathrm{d}\alpha = \frac{h^{n+1}}{n+1}$, we obtain

$$\Gamma_0 \;=\; \sum_{n=0}^{\infty} \frac{A^n\,h^{\,n+1}}{n!\,(n+1}\,B \;=\; \sum_{n=0}^{\infty} \frac{A^n\,h^{\,n+1}}{(n+1)!}\,(n+1)\,B \;=\; \Big[I + A\frac{h}{2!} + A^2\frac{h^2}{3!} + \dots\Big]\,B\,h.$$

Computing Γ_1.

$$\Gamma_1 \;=\; \int_0^h \alpha\,e^{A\,\alpha}\,B\,\mathrm{d}\alpha \;=\; \sum_{n=0}^{\infty} \frac{A^n}{n!} \int_0^h \alpha^{n+1}\,\mathrm{d}\alpha\;B.$$

And $\int_0^h \alpha^{n+1}\,\mathrm{d}\alpha = \frac{h^{\,n+2}}{n+2}$. So

$$\Gamma_1 \;=\; \sum_{n=0}^{\infty} \frac{A^n\,h^{\,n+2}}{n!\,(n+2}\;B \;=\; \sum_{n=0}^{\infty} \frac{A^n\,h^{\,n+2}}{(n+2)!}\,\big(n+2\big)\big(n+1\big)\;\dots$$

We may leave it as a power series or factor it further (see below).

(b) Factorization If A Is Invertible

One standard identity is

$$\int_0^h e^{A\,\alpha}\,\mathrm{d}\alpha \;=\; A^{-1}\,\big[e^{A\,h} - I\big],$$

which immediately implies

$$\Gamma_0 \;=\; \int_0^h e^{A\,\alpha} B\,\mathrm{d}\alpha \;=\; A^{-1}\,\big[e^{A\,h} - I\big]\,B, \quad \text{(assuming } A \text{ is invertible).}$$

To compute $\Gamma_1 = \int_0^h \alpha\,e^{A\,\alpha}\,B\,\mathrm{d}\alpha$, we can use integration by parts:

$$u = \alpha\,I, \quad \mathrm{d}v = e^{A\alpha}B\,\mathrm{d}\alpha \;\Longrightarrow\; \mathrm{d}u = \mathrm{d}\alpha, \quad v(\alpha) = A^{-1}e^{A\alpha}B.$$

Hence

$$\Gamma_1 = \underbrace{\alpha\,A^{-1}e^{A\alpha}B\,\big|_{\alpha=0}^{\alpha=h}}_{=h\,A^{-1}e^{A\,h}\,B} - \int_0^h 1\cdot A^{-1}e^{A\alpha}B\,\mathrm{d}\alpha \;=\; h\,A^{-1}e^{A\,h}\,B \;-\; A^{-1}\int_0^h e^{A\alpha}B\,\mathrm{d}\alpha.$$

$$\Gamma_1 \;=\; h\,A^{-1}e^{A\,h}\,B \;-\; A^{-1}\,\Gamma_0 \;=\; h\,A^{-1}e^{A\,h}\,B \;-\; A^{-1}\,\Big[A^{-1}\big(e^{A\,h} - I\big)B\Big],$$

which can be rearranged to various forms.

Details of Change of Variable

Sometimes we prefer to rewrite the integral from 0 to h rather than from t to $t+h$. The change of variable is:

$$\alpha \;=\; \tau \;-\; t, \quad \mathrm{d}\alpha \;=\; \mathrm{d}\tau, \quad \text{and the bounds: } \tau = t \implies \alpha = 0, \quad \tau = t+h \implies \alpha = h.$$

Then, for the exponent,

$$t + h - \tau \;=\; t + h - (t + \alpha) \;=\; h \;-\; \alpha,$$

so the integral

$$\int_t^{t+h} e^{A\,[\,t+h-\tau\,]} B\,u(\tau)\,\mathrm{d}\tau \;\;=\;\; \int_0^h e^{A\,(h-\alpha)}\,B\,u\big(t+\alpha\big)\,\mathrm{d}\alpha.$$

Problem 5

Given System:

$$\dot{x}(t) \;=\; A(t)\,x(t) \;+\; B(t)\,u(t),$$

where $A(t)$ and $B(t)$ are piecewise continuous, and the input $u(t)$ is held constant over each sampling interval $[t,\, t+h)$. We assume that for $t \in [k\,h,\, (k+1)\,h)$,

$$A(\tau) = A_k, \quad B(\tau) = B_k, \quad u(\tau) = u_k.$$

Hence, for $\tau \in [k\,h,\, (k+1)\,h)$, the system becomes

$$\dot{x}(\tau) \;=\; A_k\,x(\tau) \;+\; B_k\,u_k.$$

Let $t = k\,h$. Over the interval $[t,\, t+h]$, the state satisfies

$$\dot{x}(\tau) = A_k\,x(\tau) + B_k\,u_k, \quad t \le \tau < t+h.$$

This is a *constant-coefficient* linear ODE in τ. The solution from $\tau = t$ to $\tau = t+h$ is given by the standard matrix exponential formula:

$$x(t+h) \;=\; e^{A_k\,h}\,x(t) \;+\; \int_0^h e^{A_k\,(h-\sigma)}\,B_k\,u_k\,\mathrm{d}\sigma.$$

Since u_k is constant over $[t,\, t+h)$, we can factor it out of the integral:

$$x(t+h) \;=\; e^{A_k\,h}\,x(t) \;+\; \left(\int_0^h e^{A_k\,(h-\sigma)}\,B_k\,\mathrm{d}\sigma\right) u_k.$$

Let us define the following matrices (constants over the interval since A_k, B_k are fixed for that interval):

$$F_k \;=\; e^{A_k h}, \qquad G_k \;=\; \int_0^h e^{A_k\,(h-\sigma)}\,B_k\,\mathrm{d}\sigma.$$