第十二讲留数定理及其应用(二)

北京大学 物理学院 数学物理方法课程组

2007年春



讲授要点

- ❶ 留数定理计算定积分(续)
 - 含三角函数的无穷积分
 - 实轴上有奇点的情形
 - 多值函数的积分





References

► 吴崇试, 《数学物理方法》, §7.4 — 7.6

№ 梁昆淼,《数学物理方法》,§4.2,4.3

● 胡嗣柱、倪光炯、《数学物理方法》, §5.3, 5.4, 5.5



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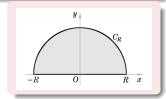


预备知识: Jordan引理

设在 $0 \le \arg z \le \pi$ 范围内, $\exists |z| \to \infty$ 时 $Q(z) \rightrightarrows 0$,则

$$\lim_{R \to \infty} \int_{C_R} Q(z) \mathrm{e}^{\mathrm{i}pz} \mathrm{d}z = 0$$

其中 $p > 0$, C_R 是以原点为圆
心、 R 为半径的上半圆



【证】当
$$z$$
在 C_R 上时, $z = Re^{i\theta}$

$$\left| \int_{C_R} Q(z) e^{ipz} dz \right| = \left| \int_0^{\pi} Q(Re^{i\theta}) e^{ipR(\cos\theta + i\sin\theta)} Re^{i\theta} id\theta \right|$$

$$\leq \int_0^{\pi} |Q(Re^{i\theta})| e^{-pR\sin\theta} Rd\theta$$

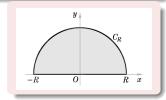


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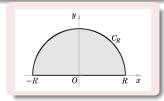


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【证】当
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$$\begin{split} \left| \int_{C_R} & Q(z) \mathrm{e}^{\mathrm{i} p z} \mathrm{d}z \right| = \left| \int_0^\pi & Q(R \mathrm{e}^{\mathrm{i} \theta}) \mathrm{e}^{\mathrm{i} p R(\cos \theta + \mathrm{i} \sin \theta)} R \mathrm{e}^{\mathrm{i} \theta} \mathrm{i} \mathrm{d}\theta \right| \\ & \leq \int_0^\pi \left| Q(R \mathrm{e}^{\mathrm{i} \theta}) \right| \mathrm{e}^{-pR \sin \theta} R \mathrm{d}\theta \end{split}$$



设在
$$0 \leq \arg z \leq \pi$$
范围内,当 $|z| \to \infty$ 时 $Q(z) \Rightarrow 0$,则 $\lim_{R \to \infty} \int_{C_R} Q(z) \mathrm{e}^{\mathrm{i} p z} \mathrm{d}z = 0$

$$C_R$$
 C_R
 R
 x

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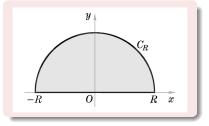
$$\left| \int_{C_R} Q(z) \mathrm{e}^{\mathrm{i} p z} \mathrm{d}z \right| \leq \int_0^\pi \left| Q(R \mathrm{e}^{\mathrm{i} \theta}) \right| \mathrm{e}^{-pR \sin \theta} R \mathrm{d}\theta$$

$$\leq \varepsilon R \int_0^{\pi} e^{-pR\sin\theta} d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-pR\sin\theta} d\theta$$

证明的关键在于精确估计 $\sin heta$ 值



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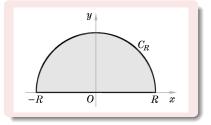
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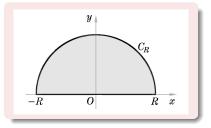
【证】当
$$z$$
在 C_R 上时, $z=R$ e $^{\mathrm{i} heta}$

$$\left| \int_{C_R} Q(z) e^{ipz} dz \right| \leq \int_0^{\pi} \left| Q(Re^{i\theta}) \right| e^{-pR\sin\theta} R d\theta$$
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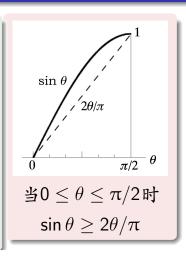
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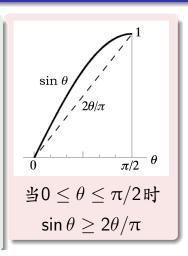
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 $\lim_{R o\infty}\int_{C_D}Q(z)\mathrm{e}^{\mathrm{i}pz}\mathrm{d}z=0$ \square



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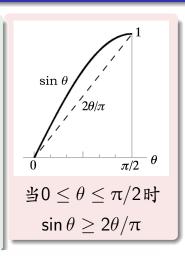








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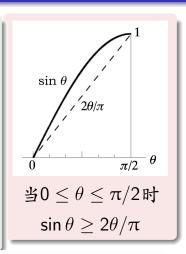


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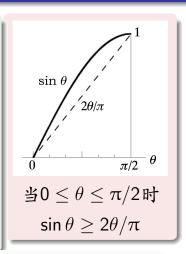








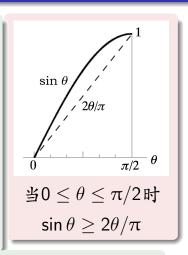
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$$\lim_{R \to \infty} \int_{C_R} Q(z) \mathrm{e}^{\mathrm{i} p z} \mathsf{d} z = 0 \quad \square$$



这类积分的标准形式

$$(不妨设 $p > 0)$$$

$$I = \int_{-\infty}^{\infty} f(x) \cos px dx$$
 或 $I = \int_{-\infty}^{\infty} f(x) \sin px dx$

处理这种类型的积分,仍可以采用半圆形的围道

原因: 不便于直接计算
$$\lim_{R\to\infty}\int_{C_R}f(z)\cos pz\mathrm{d}z \quad \mathrm{g} \quad \lim_{R\to\infty}\int_{C_R}f(z)\sin pz\mathrm{d}z$$

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- 正确的做法是将被积函数取为 $f(z)e^{ipz}$
- 如果函数 $f(z)e^{ipz}$ 在上半平面内只有有限个奇点,则

$$\oint_C f(z)e^{ipz}dz = \int_{-R}^R f(x)e^{ipx}dx + \int_{C_R} f(z)e^{ipz}dz$$

$$= 2\pi i \sum_{\substack{\pm \neq \pi \text{od}}} \text{res } \left\{ f(z)e^{ipz} \right\}$$

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$$I = \int_{-\infty}^{\infty} f(x) \cos px dx \quad \text{if} \quad I = \int_{-\infty}^{\infty} f(x) \sin px dx$$

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$$\int_{-\infty}^{\infty} f(x) \mathrm{e}^{\mathrm{i} p x} \mathrm{d} x = 2 \pi \mathrm{i} \sum_{\pm \neq \pi} \mathrm{res} \left\{ f(z) \mathrm{e}^{\mathrm{i} p z} \right\}$$

• 分别比较实部和虚部,就可以求得
$$\int_{-\infty}^{\infty} f(x) \cos px dx \quad \pi \quad \int_{-\infty}^{\infty} f(x) \sin px dx$$





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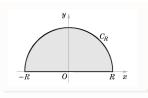
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$$\int_{-\infty}^{\infty} f(x) \cos px dx$$
 \Leftrightarrow $\int_{-\infty}^{\infty} f(x) \sin px dx$

例12.1 计算积分
$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$$
 $a > 0$

【解】考虑复变积分
$$\oint_C \frac{z}{z^2 + a^2} e^{iz} dz$$

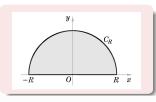


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= 2\pi i \operatorname{res} \left\{ \frac{z}{z^2 + a^2} e^{iz} \right\}_{z = ia} = 2\pi i \cdot \frac{1}{2} e^{-a} = \pi i e^{-a}$$



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 围道 C 如右图

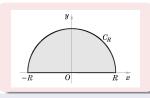


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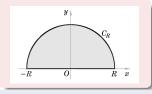
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围道C如右图

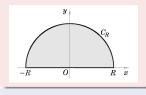
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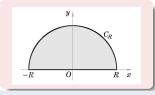
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取极限 $R \to \infty$

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围道C如右图

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= 2\pi i \operatorname{res} \left\{ \frac{z}{z^2 + a^2} e^{iz} \right\}_{z = ia} = 2\pi i \cdot \frac{1}{2} e^{-a} = \pi i e^{-a}$$

取极限 $R \to \infty$

例12.1 计算积分
$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$$
 $a > 0$

$$\int_{-\infty}^{\infty} \frac{x}{x^2 + a^2} \mathrm{e}^{\mathrm{i}x} \mathrm{d}x + \lim_{R \to \infty} \int_{C_R} \frac{z}{z^2 + a^2} \mathrm{e}^{\mathrm{i}z} \mathrm{d}z = \pi \mathrm{i} \, \mathrm{e}^{-a}$$

$$\lim_{z \to \infty} \frac{z}{z^2 + a^2} = 0$$

$$\lim_{R \to \infty} \int_{C_R} \frac{z}{z^2 + a^2} e^{iz} dz = 0 \qquad (Jordan \, \mathfrak{f}) \, \mathfrak{P}$$

$$\int_{-\infty}^{\infty} \frac{x}{x^2 + a^2} e^{iz} dx = \pi i e^{-a}$$



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$$\vdots \qquad \lim_{z \to \infty} \frac{z}{z^2 + a^2} = 0$$

$$\vdots \qquad \lim_{R \to \infty} \int_{C_R} \frac{z}{z^2 + a^2} e^{iz} dz = 0 \qquad \text{(Jordan 引 理)}$$

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$$\lim_{z\to\infty}\frac{z}{z^2+a^2}=0$$

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例12.1 计算积分 $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$ a > 0

分别比较实部和虚部,即得

$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + a^2} \mathrm{d}x = 0$$

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}$$

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$$

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讲授要点

- 留数定理计算定积分(续)
 - 含三角函数的无穷积分
 - 实轴上有奇点的情形
 - 多值函数的积分



瑕积分(设瑕点为c)的定义是

$$\int_a^b f(x) \mathrm{d}x = \lim_{\delta_1 \to 0} \int_a^{c-\delta_1} f(x) \mathrm{d}x + \lim_{\delta_2 \to 0} \int_{c+\delta_2}^b f(x) \mathrm{d}x$$

如果这两个极限单独都不存在, 但是

$$\lim_{\delta \to 0} \left[\int_{a}^{c-\delta} f(x) dx + \int_{c+\delta}^{b} f(x) dx \right]$$

存在,则称为瑕积分的主值存在,记为

v.p.
$$\int_{a}^{b} f(x) dx = \lim_{\delta \to 0} \left[\int_{a}^{c-\delta} f(x) dx + \int_{c+\delta}^{b} f(x) dx \right]$$

当然,如果瑕积分及其主值都存在,那么它们一

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当然,如果瑕积分及其主值都存在,那么它们-定相等

- •如果实变积分是一个瑕积分,在处理相应的复变积分 $\oint_C f(z) dz$ 时,实轴上的瑕点也是被积函数的奇点,必须绕开奇点而构成闭合的积分围道
- 由于实变积分的被积函数与复变积分的被积 函数形式不见得相同,不排除实变积分不是 瑕积分,但复变积分的被积函数在实轴上有 奇点
- 下面通过两个例子来具体说明处理这类积分的基本精神



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例12.2 计算积分 $\int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)}$

【分析】这是一个反常积分,反常性既表现在积分限为 $\pm \infty$, 又表现为被积函数在x=0点不连续(x=0点为瑕点)。此积分在主值意义下存在

$$\begin{aligned} & \text{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} \\ &= \lim_{R_1 \to \infty} \int_{-R_1}^{-1} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} + \lim_{R_2 \to \infty} \int_{1}^{R_2} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} \\ &\quad + \lim_{\delta \to 0} \left[\int_{-1}^{-\delta} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} + \int_{\delta}^{1} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} \right] \end{aligned}$$



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【分析】这是一个反常积分,反常性既表现在积分限为 $\pm \infty$, 又表现为被积函数在x=0点不连续(x=0点为瑕点). 此积分在主值意义下存在

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x(1+x+x^2)}$$

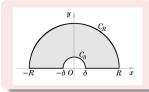
$$= \lim_{R_1 \to \infty} \int_{-R_1}^{-1} \frac{\mathrm{d}x}{x(1+x+x^2)} + \lim_{R_2 \to \infty} \int_{1}^{R_2} \frac{\mathrm{d}x}{x(1+x+x^2)}$$

$$+\lim_{\delta\to 0}\left[\int_{-1}^{-\delta}\frac{\mathrm{d}x}{x(1+x+x^2)}+\int_{\delta}^{1}\frac{\mathrm{d}x}{x(1+x+x^2)}\right]$$



例12.2 计算积分
$$\int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)}$$

【解】取积分路径C如右图,计算复变积分 $\oint_C \frac{dz}{z(1+z+z^2)}$

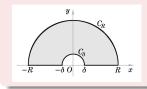


$$\oint_C \frac{dz}{z(1+z+z^2)} = \int_{-R}^{-\delta} \frac{dx}{x(1+x+x^2)} + \int_{C_{\delta}} \frac{dz}{z(1+z+z^2)} + \int_{\delta} \frac{dz}{z(1+z+z^2)} + \int_{\delta} \frac{dz}{z(1+z+z^2)} = 2\pi i \times \text{res } \frac{1}{z(1+z+z^2)} \Big|_{z=e^{i2\pi/3}} = -\frac{\pi}{\sqrt{3}} - \pi i$$

例12.2 计算积分
$$\int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)}$$

【解】取积分路径C如右图,计

算复变积分
$$\oint_C \frac{\mathrm{d}z}{z(1+z+z^2)}$$



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例12.2 计算积分
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【解】取积分路径C如右图,计

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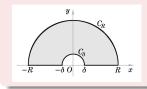
$$C_{R}$$
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$$\oint_{C} \frac{\mathrm{d}z}{z(1+z+z^{2})} = \int_{-R}^{-\delta} \frac{\mathrm{d}x}{x(1+x+x^{2})} + \int_{C_{\delta}} \frac{\mathrm{d}z}{z(1+z+z^{2})} + \int_{\delta} \frac{\mathrm{d}z}{z(1+z+z^{2})} + \int_{\delta} \frac{\mathrm{d}z}{x(1+x+x^{2})} + \int_{C_{R}} \frac{\mathrm{d}z}{z(1+z+z^{2})} = 2\pi i \times \text{res } \frac{1}{z(1+z+z^{2})} \Big|_{z=\mathrm{e}^{\mathrm{i}2\pi/3}} = -\frac{\pi}{\sqrt{3}} - \pi i$$

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$$\begin{split} & \int_{-R}^{-\delta} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} + \int_{\delta}^{R} \frac{\mathrm{d}x}{x(1\!+\!x\!+\!x^2)} \\ & + \int_{C_R} \frac{\mathrm{d}z}{z(1\!+\!z\!+\!z^2)} + \int_{C_{\delta}} \frac{\mathrm{d}z}{z(1\!+\!z\!+\!z^2)} = -\frac{\pi}{\sqrt{3}} - \pi \mathrm{i} \end{split}$$

$$\lim_{z\to\infty}z\cdot\frac{1}{z(1+z+z^2)}=0$$

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从上面的计算中可以看出,对于积分路径上有奇 点的情形,总要计算围绕奇点的小圆弧积分的极 限值

思考题

•如果积分围道中的小半圆弧是从下半平面绕 过z=0点,因而把z=0点包围在围道内,

是否会得到不同的结果? 为什么?

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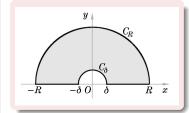
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例12.3 计算积分 $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

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计算复变积分 $\oint_C \frac{e^{iz}}{z} dz$



$$\oint_C \frac{e^{iz}}{z} dz = \int_{-R}^{-\sigma} \frac{e^{ix}}{x} dx + \int_{C_{\delta}} \frac{e^{iz}}{z} dz + \int_{\delta} \frac{e^{iz}}{x} dx + \int_{C_R} \frac{e^{iz}}{z} dz$$

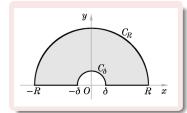
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(因为积分围道内无奇点)

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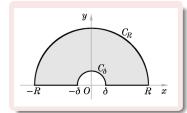
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(因为积分围道内无奇点)

$$\int_{-R}^{-\delta} \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x + \int_{\delta}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x + \int_{C_{R}} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z + \int_{C_{\delta}} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z = 0$$

$$\lim_{z \to \infty} \frac{1}{z} = 0$$

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0 \qquad \text{(Jordan引理)}$$

$$\lim_{z \to 0} z \cdot \frac{\mathsf{e}^{\mathsf{i}z}}{z} = 1$$

$$\lim_{\delta \to 0} \int_{C} \frac{e}{z} dz = -\pi i \qquad (小圆弧引理)$$



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取极限
$$R \to \infty, \delta \to 0$$

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取极限
$$R \to \infty, \delta \to 0$$

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分别比较等式两端的实部和虚部,即得

v.p.
$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0$$
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$



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分别比较等式两端的实部和虚部,即得 $f^{\infty} \cos x$

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$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0$$
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计算这些积分,关键在于正确地选择复变积分的被积函数

• 例如, 为了计算积分

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \mathrm{d}x$$

就应该考虑复变积分

$$\oint_C \frac{1 - e^{i2z}}{z^2} dz$$



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• 就复变积分而言, 在实轴上可以有奇点

- 但这种奇点,一般说来,只能是可去奇点或 一阶极点.这从小圆弧引理就可以看出
- •如果是二阶或二阶以上的极点,或是本性奇点,沿小圆弧 C_δ 的积分就可能趋于 ∞



评述

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讲授要点

- 留数定理计算定积分(续)
 - 含三角函数的无穷积分
 - 实轴上有奇点的情形
 - 多值函数的积分



基本约定

☞ 这里所说的多值函数的积分,是从复变函数的角度说的

避 从复数域来看,实变定积分中的积分变量x 在x > 0时应该理解为x = 0



含幂函数的积分

$$I = \int_0^\infty x^{s-1} Q(x) \mathrm{d}x$$

其中s为实数,Q(x)单值,在正实轴上没有奇点

为了保证积分收敛,要求

$$\lim_{x \to \infty} x \cdot x^{s-1} Q(x) = \lim_{x \to 0} x \cdot x^{s-1} Q(x) = 0$$

考虑相应的复变积分 $\oint_C z^{s-1}Q(z)dz$



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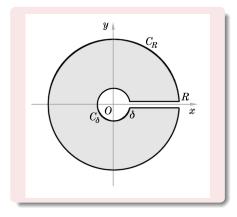
$$\lim_{x \to \infty} x \cdot x^{s-1} Q(x) = \lim_{x \to 0} x \cdot x^{s-1} Q(x) = 0$$

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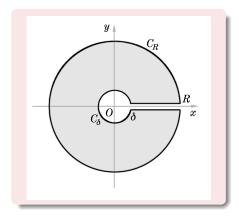
由于z = 0及 $z = \infty$ 是被积函数的枝点,所以需要将平面沿正实轴割开,并规定沿割线上岸 $\arg z = 0$

这时的积分路径由割开 的大小圆弧(半径分别为 $R和\delta$)及割线上下岸组成



沿割线上下岸的积分显然直接与所要计算的实变 积分有关 由于z = 0及 $z = \infty$ 是被积函数的枝点,所以需要将平面沿正实轴割开,并规定沿割线上岸 $\arg z = 0$

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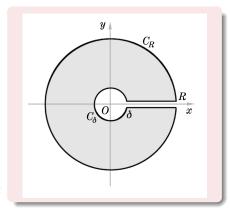
$$\oint_C z^{s-1}Q(z)dz$$

$$= \int_{\delta}^R x^{s-1}Q(x)dx$$

$$+ \int_{C_R} z^{s-1}Q(z)dz$$

$$+ \int_{R} (xe^{2\pi i})^{s-1}Q(x)dx$$

$$+ \int_{C} z^{s-1}Q(z)dz$$







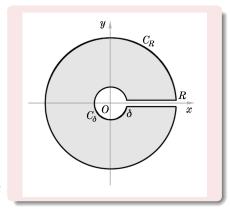
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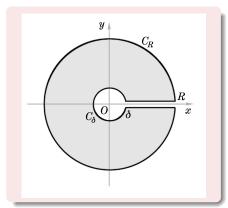
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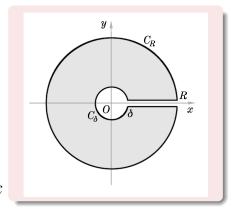
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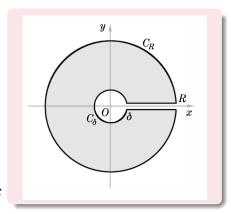
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$$\oint_C z^{s-1}Q(z)dz = \int_{\delta}^R x^{s-1}Q(x)dx + \int_R^{\delta} (xe^{2\pi i})^{s-1}Q(x)dx + \int_{C_R} z^{s-1}Q(z)dz + \int_{C_{\delta}} z^{s-1}Q(z)dz$$

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则根据大圆弧引理与小圆弧引理,有

$$\lim_{R\to\infty}\int_{C_R}z^{s-1}Q(z)\mathrm{d}z=0\quad\lim_{\delta\to0}\int_{C_\delta}z^{s-1}Q(z)\mathrm{d}z=0$$

②如果Q(z)在全平面上除了有限个孤立奇点(不 在正实轴上)外,是单值解析的,因而可以应 用留数定理

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②如果Q(2)在全平面上除了有限个孤立奇点(不 在正实轴上)外,是单值解析的,因而可以应 用留数定理

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在取极限 $\delta \to 0, R \to \infty$ 后, 就得到

$$\left(1 - \mathrm{e}^{2\pi \mathrm{i} s}\right) \int_0^\infty x^{s-1} Q(x) \mathrm{d} x = 2\pi \mathrm{i} \sum_{\text{$x = \mathfrak{A}}} \mathrm{res} \left\{ z^{s-1} Q(z) \right\}$$

$$\int_0^\infty x^{s-1}Q(x)\mathrm{d}x = \frac{2\pi\mathrm{i}}{1-\mathrm{e}^{2\pi\mathrm{i}s}}\sum_{\underline{x}\in\Phi}\mathrm{res}\left\{z^{s-1}Q(z)\right\}$$

需要注意,在计算留数时,要遵守上面对于多值函数 z^s 所作的限制,即 $0 \le \arg z \le 2\pi$

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思考题

• 如果规定在割线上岸 $\arg z = 2\pi$,是否影响最后结果?

•如果Q(x)具有一定的对称性质,例如是x的奇函数或偶函数,是否可以取其他形式的围道?



思考题

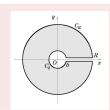
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例12.4 计算积分
$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx$$
, $0 < \alpha < 1$

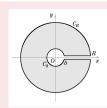
【解】计算复变积分 $\int_C \frac{z^{\alpha-1}}{1+z} \mathrm{d}z$,取积分路径C如右图,并规定割线上岸z=0



$$\oint_C \frac{z^{\alpha-1}}{1+z} dz = \int_{\delta}^R \frac{x^{\alpha-1}}{1+x} dx + \int_{C_R} \frac{z^{\alpha-1}}{1+z} dz + \int_{R} \frac{(xe^{2\pi i})^{\alpha-1}}{1+x} dx + \int_{C_{\delta}} \frac{z^{\alpha-1}}{1+z} dz$$

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取极限
$$\delta \to 0, R \to \infty$$
, 就得到

$$\begin{split} \left(1 - \mathrm{e}^{2\pi\mathrm{i}\alpha}\right) \int_0^\infty \frac{x^{\alpha-1}}{1+x} \mathrm{d}x &= 2\pi\mathrm{i} \times \mathrm{res} \left\{\frac{z^{\alpha-1}}{1+z}\right\}_{z=\mathrm{e}^{\mathrm{i}\pi}} \\ &= 2\pi\mathrm{i} \times \mathrm{e}^{\mathrm{i}\pi(\alpha-1)} \end{split}$$

$$\int_0^\infty \frac{x^{\alpha - 1}}{1 + x} dx = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \times e^{i\pi(\alpha - 1)}$$

$$=\frac{2\pi i}{e^{2\pi i\alpha}-1}\times e^{i\pi\alpha}$$

 $e^{\pi i \alpha} - e^{-\pi i \alpha}$





取极限
$$\delta \to 0, R \to \infty$$
, 就得到

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$$= 2\pi\mathrm{i} \times \mathrm{e}^{\mathrm{i}\pi(\alpha-1)}$$

$$\int_0^\infty \frac{x^{\alpha - 1}}{1 + x} dx = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \times e^{i\pi(\alpha - 1)}$$
$$= \frac{2\pi i}{e^{2\pi i \alpha} - 1} \times e^{i\pi \alpha}$$

$$= \frac{2711}{e^{\pi i \alpha} - e^{-\pi i \alpha}}$$





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$$\therefore \qquad \int_0^\infty \frac{x^{\alpha - 1}}{1 + x} \mathrm{d}x = \frac{2\pi \mathrm{i}}{1 - \mathrm{e}^{2\pi \mathrm{i}\alpha}} \times \mathrm{e}^{\mathrm{i}\pi(\alpha - 1)} = \frac{2\pi \mathrm{i}}{\mathrm{e}^{2\pi \mathrm{i}\alpha} - 1} \times \mathrm{e}^{\mathrm{i}\pi\alpha} = \frac{2\pi \mathrm{i}}{\mathrm{e}^{\pi \mathrm{i}\alpha} - \mathrm{e}^{-\pi \mathrm{i}\alpha}} = \frac{2\pi \mathrm{i}}{\mathrm{e}^{\pi \mathrm{i}\alpha} - \mathrm{e}^{-\pi \mathrm{i}\alpha}}$$

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$$\therefore \qquad \int_0^\infty \frac{x^{\alpha-1}}{1+x} \mathrm{d}x = \frac{2\pi\mathrm{i}}{1-\mathrm{e}^{2\pi\mathrm{i}\alpha}} \times \mathrm{e}^{\mathrm{i}\pi(\alpha-1)} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{2\pi\mathrm{i}\alpha}-1} \times \mathrm{e}^{\mathrm{i}\pi\alpha} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\pi\mathrm{i}\alpha}-\mathrm{e}^{-\pi\mathrm{i}\alpha}} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\pi\mathrm{i}\alpha}-\mathrm{e}^{-\pi\mathrm{i}\alpha}} = \frac{\pi}{\mathrm{sin}\,\pi\alpha}$$

C. S. Wu

含对数函数的积分

$$I = \int_0^\infty Q(x) \ln x \mathrm{d}x$$

其中Q(x)单值,在正实轴上没有奇点

为了保证积分收敛,要求

$$\lim_{x \to \infty} x \cdot Q(x) \ln x = \lim_{x \to 0} x \cdot Q(x) \ln x = 0$$

考虑相应的复变积分 $\oint_C Q(z) \ln z dz$?



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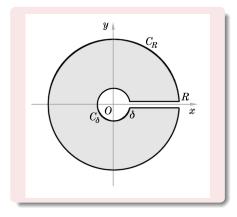
$$\lim_{x \to \infty} x \cdot Q(x) \ln x = \lim_{x \to 0} x \cdot Q(x) \ln x = 0$$

考虑相应的复变积分 $\oint_C Q(z) \ln z dz$?



由于z = 0及 $z = \infty$ 是被积函数的枝点,所以需要将平面沿正实轴割开,并规定沿割线上岸 $\arg z = 0$

积分路径仍由割开的大 小圆弧(半径分别为R和 δ)及割线上下岸组成

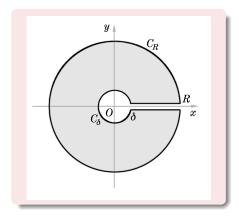


沿割线上下岸的积分显然直接与所要计算的实变 积分有关



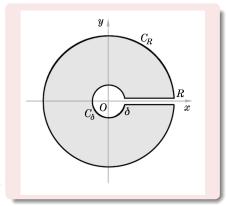
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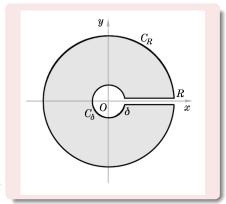
$$egin{aligned} \oint_C Q(z) \ln z \mathrm{d}z \ &= \int_\delta^R Q(x) \ln x \mathrm{d}x \ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z \ &+ \int_R^\delta Q(x) \ln \left(x \mathrm{e}^{2\pi \mathrm{i}}
ight) \mathrm{d}x \ &+ \int_C Q(z) \ln z \mathrm{d}z \end{aligned}$$







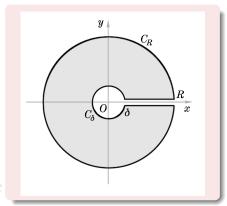
$$\begin{split} \oint_C Q(z) \ln z \mathrm{d}z \\ &= \int_{\delta}^R Q(x) \ln x \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z \\ &+ \int_{R}^{\delta} Q(x) \ln \left(x \mathrm{e}^{2\pi \mathrm{i}}\right) \mathrm{d}x \\ &+ \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \end{split}$$







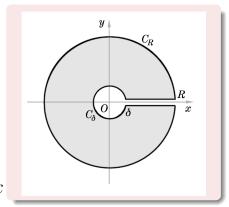
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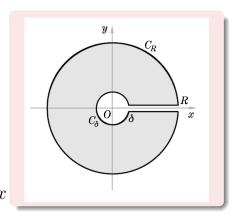
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$$\begin{split} \oint_C Q(z) \ln z \mathrm{d}z \\ &= \int_{\delta}^R Q(x) \ln x \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z \\ &+ \int_{R}^{\delta} Q(x) \ln \left(x \mathrm{e}^{2\pi \mathrm{i}}\right) \mathrm{d}x \\ &+ \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \end{split}$$







$$\begin{split} \oint_C Q(z) \ln z \mathrm{d}z = & \int_{\delta}^{R} Q(x) \ln x \mathrm{d}x + \int_{R}^{\delta} Q(x) [\ln x + 2\pi \mathrm{i}] \mathrm{d}x \\ & + \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \\ & = -2\pi \mathrm{i} \int_{\delta}^{R} Q(x) \mathrm{d}x \\ & + \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \end{split}$$





$$\begin{split} \oint_C Q(z) \ln z \mathrm{d}z &= \int_{\delta}^R Q(x) \ln x \mathrm{d}x + \int_R^{\delta} Q(x) [\ln x + 2\pi \mathrm{i}] \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \\ &= -2\pi \mathrm{i} \int_{\delta}^R Q(x) \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \end{split}$$



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• 计算失败! 尽管现在沿割线上下岸的积分都与所要计算的积分有关, 但是非常不巧,它们却相互抵消掉了,而只剩下一个并非所要计算的定积分 $\int_{\delta}^{R} Q(x) dx \rightarrow \int_{0}^{\infty} Q(x) dx$

$$\begin{split} \oint_C Q(z) \ln z \mathrm{d}z = & \int_{\delta}^{R} Q(x) \ln x \mathrm{d}x + \int_{R}^{\delta} Q(x) [\ln x + 2\pi \mathrm{i}] \mathrm{d}x \\ & + \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \\ & = -2\pi \mathrm{i} \int_{\delta}^{R} Q(x) \mathrm{d}x \\ & + \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \end{split}$$

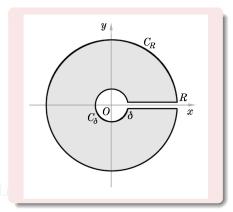
• 失败的原因是,和根式函数不同,对数函数ln z 的多值性表现在虚部上, 因此沿割线上下岸积分时,其实部(即ln x)互相抵消

$$\begin{split} \oint_C Q(z) \ln z \mathrm{d}z &= \int_{\delta}^{R} Q(x) \ln x \mathrm{d}x + \int_{R}^{\delta} Q(x) [\ln x + 2\pi \mathrm{i}] \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \\ &= -2\pi \mathrm{i} \int_{\delta}^{R} Q(x) \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln z \mathrm{d}z \end{split}$$

• 但也提示我们,如果要计算 $\int_0^\infty f(x) \ln x dx$,则可以考虑复变积分 $\oint_C f(z) \ln^2 z dz$



$$\begin{split} \oint_C Q(z) \ln^2 z \mathrm{d}z \\ &= \int_\delta^R Q(x) \ln^2 x \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln^2 z \mathrm{d}z \\ &+ \int_R^\delta Q(x) \ln^2 \left(x \mathrm{e}^{2\pi \mathrm{i}}\right) \mathrm{d}x \\ &+ \int_{C_\delta} Q(z) \ln^2 z \mathrm{d}z \end{split}$$







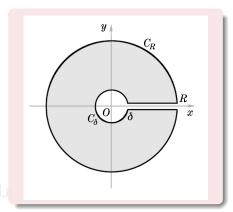
$$\oint_{C} Q(z) \ln^{2} z dz$$

$$= \int_{\delta}^{R} Q(x) \ln^{2} x dx$$

$$+ \int_{C_{R}} Q(z) \ln^{2} z dz$$

$$+ \int_{R}^{\delta} Q(x) \ln^{2} (x e^{2\pi i}) dx$$

$$+ \int_{C_{\delta}} Q(z) \ln^{2} z dz$$







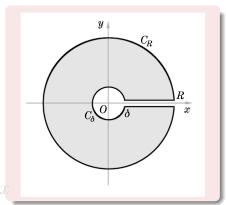
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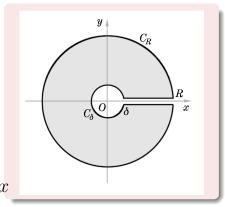
$$\oint_{C} Q(z) \ln^{2} z dz$$

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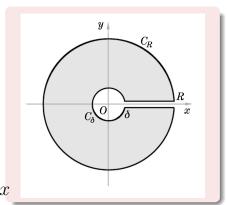
$$+ \int_{R} Q(z) \ln^{2} z dz$$







$$\begin{split} \oint_C Q(z) \ln^2 z \mathrm{d}z \\ &= \int_{\delta}^R Q(x) \ln^2 x \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln^2 z \mathrm{d}z \\ &+ \int_{R}^{\delta} Q(x) \ln^2 \left(x \mathrm{e}^{2\pi \mathrm{i}} \right) \mathrm{d}x \\ &+ \int_{C_{\delta}} Q(z) \ln^2 z \mathrm{d}z \end{split}$$







$$\begin{split} \oint_C Q(z) \ln^2 z \mathrm{d}z \\ &= \int_{\delta}^R Q(x) \ln^2 x \mathrm{d}x + \int_{R}^{\delta} Q(x) [\ln x + 2\pi \mathrm{i}]^2 \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln^2 z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln^2 z \mathrm{d}z \\ &= -4\pi \mathrm{i} \int_{\delta}^R Q(x) \ln x \mathrm{d}x + 4\pi^2 \int_{\delta}^R Q(x) \mathrm{d}x \\ &+ \int_{C_R} Q(z) \ln^2 z \mathrm{d}z + \int_{C_{\delta}} Q(z) \ln^2 z \mathrm{d}z \end{split}$$





① 如果在 $0 < \arg z < 2\pi$ 的范围内

$$\lim_{z \to \infty} z \cdot Q(z) \ln z = 0 \qquad \lim_{z \to 0} z \cdot Q(z) \ln z = 0$$

则根据大圆弧引理与小圆弧引理,有

$$\lim_{R o\infty}\int_{C_R}Q(z)\ln z\mathrm{d}z=0$$
 $\lim_{\delta o0}\int_{C_s}Q(z)\ln z\mathrm{d}z=0$

②如果Q(z)在全平面上除了有限个孤立奇点(不 在正实轴上)外,是单值解析的,因而可以应 用留数定理



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②如果Q(z)在全平面上除了有限个孤立奇点(不 在正实轴上)外,是单值解析的,因而可以应 用留数定理



取极限
$$\delta o 0,\,R o \infty$$
,就有
$$-4\pi \mathrm{i} \int_0^\infty Q(x) \ln x \mathrm{d}x + 4\pi^2 \int_0^\infty Q(x) \mathrm{d}x$$
$$= 2\pi \mathrm{i} \, \sum_{\underline{x} \in \Psi_0} \mathrm{res} \, \Big\{ Q(z) \ln z \Big\}$$

分别比较实部和虚部,即得

$$\int_0^\infty Q(x) \ln x dx = -\frac{1}{2} \operatorname{Re} \left\{ \sum_{\underline{x} = \underline{u}} \operatorname{res} \left[Q(z) \ln z \right] \right\}$$

$$\int_0^\infty Q(x)\mathrm{d}x = -\frac{1}{2\pi}\mathrm{Im}\left\{\sum_{\underline{x}\in\Phi}\mathrm{res}\left[Q(z)\ln z\right]\right\}$$

取极限
$$\delta o 0,\,R o \infty$$
,就有 $-4\pi \mathrm{i} \int_0^\infty Q(x) \ln x \mathrm{d}x + 4\pi^2 \int_0^\infty Q(x) \mathrm{d}x$ $=2\pi \mathrm{i} \sum_{\Delta \in \Phi} \mathrm{res} \left\{ Q(z) \ln z \right\}$

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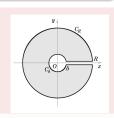
$$\int_0^\infty Q(x) \ln x dx = -\frac{1}{2} \operatorname{Re} \left\{ \sum_{\underline{x} \neq \underline{a}} \operatorname{res} \left[Q(z) \ln z \right] \right\}$$

$$\int_0^\infty Q(x)\mathrm{d}x = -\frac{1}{2\pi}\mathrm{Im}\left\{\sum_{\underline{x} \neq \underline{\phi}}\mathrm{res}\left[Q(z)\ln z\right]\right\}$$



例12.5 计算积分
$$\int_0^\infty \frac{\ln x}{1+x+x^2} dx$$

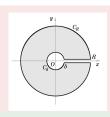
【解】取积分路径C如右图,并规定割线上岸 $\arg z=0$, 计算复变积分 $\oint_C \frac{\ln^2 z}{1+z+z^2} \mathrm{d}z$



$$\oint_{C} \frac{\ln^{2} z}{1+z+z^{2}} dz = \int_{\delta}^{R} \frac{\ln^{2} x}{1+x+x^{2}} dx + \int_{C_{R}} \frac{\ln^{2} z}{1+z+z^{2}} dz + \int_{R}^{\delta} \frac{(\ln x + 2\pi i)^{2}}{1+x+x^{2}} dx + \int_{C_{\delta}} \frac{\ln^{2} z}{1+z+z^{2}} dz$$

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$$\oint_{C} \frac{\ln^{2} z}{1+z+z^{2}} dz = \int_{\delta}^{R} \frac{\ln^{2} x}{1+x+x^{2}} dx + \int_{C_{R}} \frac{\ln^{2} z}{1+z+z^{2}} dz + \int_{R}^{\delta} \frac{(\ln x + 2\pi i)^{2}}{1+x+x^{2}} dx + \int_{C_{\delta}} \frac{\ln^{2} z}{1+z+z^{2}} dz$$

$$\lim_{z \to \infty} z \cdot \frac{\ln z}{1 + z + z^2} = 0 \qquad \lim_{z \to 0} z \cdot \frac{\ln z}{1 + z + z^2} = 0$$

$$\therefore \qquad \lim_{R \to \infty} \int_{C_R} \frac{\ln^2 z}{1 + z + z^2} dz = 0 \qquad (大圆弧引理)$$

$$\lim_{\delta \to 0} \int_{C_\delta} \frac{\ln^2 z}{1 + z + z^2} dz = 0 \qquad (小圆弧引理)$$

$$\oint_{C} \frac{\ln^{2} z}{1+z+z^{2}} dz = \int_{\delta}^{R} \frac{\ln^{2} x}{1+x+x^{2}} dx + \int_{C_{R}} \frac{\ln^{2} z}{1+z+z^{2}} dz + \int_{R}^{\delta} \frac{(\ln x + 2\pi i)^{2}}{1+x+x^{2}} dx + \int_{C_{\delta}} \frac{\ln^{2} z}{1+z+z^{2}} dz$$

$$\lim_{z \to \infty} z \cdot \frac{\ln^2 z}{1 + z + z^2} = 0 \qquad \lim_{z \to 0} z \cdot \frac{\ln^2 z}{1 + z + z^2} = 0$$

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$$\oint_{C} \frac{\ln^{2} z}{1+z+z^{2}} dz = \int_{\delta}^{R} \frac{\ln^{2} x}{1+x+x^{2}} dx + \int_{C_{R}} \frac{\ln^{2} z}{1+z+z^{2}} dz + \int_{R}^{\delta} \frac{(\ln x + 2\pi i)^{2}}{1+x+x^{2}} dx + \int_{C_{\delta}} \frac{\ln^{2} z}{1+z+z^{2}} dz$$

取极限
$$\delta \to 0, R \to \infty$$
, 就得到

$$- \, 4\pi \mathrm{i} \int_0^\infty \frac{\ln x}{1\!+\!x\!+\!x^2} \mathrm{d} x + 4\pi^2 \int_0^\infty \frac{1}{1\!+\!x\!+\!x^2} \mathrm{d} x$$

$$=2\pi\mathrm{i}\sum_{\mathbf{\hat{x}}\mathbf{\hat{x}}\mathbf{\hat{n}}}\mathrm{res}\,rac{\ln^2z}{1\!+\!z\!+\!z^2}=rac{8}{3\sqrt{3}}\pi^3$$

$$\int_0^\infty \frac{\ln x}{1 + x + x^2} \mathrm{d}x = 0$$

$$\int_0^\infty \frac{\mathrm{d}x}{1+x+x^2} = \frac{2\pi}{3\sqrt{3}}$$



取极限
$$\delta \to 0, R \to \infty$$
, 就得到

$$-4\pi i \int_0^\infty \frac{\ln x}{1+x+x^2} dx + 4\pi^2 \int_0^\infty \frac{1}{1+x+x^2} dx$$
$$= 2\pi i \sum_{n=1}^\infty \operatorname{res} \frac{\ln^2 z}{1+x+x^2} = \frac{8}{1+x+x^2} \pi^3$$

$$=2\pi\mathrm{i}\sum_{\text{$\underline{\pm}$+\underline{a}}}\operatorname{res}\frac{\ln^2z}{1\!+\!z\!+\!z^2}=\frac{8}{3\sqrt{3}}\pi^3$$

$$\int_0^\infty \frac{\ln x}{1+x+x^2} \mathrm{d}x = 1$$

同时还求得
$$\int_0^\infty \frac{\mathrm{d}x}{1+x+x^2} = \frac{2x}{3x}$$



取极限
$$\delta \to 0, R \to \infty$$
, 就得到

$$- 4\pi i \int_0^\infty \frac{\ln x}{1 + x + x^2} dx + 4\pi^2 \int_0^\infty \frac{1}{1 + x + x^2} dx$$

$$=2\pi\mathrm{i}\sum_{ extstyle 2+ ilde{\mathfrak{m}}}\mathrm{res}\,rac{\ln^2z}{1\!+\!z\!+\!z^2}=rac{8}{3\sqrt{3}}\pi^3$$

··.

$$\int_0^\infty \frac{\ln x}{1 + x + x^2} \mathrm{d}x = 0$$

同时还求得

$$\int_0^\infty \frac{\mathrm{d}x}{1+x+x^2} = \frac{2\pi}{3\sqrt{3}}$$



取极限
$$\delta \to 0, R \to \infty$$
, 就得到

$$-4\pi i \int_0^\infty \frac{\ln x}{1+x+x^2} dx + 4\pi^2 \int_0^\infty \frac{1}{1+x+x^2} dx$$
$$= 2\pi i \sum_{x \in \mathbb{F}_0} \operatorname{res} \frac{\ln^2 z}{1+z+z^2} = \frac{8}{3\sqrt{3}} \pi^3$$

$$\int_{0}^{\infty} \frac{\ln x}{1 + x + x^2} dx = 0$$

同时还求得
$$\int_0^\infty \frac{\mathrm{d}x}{1+x+x^2} = \frac{2\pi}{3\sqrt{3}}$$

