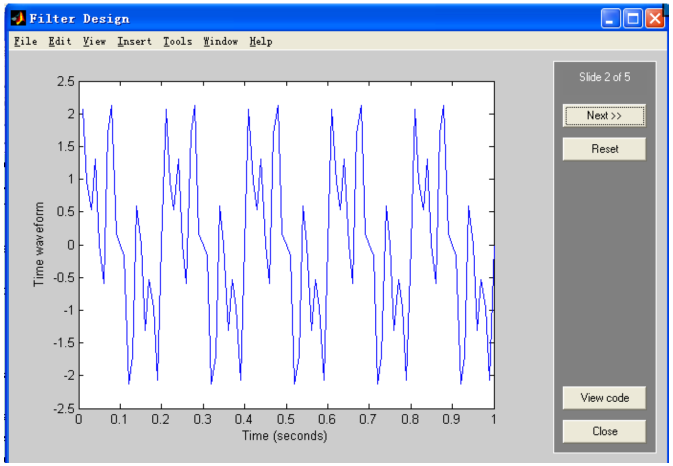
# 第三章　　连续信号的正交分解

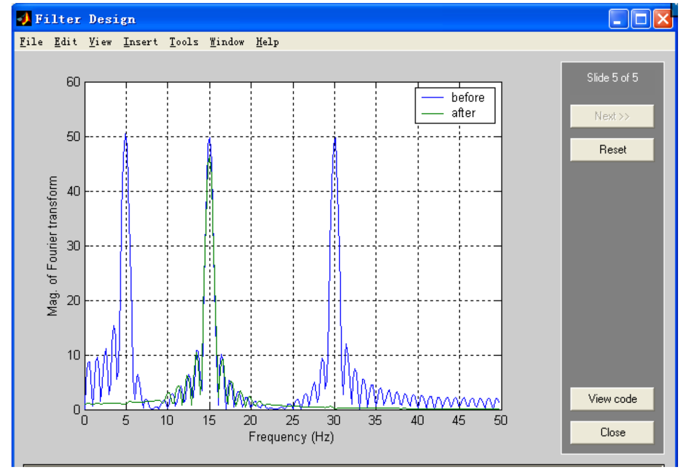
## §3.1　　引　　言

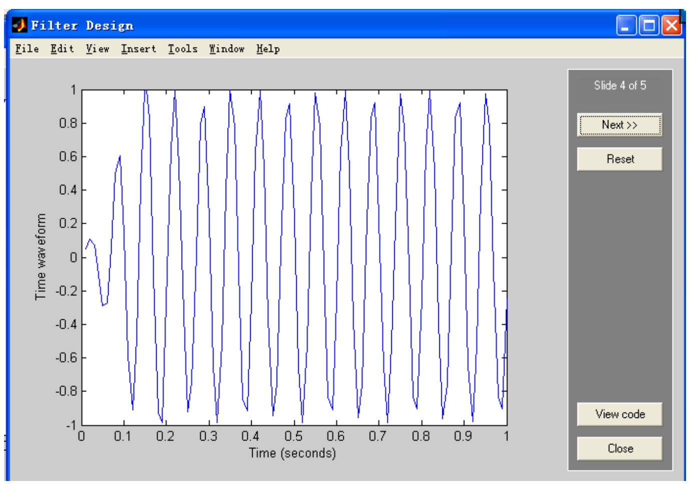
信号具有时域特性和频域特性，本章讨论信号的频域特性，其目的一是掌握信号频域特性的分析，二是为系统的频域分析方法作准备。

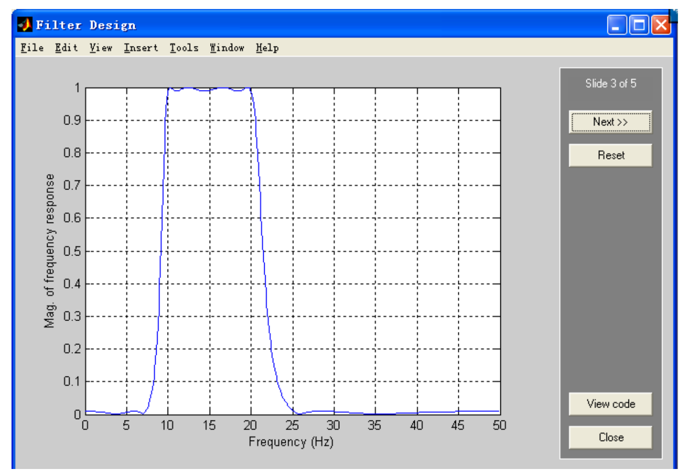
# 信号处理例题

$$f(t)=50\cos(5\times 2\pi t)+50\cos(15\times 2\pi t)+50\cos(30\times 2\pi t)$$

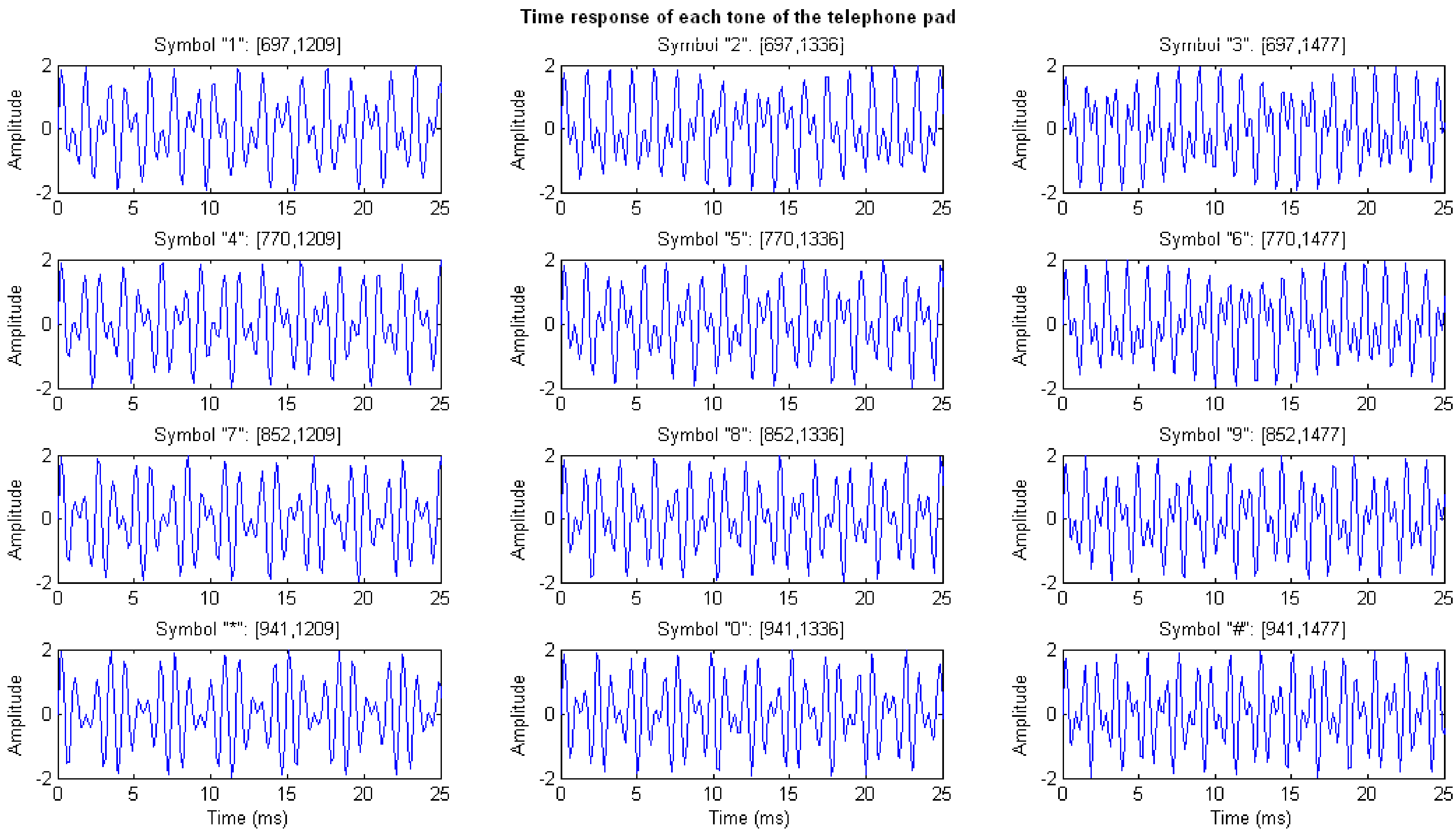


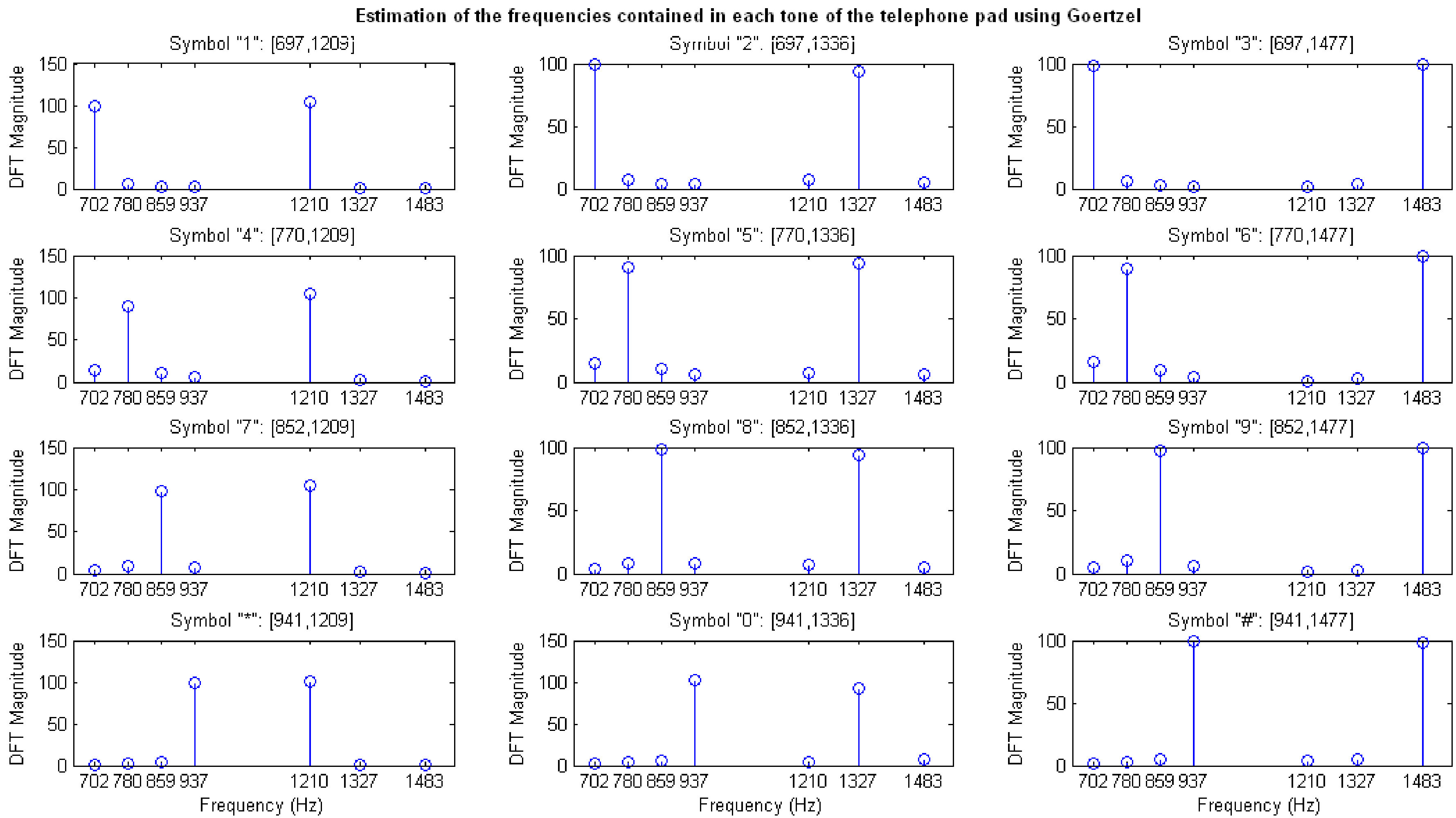



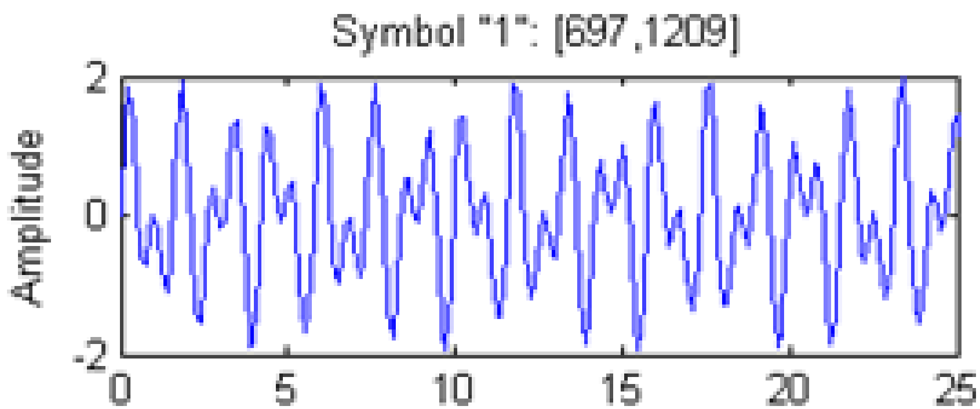
# 电话按键的双音频信号


Time response of each tone of the telephone pad
Symbol "1": [697,1209]
Symbol "2": [697,1336]
Symbol "3": [697,1477]
Symbol "4": [770,1209]
Symbol "5": [770,1336]
Symbol "6": [770,1477]
Symbol "7": [852,1209]
Symbol "8": [852,1336]
Symbol "9": [852,1477]
Symbol "*": [941,1209]
Symbol "0": [941,1336]
Symbol "#": [941,1477]
Amplitude
Time (ms)


Estimation of the frequencies contained in each tone of the telephone pad using Goertzel
Symbol "1": [697,1209]
Symbol "2": [697,1336]
Symbol "3": [697,1477]
Symbol "4": [770,1209]
Symbol "5": [770,1336]
Symbol "6": [770,1477]
Symbol "7": [852,1209]
Symbol "8": [852,1336]
Symbol "9": [852,1477]
Symbol "*": [941,1209]
Symbol "0": [941,1336]
Symbol "#": [941,1477]
DFT Magnitude
Frequency (Hz)
702 780 859 937 1210 1327 1483

$$f(t) = 100\cos\Omega_1 t + 100\cos\Omega_2 t,$$

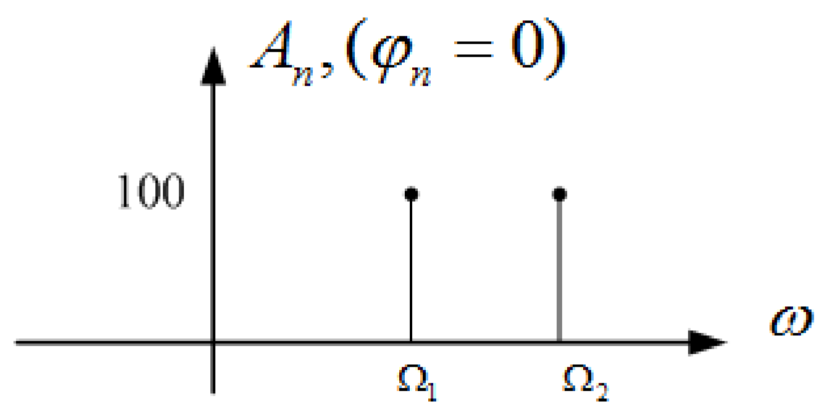
$$\Omega_1 = 2\pi\times 697 (rad/s), \Omega_2 = 2\pi\times 1209 (rad/s)$$




$$\Omega_1 = 2\pi\times 697 (rad/s), \Omega_2 = 2\pi\times 1209 (rad/s)$$

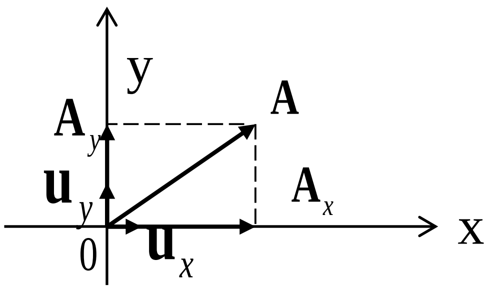
$$A_1 = 100, A_2 = 100$$

$$\varphi_1 = 0, \varphi_2 = 0$$



# §3.2　周期信号表示为傅里叶级数

## 矢量的正交分解：




$$\mathbf{A}=\mathbf{A}_x+\mathbf{A}_y$$

$$=\left|\mathbf{A}_x\right|\mathbf{u}_x+\left|\mathbf{A}_y\right|\mathbf{u}_y$$

$$\mathbf{u}_x\bullet\mathbf{u}_y=0$$

$$\mathbf{u}_x\bullet\mathbf{u}_x=\mathbf{u}_y\bullet\mathbf{u}_y=1$$

## 信号的正交函数分解：

构成一函数集合 $g_1(t),g_2(t),\cdots,g_n(t)$ ，这些函数在区间 $(t_1,t_2)$ 内满足以下的正交特性：

$$\int_{t_1}^{t_2} g_i(t)\cdot g_j(t)dt=\begin{cases}k_i & i=j\\ 0 & i\neq j\end{cases}$$

则称此函数集为正交函数集。(Orthogonal Function)

$f(t)$ 是区间 $(t_1, t_2)$ 上的任意能量有限信号，用正交函数集合中的函数的线性组合来近似表示，即：

$$f(t) \approx C_1 g_1(t) + C_2 g_2(t) + \cdots + C_n g_n(t)$$

如何求系数？

两边同乘 $g_i(t)$ ，并在区间 $(t_1, t_2)$ 上积分有：

$$\int_{t_1}^{t_2} f(t) g_i(t) dt \approx \int_{t_1}^{t_2} [C_1 g_1(t) + C_2 g_2(t) + \cdots + C_n g_n(t)] g_i(t) dt = C_i k_i$$

$$C_i = \frac{1}{k_i} \int_{t_1}^{t_2} f(t) g_i(t) dt, \quad i = 1, 2, \cdots$$

**可以证明：**

**三角函数集合**

$$\{1, \cos\Omega t, \cos 2\Omega t, \cdots, \cos n\Omega t, \cdots, \sin\Omega t, \sin 2\Omega t, \cdots, \sin n\Omega t, \cdots\}$$

**在区间** $(t_1, t_1+T)$ **上构成一完备正交函数集。其中** $T = \dfrac{2\pi}{\Omega}$

**复指数函数集合** $\{e^{jn\Omega t} \mid n = -\infty \sim \infty\}$ **在区间** $(t_1, t_1+T)$ **上构成一完备正交函数集。其中** $T = \dfrac{2\pi}{\Omega}$

**完备正交函数集**

对信号进行正交函数集分解时，应从两方面考虑：1、信号分解后的表达式应能反映信号本身的某些物理特性；2、分解简便。

三角函数集合和复指数函数集合恰好能够满足这两个条件，信号在这两个函数集合中分解得到的级数叫做傅立叶级数（Fourier series — FS）。

## 周期信号进行傅立叶分解应满足的狄里赫利条件：

1. 在一周期内，$f(t)$ 必须满足绝对可积条件，即

$$\int_T |f(t)| dt < \infty$$

（这一条件保证了每一系数都是有限值。）

2. 在一周期内有有限个极值点；
3. 在一周期内有有限个第一类间断点。

（即左右极限存在但不等的间断点）。

## 1、三角傅立叶级数(trigonometric Fourier series)

设 $f(t)$ 是一周期为 $T$ 的周期信号，且满足狄氏条件，三角函数集合

$$\{1,\cos\Omega t,\cos 2\Omega t,\cdots,\cos n\Omega t,\cdots,$$
$$\sin\Omega t,\sin 2\Omega t,\cdots,\sin n\Omega t,\cdots\}$$

在区间 $(t_1,t_1+T)$上是一完备的正交函数集合，其中 $T=\dfrac{2\pi}{\Omega}$ ，则$f(t)$可在此区间上分解为：

$$f(t)=\frac{a_0}{2}+a_1\cos\Omega t+a_2\cos 2\Omega t+\cdots+a_n\cos n\Omega t+\cdots$$
$$+b_1\sin\Omega t+b_2\sin 2\Omega t+\cdots+b_n\sin n\Omega t+\cdots$$

$$f(t)=\frac{a_0}{2}+a_1\cos\Omega t+a_2\cos 2\Omega t+\cdots+a_n\cos n\Omega t+\cdots$$
$$+b_1\sin\Omega t+b_2\sin 2\Omega t+\cdots+b_n\sin n\Omega t+\cdots$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\Omega t+b_n\sin n\Omega t)$$

$$\boxed{C_i=\frac{1}{k_i}\int_{t_1}^{t_2}f(t)g_i(t)dt}$$

**其中：**

$$\frac{a_0}{2}=\frac{1}{T}\int_{t_1}^{t_1+T}f(t)dt$$

$$a_n=\frac{2}{T}\int_{t_1}^{t_1+T}f(t)\cdot\cos n\Omega t dt$$

$$b_n=\frac{2}{T}\int_{t_1}^{t_1+T}f(t)\cdot\sin n\Omega t dt$$

$T$ 信号的周期

$\Omega=\frac{2\pi}{T}$ 基波角频率

$2\Omega$ 二次谐波角频率

$n\Omega$ $n$次谐波角频率

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\Omega t+b_n\sin n\Omega t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\Omega t+\varphi_n)$$

**令** $a_n\cos n\Omega t+b_n\sin n\Omega t=A_n\cos(n\Omega t+\varphi_n)$

$$=A_n\cos\varphi_n\cos n\Omega t-A_n\sin\varphi_n\sin n\Omega t$$

**其中** $a_n=A_n\cos\varphi_n \qquad b_n=-A_n\sin\varphi_n$

$$A_n=\sqrt{a_n^2+b_n^2} \qquad \varphi_n=-arctg\frac{b_n}{a_n}$$

**这样可以得到余弦形式的傅立叶级数：**

**Fundamental frequency, harmonic frequency**

$$f(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} A_n \cos(n\Omega t + \varphi_n)$$

**Fundamental component**

**Harmonic component**

$$= \underbrace{\frac{a_0}{2}} + \underbrace{A_1 \cos(\Omega t + \varphi_1)} + \underbrace{A_2 \cos(2\Omega t + \varphi_2)} + \cdots + \underbrace{A_n \cos(n\Omega t + \varphi_n)} + \cdots$$

**直流分量**　**基波分量**　**二次谐波分量**　$n$ **次谐波分量**

$A_n$ —— $n$**次谐波分量的振幅**

$\varphi_n$ —— $n$**次谐波分量的初始相位**

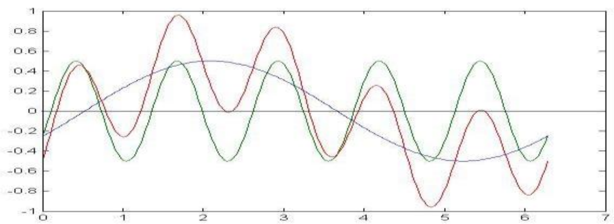
$n\Omega$、$A_n$、$\varphi_n$

**反映了信号的频域特性，给定一个信号，这些量就可以唯一的确定。反过来，给定这些量，就有一个唯一的信号与之对应。**

**例如** $$f(t)=\frac{1}{2}\cos\left(t-\frac{2\pi}{3}\right)+\frac{1}{2}\cos\left(5t-\frac{2\pi}{3}\right)$$

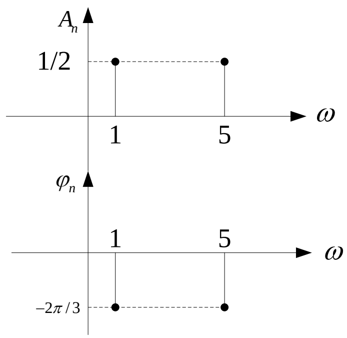
**说明:**

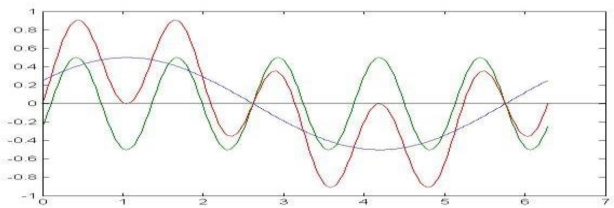
**信号无直流分量，只包含2个频率分量：基波分量和5次谐波分量**



$\Omega=1$、$A_1=\frac{1}{2}$、$\varphi_1=-\frac{2\pi}{3}$

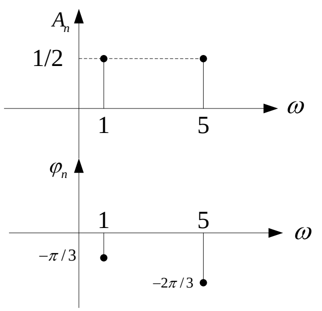
$5\Omega=5$、$A_5=\frac{1}{2}$、$\varphi_5=-\frac{2\pi}{3}$

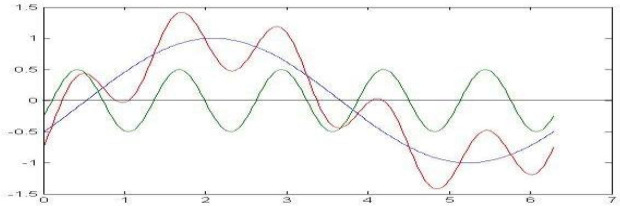


$\Omega=1$、$A_1=\frac{1}{2}$、$\varphi_1=-\frac{\pi}{3}$

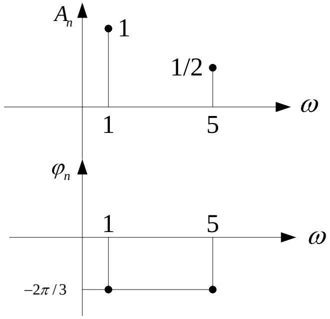
$5\Omega=5$、$A_5=\frac{1}{2}$、$\varphi_5=-\frac{2\pi}{3}$



$\Omega=1$、$A_1=1$、$\varphi_1=-\frac{2\pi}{3}$

$5\Omega=5$、$A_5=\frac{1}{2}$、$\varphi_5=-\frac{2\pi}{3}$



**傅立叶分解的核心就是要把信号分解成不同频率的正弦信号的加权和：**

$$f(t)=\cos(2t)\cdot\cos(3t-\frac{2\pi}{3})$$

$$=\frac{1}{2}\cos(t-\frac{2\pi}{3})+\frac{1}{2}\cos(5t-\frac{2\pi}{3})$$

基本型

$$f(t)=\frac{a_0}{2}+a_1\cos\Omega t+a_2\cos 2\Omega t+\mathrm{L}\ +a_n\cos n\Omega t+\mathrm{L}$$

$$+b_1\sin\Omega t+b_2\sin 2\Omega t+\mathrm{L}\ +b_n\sin n\Omega t+\mathrm{L}$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\Omega t+b_n\sin n\Omega t)$$

余弦型

$$f(t)=\frac{a_0}{2}+\sum_{i=1}^{\infty}A_n\cos(n\Omega t+\varphi_n)$$

$$=\frac{a_0}{2}+A_1\cos(\Omega t+\varphi_1)+A_2\cos(2\Omega t+\varphi_2)+\mathrm{L}\ +A_n\cos(n\Omega t+\varphi_n)+\mathrm{L}$$

$$\Omega=\frac{2\pi}{T}$$ 基波角频率

$$f=\frac{1}{T}$$ 基波频率

$$\frac{a_0}{2} = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$$

若 $f(t)$ 是实信号：

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cdot\cos n\Omega t dt,$$

角频率的偶函数

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cdot\sin n\Omega t dt$$

角频率的奇函数

偶函数

奇函数

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = -\arctan\frac{b_n}{a_n}$$

$$a_n = A_n\cos\varphi_n, \quad b_n = -A_n\sin\varphi_n$$

## 2、指数形式的傅立叶级数(exponential Fourier series)

复指数函数集合 $\{e^{jn\Omega t} \mid n = 0,\pm1,\pm2,\cdots\}$ 在区间 $(t_1,t_1+T)$ 上是一完备的正交函数集合，则周期信号 $f(t)$ 的指数形式的傅立叶级数为：

$$f(t) = \cdots + C_{-2}e^{-j2\Omega t} + C_{-1}e^{-j\Omega t} + C_0 + C_1 e^{j\Omega t} + C_2 e^{j2\Omega t} + \cdots = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}$$

$T$是周期信号$f(t)$的周期, $\Omega = \dfrac{2\pi}{T}$ 是基波频率。

补充：复函数集合正交的定义：

$$\int_{t_1}^{t_2} g_i(t)\cdot g_j^*(t)dt = \begin{cases} k_i & i = j \\ 0 & i \neq j \end{cases}$$

$$\boxed{C_n = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)\cdot e^{-jn\Omega t}dt}$$

$$\cos(n\Omega t+\varphi_n)=\frac{1}{2}[e^{j(n\Omega t+\varphi_n)}+e^{-j(n\Omega t+\varphi_n)}]$$

**从三角傅立叶级数推导：**

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\Omega t+\varphi_n)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{1}{2}A_n[e^{j(n\Omega t+\varphi_n)}+e^{-j(n\Omega t+\varphi_n)}]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{1}{2}A_n e^{j\varphi_n}e^{jn\Omega t}+\sum_{n=1}^{\infty}\frac{1}{2}A_n e^{-j\varphi_n}e^{-jn\Omega t}$$

由于$A_n$是偶函数，即$A_n = A_{-n}$，$\varphi_n$是奇函数，$\varphi_n = -\varphi_{-n}$
上式第三项变量代换 $n = -m$，得：

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2} A_n e^{j\varphi_n} e^{jn\Omega t} + \sum_{m=-1}^{-\infty} \frac{1}{2} A_{-m} e^{-j\varphi_{-m}} e^{-j(-m)\Omega t}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2} A_n e^{j\varphi_n} e^{jn\Omega t} + \sum_{n=-1}^{-\infty} \frac{1}{2} A_n e^{j\varphi_n} e^{jn\Omega t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2} A_n e^{j\varphi_n} e^{jn\Omega t} \boxed{= \sum_{n=-\infty}^{\infty} \frac{1}{2} \dot{A}_n e^{jn\Omega t}} \boxed{= \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}}$$

其中 $\dot{A}_n = A_n e^{j\varphi_n}$ $\quad \frac{a_0}{2} = \frac{1}{2} \dot{A}_0$

$\dot{A}_n$叫做 $n$ 次谐波分量的复振幅

与前式比较可得：$C_n = \frac{1}{2}\dot{A}_n$

$$\therefore \dot{A}_n = 2C_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cdot e^{-jn\Omega t}dt$$

说明：

1）负频率的出现是数学处理的结果。

2）傅立叶级数周期性说明。

3）$t_1$可以取 $0$ 或 $-\frac{T}{2}$ 。

## 3、函数的奇偶性及其与谐波含量的关系

**偶函数** $f(t) = f(-t)$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = 2\cdot\int_{0}^{\frac{T}{2}} f(t)dt$$

**奇函数** $f(-t) = -f(t)$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = 0$$

**(a) $f(t)$ 是偶函数**

$$\frac{a_0}{2}=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt=\frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)dt$$

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\cos n\Omega t dt=\frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\cdot\cos n\Omega t dt$$

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\sin n\Omega t dt=0$$

**说明其傅立叶级数展开式中只含有直流分量和余弦谐波分量，不含有正弦谐波分量。**

(b) $f(t)$是奇函数

$$\frac{a_0}{2}=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt=0$$

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\cos n\Omega t dt=0$$

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\sin n\Omega t dt=\frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\cdot\sin n\Omega t dt$$

说明其傅立叶级数展开式中只含正弦谐波分量，不含有直流分量和余弦谐波分量。

# §3.4　周期信号的频谱

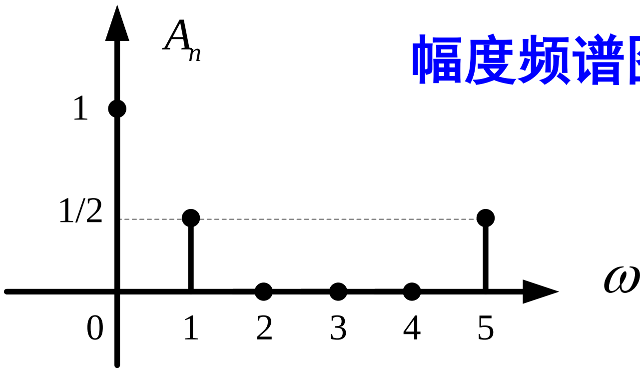
例如

$$f(t)=1+\frac{1}{2}\cos(t-\frac{2\pi}{3})+\frac{1}{2}\cos(5t-\frac{2\pi}{3})$$

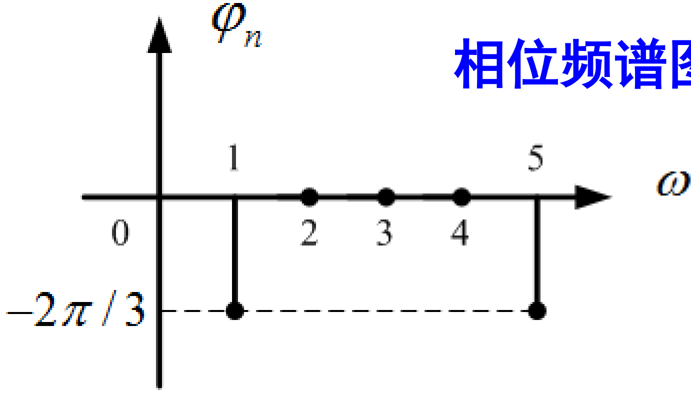
$\Omega=0$,　1

$\Omega=1$、$A_1=\frac{1}{2}$、$\varphi_1=-\frac{2\pi}{3}$

$5\Omega=5$、$A_5=\frac{1}{2}$、$\varphi_5=-\frac{2\pi}{3}$




幅度频谱图




相位频谱图

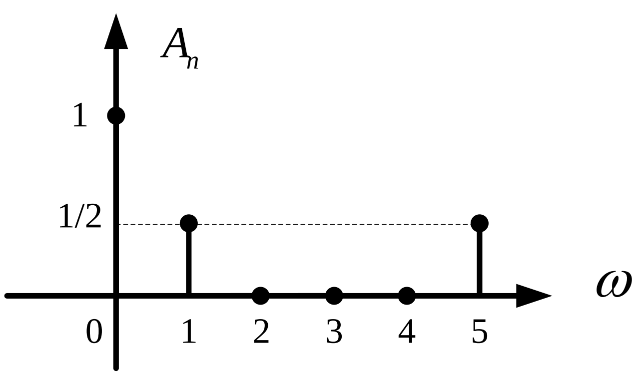
$$f(t)=1-\frac{1}{4}\cos t-\frac{1}{4}\cos 5t+\frac{\sqrt{3}}{4}\sin t+\frac{\sqrt{3}}{4}\sin 5t$$

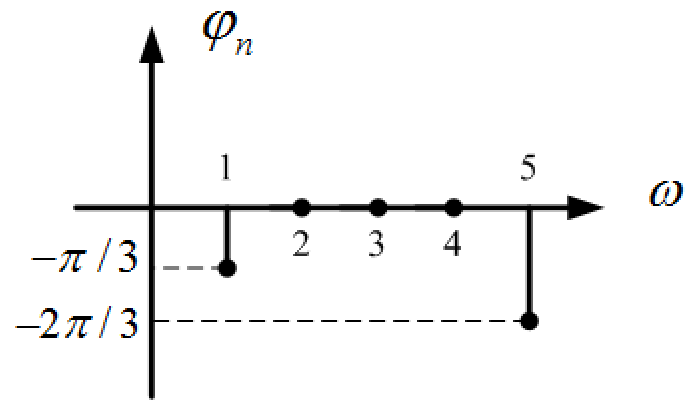
$$f(t)=1+\frac{1}{2}\cos(t-\frac{\pi}{3})+\frac{1}{2}\cos(5t-\frac{2\pi}{3})$$

$$\Omega=0,\quad 1$$

$$\Omega=1、A_1=\frac{1}{2}、\varphi_1=-\frac{\pi}{3}$$

$$5\Omega=5、A_5=\frac{1}{2}、\varphi_5=-\frac{2\pi}{3}$$



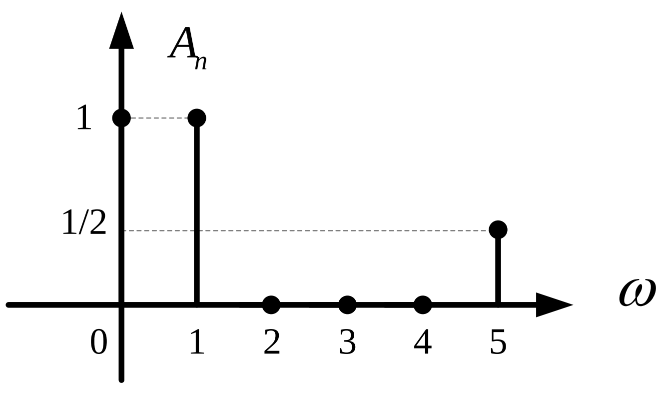



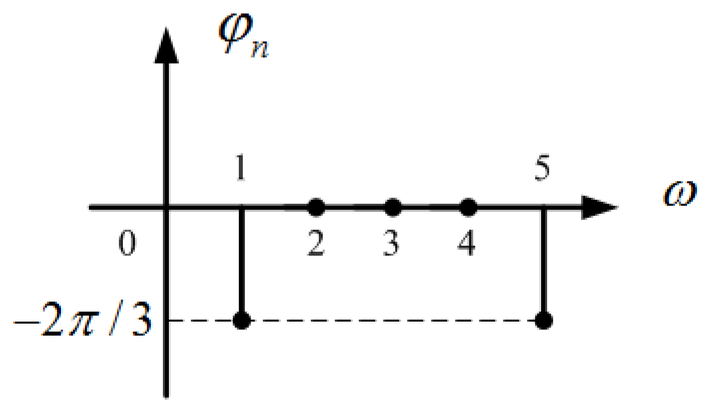
$$f(t)=1+\cos(t-\frac{2\pi}{3})+\frac{1}{2}\cos(5t-\frac{2\pi}{3})$$

$$\Omega=0,\quad 1$$

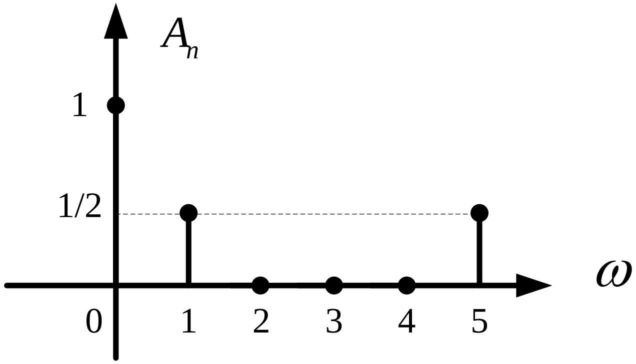
$$\Omega=1、A_1=1、\varphi_1=-\frac{2\pi}{3}$$

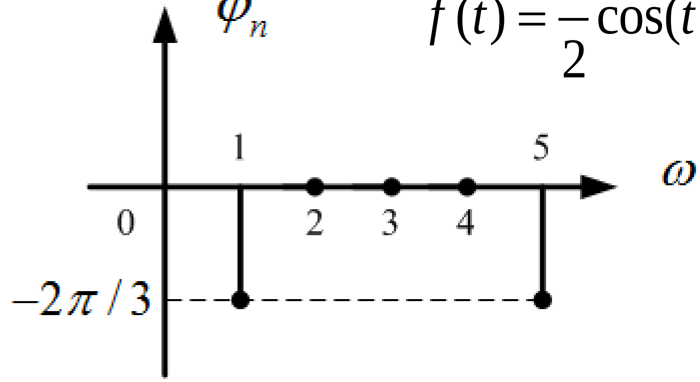
$$5\Omega=5、A_5=\frac{1}{2}、\varphi_5=-\frac{2\pi}{3}$$



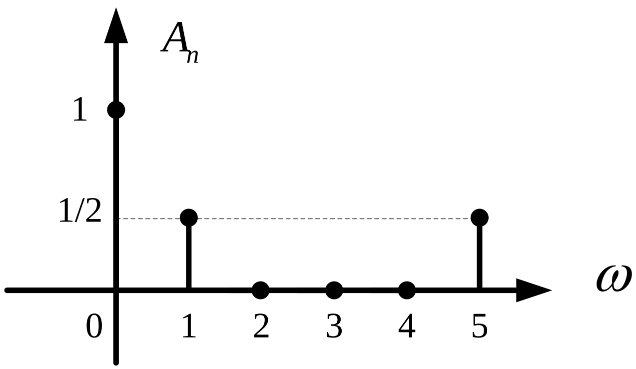



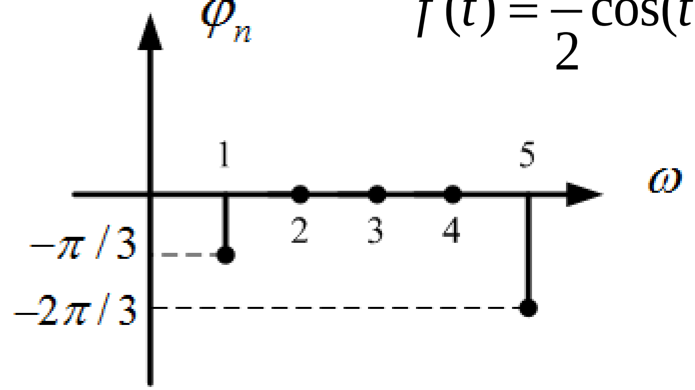
$$f(t) = \frac{1}{2}\cos\left(t - \frac{2\pi}{3}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$



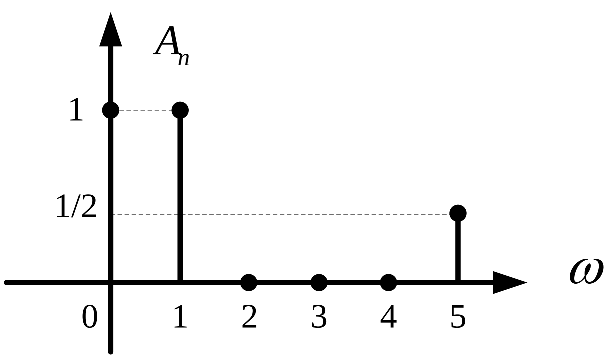



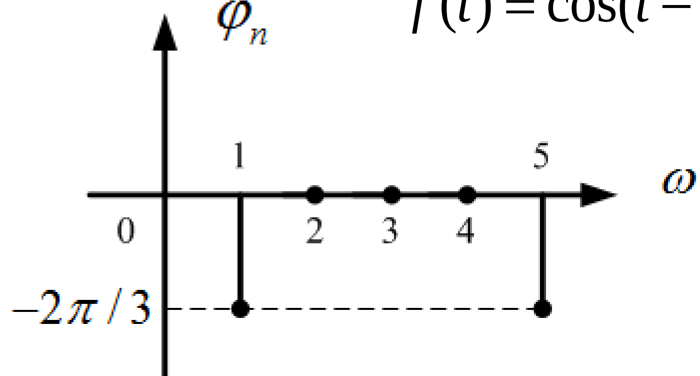

$$f(t) = \frac{1}{2}\cos\left(t - \frac{\pi}{3}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$




$$f(t) = \cos\left(t - \frac{2\pi}{3}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$



另外

$$f(t)=1+\frac{1}{2}\cos(t-\frac{2\pi}{3})+\frac{1}{2}\cos(5t-\frac{2\pi}{3})$$

$$=1+\frac{1}{2}\cdot\frac{1}{2}\left[e^{j(t-\frac{2\pi}{3})}+e^{-j(t-\frac{2\pi}{3})}\right]+\frac{1}{2}\cdot\frac{1}{2}\left[e^{j(5t-\frac{2\pi}{3})}+e^{-j(5t-\frac{2\pi}{3})}\right]$$

$$=1+\frac{1}{4}e^{j(t-\frac{2\pi}{3})}+\frac{1}{4}e^{-j(t-\frac{2\pi}{3})}+\frac{1}{4}e^{j(5t-\frac{2\pi}{3})}+\frac{1}{4}e^{-j(5t-\frac{2\pi}{3})}$$

$$=\frac{1}{4}e^{j\frac{2\pi}{3}}e^{-j5t}+\frac{1}{4}e^{j\frac{2\pi}{3}}e^{-jt}+1+\frac{1}{4}e^{-j\frac{2\pi}{3}}e^{jt}+\frac{1}{4}e^{-j\frac{2\pi}{3}}e^{j5t}$$

$$=C_{-5}e^{-j5t}+C_{-1}e^{-jt}+C_0+C_1e^{jt}+C_5e^{j5t}$$

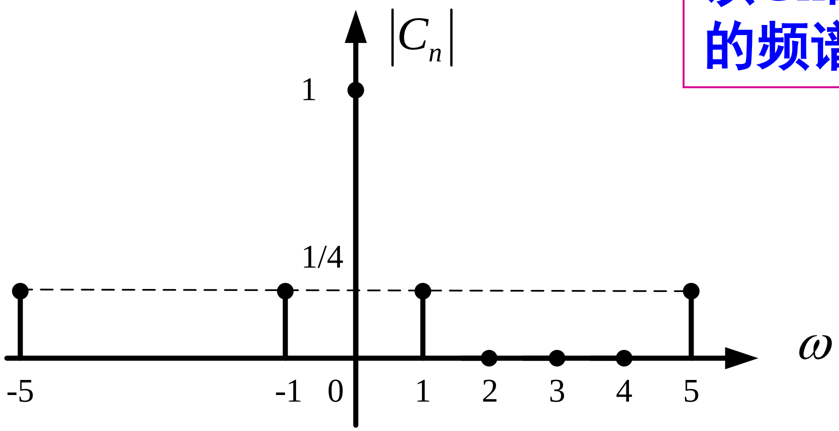
共轭

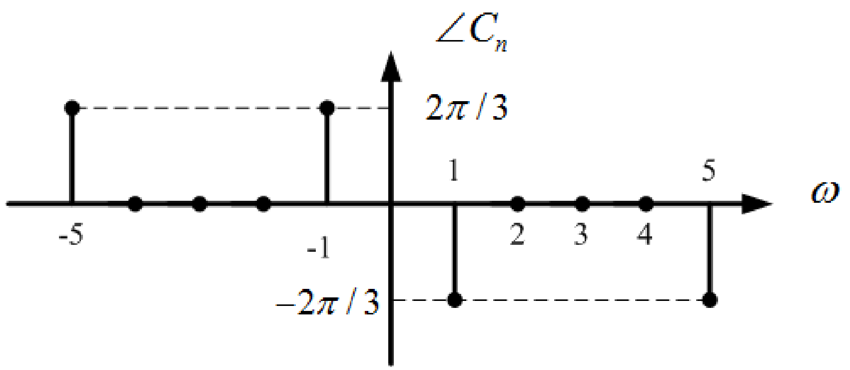
$$\boxed{C_{-5}=\frac{1}{4}e^{j\frac{2\pi}{3}},C_{-1}=\frac{1}{4}e^{j\frac{2\pi}{3}},C_0=1,C_1=\frac{1}{4}e^{-j\frac{2\pi}{3}},C_5=\frac{1}{4}e^{-j\frac{2\pi}{3}}}$$

$$f(t)=C_{-5}e^{-j5t}+C_{-1}e^{-jt}+C_0+C_1e^{jt}+C_5e^{j5t}$$

$$C_{-5}=\frac{1}{4}e^{j\frac{2\pi}{3}},C_{-1}=\frac{1}{4}e^{j\frac{2\pi}{3}},C_0=1,C_1=\frac{1}{4}e^{-j\frac{2\pi}{3}},C_5=\frac{1}{4}e^{-j\frac{2\pi}{3}}$$

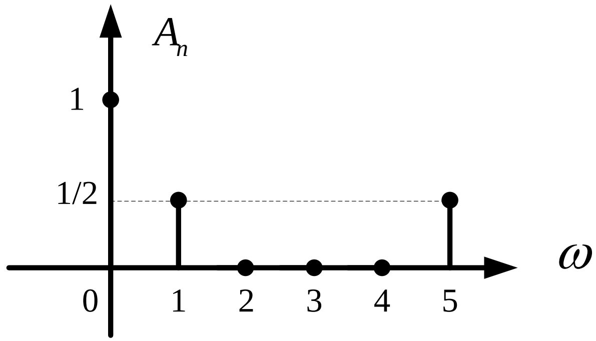
**以Cn的模值和俯角作图，得到的频谱图为双边频谱图。**

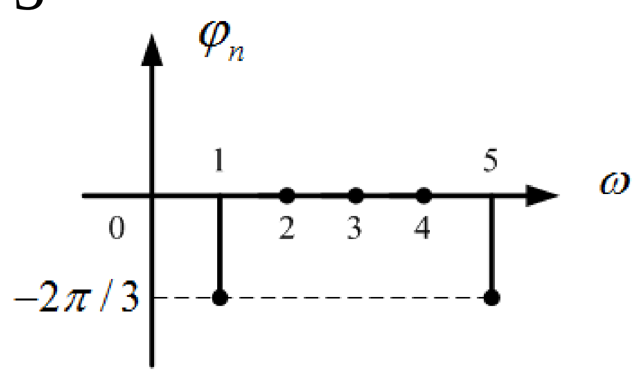


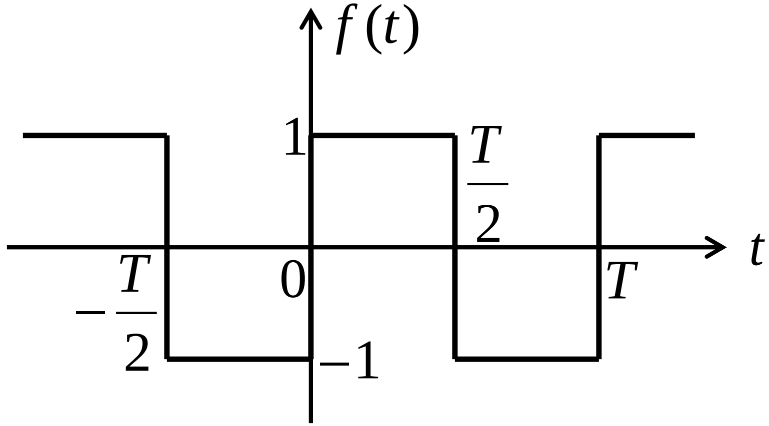

$$f(t)=\frac{1}{2}\cos(t-\frac{2\pi}{3})+\frac{1}{2}\cos(5t-\frac{2\pi}{3})$$

**单边频谱图。**



**例：一周期信号如图示，试将其展开为三角傅立叶级数。**




$$f(t)=\begin{cases}1 & 0<t<\dfrac{T}{2}\\ -1 & -\dfrac{T}{2}<t<0\end{cases}$$

**解：** **信号在** $(-\frac{T}{2},\frac{T}{2})$ **一周期内是奇函数，**

$$\therefore \frac{a_0}{2}=0 \qquad a_n=0$$

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\sin n\Omega t dt \quad =\frac{4}{T}\int_0^{\frac{T}{2}}\sin n\Omega t dt$$

$$=\frac{4}{T}\cdot\frac{1}{n\Omega}(-\cos n\Omega t)\Big|_0^{T/2}=\frac{4}{T}\cdot\frac{1}{n\Omega}(1-\cos n\Omega\frac{T}{2})$$

$$=\frac{2}{n\pi}(1-\cos n\pi) \quad =\begin{cases}\dfrac{4}{n\pi} & n\text{为奇数}\\ 0 & n\text{为偶数}\end{cases}$$

$$\therefore\ f(t)=\frac{4}{\pi}(\sin\Omega t+\frac{1}{3}\sin 3\Omega t+\frac{1}{5}\sin 5\Omega t+\cdots)$$

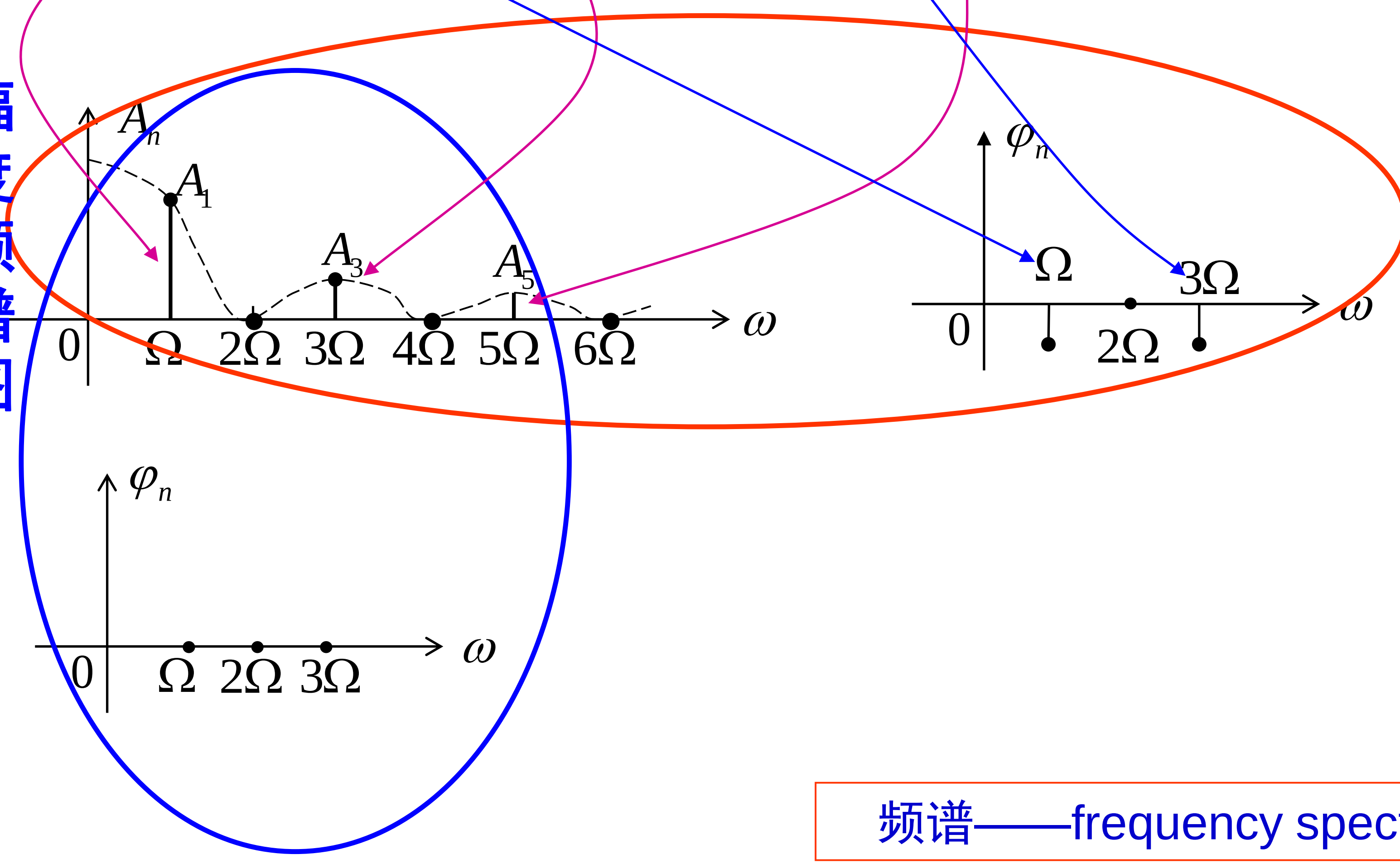
$$=\frac{4}{\pi}\mathbf{[cos(}\Omega t-\frac{\pi}{2}\mathbf{)}+\frac{1}{3}\mathbf{cos(}3\Omega t-\frac{\pi}{2}\mathbf{)}+\frac{1}{5}\mathbf{cos(}5\Omega t-\frac{\pi}{2}\mathbf{)}+\cdots\mathbf{]}$$

频谱图——用不同长短的线段表示各谐波分量振幅或相位的相对大小，然后按频率由低到高的顺序排列起来得到的图叫**信号的频谱图**。各线段顶点的连线叫做频谱图的**包络线**。

$$\therefore\ f(t)=\frac{4}{\pi}(\sin\Omega t+\frac{1}{3}\sin 3\Omega t+\frac{1}{5}\sin 5\Omega t+\cdots)$$

$$=\frac{4}{\pi}\Big[\cos(\Omega t-\frac{\pi}{2})+\frac{1}{3}\cos(3\Omega t-\frac{\pi}{2})+\frac{1}{5}\cos(5\Omega t-\frac{\pi}{2})+\cdots\Big]$$

**幅度频谱图**




**相位频谱图**

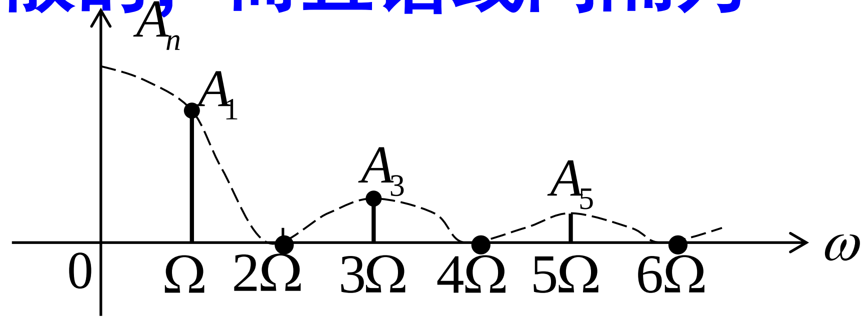
频谱——frequency spectrum

**结论：**

（1）周期信号由无穷多个频率分量构成，其频谱占有整个频率范围；

（2）周期信号的频谱是离散的，而且谱线间隔为基波频率；

（3）高频分量幅度小。




$\boxed{\text{离散性、谐波性、收敛性}}$

忽略高频分量，这样信号就是由有限个频率分量构成。有限的频率范围叫做信号的<u>有效占有频带宽度</u>，或叫 信号的<u>频宽</u>。

Frequency band width

吉伯斯现象：

**例　已知连续时间周期信号**

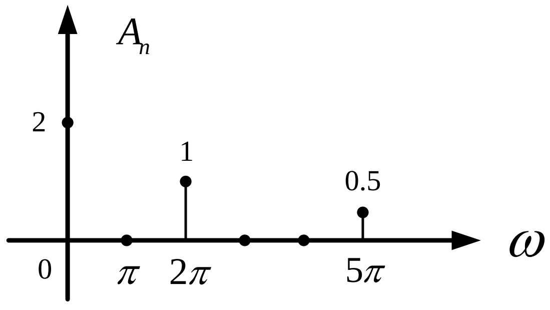
$$f(t)=2+\cos(2\pi t)\text{-}0.5\cos(5\pi t)$$

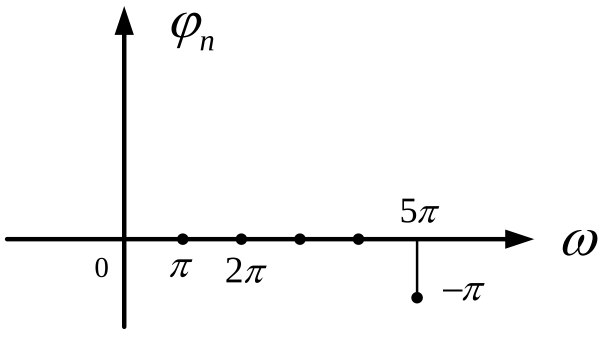
**（1）试绘出信号的幅度频谱图和相位频谱图；**
**（2）将幅度频谱图和相位频谱图画在一张图中；**
**（3）画双边频谱图；**
**（4）求信号的基波角频率和平均功率。**

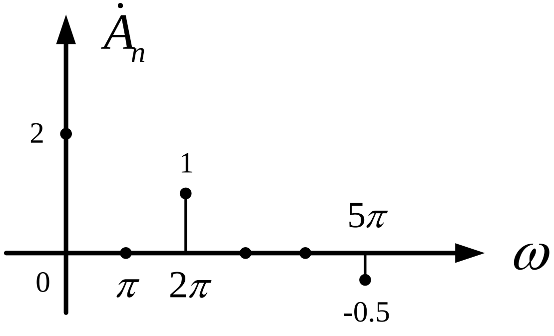
**解：　（1）首先写出标准的余弦形式傅里叶级数：**

$$f(t)=2+\cos(2\pi t)+0.5\cos(5\pi t-\pi)$$

$$\frac{a_0}{2}=2,\varphi_0=0;\ A_2=1,\varphi_2=0;\ A_5=0.5,\ \ \varphi_5=-\pi$$ **（2）**



**(3)** $$f(t)=2+\cos(2\pi t)+0.5\cos(5\pi t-\pi)$$

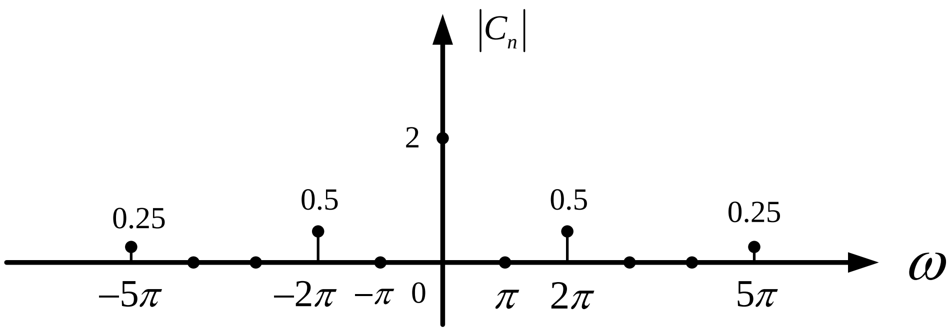
$$=0.25e^{-j(5\pi t-\pi)}+0.5e^{-j2\pi t}+2+0.5e^{j2\pi t}+0.25e^{j(5\pi t-\pi)}$$

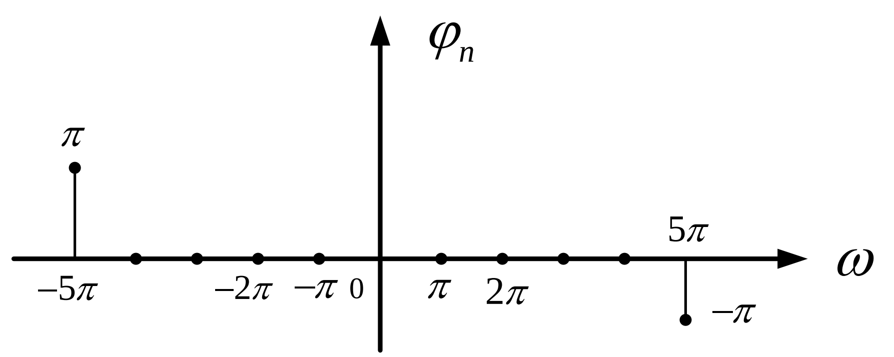
$$=0.25e^{j\pi}e^{-j5\pi t}+0.5e^{-j2\pi t}+2+0.5e^{j2\pi t}+0.25e^{-j\pi}e^{j5\pi t}$$

$$C_{-5}=0.25e^{j\pi},\ C_{-2}=0.5,\ C_0=2,\ C_2=0.5,\ C_5=0.25e^{-j\pi}$$

$$|C_{-5}|=0.25,\quad |C_{-2}|=0.5,\ |C_0|=2,\ |C_2|=0.5,\ |C_5|=0.25e^{-j\pi}$$

$$\varphi_{-5}=\pi,\quad \varphi_{-2}=0,\quad \varphi_0=0,\ \varphi_2=0,\quad \varphi_5=-\pi,$$



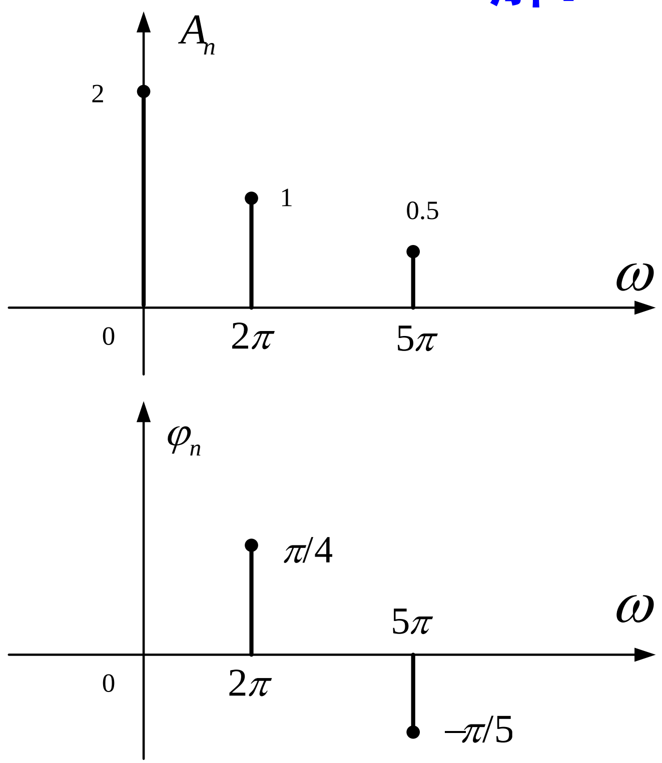




**(4)** $$P=\frac{1}{T}\int_0^T f^2(t)dt=2^2+\frac{1}{2}(A_2)^2+\frac{1}{2}(A_5)^2$$

$\Omega=\pi(rad/s)$

$$=2^2+\frac{1}{2}+\frac{1}{2}(0.5)^2=4.625(W)$$

**例　已知连续时间周期信号频谱图如图所示，试写出余弦形式的傅立叶级数和复指数形式的傅立叶级数，并求信号的基波角频率和信号带宽，绘出信号的双边谱。**



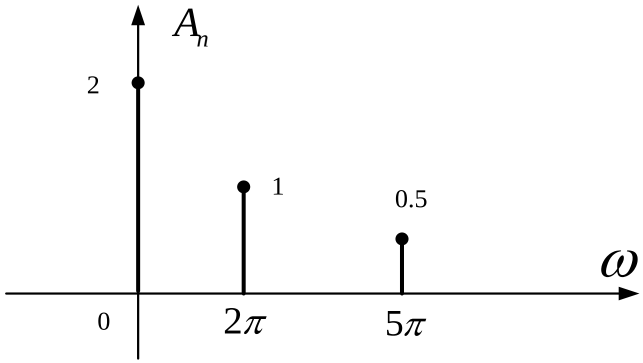

**解：**

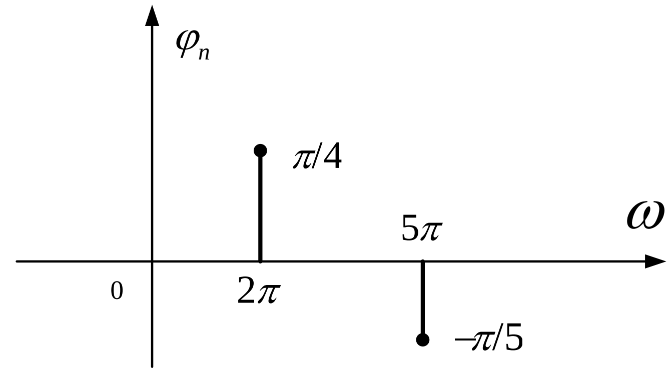
$$f(t)=2+\cos(2\pi t+\pi/4)+0.5\cos(5\pi t-\pi/5)$$

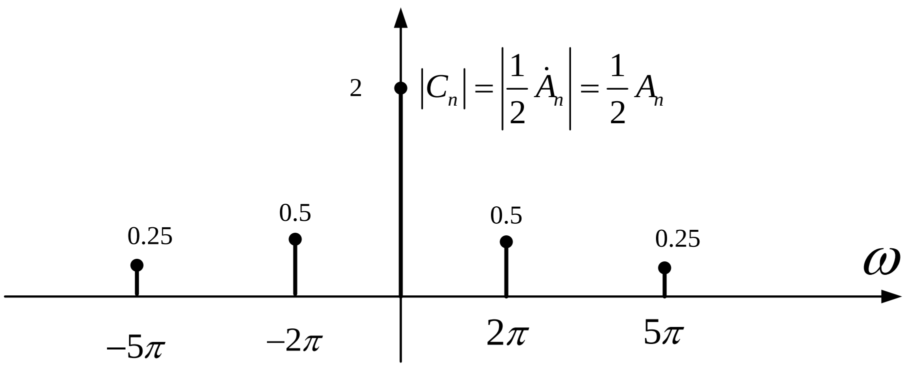
$$\begin{aligned} f(t)&=2+\frac{1}{2}e^{j(2\pi t+\pi/4)}+\frac{1}{2}e^{-j(2\pi t+\pi/4)}\\ &+\frac{1}{2}e^{j(5\pi t-\pi/5)}+\frac{1}{2}e^{-j(5\pi t-\pi/5)}\\ &=\frac{1}{2}e^{j\pi/5}e^{-j5\pi t}+\frac{1}{2}e^{-j\pi/4}e^{-j2\pi t}+2\\ &+\frac{1}{2}e^{j\pi/4}e^{j2\pi t}+\frac{1}{2}e^{-j\pi/5}e^{j5\pi t} \end{aligned}$$

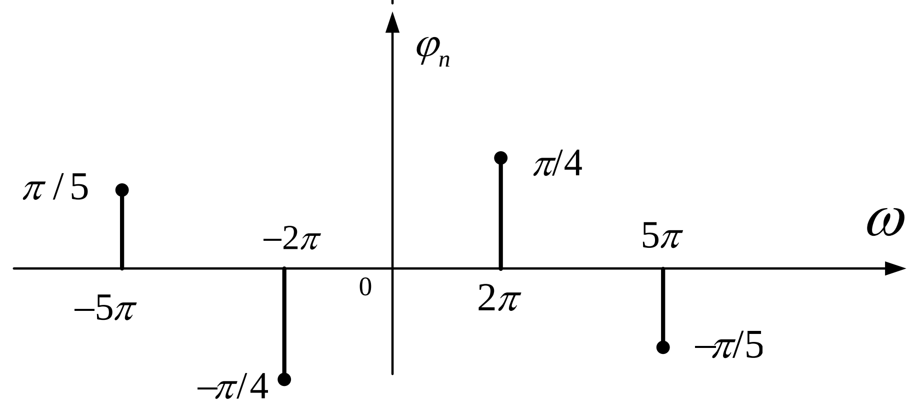
$$\Omega=\pi(rad/s),\ B_s=5\pi(rad/s)$$

$$f(t)=\frac{1}{2}e^{j\pi/5}e^{-j5\pi t}+\frac{1}{2}e^{-j\pi/4}e^{-j2\pi t}+2+\frac{1}{2}e^{j\pi/4}e^{j2\pi t}+\frac{1}{2}e^{-j\pi/5}e^{j5\pi t}$$




$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\Omega t-\varphi_n)=\sum_{n=-\infty}^{\infty}\frac{1}{2}\dot{A}_n e^{jn\Omega t}=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega t}$$

**例　已知某实的连续时间周期信号的复指数形式傅立叶级数如下，试确定式中系数a，b，并画出其单边频谱图。**

$$f(t) = 2 + \frac{1}{2}e^{j(2\pi t + \pi/4)} + ae^{-j2\pi t} + \frac{1}{2}e^{j(5\pi t)} + be^{-j(5\pi t - \pi/5)}$$

$$a = \frac{1}{2}e^{-j\pi/4}, \quad b = \frac{1}{2}e^{-j\pi/5}$$

$$f(t) = 2 + \cos(2\pi t + \pi/4) + \frac{1}{2}\cos(5\pi t)$$

$$f(t)=2+\cos(2\pi t)-0.5\cos(5\pi t)$$

$$=2+\cos(2\pi t)+0.5\cos(5\pi t-\pi)$$

**假设某周期信号的复振幅为实的：**

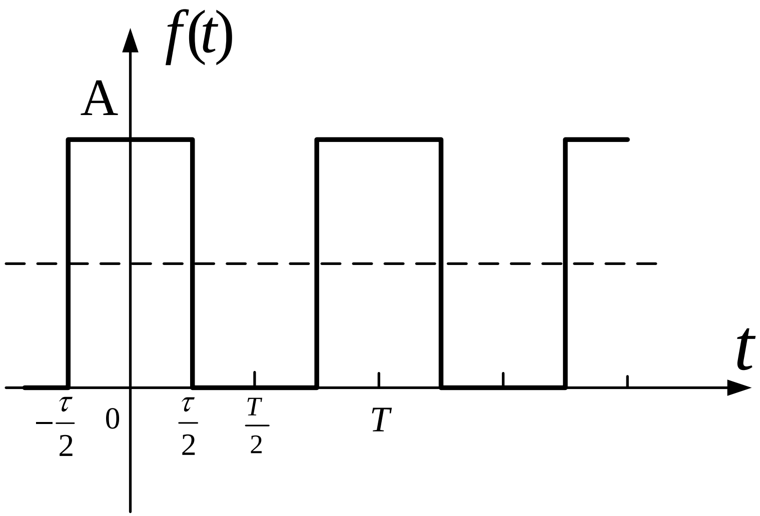
$$\dot{A}_n=(-0.5)^n \quad \rightarrow \quad C_n=\frac{1}{2}\dot{A}_n=\frac{1}{2}\cdot(-0.5)^n$$

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{1}{2}\dot{A}_n e^{jn\Omega t}=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega t}=\sum_{n=-\infty}^{\infty}\left[\frac{1}{2}\cdot(-0.5)^n e^{jn\Omega t}\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left|\dot{A}_n\right|\cos(n\Omega t+\varphi_n)=\frac{1}{2}+\sum_{n=1}^{\infty}(0.5)^n\cos(n\Omega t+\varphi_n)$$

$$\varphi_n=\begin{cases}0, & n\text{偶}\\ \pi, & n\text{奇}\end{cases}$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}(-0.5)^n\cos(n\Omega t)$$




$$f(t)=\begin{cases} A & |t|<\dfrac{\tau}{2} \\ 0 & \dfrac{\tau}{2}<|t|<\dfrac{T}{2} \end{cases}$$

**首先将其展开成指数形式的傅立叶级数**

$$f(t)=\cdots+C_{-2}e^{-j2\Omega t}+C_{-1}e^{-j\Omega t}+C_0+C_1e^{j\Omega t}+C_2e^{j2\Omega t}+\cdots$$

$$=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega t}=\sum_{n=-\infty}^{\infty}\frac{1}{2}\dot{A}_n e^{jn\Omega t}$$

**复振幅**

$$\dot{A}_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-jn\Omega t}dt$$

$$= \frac{2}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-jn\Omega t}dt = \frac{2A}{T}\left(\frac{1}{-jn\Omega}\right)e^{-jn\Omega t}\Bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= \frac{2A}{T}\left(\frac{1}{-jn\Omega}\right)\left(e^{-jn\Omega\frac{\tau}{2}} - e^{jn\Omega\frac{\tau}{2}}\right) = \frac{4A}{Tn\Omega}\sin\left(n\Omega\frac{\tau}{2}\right) = \frac{2A\tau}{T}\cdot\frac{\sin\left(n\Omega\frac{\tau}{2}\right)}{n\Omega\frac{\tau}{2}}$$

$$= \frac{2A\tau}{T}Sa\left(n\Omega\frac{\tau}{2}\right) = \frac{2A\tau}{T}Sa\left(n\frac{2\pi}{T}\frac{\tau}{2}\right) = \frac{2A\tau}{T}Sa\left(n\frac{\tau\pi}{T}\right)$$

**直流分量** $$\frac{a_0}{2} = \frac{\dot{A}_0}{2} = \frac{A\tau}{T}$$

$$\therefore\ f(t)=\sum_{n=-\infty}^{\infty}\frac{1}{2}\dot{A}_n e^{jn\Omega t}=\sum_{n=-\infty}^{\infty}\frac{A\tau}{T}\frac{\sin(n\Omega\frac{\tau}{2})}{n\Omega\frac{\tau}{2}}e^{jn\Omega t}$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\Omega t-\varphi_n)$$

$$\boxed{\dot{A}_n=\frac{2A\tau}{T}Sa(n\frac{\tau\pi}{T})}$$

$$=\frac{A\tau}{T}+\sum_{n=1}^{\infty}\frac{2A\tau}{T}\frac{\sin(n\Omega\frac{\tau}{2})}{n\Omega\frac{\tau}{2}}\cos n\Omega t$$

$$\boxed{\therefore\quad A_n=\frac{2A\tau}{T}\left|Sa(n\Omega\frac{\tau}{2})\right|}$$

**讨论：**

**1、 $\tau$ 不变、$T$ 变**

$\Omega=\dfrac{2\pi}{T}$，$T$变大，$\Omega$、$A_n$变小；反之，$\Omega$、$A_n$变大。

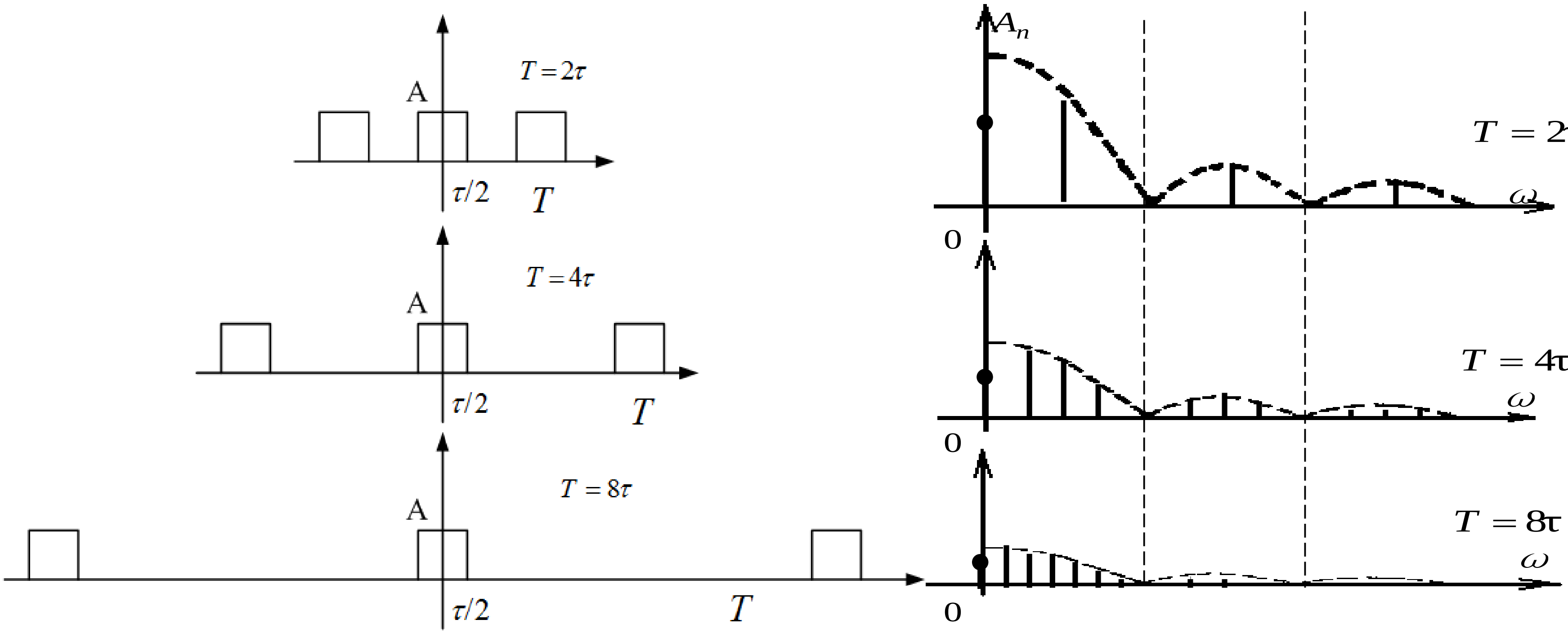
$$T = 2\tau, \Omega = \frac{2\pi}{T} = \frac{\pi}{\tau}, \quad A_n = A\left|\frac{\sin(n\frac{\pi}{2})}{n\frac{\pi}{2}}\right|$$

令 $n = 0, 1, 2, \ldots$ $\frac{A_0}{2} = \frac{a_0}{2} = \frac{A}{2}$

$$A_1 = \frac{2A}{\pi} \quad A_2 = 0 \quad A_3 = \frac{2A}{3\pi} \quad A_4 = 0 \quad \cdots$$

$$T = 4\tau, \Omega = \frac{2\pi}{T} = \frac{\pi}{2\tau}, \quad A_n = \frac{A}{2}\left|\frac{\sin(n\frac{\pi}{4})}{n\frac{\pi}{4}}\right|$$

$\vdots$




$$A_n=\frac{2A\tau}{T}\left|Sa(n\Omega\frac{\tau}{2})\right|$$

$\tau$不变、$T$变：

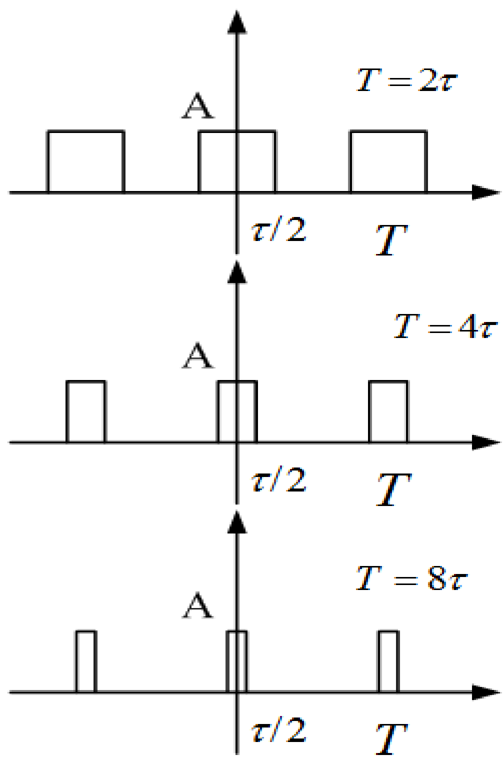
（1）幅度频谱图的包络线不变；

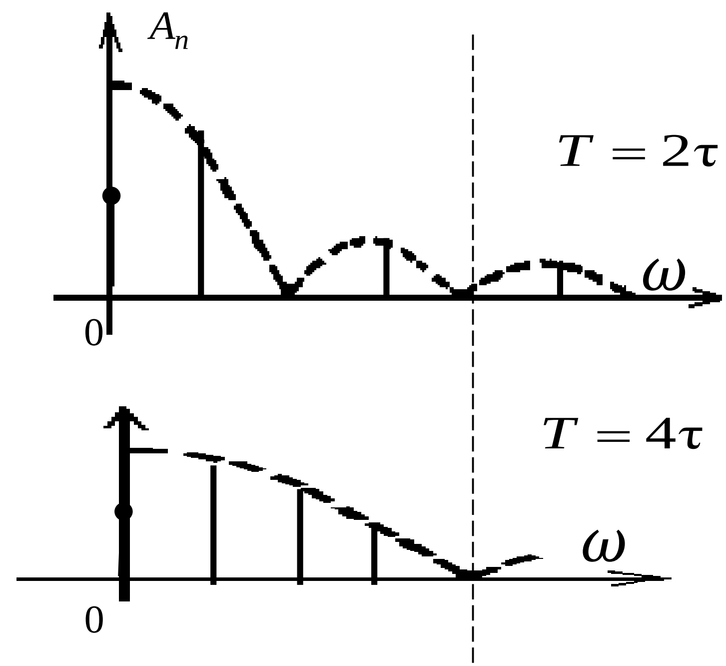
（2）第一个过零点对应的频率 $\dfrac{2\pi}{\tau}$ 不变，信号频宽度不变；

（3）$T$ 增加， 基波频率$\Omega$ 减小，谱线加密，振幅减小。

## 2、$\tau$变、$T$不变

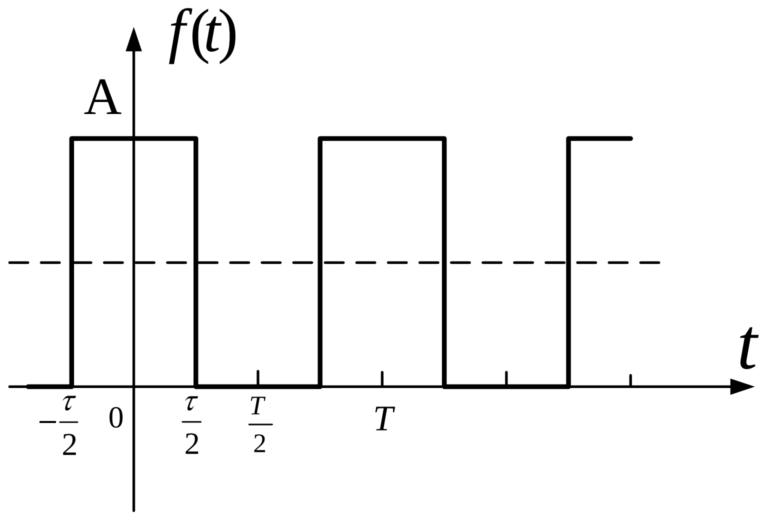
$$A_n = \frac{2A\tau}{T}\left|Sa(n\Omega\frac{\tau}{2})\right|$$




**（1）$T$不变，基波频率不变，谱线密度不变；**

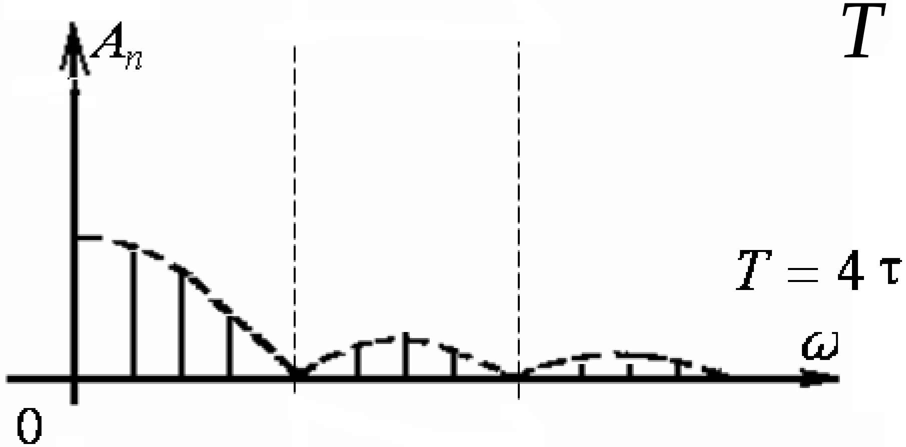
**（2）$\tau$减小，幅度减小，第一个过零点对应的频率 $\frac{2\pi}{\tau}$ 增加，信号的有效占有频带宽度增加。**




$$\dot{A}_n = \frac{2A\tau}{T} Sa\left(n\frac{\tau\pi}{T}\right)$$

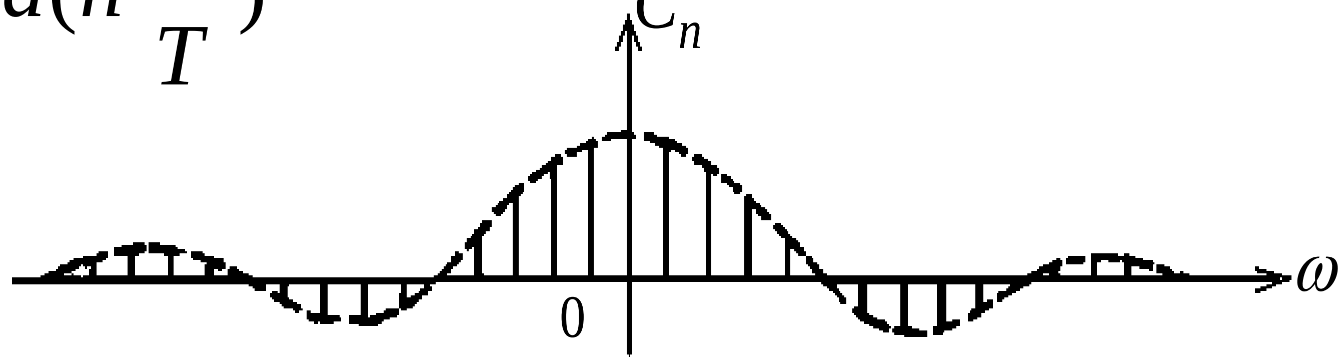
$$f(t)=\sum_{n-\infty}^{\infty} C_n e^{jn\Omega t} = \sum_{n-\infty}^{\infty}\frac{1}{2} \dot{A}_n e^{jn\Omega t} = \sum_{n-\infty}^{\infty}\frac{A\tau}{T}\frac{\sin(n\Omega\frac{\tau}{2})}{n\Omega\frac{\tau}{2}}e^{jn\Omega t}$$

$$=\frac{A\tau}{T}+\sum_{n=1}^{\infty}\frac{2A\tau}{T}\frac{\sin(n\Omega\frac{\tau}{2})}{n\Omega\frac{\tau}{2}}\cos n\Omega t$$



**当复振幅为实函数时，幅度谱和相位谱可以共同画在一张图上。**

$$\dot{A}_n = \frac{2A\tau}{T} Sa(n\frac{\tau\pi}{T})$$




$$\dot{A}_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-jn\Omega t}dt$$

$$\frac{T}{2}\dot{A}_n = TC_n = A\tau Sa(n\frac{\tau\pi}{T})$$

## §3.5　傅里叶变换与非周期信号的频谱（Fourier transform）

周期信号的频谱，具有离散性、收敛性，但当周期趋于无穷大时，周期信号趋于非周期信号，此时的谱也由离散谱变为连续谱，而其幅度趋于无穷小，但各频率分量振幅间的相对大小关系没有变。

$$\because \quad \dot{A}_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-jn\Omega t}dt$$

**两边同乘** $\frac{T}{2}$ **有** $$\frac{T}{2}\dot{A}_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-jn\Omega t}dt$$

**当** $T \to \infty$ **时，** $n\Omega \to \omega$

$$\lim_{T\to\infty}\frac{T}{2}\dot{A}_n = \int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt \triangleq F(j\omega)$$

**即** $$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt}$$

**非周期信号** $f(t)$ **的傅立叶变换**

## $F(j\omega)$ 的量纲：

$$F(j\omega)=\lim_{T\to\infty}\frac{T}{2}\dot{A}_n=\lim_{\Omega\to 0}\pi\frac{\dot{A}_n}{\Omega} \Rightarrow \text{单位频带内的振幅}$$

**频谱密度函数（频谱函数）**

$\boxed{F(j\omega)d\omega}$

$$F(j\omega)=\left|F(j\omega)\right|e^{j\varphi(\omega)}$$

幅度频谱密度函数

相位频谱函数

若信号 $f(t)$ 是实信号，$\left|F(j\omega)\right|$ 就是频率的偶函数，

$\varphi(\omega)$ 是频率的奇函数。

## 由周期信号的指数形式傅立叶级数：

$$F(j\omega)=\lim_{T\to\infty}\frac{T}{2}\dot{A}_n=\lim_{\Omega\to 0}\pi\frac{\dot{A}_n}{\Omega}$$

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{1}{2}\dot{A}_n e^{jn\Omega t}$$

$$\text{当}T\to\infty\text{时},\ \Omega\to d\omega\quad n\Omega\to\omega\quad \dot{A}_n\to\frac{F(j\omega)}{\pi}d\omega$$

$$\sum_{n=-\infty}^{\infty}\to\int_{-\infty}^{\infty}$$

$$\boxed{\therefore\ f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega}$$

**非周期信号 $f(t)$ 的傅立叶积分表示式或傅立叶反变换式。**

**正变换：**

$$\boxed{F(j\omega) = \mathscr{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

**反变换：**

$$\boxed{f(t) = \mathscr{F}^{-1}\{F(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$$

$$f(t) \leftrightarrow F(j\omega)$$

**傅立叶变换的余弦形式：**

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|e^{j[\omega t+\varphi(\omega)]}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|\cos[\omega t+\varphi(\omega)]d\omega$$

$$+j\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|\sin[\omega t+\varphi(\omega)]d\omega$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\left|F(j\omega)\right|\cos[\omega t+\varphi(\omega)]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|\cos[\omega t+\varphi(\omega)]d\omega$$

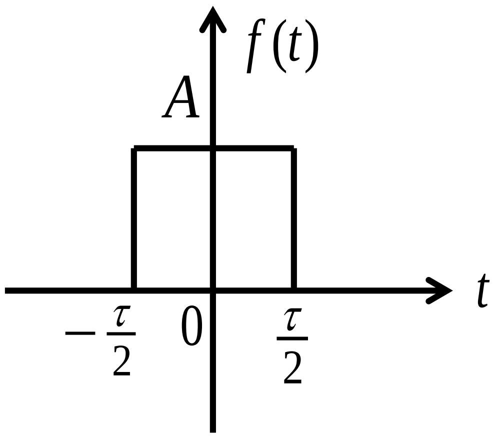
## 非周期信号傅立叶变换存在的充分条件：

$$\boxed{\int_{-\infty}^{\infty} |f(t)| dt < \infty}$$

**绝对可积**

**仅仅是充分条件，不是必要条件。**

**例：**



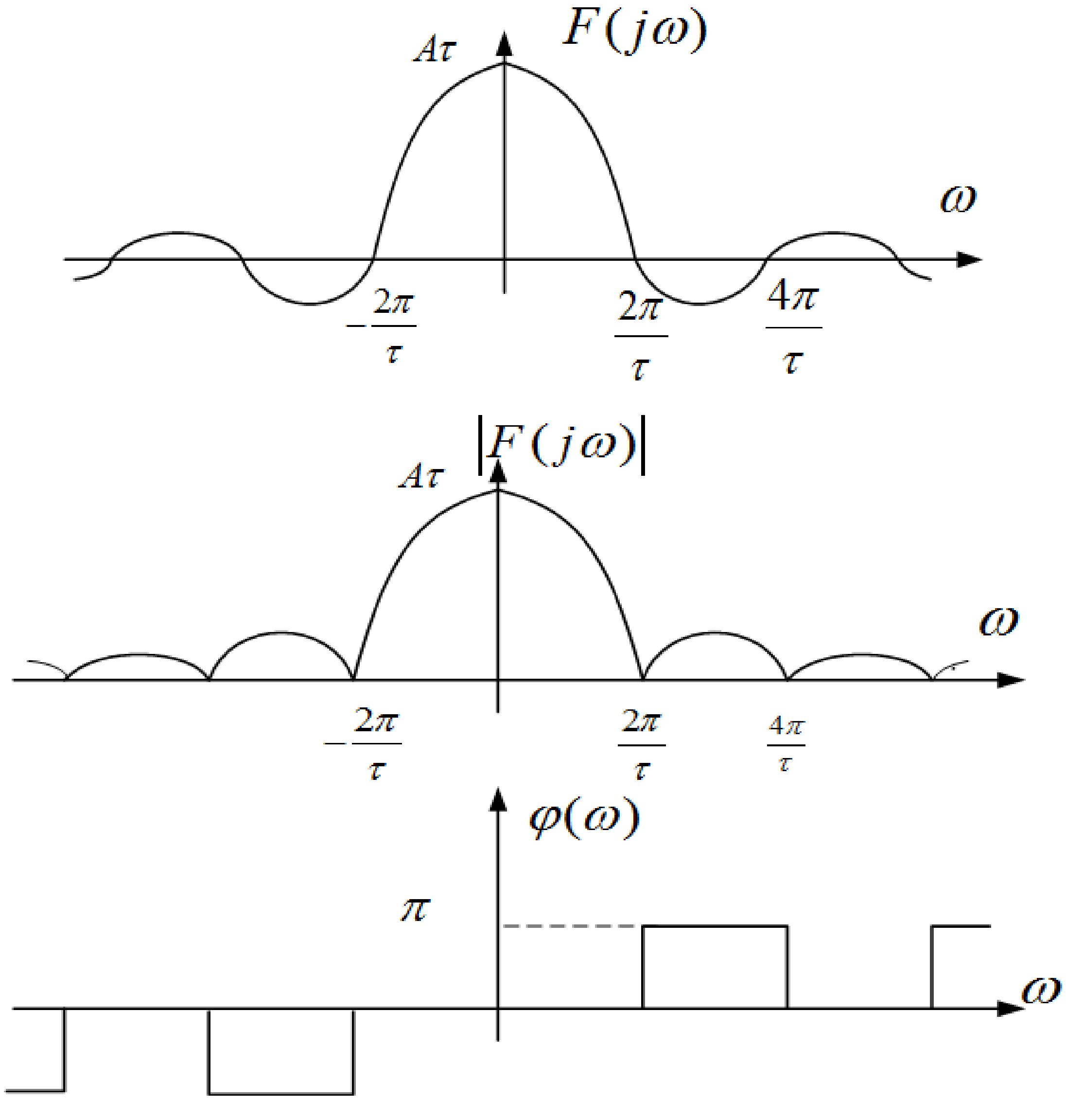

$$f(t)=\begin{cases} A & t<\left|\frac{\tau}{2}\right| \\ 0 & t>\left|\frac{\tau}{2}\right| \end{cases}$$

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cdot e^{-j\omega t}dt$$

$$=\frac{A}{-j\omega}\cdot e^{-j\omega t}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}=\frac{A}{j\omega}(e^{j\omega\frac{\tau}{2}}-e^{-j\omega\frac{\tau}{2}})$$

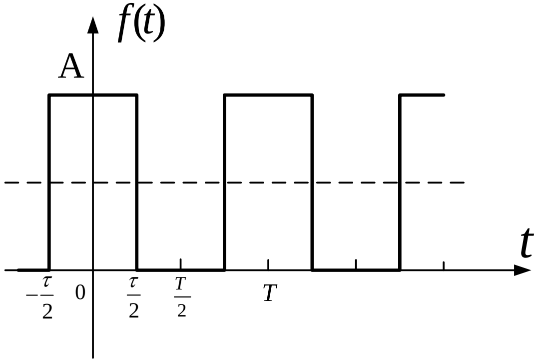
$$=\frac{2A}{\omega}\sin(\omega\cdot\frac{\tau}{2})=A\tau\cdot\frac{\sin(\omega\cdot\frac{\tau}{2})}{\omega\cdot\frac{\tau}{2}}=A\tau\cdot Sa(\omega\cdot\frac{\tau}{2})$$

$$F(j\omega) = A\tau \cdot Sa(\omega \cdot \frac{\tau}{2})$$



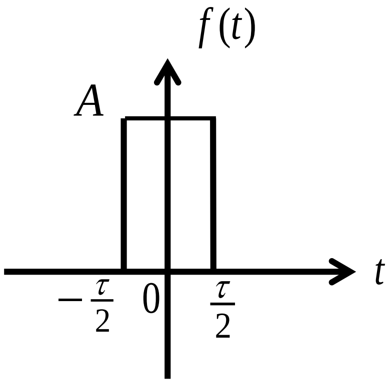
$$\therefore \left|F(j\omega)\right| = A\tau\left|\frac{\sin(\omega\frac{\tau}{2})}{\omega\frac{\tau}{2}}\right| = A\tau\left|Sa(\omega\frac{\tau}{2})\right|$$

$$\varphi(\omega) = \begin{cases} 0 & \dfrac{4n\pi}{\tau} < \omega < \dfrac{2(2n+1)\pi}{\tau} \\ \pi & \dfrac{2(2n+1)\pi}{\tau} < \omega < \dfrac{2(2n+2)\pi}{\tau} \end{cases} \qquad n = 0,1,2,\cdots$$



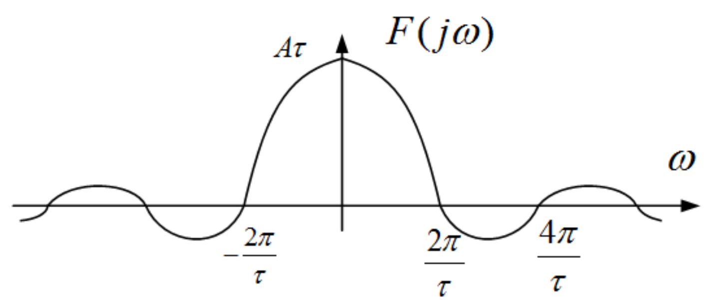

$$f(t)=\sum_{n-\infty}^{\infty}\frac{1}{2}\dot{A}_n e^{jn\Omega t}$$

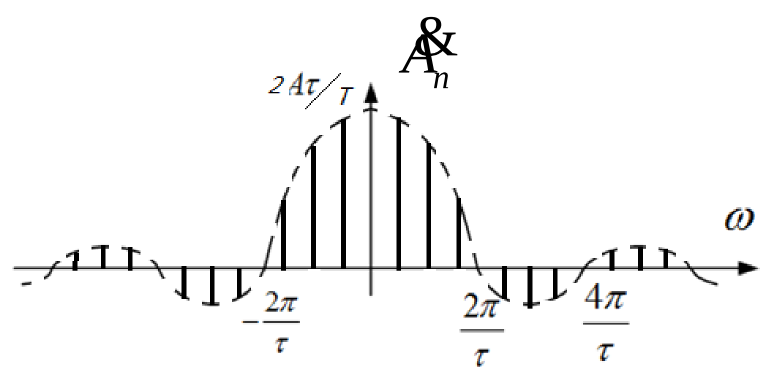
$$\boxed{\dot{A}_n=\frac{2A\tau}{T}Sa(n\Omega\frac{\tau}{2})}$$

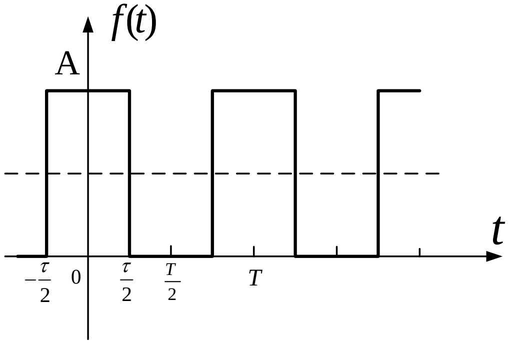


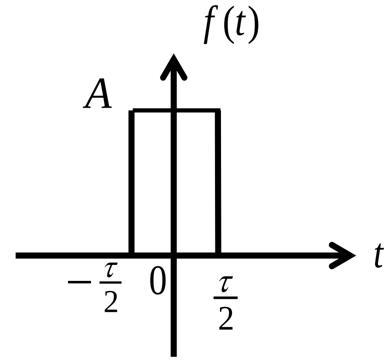

$$\boxed{F(j\omega)=A\tau\cdot Sa(\omega\cdot\frac{\tau}{2})}$$

$$\boxed{\dot{A}_n=\frac{2}{T}F(j\omega)\Big|_{\omega=n\Omega}}$$






**单个矩形脉冲与周期矩形脉冲相频谱比较：**

**(1) 包络线相同，同为抽样函数的形式；**

**(2) $F(j\omega)$ 中自变量连续取值，而 $\dot{A}_n$ 离散取值**

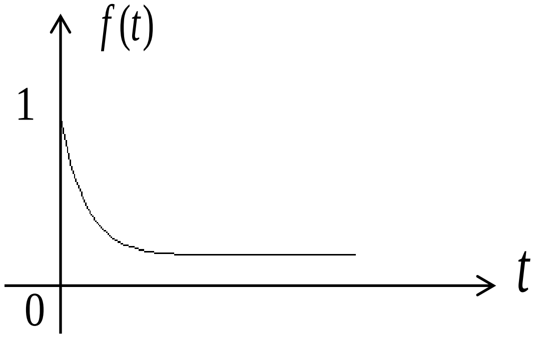
**(3) 在把 $F(j\omega)$ 离散化后，$F(j\omega)$ 与 $\dot{A}_n$ 只相差一系数 $\frac{2}{T}$ ；**

$$\boxed{\dot{A}_n = \frac{2}{T} F(j\omega)\Big|_{\omega=n\Omega}}$$

# §3.6 常见信号频谱函数举例

## 一、单边指数信号

$$f(t)=\begin{cases} e^{-\alpha t} & t>0 \\ 0 & t<0 \end{cases} = e^{-\alpha t}\varepsilon(t)$$ 其中 $\alpha>0$

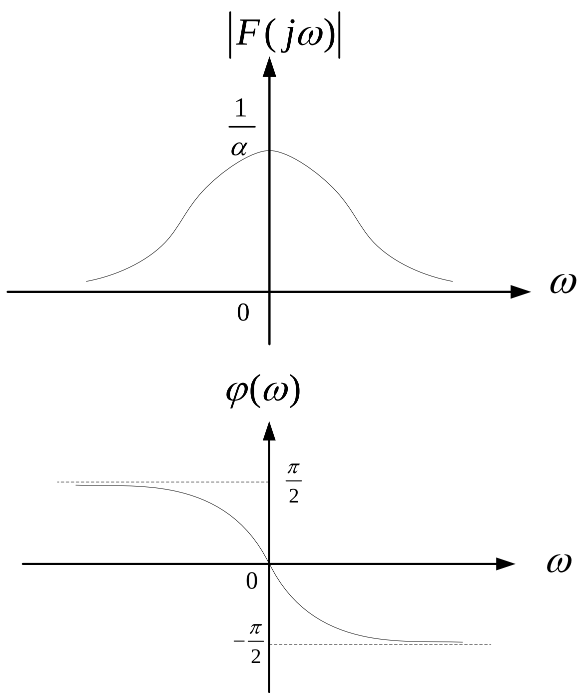



则 $F(j\omega)=\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt=\int_{0}^{\infty} e^{-\alpha t}\cdot e^{-j\omega t}dt$

$$=\int_{0}^{\infty} e^{-(\alpha+j\omega)t}dt=\frac{1}{\alpha+j\omega}$$

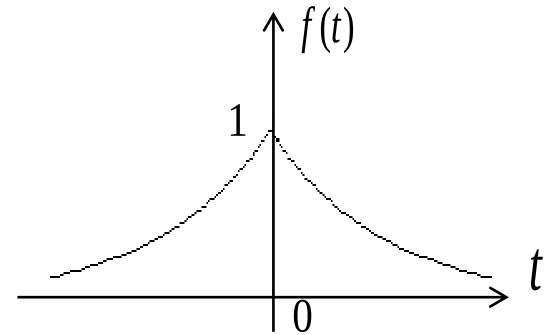
$$|F(j\omega)|=\frac{1}{(\alpha^2+\omega^2)^{1/2}}$$

$$\varphi(\omega)=-arctg(\frac{\omega}{\alpha})$$



## 二、双边指数信号

$$f(t) = e^{-\alpha|t|} \qquad -\infty < t < \infty$$

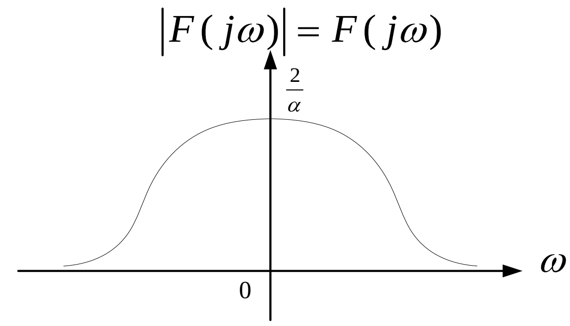



$$F(j\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-\alpha|t|} \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{\alpha t} \cdot e^{-j\omega t} dt + \int_{0}^{\infty} e^{-\alpha t} \cdot e^{-j\omega t} dt$$

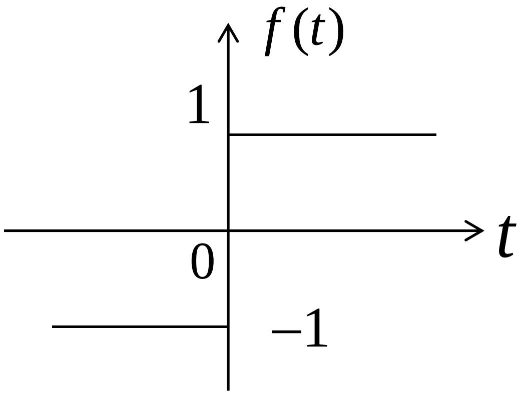
$$= \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$\begin{cases} |F(j\omega)| = F(j\omega) = \dfrac{2\alpha}{\alpha^2 + \omega^2} \\ \varphi(\omega) = 0 \end{cases}$$



# 三、符号函数

$$f(t)=\mathrm{sgn}(t)=\begin{cases}1 & t>0\\ -1 & t<0\end{cases}$$



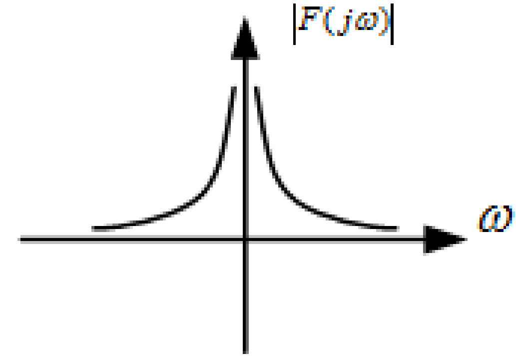

令 $$f_1(t)=f(t)\cdot e^{-\alpha|t|}=\mathrm{sgn}(t)\cdot e^{-\alpha|t|}$$

则 $$f(t)=\lim_{\alpha\to 0} f_1(t),\quad F(j\omega)=F_1(j\omega)$$

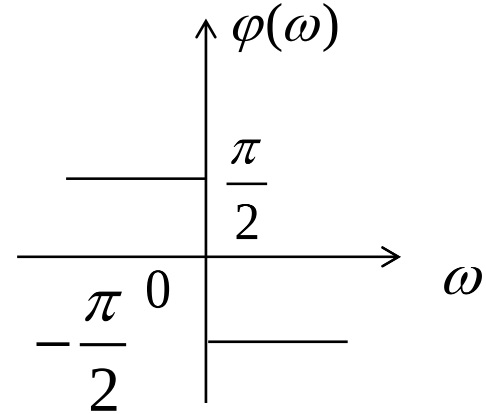
$$F_1(j\omega)=\int_{-\infty}^{\infty} f_1(t)\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}\mathrm{sgn}(t)e^{-\alpha|t|}\cdot e^{-j\omega t}dt$$

$$=-\int_{-\infty}^{0}e^{\alpha t}\cdot e^{-j\omega t}dt+\int_{0}^{\infty}e^{-\alpha t}\cdot e^{-j\omega t}dt=\frac{-j2\omega}{\alpha^2+\omega^2}$$

$$\therefore F(j\omega)=\lim_{\alpha\to 0}F_1(j\omega)=\frac{2}{j\omega}=\begin{cases}\dfrac{2}{\omega}e^{-j\frac{\pi}{2}} & \omega>0\\ \dfrac{2}{|\omega|}e^{j\frac{\pi}{2}} & \omega<0\end{cases}$$



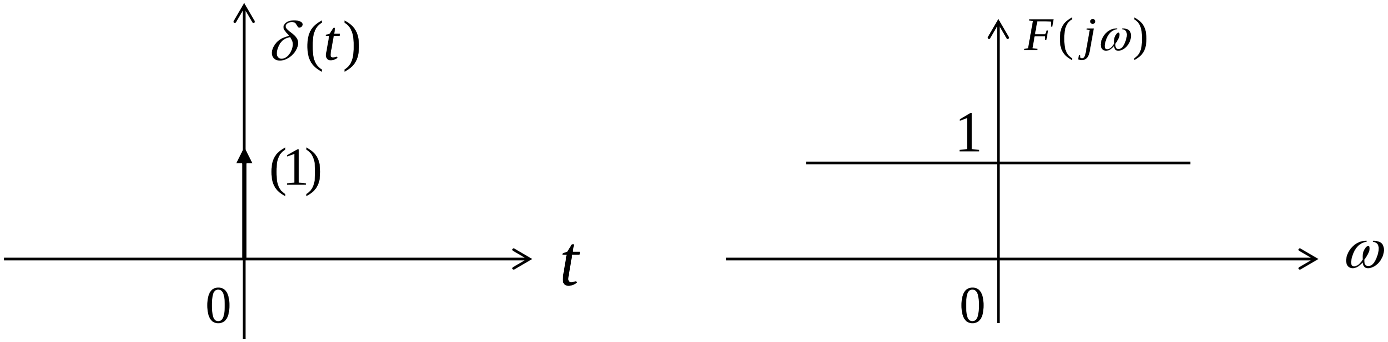

$$\begin{cases}|F(j\omega)|=\dfrac{2}{|\omega|}\\ \varphi(\omega)=\begin{cases}\dfrac{\pi}{2} & \omega>0\\ -\dfrac{\pi}{2} & \omega<0\end{cases}\end{cases}$$



# 四、冲激信号

$$f(t)=\delta(t)$$

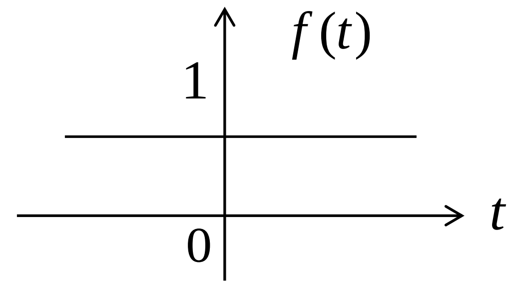
则 $$F(j\omega)=\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}\delta(t)\cdot e^{-j\omega t}dt=1$$




**冲激信号的频宽为无穷宽。**

# 五、直流信号

$$f(t)=1,\quad -\infty<t<\infty$$




$$\text{Q}\quad \delta(t)\leftrightarrow 1$$

$$\therefore\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}1\cdot e^{j\omega t}d\omega=\delta(t)$$

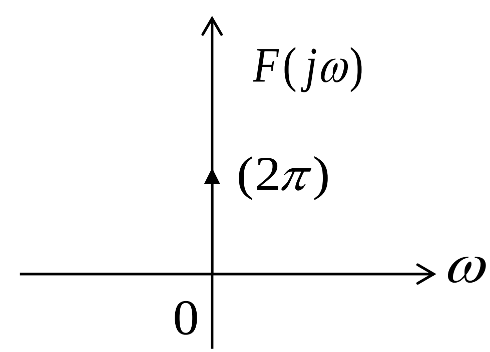
或 $$\int_{-\infty}^{\infty}1\cdot e^{j\omega t}d\omega=2\pi\cdot\delta(t)$$

**变量代换：** $\omega\rightarrow\tau\qquad t\rightarrow\omega$

$$\int_{-\infty}^{\infty}1\cdot e^{j\tau\omega}d\tau=2\pi\cdot\delta(\omega)$$

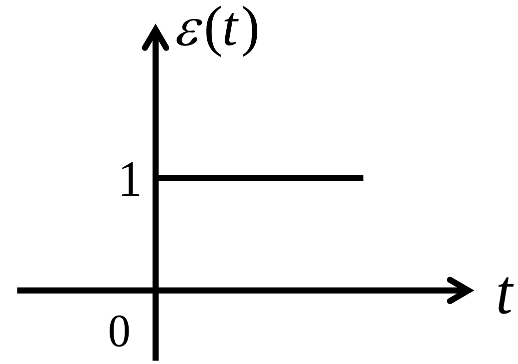
**再令：** $\tau=-t$ 则 $d\tau=-dt$

$$\therefore\int_{-\infty}^{\infty}1\cdot e^{-j\omega t}dt=2\pi\cdot\delta(\omega)$$



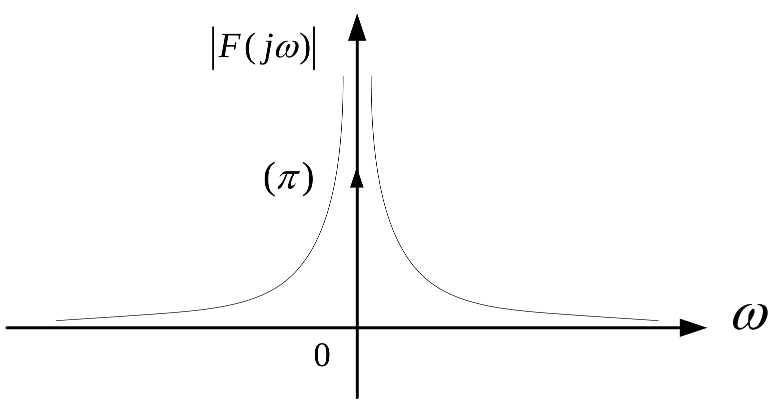

$$\boxed{\therefore\quad 1\leftrightarrow 2\pi\delta(\omega)}$$

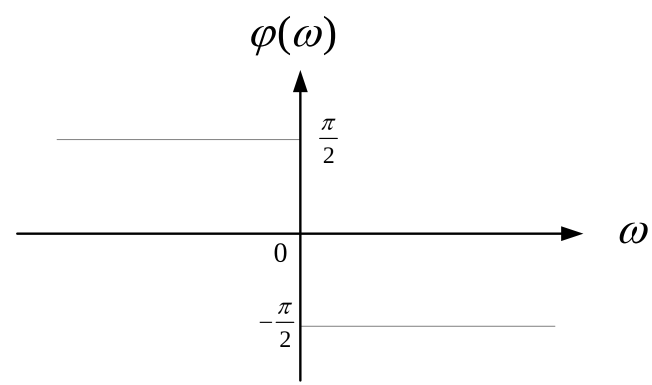
# 六、单位阶跃信号 $\varepsilon(t)$




$$\text{Q} \quad \varepsilon(t) = \frac{1}{2}[\mathrm{sgn}(t) + 1(t)]$$

$$\therefore \quad \varepsilon(t) \leftrightarrow \frac{1}{2}\left[\frac{2}{j\omega} + 2\pi\delta(\omega)\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$



## 七、复指数信号

$$f(t) = e^{j\omega_c t}$$

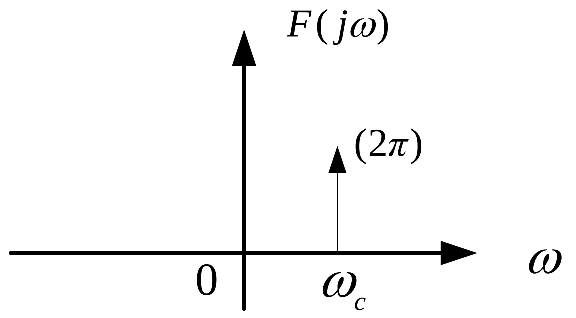
$$F(j\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{j\omega_c t} \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j(\omega - \omega_c)t} dt$$

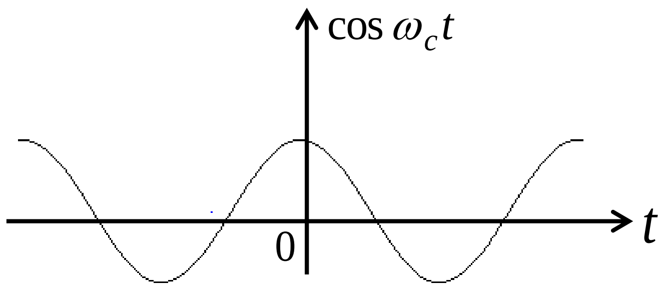
**利用直流信号傅立叶变换的结果，可求得：**

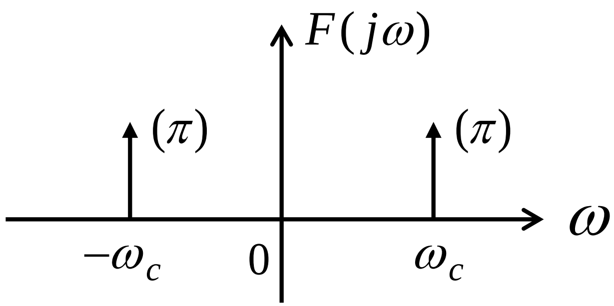
$$F(j\omega) = 2\pi\delta(\omega - \omega_c)$$

**即** $$e^{i\omega_c t} \leftrightarrow 2\pi\delta(\omega - \omega_c)$$



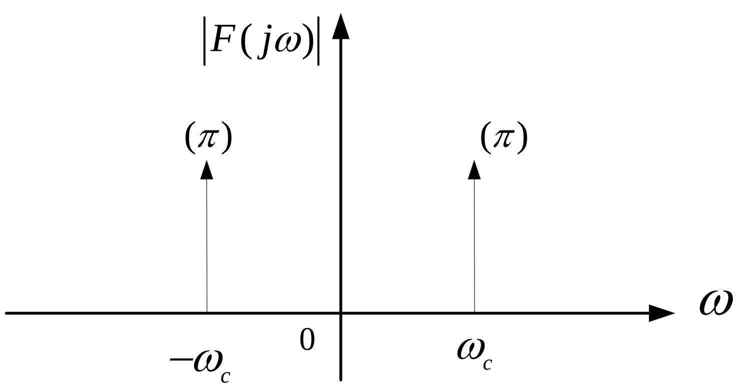
$$\cos\omega_c t=\frac{1}{2}(e^{j\omega_c t}+e^{-j\omega_c t}) \longleftrightarrow \pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)$$

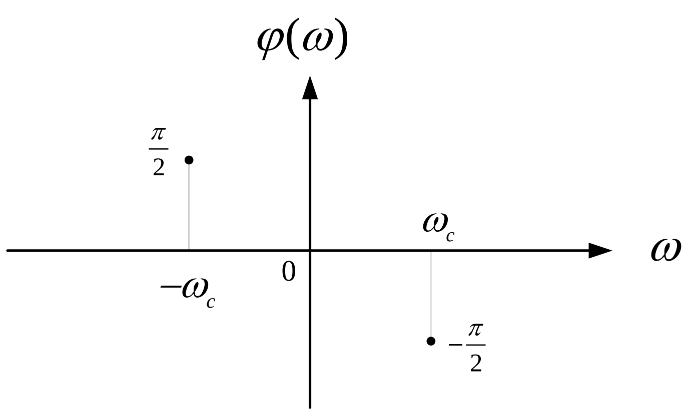



**同理可得正弦信号的频谱函数为**

$$\sin\omega_c t \longleftrightarrow j\pi[\delta(\omega+\omega_c)-\delta(\omega-\omega_c)]$$



# 八、周期信号与冲激序列

周期信号 $f(t)$ 可展开成指数傅立叶级数为

$$\boxed{f(t)=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega t}}$$

则其傅立叶变换为：

$$F(j\omega)=\mathcal{F}\{f(t)\}=\mathcal{F}\{\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega t}\}=\sum_{n=-\infty}^{\infty}C_n\mathcal{F}\{e^{jn\Omega t}\}$$

$$=\sum_{n=-\infty}^{\infty}C_n\cdot 2\pi\delta(\omega-n\Omega)=2\pi\sum_{n=-\infty}^{\infty}C_n\delta(\omega-n\Omega)$$

$$\boxed{F(j\omega)=2\pi\sum_{n=-\infty}^{\infty}C_n\delta(\omega-n\Omega)=\pi\sum_{n=-\infty}^{\infty}\dot{A}_n\delta(\omega-n\Omega)}$$

**例如周期信号** $f(t)=1+\frac{1}{2}\cos t+\frac{1}{4}\cos 5t$

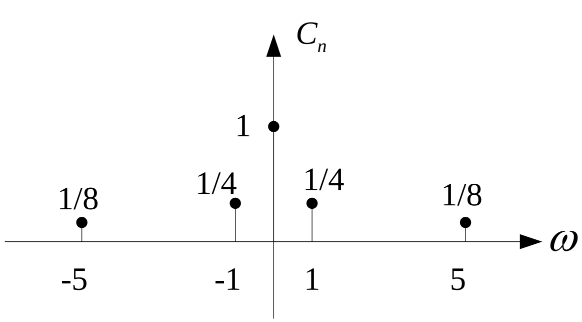
$$F(j\omega)=2\pi\delta(\omega)+\frac{\pi}{2}\delta(\omega+1)+\frac{\pi}{2}\delta(\omega-1)+\frac{\pi}{4}\delta(\omega+5)+\frac{\pi}{4}\delta(\omega-5)$$

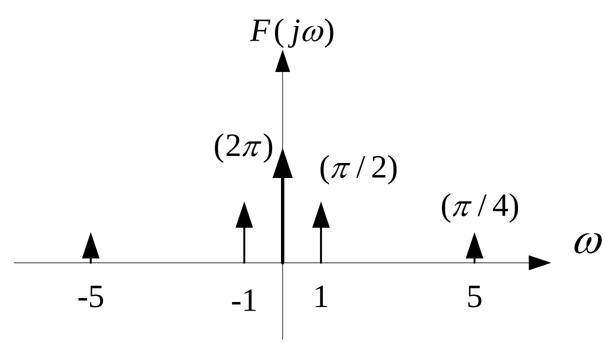
**复振幅** $\frac{\dot{A}_0}{2}=1$, $\dot{A}_1=\frac{1}{2}$, $\dot{A}_5=\frac{1}{4}$

$$C_0=\frac{\dot{A}_0}{2}=1,\quad C_1=\frac{1}{2}\dot{A}_1=\frac{1}{4},\quad C_5=\frac{1}{2}\dot{A}_5=\frac{1}{8},\quad C_{-1}=C_1^*=\frac{1}{4},\quad C_{-5}=C_5^*=\frac{1}{8}$$

**或** $F(j\omega)=2\pi\sum_{n=-\infty}^{\infty}C_n\delta(\omega-n\Omega)$

$$=2\pi\delta(\omega)+\frac{\pi}{2}\delta(\omega+1)+\frac{\pi}{2}\delta(\omega-1)+\frac{\pi}{4}\delta(\omega+5)+\frac{\pi}{4}\delta(\omega-5)$$



**例如周期信号** $$f(t)=1+\frac{1}{2}\cos(t-\frac{2\pi}{3})+\frac{1}{4}\cos(5t+\frac{\pi}{4})$$

$$=1-\frac{1}{4}\cos t+\frac{\sqrt{3}}{4}\sin t+\frac{\sqrt{2}}{8}\cos 5t-\frac{\sqrt{2}}{8}\sin 5t$$

$$F(j\omega)=2\pi\delta(\omega)-\frac{\pi}{4}\delta(\omega+1)-\frac{\pi}{4}\delta(\omega-1)+\frac{\sqrt{2}\pi}{8}\delta(\omega+5)+\frac{\sqrt{2}\pi}{8}\delta(\omega-5)$$

$$+\frac{\sqrt{3}\pi}{4}j\delta(\omega+1)-\frac{\sqrt{3}\pi}{4}j\delta(\omega-1)-\frac{\sqrt{2}\pi}{8}j\delta(\omega+5)+\frac{\sqrt{2}\pi}{8}j\delta(\omega-5)$$

$$=2\pi\delta(\omega)-(\frac{\pi}{4}-\frac{\sqrt{3}\pi}{4}j)\delta(\omega+1)-(\frac{\pi}{4}+\frac{\sqrt{3}\pi}{4}j)\delta(\omega-1)$$

$$+(\frac{\sqrt{2}\pi}{8}-\frac{\sqrt{2}\pi}{8}j)\delta(\omega+5)+(\frac{\sqrt{2}\pi}{8}+\frac{\sqrt{2}\pi}{8}j)\delta(\omega-5)$$

$$=2\pi\delta(\omega)+\frac{\pi}{2}e^{j\frac{2\pi}{3}}\delta(\omega+1)+\frac{\pi}{2}e^{-j\frac{2\pi}{3}}\delta(\omega-1)+\frac{\pi}{4}e^{-j\frac{\pi}{4}}\delta(\omega+5)+\frac{\pi}{4}e^{j\frac{\pi}{4}}\delta(\omega-5)$$

$$f(t)=1+\frac{1}{2}\cos(t-\frac{2\pi}{3})+\frac{1}{4}\cos(5t+\frac{\pi}{4})$$

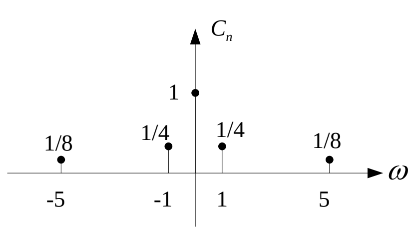
$$F(j\omega)=2\pi\delta(\omega)+\frac{\pi}{2}e^{j\frac{2\pi}{3}}\delta(\omega+1)+\frac{\pi}{2}e^{-j\frac{2\pi}{3}}\delta(\omega-1)+\frac{\pi}{4}e^{-j\frac{\pi}{4}}\delta(\omega+5)+\frac{\pi}{4}e^{j\frac{\pi}{4}}\delta(\omega-5)$$

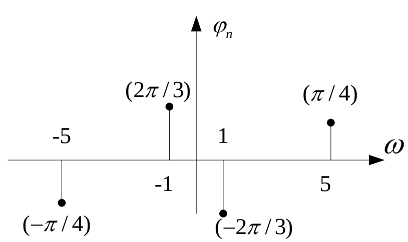
**复振幅** $\frac{\dot{A}_0}{2}=1,\quad \dot{A}_1=\frac{1}{2}e^{-j\frac{2\pi}{3}},\quad \dot{A}_5=\frac{1}{4}e^{j\frac{\pi}{4}}$

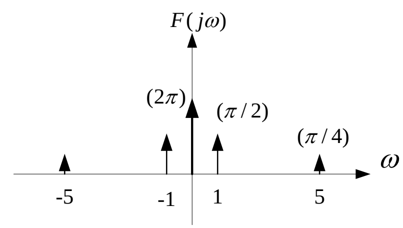
$$C_0=\frac{\dot{A}_0}{2}=1,\quad C_1=\frac{1}{2}\dot{A}_1=\frac{1}{4}e^{-j\frac{2\pi}{3}},\quad C_5=\frac{1}{2}\dot{A}_5=\frac{1}{8}e^{j\frac{\pi}{4}},\quad C_{-1}=C_1^*=\frac{1}{4}e^{j\frac{2\pi}{3}},\quad C_{-5}=C_5^*=\frac{1}{8}e^{-j\frac{\pi}{4}}$$

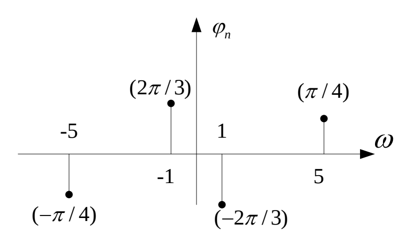
$$F(j\omega)=2\pi\sum_{n=-\infty}^{\infty}C_n\delta(\omega-n\Omega)$$

$$=2\pi\delta(\omega)+\frac{\pi}{2}e^{j\frac{2\pi}{3}}\delta(\omega+1)+\frac{\pi}{2}e^{-j\frac{2\pi}{3}}\delta(\omega-1)+\frac{\pi}{4}e^{-j\frac{\pi}{4}}\delta(\omega+5)+\frac{\pi}{4}e^{j\frac{\pi}{4}}\delta(\omega-5)$$



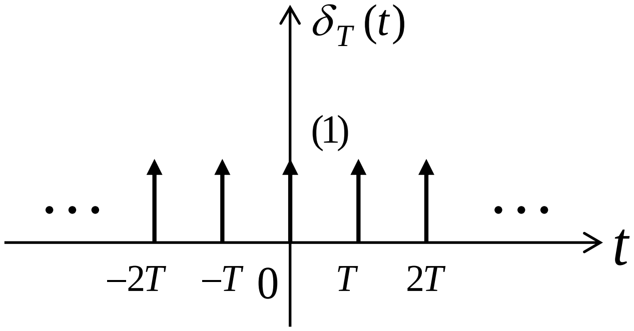



# 冲激序列

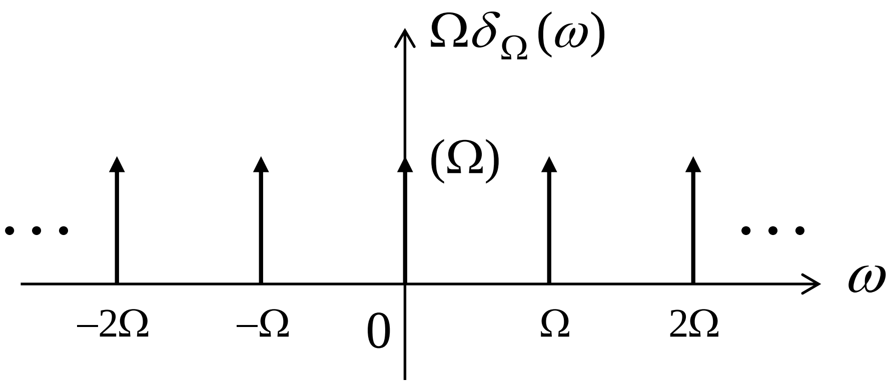
$$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$




## 是一周期信号，其复振幅

$$\dot{A}_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)\cdot e^{-jn\Omega t}dt=\frac{2}{T}$$

$$\therefore F(j\omega)=\pi\sum_{n=-\infty}^{\infty}\frac{2}{T}\delta(\omega-n\Omega)\quad=\Omega\sum_{n=-\infty}^{\infty}\delta(\omega-n\Omega)=\Omega\delta_{\Omega}(\omega)$$



# §3.7 傅立叶变换的性质

## 一、线性特性

若 $f_1(t) \leftrightarrow F_1(j\omega),\quad f_2(t) \leftrightarrow F_2(j\omega)$

则 $a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(j\omega) + a_2 F_2(j\omega)$

其中 $\alpha_1$，$\alpha_2$ 为常量。

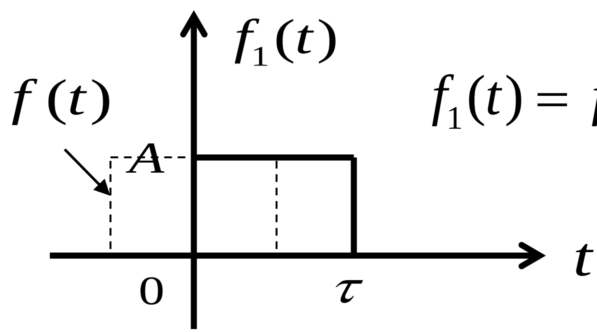
# 二、时移特性（延时特性）

若 $$f(t) \leftrightarrow F(j\omega) = |F(j\omega)| e^{j\varphi(\omega)}$$

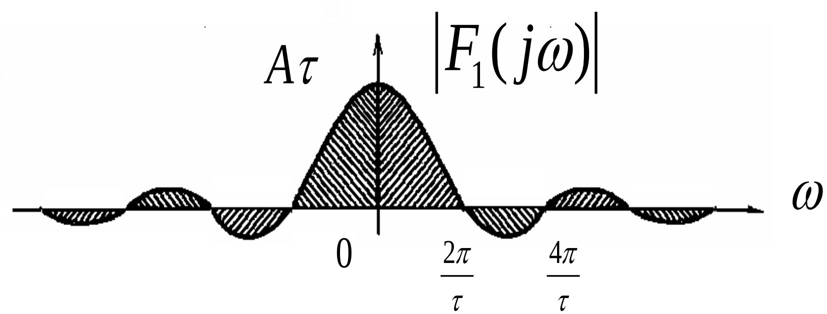
则 $$f(t-t_0) \leftrightarrow F(j\omega)e^{-j\omega t_0} = |F(j\omega)| e^{j[\varphi(\omega)-\omega t_0]}$$

**信号在时域中的时移，对应频谱函数在频域中产生的附加相移，而幅度频谱保持不变。**

## 例如延时的单个脉冲

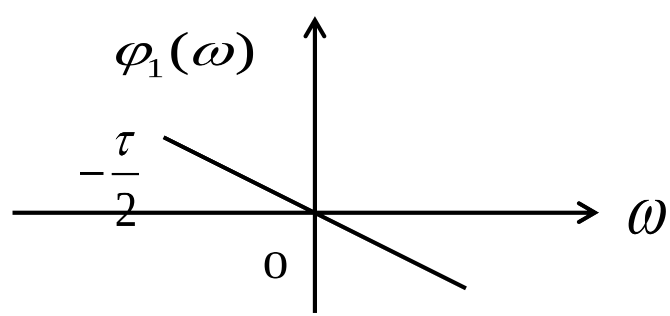



$$f_1(t) = f(t-\tau/2)$$




$$F_1(j\omega) = F(j\omega)e^{-j\omega\tau/2}$$

$$|F_1(j\omega)| = F(j\omega), \quad \varphi(\omega) = -\frac{\tau}{2}\omega$$



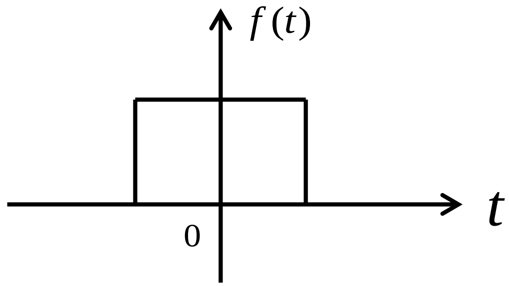
# 三、移频特性

**若** $f(t) \leftrightarrow F(j\omega)$ **则** $f(t)\cdot e^{j\omega_c t} \leftrightarrow F(j\omega - j\omega_c)$

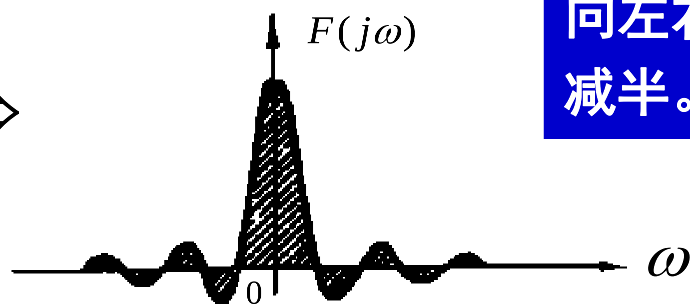
**实际中是信号与正弦信号相乘，则其频谱为：**

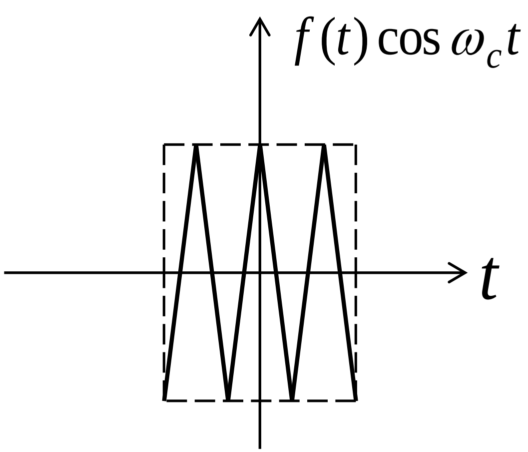
$$\boxed{f(t)\cdot\cos\omega_c t \leftrightarrow \frac{1}{2}[F(j\omega + j\omega_c) + F(j\omega - j\omega_c)]}$$

**信号$x(t)$与余弦信号$\cos w_c\, t$相乘后，其频谱是将原来信号频谱向左右搬移$w_c$，幅度减半。**

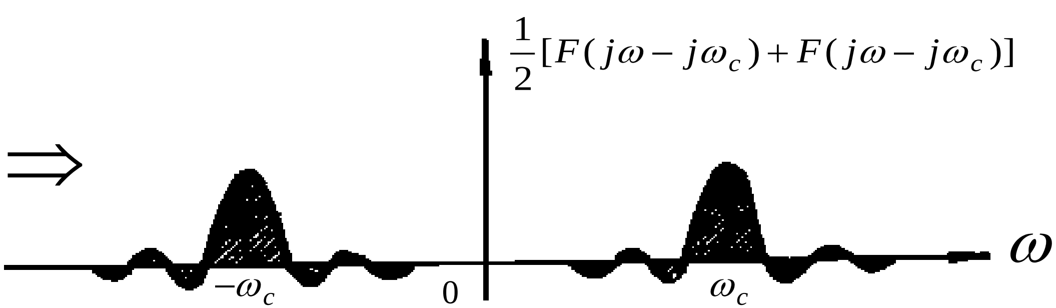



$\Rightarrow$




$\Rightarrow$

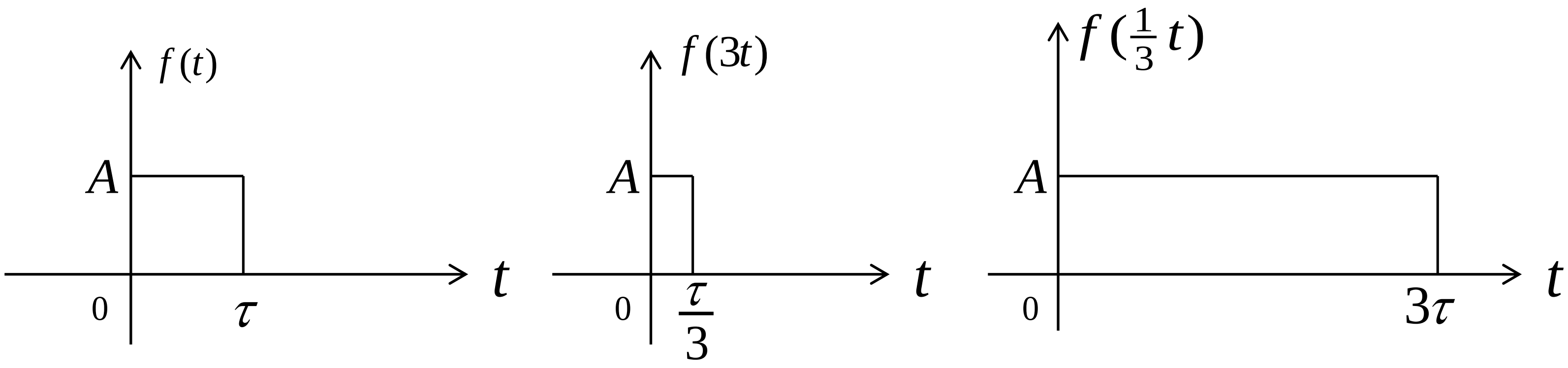


同理

$$f(t)\sin\omega_c t \quad \leftrightarrow \quad \frac{1}{2\mathrm{j}}\mathcal{F}[f(t)\mathrm{e}^{j\omega_\mathrm{c}t}]-\frac{1}{2\mathrm{j}}\mathcal{F}[f(t)\mathrm{e}^{-j\omega_\mathrm{c}t}]$$

$$=\frac{\mathrm{j}}{2}F[\mathrm{j}(\omega+\omega_c)]-\frac{\mathrm{j}}{2}F[\mathrm{j}(\omega-\omega_c)]$$

# 四、尺度变换特性

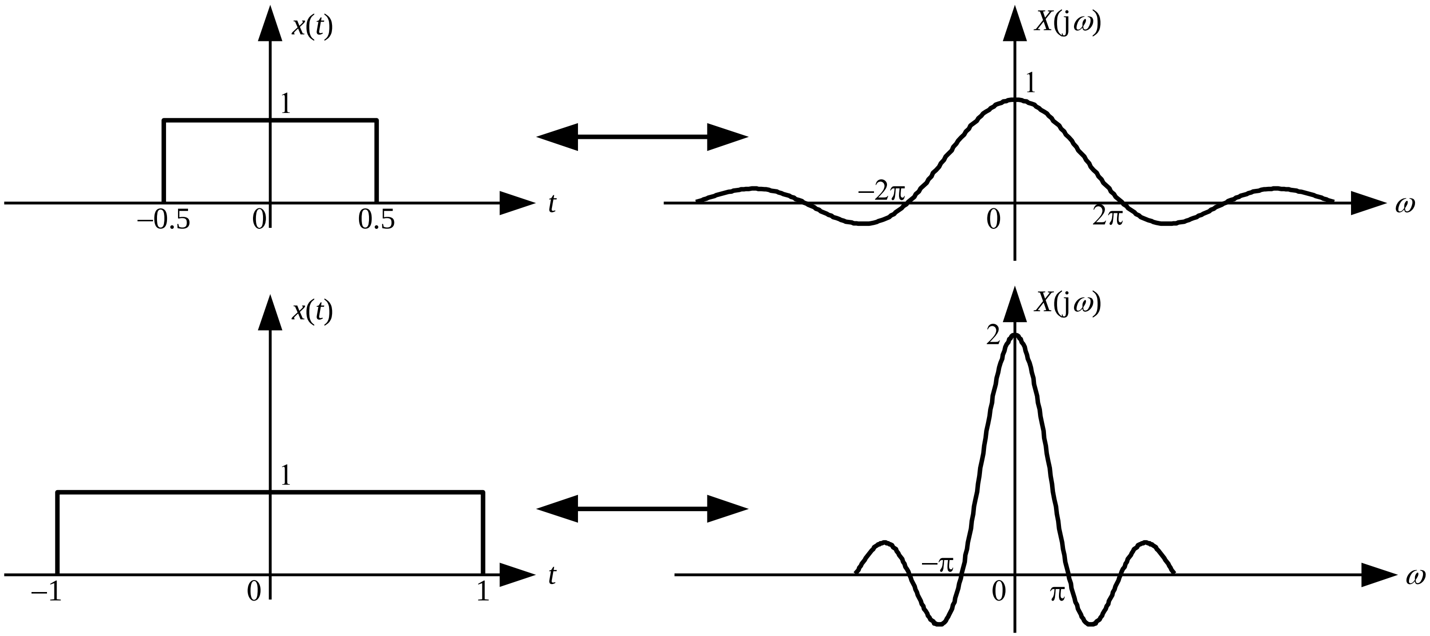



**一般** $f(at)$ $a>1$ **表示压缩** $a$ **倍**

$a<1$ **表示展宽** $\frac{1}{a}$ **倍**

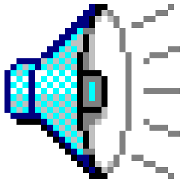
**若** $f(t) \leftrightarrow F(j\omega)$ **则** $f(at) \leftrightarrow \frac{1}{|a|} F(j\frac{\omega}{a})$

**时域压缩，则频域展宽；展宽时域，则频域压缩。**

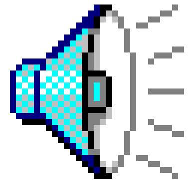


x(t)
1
−0.5
0
0.5
t
X(jω)
1
−2π
0
2π
ω
x(t)
1
−1
0
1
t
X(jω)
2
−π
0
π
ω

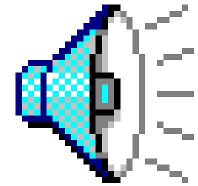
# 例　尺度变换后语音信号的变化



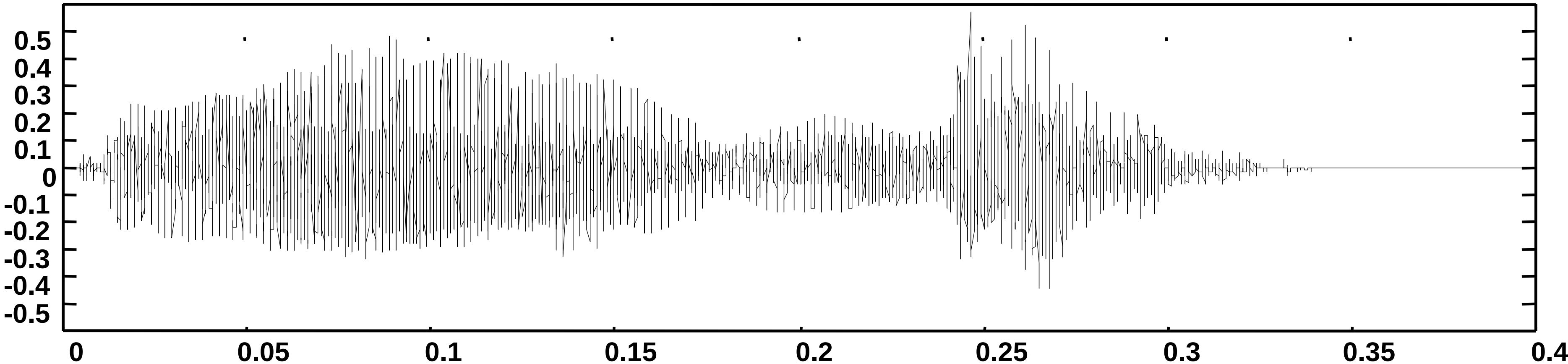
$f(t)$



$f(2t)$



$f(t/2)$




一段语音信号（“对了”）。抽样频率 = 22050Hz

# 五、奇偶特性

**假设信号是实函数**

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt \quad =\int_{-\infty}^{\infty} f(t)\cdot \cos\omega t dt - j\int_{-\infty}^{\infty} f(t)\cdot \sin\omega t dt$$

$$=R(\omega)+jX(\omega)=\left|F(j\omega)\right|e^{j\varphi(\omega)}$$

**其中** $R(\omega)=\int_{-\infty}^{\infty} f(t)\cdot \cos\omega t dt$ ——**偶函数**

$X(\omega)=-\int_{-\infty}^{\infty} f(t)\cdot \sin\omega t dt$ ——**奇函数**

$\left|F(j\omega)\right|=[R^2(\omega)+X^2(\omega)]^{1/2}$ ——**偶函数**

$\varphi(\omega)=arctg\dfrac{X(\omega)}{R(\omega)}$ ——**奇函数**

$\boxed{\therefore F(-j\omega)=\left|F(-j\omega)\right|e^{j\varphi(-\omega)}=F^*(j\omega)}$ ——**实信号的充分必要条件**

**若** $f(t)=f(-t)$ **偶函数**

**则** $R(\omega)=2\int_0^{\infty} f(t)\cos\omega t dt,\qquad X(\omega)=0$

$$\boxed{F(j\omega)=R(\omega)}$$

**实偶函数的傅立叶变换必为实的，且是偶函数，反之也成立。**

**若** $f(t)=-f(-t)$ **奇函数**

**则** $R(\omega)=0,\qquad X(\omega)=-2\int_0^{\infty} f(t)\sin\omega t dt$

$$\boxed{F(j\omega)=jX(\omega)}$$

**实奇函数的傅立叶变换必为纯虚函数，反之也成立。**

**任一实信号还可分解为一个偶分量和一个奇分量和的形式，即**

$$f(t) = f_e(t) + f_o(t)$$

**其中** $f_e(t) = \frac{1}{2}[f(t) + f(-t)]$ $f_o(t) = \frac{1}{2}[f(t) - f(-t)]$

**偶分量** **奇分量**

$$F_e(j\omega) = \frac{1}{2}[F(j\omega) + F(-j\omega)]$$

$$= \frac{1}{2}\left[F(j\omega) + F^*(j\omega)\right] = \mathrm{Re}\{F(j\omega)\} = R(\omega)$$

$$F_o(j\omega) = \frac{1}{2}[F(j\omega) - F(-j\omega)]$$

$$= \frac{1}{2}\left[F(j\omega) - F^*(j\omega)\right] = j\,\mathrm{Im}\{F(j\omega)\} = jX(\omega)$$

## 六、对称特性

若 $f(t) \leftrightarrow F(j\omega)$

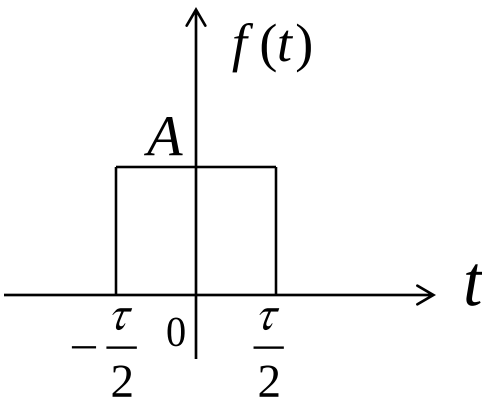
则 $F(jt) \leftrightarrow 2\pi \cdot f(-\omega)$

若 $f(t)$ 为偶函数，则 $F(jt) \leftrightarrow 2\pi \cdot f(\omega)$

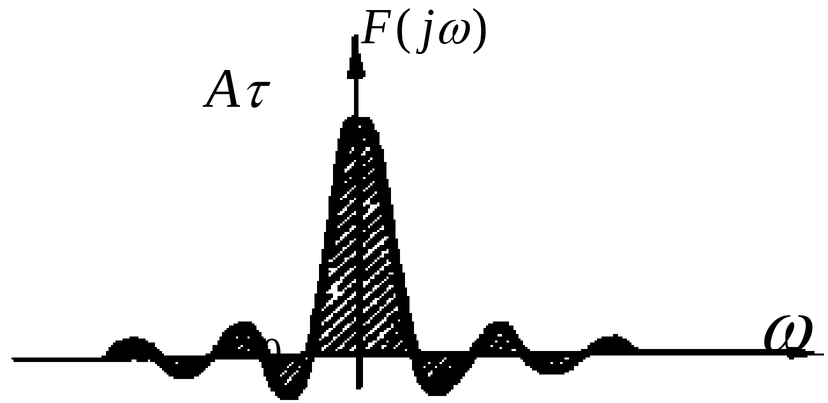
若 $f(t)$ 为奇函数，则 $F(jt) \leftrightarrow -2\pi \cdot f(\omega)$

# 利用对称特性

$\delta(t) \leftrightarrow 1$ **则** $1 \leftrightarrow 2\pi\delta(\omega)$

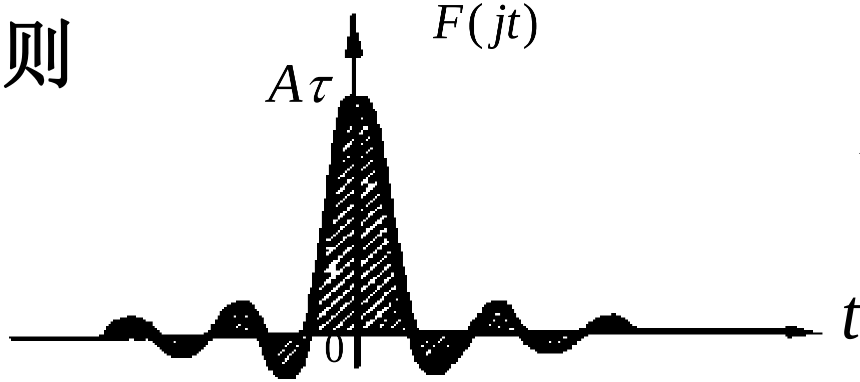



$\longleftrightarrow$

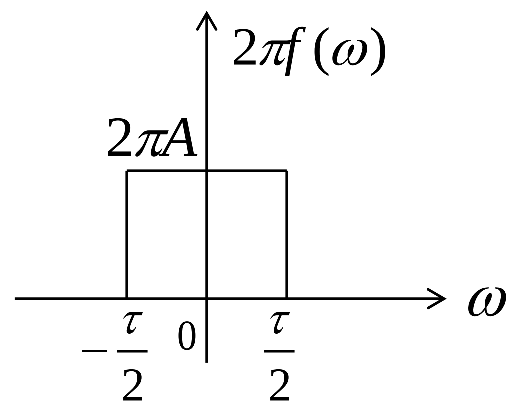



$$F(jt) \leftrightarrow 2\pi \cdot f(-\omega)$$

则




$\longleftrightarrow$



## 七、微分特性

若 $f(t) \leftrightarrow F(j\omega)$

则 $$\boxed{\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega)}$$

$$\boxed{\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)}$$

## 八、积分特性

若 $f(t) \leftrightarrow F(j\omega)$

则 $$\boxed{\int_{-\infty}^{t} f(\tau)d\tau \leftrightarrow \pi F(j0)\delta(\omega) + \frac{F(j\omega)}{j\omega}}$$

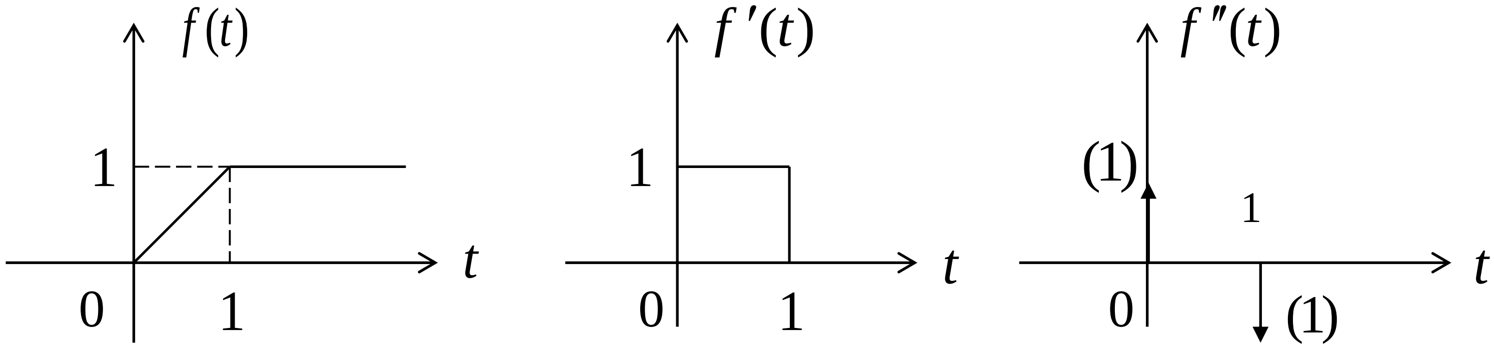
当 $F(j0)=0$，或不计 $\omega=0$ 点的频谱值，则积分特性为

$$\boxed{\int_{-\infty}^{t} f(\tau)d\tau \leftrightarrow \frac{F(j\omega)}{j\omega}}$$

积分特性使用的条件：

$$f(-\infty)=0$$

**例 求$f(t)$的傅立叶变换。**




**由图知** $f''(t)=\delta(t)-\delta(t-1)$ **令** $f_2(t)=f''(t)$

**则** $F_2(j\omega)=1-e^{-j\omega}$ **再令** $f_1(t)=f'(t)$

$$\boxed{\begin{aligned} f_1(t) &= \int_{-\infty}^{t} f_2(\tau)d\tau \\ f(t) &= \int_{-\infty}^{t} f_1(\tau)d\tau \end{aligned}}$$

**则利用微积分特性有：**

$$F_1(j\omega)=\pi F_2(j0)\delta(\omega)+\frac{F_2(j\omega)}{j\omega}=\frac{1-e^{-j\omega}}{j\omega}=\frac{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}})}{j\omega}$$

$$=\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}e^{-j\frac{\omega}{2}}=Sa(\frac{\omega}{2})e^{-j\frac{\omega}{2}}$$

**同理**

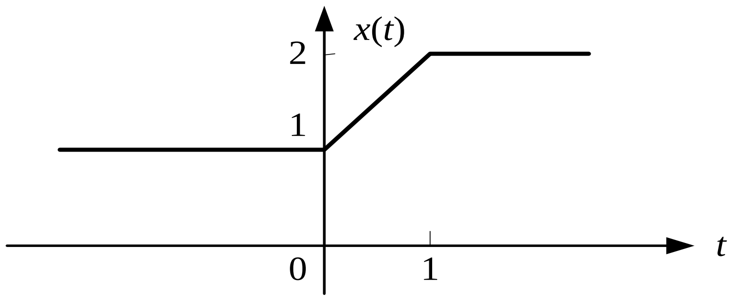
$$F(j\omega)=\pi F_1(j0)\delta(\omega)+\frac{F_1(j\omega)}{j\omega}\quad=\pi\delta(\omega)+\frac{Sa(\frac{\omega}{2})e^{-j\frac{\omega}{2}}}{j\omega}$$

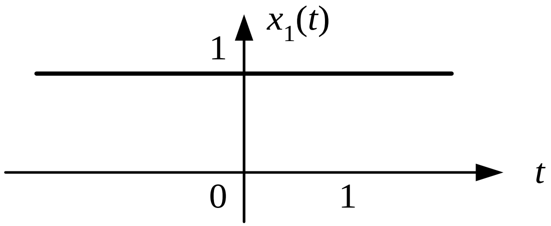
$$=\pi\delta(\omega)+\frac{1}{\omega}Sa(\frac{\omega}{2})e^{-j(\frac{\omega}{2}+\frac{\pi}{2})}$$

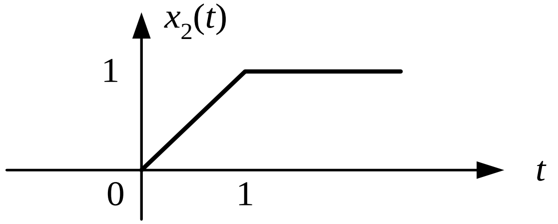
**其中** $F_1(j0)=1$

## 例4　试利用积分特性求图示信号$x(t)$的频谱函数。

解：将$x(t)$表示为$x_1(t)+x_2(t)$



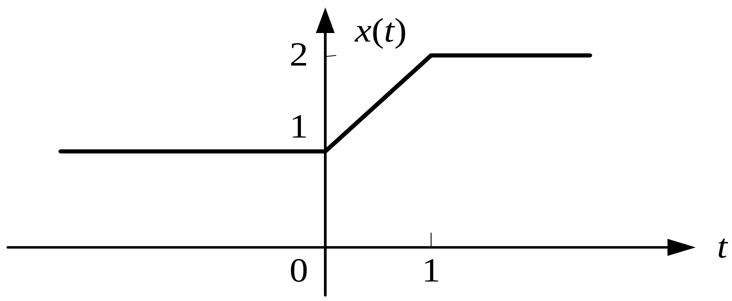



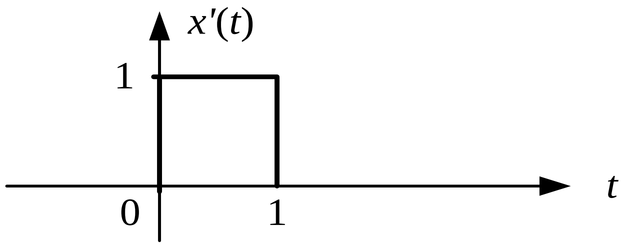

即

$$x(t)=1+x_2(t)$$

$$X(\mathrm{j}\omega)=\frac{1}{\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}(\frac{\omega}{2}+\frac{\pi}{2})}+3\pi\delta(\omega)$$

## 利用微积分特性：




$$x'(t) \longleftrightarrow \mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega}$$

**利用时域微分特性，可得**

$$X(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega} + \pi\delta(\omega)$$

$$\neq \frac{1}{\mathrm{j}\omega}\mathrm{Sa}(0.5\omega)\mathrm{e}^{-\mathrm{j}0.5\omega} + 3\pi\delta(\omega)$$

**信号的时域微分，使信号中的直流分量丢失。**

# 时域微分特性—修正的时域微分特性

记 $f'(t) = f_1(t)$

若 $f(t) \longleftrightarrow F(\mathrm{j}\omega)$ $\quad f_1(t) \longleftrightarrow F_1(\mathrm{j}\omega)$

则 $$F(\mathrm{j}\omega) = \pi[f(\infty) + f(-\infty)]\delta(\omega) + \frac{F_1(\mathrm{j}\omega)}{\mathrm{j}\omega}$$

再利用此性质推导上例。

## 九、频域的微分与积分

若 $f(t) \leftrightarrow F(j\omega)$ 则 $\boxed{t \cdot f(t) \leftrightarrow j\frac{dF(j\omega)}{d\omega}}$

$$\boxed{\pi f(0)\delta(t) + j\frac{f(t)}{t} \leftrightarrow \int_{-\infty}^{\omega} F(j\Omega)d\Omega}$$

## 十、卷积定理

### 1、时域卷积定理

若 $f_1(t) \leftrightarrow F_1(j\omega),\quad f_2(t) \leftrightarrow F_2(j\omega)$

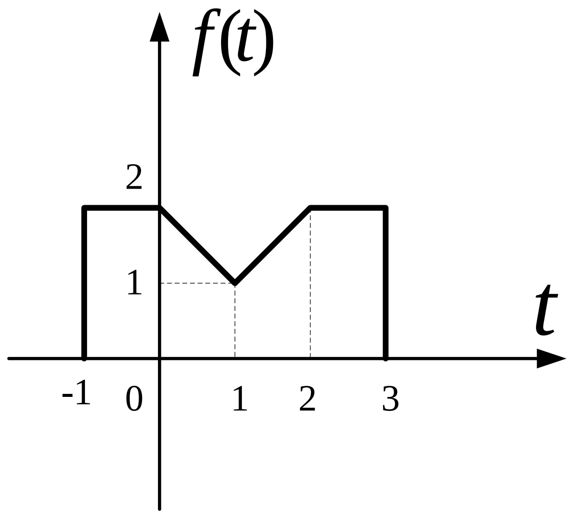
则 $\boxed{f_1(t) * f_2(t) \leftrightarrow F_1(j\omega) \cdot F_2(j\omega)}$

### 2、频域卷积定理

若 $f_1(t) \leftrightarrow F_1(j\omega),\quad f_2(t) \leftrightarrow F_2(j\omega)$

则 $\boxed{f_1(t) \cdot f_2(t) \leftrightarrow \frac{1}{2\pi} F_1(j\omega) * F_2(j\omega)}$

4　已知信号波形如图示，其傅立叶变换为 $F(j\omega)$ 计算下列各题：




（1）求 $\angle F(j\omega)$

（2）求 $F(j0)$

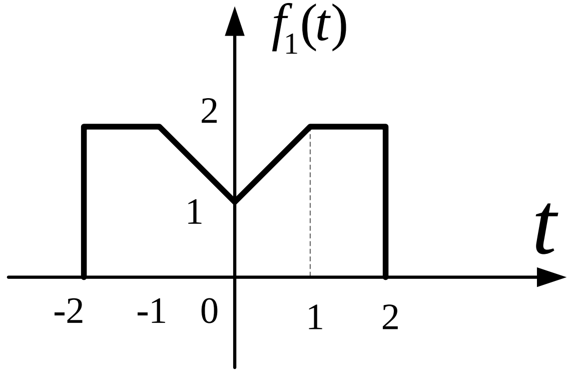
（3）求 $\int_{-\infty}^{\infty} F(j\omega)d\omega$

（4）求 $\int_{-\infty}^{\infty} F(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega$

（5）求 $\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$

（6）求 $\mathrm{Re}\{F(j\omega)\}$ 的傅立叶反变换。

解：　（1）构造 $f_1(t)$




$f_1(t) \leftrightarrow F_1(j\omega)$ 为实偶函数

$$f(t) = f_1(t-1)$$

$F(j\omega) = F_1(j\omega)e^{-j\omega}$　　$\therefore \angle F(j\omega) = \omega$

（2） $F(j0)=F(j\omega)\Big|_{\omega=0}=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\Big|_{\omega=0} \quad =\int_{-\infty}^{\infty} f(t)dt=7$

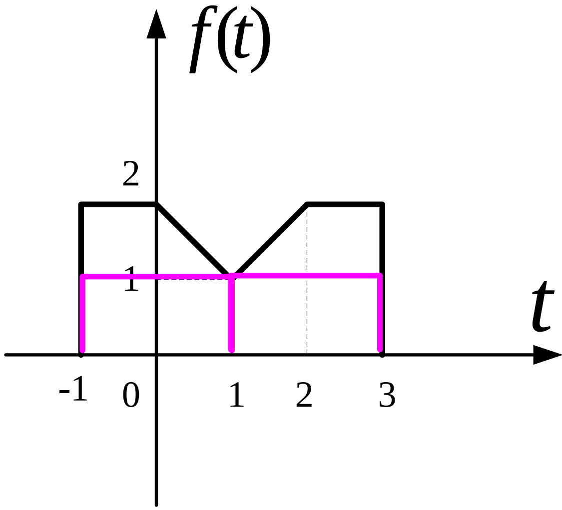
（3） $\int_{-\infty}^{\infty} F(j\omega)d\omega=2\pi[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega]_{t=0}=2\pi f(t)\Big|_{t=0}$

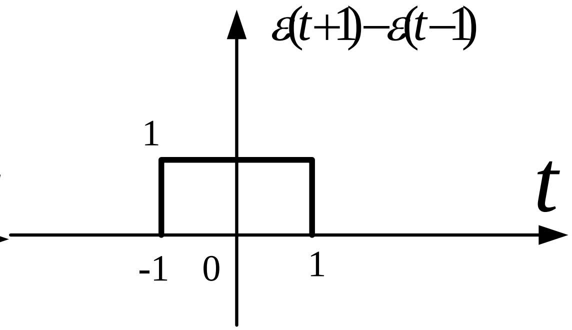
$$=2\pi f(0)=4\pi$$

（4） $\int_{-\infty}^{\infty} F(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega=2\pi[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{2\sin\omega}{\omega}e^{jt\omega}d\omega]_{t=2}$

$$=2\pi\mathcal{F}^{-1}\{F(j\omega)\frac{2\sin\omega}{\omega}\}\Big|_{t=2}=2\pi f(t)*\mathcal{F}^{-1}\{\frac{2\sin\omega}{\omega}\}\Big|_{t=2}$$

$$=2\pi f(t)*[\varepsilon(t+1)-\varepsilon(t-1)]\Big|_{t=2}$$

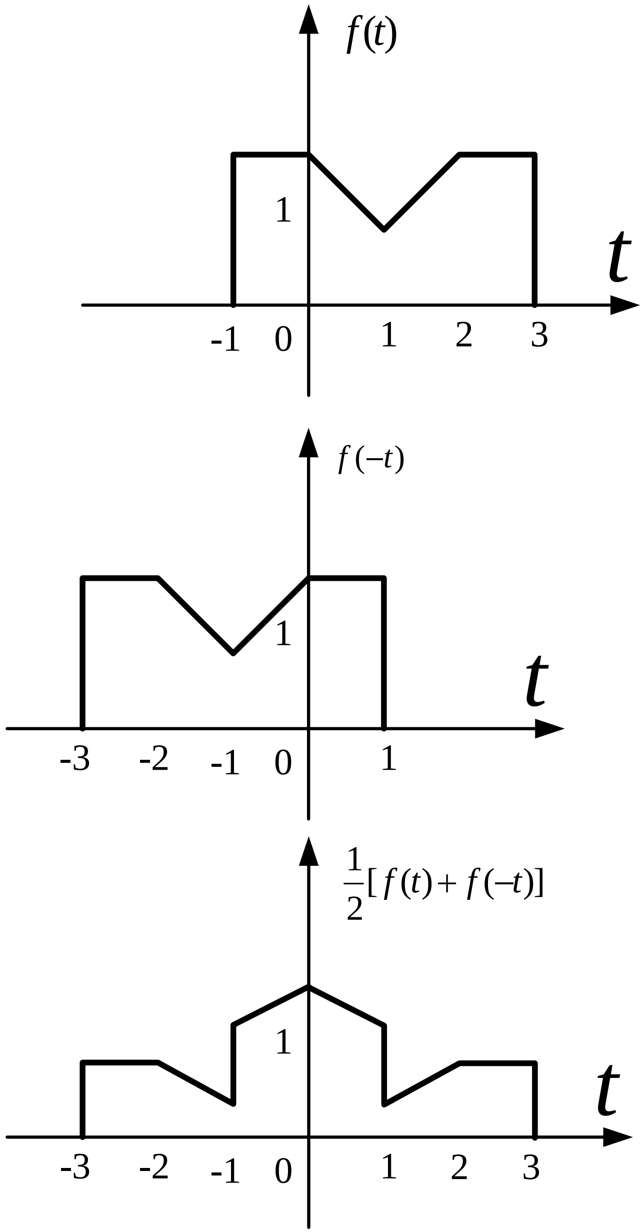



$$\therefore \quad \int_{-\infty}^{\infty} F(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega = 2\pi\cdot 3.5 = 7\pi$$

**（5）** $\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega = 2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty} f^2(t)dt = \frac{72}{3}\pi$

**（6）** $\mathcal{F}^{-1}\{\mathrm{Re}\{F(j\omega)\}\} = \frac{1}{2}[f(t)+f(-t)]$



$f(t)$
1
$t$
-1
0
1
2
3
$f(-t)$
1
$t$
-3
-2
-1
0
1
$\frac{1}{2}[f(t)+f(-t)]$
1
$t$
-3
-2
-1
0
1
2
3

例　若信号$f(t)$ 的带宽为W，求下列信号的带宽：

| | | | |
|---|---|---|---|
| $f(2t)$ | $f(t)+f(t/2)$ | $f(t)*f(2t)$ | $f(t)\cdot f(2t)$ |
| $2W$ | $W$ | $W$ | $3W$ |

## §3.8 帕塞瓦尔定理与能量频谱

(Parseval's theorem, energy frequency spectrum)

能量：信号在全部时间内耗于 1Ω 电阻中的总能量。

$$W=\int_{-\infty}^{\infty} f^2(t)dt$$

功率:信号在单位时间内耗于1Ω 电阻中的能量。

$$P=\overline{f^2(t)}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f^2(t)dt$$

周期信号是功率信号，其平均功率为

$$P=\overline{f^2(t)}=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt$$

# 一、帕塞瓦尔定理(Parseval's theorem)

**周期信号在完备正交函数集中的正交分解：**

$$f(t)=\sum_{i=1}^{\infty}C_i g_i(t)$$

**此时的均方误差=0，即**

$$\overline{e_\Delta^2(t)}=\frac{1}{T}\int_0^T e_\Delta(t)\cdot e_\Delta^*(t)dt$$

$$=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt-\sum_{i=1}^{n}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}[C_i g_i(t)]^2 dt=0$$

$$\overline{f^2(t)} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt = \sum_{i=1}^{\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}[C_i g_i(t)]^2 dt$$

**帕塞瓦尔(Parseval)功率守恒定理**

**说明：** **周期信号的方均值等于该周期信号在完备正交函数集合中各分量信号方均值的和。**

**对于周期信号的傅里叶分解，说明信号的平均功率等于各谐波分量信号功率之和，即**

$$\overline{f^2(t)} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt \quad = (\frac{a_0}{2})^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2 = \sum_{n=-\infty}^{\infty}(\frac{A_n}{2})^2 = \sum_{n=-\infty}^{\infty}|C_n|^2$$

$$\boxed{C_n = \frac{1}{2}\dot{A}_n}$$

## 二、能量谱(energy frequency spectrum)

$$W=\int_{-\infty}^{\infty} f^2(t)dt$$

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

$$\therefore\quad W=\int_{-\infty}^{\infty} f(t)\cdot[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega]dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)[\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\cdot F(-j\omega)d\omega$$

又因 $f(t)$ 是实信号　所以 $F(-j\omega)=F^{*}(j\omega)$

$$\therefore W=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\cdot F^{*}(j\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|^{2}d\omega$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\left|F(j\omega)\right|^{2}d\omega=\int_{0}^{\infty}G(\omega)d\omega$$

或 $$W=\int_{-\infty}^{\infty}f^{2}(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|^{2}d\omega=\int_{-\infty}^{\infty}\left|F(j2\pi f)\right|^{2}df$$

$$=\int_{-\infty}^{\infty}G(\omega)d\omega$$

上式是Parseval定理在非周期信号中的表示形式，也称雷利定理(Rayleigh’s Theorem)。表明信号在时域中求能量等于在频域中求能量。

定义 $G(\omega)=\dfrac{1}{\pi}\left|F(j\omega)\right|^2$ —— 焦耳 秒/弧度

为信号的能量频谱密度函数，简称能量频谱

或 $G_f(f)=2\left|F(j2\pi f)\right|^2=2\pi G(\omega)$ —— 焦耳/赫

则 $W=\int_0^\infty G(\omega)d\omega=\int_0^\infty G_f(f)df=\int_0^\infty \dfrac{1}{\pi}\left|F(j\omega)\right|^2 d\omega=\int_0^\infty 2\left|F(j2\pi f)\right|^2 df$

可以看出，信号的能量频谱仅仅与信号的幅度谱有关，与相位谱无关。

从能量的角度定义信号的有效占有频带宽度：

$$\frac{1}{\pi}\int_0^{B_s}\left|F(j\omega)\right|^2 d\omega=\eta\cdot\frac{1}{\pi}\int_0^\infty\left|F(j\omega)\right|^2 d\omega=\eta W$$

通常 $\eta=0.9$

# 本章小结

<u>基本概念</u>：正交函数集合、傅立叶级数、傅立叶变换、频谱、有效占有频带宽度(频宽、带宽)、 能量频谱密度函数。

<u>基本运算</u>：周期信号的傅立叶级数、周期信号的频谱及特点、非周期信号的傅立叶变换、非周期信号的频谱、傅立叶变换性质的应用、帕色伐尔定理。