Strategies for the Shannon switching game

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June 16, 2000

We present a proof that the Shannon switching game on a graph with distinguished vertices A and B has a winning strategy for player Short iff the graph has a subgraph connecting A to B with two edge disjoint spanning trees. This theorem was first proved by Lehman in [11]. The difficulty is that his proof is phrased in terms of matroids and appears very difficult. However, I was able to cull a neat and short but elegant proof out of Lehman's work and that is what I want to present here. Using this proof, we will be able to give a strategy for player Cut for the graphs in which he has a win. This strategy is not be quite as neat as Short's strategy, and does not appear to be computationally feasible, but at least it does not involve any look ahead analysis.

We are given a multi-graph G with two distinguished vertices, A and B. The two players, Cut and Short play alternately. Short's move consists of marking an edge while Cut may remove any unmarked edge from the graph. If Short can succeed at marking an entire path from A to B, he wins. Otherwise Cut wins. As we shall see shortly, if G contains two trees connecting A to B having no edges in common but sharing the same vertex set, then Short has a winning strategy even if he goes second. Our main theorem is the converse, that if he has a winning strategy from the second position, then there must be two such trees. Brualdi has written an excellent expository article about the switching game. See [1]. This article has been essentially repeated as a section in his book [2, Section 11.6].

Let us say that a graph is *positive* if it has an edge disjoint pair of spanning trees. Then what we wish to show is that G has a positive subgraph containing both A and B if and only if Short has a winning strategy even if he plays second. The reader may easily verify that the union of two positive graphs is either disconnected or positive.

The reader may also also easily verify that if there is a positive subgraph connecting A to B, then Short has a winning strategy. He proceeds as follows: If Cut removes an edge from one of the two trees, Short finds an edge in the other tree which reconnects the broken tree and marks it. He then has two spanning trees for the subgraph with only marked edges in common.

Let us prove the converse. Suppose Short goes second and has a winning strategy. We must prove the existence of a positive subgraph. So Cut goes first and deletes an edge. Short then marks an edge, call it a. By induction, we may assume that there are two trees, S_1 and T_1 from A to B having a common vertex set but only the edge a in common.

We now need a lemma. We shall state and use the lemma and only when the main proof is finished will we come back and prove it. The lemma is the following: *If S and T are two trees with a common vertex set and at most one edge in common and if P and Q are two distinct vertices of these trees, then either $S \cup T$ has a positive subgraph connecting P to Q or it has two spanning trees with only one edge in common having the additional property that in at least one of the trees the common edge lies on the path from P to Q.* As a first use of this lemma, we may as well assume that a lies on the path in S_1 from A to B.

If every edge lying on the path from A to B in T_1 was spanned by a positive subgraph of G, then the union of these positive subgraphs would be a positive graph connecting A to B. So choose an edge b not spanned by any positive subgraph of G and lying on the path in T_1 from A to B. Consider the fact that Cut could have chosen b on his first move instead of the one he did choose. Again, by induction, we get trees S_2 and T_2 avoiding b with a common vertex set and at most one edge in common joining A to B.

Deleting the edge a from S_1 splits S_1 into two connected components with A in one of the components and B in the other. Deleting the edge b from T_1 has a similar effect. Thus, since S_2 and T_2 are both connected graphs containing both A and B, the graphs $T_2 \cup (T_1 \setminus \{b\})$ and $S_2 \cup (S_1 \setminus \{a\})$ are both connected. In order to apply the Lemma, we also need the condition that the graphs have only one edge in common. For these graphs this need not be the case since S_1 may overlap T_2 and T_1 may overlap S_2. However, since T_2 spans every edge in S_2, we may further take all the edges of $S_2 \setminus T_2$ away from $T_2 \cup (T_1 \setminus \{b\})$ and still have a connected graph. Similarly, we may delete $T_2 \setminus S_2$ from $S_2 \cup (S_1 \setminus \{a\})$ without losing connectivity. Thus, the two graphs, $T_3 = T_2 \cup (T_1 \setminus (S_2 \cup \{b\}))$ and $S_3 = S_2 \cup (S_1 \setminus (T_2 \cup \{a\}))$ are both connected. Both these graphs span the edge b, but neither contain

b. Furthermore, they have at most one edge in common. Using the lemma again, we see that $S_3 \cup T_3$ must have two spanning trees whose only common edge lies on the path in one of the two trees between the two ends of b. That tree may then be disconnected by deleting the common edge from it and then reconnected by adding the edge b. This proves the theorem and we are left with proving the lemma.

To this end, let S and T be the two given trees. A *spanning chain* is a sequence $C_0, C_1, \ldots, C_n$ such that C_0 is the path in S from P to Q and for each $i < n/2$, the path C_{2i+1} is the path in T spanning an edge in C_{2i} and C_{2i+2} is the path in S spanning an edge in C_{2i+1}. If the common edge is not on any path in any spanning chain, then let S' to be all the edges e in S for which there is a spanning chain with a path containing e, and let T' be all the edges e in T for which there is a spanning chain with a path containing e. We can easily see that S' and T' are two edge disjoint trees with a common vertex set connecting the two given vertices and thus the first possibility holds.

We complete the lemma by induction on the length of the shortest chain containing the common edge. Let $C_0, \ldots, C_{n-1}, C_n$ be such a shortest chain. If n is 0, there is nothing to prove. If n is even, then C_n is a path in S, otherwise it is a path in T. In any event, the common edge lies in C_n. Proceed as follows: Delete the common edge from the appropriate tree. (S if n is even, T if n is odd.) Then reconnect the tree by adding to it the edge in C_{n-1} spanned by C_n. This new doubled edge lies on a shorter spanning chain and so the lemma follows by induction.

The above proof yields a strategy for player Cut. Suppose it is Cut's turn and Short does not have a forced win. Cut should then find a pair of trees S and T connecting A to B with a common vertex set but only one unmarked edge in common. If there are no such trees, he may play at random. Finding such a pair however, he should perform the construction given in the lemma to ensure that the common edge lies on the path in S from A to B. He should then find an edge on the path in T from A to B which is not in any positive subgraph of G and remove it. Our proof has just shown that if this procedure allows Short to get a forced win, then Short already had a forced win before Cut made his move.

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