1. $$\begin{vmatrix}1+x & 1 & 1 & 1\\ 1 & 1-x & 1 & 1\\ 1 & 1 & 1+y & 1\\ 1 & 1 & 1 & 1-y\end{vmatrix} \overset{\text{都减第一行}}{=} \begin{vmatrix}1+x & 1 & 1 & 1\\ -x & -x & 0 & 0\\ -x & 0 & y & 0\\ -x & 0 & 0 & -y\end{vmatrix}$$

$$\overset{\text{做列变换}}{=} \begin{vmatrix}x & 1 & 1 & 1\\ 0 & -x & 0 & 0\\ 0 & 0 & y & 0\\ 0 & 0 & 0 & -y\end{vmatrix} = x^2y^2$$

2. $$\begin{vmatrix}a^2+\frac{1}{a^2} & a & \frac{1}{a} & 1\\ b^2+\frac{1}{b^2} & b & \frac{1}{b} & 1\\ c^2+\frac{1}{c^2} & c & \frac{1}{c} & 1\\ d^2+\frac{1}{d^2} & d & \frac{1}{d} & 1\end{vmatrix} = \begin{vmatrix}a^2 & a & \frac{1}{a} & 1\\ b^2 & b & \frac{1}{b} & 1\\ c^2 & c & \frac{1}{c} & 1\\ d^2 & d & \frac{1}{d} & 1\end{vmatrix} + \begin{vmatrix}\frac{1}{a^2} & a & \frac{1}{a} & 1\\ \frac{1}{b^2} & b & \frac{1}{b} & 1\\ \frac{1}{c^2} & c & \frac{1}{c} & 1\\ \frac{1}{d^2} & d & \frac{1}{d} & 1\end{vmatrix}$$

$$\overset{\text{往外提公因式}}{=} \frac{1}{abcd}\begin{vmatrix}a^3 & a^2 & 1 & a\\ b^3 & b^2 & 1 & b\\ c^3 & c^2 & 1 & c\\ d^3 & d^2 & 1 & d\end{vmatrix} + \frac{1}{(abcd)^2}\begin{vmatrix}1 & a^3 & a & a^2\\ 1 & b^3 & b & b^2\\ 1 & c^3 & c & c^2\\ 1 & d^3 & d & d^2\end{vmatrix}$$

$$\overset{\text{通过对调调整}}{=} -\frac{1}{abcd}\begin{vmatrix}1 & a & a^2 & a^3\\ 1 & b & b^2 & b^3\\ 1 & c & c^2 & c^3\\ 1 & d & d^2 & d^3\end{vmatrix} + \frac{1}{(abcd)^2}\begin{vmatrix}1 & a & a^2 & a^3\\ 1 & b & b^2 & b^3\\ 1 & c & c^2 & c^3\\ 1 & d & d^2 & d^3\end{vmatrix}$$

$$\overset{abcd=1}{=} 0$$

3． $\boldsymbol{B}=(a_1,a_2,a_3,\cdots,a_{n-1},a_n)\begin{pmatrix}1&0&0&\cdots&0&2\\2&1&0&\cdots&0&0\\0&2&1&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&1&0\\0&0&0&\cdots&2&1\end{pmatrix}$

$=\boldsymbol{A}\begin{pmatrix}1&0&0&\cdots&0&2\\2&1&0&\cdots&0&0\\0&2&1&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&1&0\\0&0&0&\cdots&2&1\end{pmatrix}$

$$|\boldsymbol{B}|=|\boldsymbol{A}|\begin{vmatrix}1&0&0&\cdots&0&2\\2&1&0&\cdots&0&0\\0&2&1&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&1&0\\0&0&0&\cdots&2&1\end{vmatrix}=1+(-1)^{n+1}2^n$$

4. $$\begin{vmatrix}a&b&b&\cdots&b&b\\0&a&b&\cdots&b&b\\0&0&a&\cdots&b&b\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&a&b\\b&0&0&\cdots&0&a\end{vmatrix}\overset{\text{按第1列展开}}{=}a^n+b\cdot(-1)^{n+1}\begin{vmatrix}b&b&b&\cdots&b&b\\a&b&b&\cdots&b&b\\0&a&b&\cdots&b&b\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&b&b\\0&0&0&\cdots&a&b\end{vmatrix}$$

$$\overset{\substack{\text{从第一行开始，}\\\text{每行减下一行}}}{=}a^n+b\cdot(-1)^{n+1}\begin{vmatrix}b-a&0&0&\cdots&0&0\\a&b-a&0&\cdots&0&0\\0&a&b-a&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&b-a&0\\0&0&0&\cdots&a&b\end{vmatrix}$$

$=a^n+(-1)^{n+1}b^2(b-a)^{n-2}$

$$5.\ D_n=\begin{vmatrix} k & 1 & 1 & \cdots & 1 \\ 2 & k & 1 & \cdots & 1 \\ 2 & 2 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & k \end{vmatrix}=\begin{vmatrix} 1+(k-1) & 1+0 & 1+0 & \cdots & 1+0 \\ 2 & k & 1 & \cdots & 1 \\ 2 & 2 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & k \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & k & 1 & \cdots & 1 \\ 2 & 2 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & k \end{vmatrix}+\begin{vmatrix} k-1 & 0 & 0 & \cdots & 0 \\ 2 & k & 1 & \cdots & 1 \\ 2 & 2 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & k \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & k-2 & -1 & \cdots & -1 \\ 0 & 0 & k-2 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k-2 \end{vmatrix}+(k-1)D_{n-1}$$

$$=(k-2)^{n-1}+(k-1)D_{n-1}$$

$$D_n\overset{\text{转置}}{=}\begin{vmatrix} k & 2 & 2 & \cdots & 2 \\ 1 & k & 2 & \cdots & 2 \\ 1 & 1 & k & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & k \end{vmatrix}=\begin{vmatrix} 2+(k-2) & 2+0 & 2+0 & \cdots & 2+0 \\ 1 & k & 2 & \cdots & 2 \\ 1 & 1 & k & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & k \end{vmatrix}$$

$$=\begin{vmatrix} 2 & 2 & 2 & \cdots & 2 \\ 1 & k & 2 & \cdots & 2 \\ 1 & 1 & k & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & k \end{vmatrix}+\begin{vmatrix} k-2 & 0 & 0 & \cdots & 0 \\ 1 & k & 2 & \cdots & 2 \\ 1 & 1 & k & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & k \end{vmatrix}$$

$$=\begin{vmatrix} 2 & 2 & 2 & \cdots & 2 \\ 0 & k-1 & 1 & \cdots & 1 \\ 0 & 0 & k-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k-1 \end{vmatrix}+(k-2)D_{n-1}$$

$$=2(k-1)^{n-1}+(k-2)D_{n-1}$$

$$\begin{cases} D_n=(k-2)^{n-1}+(k-1)D_{n-1} \\ D_n=2(k-1)^{n-1}+(k-2)D_{n-1} \end{cases}$$

消去 D_{n-1},得 $D_n=2(k-1)^n-(k-2)^n$

6. $\left|\boldsymbol{E}+\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}\right|=\begin{vmatrix}1+k_1 & k_2 & k_3 & k_4\\ 2k_1 & 1+2k_2 & 2k_3 & 2k_4\\ 3k_1 & 3k_2 & 1+3k_3 & 3k_4\\ 4k_1 & 4k_2 & 4k_3 & 1+4k_4\end{vmatrix}$

$\overset{\substack{r_2-2r_1\\ r_3-3r_1}}{\underset{r_4-4r_1}{=}}\begin{vmatrix}1+k_1 & k_2 & k_3 & k_4\\ -2 & 1 & 0 & 0\\ -3 & 0 & 1 & 0\\ -4 & 0 & 0 & 1\end{vmatrix}=1+k_1+2k_2+3k_3+4k_4$

7.证： $D_n=\begin{vmatrix}x & -1 & 0 & \cdots & 0 & 0\\ 0 & x & -1 & \cdots & 0 & 0\\ 0 & 0 & x & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & x & -1\\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1\end{vmatrix}$，用数学归纳法证明.

当 $n=2$ 时，$D_2=\begin{vmatrix}x & -1\\ a_2 & x+a_1\end{vmatrix}=x^2+a_1x+a_2$，结论成立.

假设结论对 $n-1$ 成立，下面对 n 的情况加以证明.

按第一列展开，得

$D_n=xD_{n-1}+a_n$

$\overset{\text{归纳法假设}}{=}x(x^{n-1}+a_1x^{n-2}+\cdots+a_{n-2}x+a_{n-1})+a_n$

$=x^n+a_1x^{n-1}+\cdots+a_{n-2}x^2+a_{n-1}x+a_n$

8. $$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & x \\ a_1^2 & a_2^2 & \cdots & a_n^2 & x^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} & x^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} & x^{n-1} \\ a_1^n & a_2^n & \cdots & a_n^n & x^n \end{vmatrix}$$

$$\overset{\text{按第}n+1\text{列展开}}{=} A_{1,n+1} + xA_{2,n+1} + x^2A_{3,n+1} + \cdots + x^{n-2}A_{n-1,n+1} + x^{n-1}A_{n,n+1} + x^nA_{n+1,n+1}$$

其中$A_{n,n+1} = (-1)^{n+(n+1)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix}$

可见算出开头那个行列式的值，找到x^{n-1}的项就可求出要求的行列式的值。

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & x \\ a_1^2 & a_2^2 & \cdots & a_n^2 & x^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} & x^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} & x^{n-1} \\ a_1^n & a_2^n & \cdots & a_n^n & x^n \end{vmatrix} = (x-a_1)(x-a_2)\cdots(x-a_n)\prod_{1\le i<j\le n}(a_j-a_i)$$

上式中x^{n-1}的项为$-(a_1+a_2+\cdots+a_n)x^{n-1}\prod_{1\le i<j\le n}(a_j-a_i)$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = (a_1+a_2+\cdots+a_n)\prod_{1\le i<j\le n}(a_j-a_i)$$

$$9.\begin{vmatrix} 2+x_1 & 2+x_1^2 & \cdots & 2+x_1^n \\ 2+x_2 & 2+x_2^2 & \cdots & 2+x_2^n \\ \vdots & \vdots & & \vdots \\ 2+x_n & 2+x_n^2 & \cdots & 2+x_n^n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2+x_1 & 2+x_1^2 & \cdots & 2+x_1^n \\ 2 & 2+x_2 & 2+x_2^2 & \cdots & 2+x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2+x_n & 2+x_n^2 & \cdots & 2+x_n^n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 2 & x_1 & x_1^2 & \cdots & x_1^n \\ 2 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = -2 \begin{vmatrix} -\frac{1}{2} & 1 & 1 & \cdots & 1 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1+(-\frac{3}{2}) & 1+0 & 1+0 & \cdots & 1+0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} - 2 \begin{vmatrix} -\frac{3}{2} & 0 & 0 & \cdots & 0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

$$= -2\prod_{i=1}^{n}(x_i - 1)\prod_{1\le i<j\le n}(x_j - x_i) + 3x_1x_2\cdots x_n \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix}$$

$$= -2\prod_{i=1}^{n}(x_i - 1)\prod_{1\le i<j\le n}(x_j - x_i) + 3x_1x_2\cdots x_n \prod_{1\le i<j\le n}(x_j - x_i)$$

$$=[3x_1x_2\cdots x_n - 2\prod_{i=1}^{n}(x_i - 1)]\prod_{1\le i<j\le n}(x_j - x_i)$$